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SUM OF HOLLOW MODULES

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The object of this paper is to study the relationship between the module which is a sum of hollow modules and the supplement of the module. Supplemented modules were first introduced by Y. Miyashita [4] and F. Kasch and E. Mares [3]. If a module M is projective, M is a supplemented module if and only if M is a semiperfect module. And it is a wellknown result that every semiperfect module is a direct sum of local (projective) modules. By P. Fleury [1], modules with finite spanning dimension are strongly supplemented and represented as a sum of hollow modules. So we have the natural question whether every supplemented module is a sum of hollow modules. In Theorem 8, we give equivalent conditions for a module to be represented as a sum of hollow modules, in the case where the Jacobson radical of the module is small. And we shall prove that the question is true for the module with small radical (Proposition 9).

1. Preliminaries

Throughout this paper, R will denote an associative ring with unit and all modules will be unital right R-modules. For an arbitrary module M, we shall denote by $J(M)$ the Jacobson radical of M.

Let A, B and X be submodules of a module M with $A+X=M$ and $B\subset X$. We call B is a *supplement* of A in X, or A has a supplement B in X, if the following two conditions are satisfied.

(i) $A+B=M$.

(ii) If $B'\subset B$ and $A+B'=M$ then $B'=B$.

If every submodule of M has a supplement in M, then M is called a *supplemented* module. And if for every pair A, X of submodules of M with $A+X=M$, A has a supplement in X, then M is called a *strongly supplemented* module.

A submodule K of M is said to be *small* in M if $K+X=M$ implies $X=M$ for any submodule X of M. If every proper submodule of M is small in M, we call M a *hollow* module [1]. We call M a *local* module if M has a unique maximal submodule N which contains every proper submodule of M (consequently $N=J(M)$). We notice that a local module is just the same with a cyclic hollow module.

We first begin with elementary properties for supplement and smallness.

Lemma 1 (Y. Miyashita [4]). *Let $M=A+B$. Then B is a supplement of A in M if and only if $A\cap B$ is small in B.*

Lemma 2. *Let $M\supset A\supset B\supset C$. Then A/C is small in M/C if and only if A/B is small in M/B and B/C is small in M/C.*

Proof. The "only if" part is clear. We show the "if" part. Suppose $(A/C)+(X/C)=M/C$ for some submodule X of M. Then $(A/B)+((X+B)/B)=M/B$. Since A/B is small in M/B, we have $X+B=M$. So $(B/C)+(X/C)=M/C$. And $X=M$. Therefore A/C is small in M/C.

2. Coclosed submodules

DEFINITION (J.S. Golan [2]). Let A be a submodule of M. Then A is called a *coclosed* submodule of M if A/B is not small in M/B for any proper submodule B of A.

The relationship between the coclosed submodule and the supplement is contained in the following proposition.

Proposition 3. *Let $M=A+B$ and B be a supplement of A in M. Then we have following properties.*

(1) *B is a coclosed submodule of M.*

(2) *For $C\subset A$, $M=C+B$ if and only if A/C is small in M/C.*

(3) *A is coclosed submodule of M if and only if A is also the supplement of B in M.*

Proof. Since (1) can be easily verified and (3) is the consequence of (2), we only show (2). Now

$$(A/C)+((B+C)/C) = (A+B)/C = M/C\,.$$

Hence, if A/C is small in M/C, then $B+C=M$. Conversely, let $M=C+B$ ($C\subset A$) and $(A/C)+(X/C)=M/C\,(C\subset X\subset M)$. Then,

$$A+(B\cap X) = A+C+(B\cap X) = A+[(C+B)\cap X] = A+X = M\,.$$

By the minimality of B, $B=B\cap X$. Hence $X\supset B$, and $X=M$ since $X\supset B+C=M$. Therefore A/C is small in M/C.

If M is a supplemented module, then, for every submodule A of M, A is coclosed in M if and only if A is a supplement of some submodule in M (cf. [2]).

Proposition 4. *Let N be a coclosed submodule of M. Then,*

$$J(N) = N \cap J(M)\,.$$

Proof. Clearly $J(N) \subset N \cap J(M)$. So we only show $J(N) \supset N \cap J(M)$. Suppose K be a submodule of N and small in M. Then, we shall prove that K is also small in N. If K is not small in N, there is a proper submodule X of N such that $N = K + X$. Since N/X is not small in M/X, we have $(N/X) + (Y/X) = M/X$ for some proper submodule Y of M. Then $K + X + Y = N + Y = M$. Hence $Y = M$, since K is small in M and X is a submodule of Y. We have a contradiction. So K is small in N. For $x \in N \cap J(M)$, $xR \subset N$ and xR is small in M. Hence we have xR is small in N, so x is in $J(N)$. Therefore, $J(N) \supset N \cap J(M)$.

Proposition 5. *Every coclosed submodule of a strongly supplemented module M is also strongly supplemented.*

Proof. Suppose N a coclosed submodule of M. Let $A + X = N$ for A, $X \subset N$. There is a supplement N' of N in M. Then, $M = N + N' = A + X + N'$. Since M is a strongly supplemented, we can take a supplement B of $(A + N')$ in X. Hence, $A + B + N' = M$ and $A + B \subset N$. While N and N' are supplement of each other by Proposition 3. So we get $A + B = N$. Since $A \cap B \subset (A + N') \cap B$, B is the supplement of A in X by Lemma 1. Therefore N is strongly supplemented.

Proposition 6. *Let L be a hollow module and $L \subset M$. Then, L is small in M, or coclosed in M.*

Proof. If L is not coclosed in M, there is a proper submodule K of L such that L/K is small in M/K. Then, since L is hollow, K is small in L and hence small in M. Therefore L is small in M by Lemma 2.

3. Sum of hollow modules

Lemma 7. *Let $M/J(M) \supset \sum_{i \in I} \oplus [(N_i + J(M))/J(M)]$ and each N_i is coclosed submodule of M for $i \in I$. Then,*

$$J(\sum_{i \in I} N_i) = (\sum_{i \in I} N_i) \cap J(M) = \sum_{i \in I} J(N_i)\,.$$

Proof. It is clear that $\sum J(N_i) \subset J(\sum N_i) \subset (\sum N_i) \cap J(M)$. For an arbitrary element $x \in (\sum N_i) \cap J(M)$, we can find finite subset $\{1, 2, \cdots, n\} \subset I$ and $x = x_1 + \cdots + x_n$ where $x_j \in N_j$ for $j = 1, 2, \cdots, n$. Then, for any $j = 1, 2, \cdots, n$,

$$\begin{aligned} x - x_j &= x_1 + \cdots + x_{j-1} + x_{j+1} + \cdots + x_n \\ &\in (N_j + J(M)) \cap (N_1 + \cdots + N_{j-1} + N_{j+1} + \cdots + N_n) \subset J(M)\,. \end{aligned}$$

Since $x \in J(M)$, $x_j \in J(M) \cap N_j = J(N_j)$ by Proposition 4. Hence,

$$x = x_1+\cdots+x_n \in J(N_1)+\cdots+J(N_n) \subset \sum_{i\in I} J(N_i)\,.$$

Theorem 8. *For a right R-module M the following statements are equivalent.*

(1) *M is a sum of hollow mldules and J(M) is small in M.*

(2) *Every maximal submodule of M has a supplement and every proper submodule of M is contained in some maximal submodule.*

(3) *Every submodule A of M, whose factor module M/A is finitely generated, has a supplement and every proper submodule of M is contained in some maximal submodule.*

(4) *M can be written as an irredundant sum* $M=\sum_{k\in K} L_k$ *where each* L_k *is local and J(M) is small in M.*

Proof. (3)⇨(2): Obvious.

(2)⇨(1): Let H be the sum of all hollow submodules of M. If H is a proper submodule of M, there exists a maximal submodule N of M with $H\subset N$. Let L be a supplement of N in M. Then, for any proper submodule X of L, X is contained in N, since N is a maximal submodule and $N+X$ is a proper submodule of M by minimality of L. Hence $X\subset N\cap L$ and X is small in L by Lemma 1. Thus L is a hollow module. Therefore L is contained in H, so $M=L+N\subset H+N=N$. This is a contradiction. Hence we have $H=M$. Now assume A is an arbitrary proper submodule of M. There exists a maximal submodule B of M with $A\subset B$. Then we have $A+J(M)\subset B+J(M)=B$, so $A+J(M)\neq M$. Hence, $J(M)$ is small in M.

(1)⇨(4): Let $M=\sum_{i\in I} L_i$ where each L_i is a hollow module. Then, $M/J(M)=\sum_i [(L_i+J(M))/J(M)]$ and each $(L_i+J(M))/J(M)\cong L_i/(L_i\cap J(M))$ is simple or zero. Hence, $M/J(M)=\sum_{k\in K}\oplus [(L_k+J(M))/J(M)]$ for some subset $K\subset I$. Therefore $M=\sum_{k\in K} L_k$ since $J(M)$ is small. And it is easily verified that the sum $\sum L_k$ is irredundant. Since L_k is not small, so L_k is coclosed in M by Proposition 6, and L_k is not contained in $J(M)$ for every $k\in K$. Hence $J(L_k)=L_k\cap J(M)\neq L_k$. Therefore L_k is local.

(4)⇨(3): Assume $M=\sum_{k\in K} L_k$ an irredundant sum of local modules and $J(M)$ is small in M. Then,

$$M/J(M) = \sum_{k\in K}\oplus\, [(L_k+J(M))/J(M)]\,.$$

Let A be a proper submodule of M. Since $A+J(M)$ is also proper submodule, there is a nonempty subset $K'\subset K$ such that

$$M/J(M) = [(A+J(M))/J(M)]\oplus\Big(\sum_{k'\in K'}[(L_{k'}+J(M))/J(M)]\Big)\,.$$

Take one element $k_0 \in K'$ and put $K'' = K' \setminus \{k_0\}$. Then, $A + (\sum_{k'' \in K''} L_{k''}) + J(M)$ is a proper submodule of M and the factor module $M/[A + (\sum_{k'' \in K''} L_{k''}) + J(M)]$ is a canonical homomorphic image of simple module $(L_{k_0} + J(M))/J(M)$. Thus $A + (\sum L_{k''}) + J(M)$ is a maximal submodule of M. Therefore every proper submodule is contained in some maximal submodule. Furthermore, we assume that M/A is finitely generated. Then K' is a finite set, say $K' = \{1, ?, \cdots, n\}$, and we have

$$M = A + L_1 + \cdots + L_n .$$

By Lemma 7,

$$A \cap (L_1 + \cdots + L_n) \subset J(M) \cap (L_1 + \cdots + L_n) = J(L_1) + \vdots + J(L_n) .$$

Since each $J(L_j)$ is small in L_j, $J(L_1) + \cdots + J(L_n)$ is small in $L_1 + \cdots + L_n$ and so is $A \cap (L_1 + \cdots + L_n)$. Therefore $L_1 + \cdots + L_n$ is a supplement of A in M by Lemma 1.

4. Supplemented modules

Proposition 9. *If M is a supplemented module and $J(M)$ is small in M, then M is written as a irredundant sum of local modules.*

Proof. We want to show that every proper submodule A is contained in some maximal submodule of M. There is a supplement B of A in M. Then $J(B) = B \cap J(M) \neq B$, since B is coclosed by Proposition 3 and $J(M)$ is small. Hence there is a maximal submodule C of B. Now we remark that $A + C \neq M$. And consider the canonical epimorphism $B/C \rightarrow (A+B)/(A+C) = M/(A+C)$. So $A + C$ is a maximal submodule of M.

If M is strongly supplemented, then there is a supplement M' of $J(M)$ in M. Then, by Lemma 1 and Proposition 4, $J(M') = M' \cap J(M)$ is small in M' and M' is also strongly supplemented by Proposition 5. Take a supplement K of M' in $J(M)$. Then $J(K) = K \cap J(M) = K$. So K is semihollow (see [5]) and coclosed. Hence we get the following proposition.

Proposition 10. *Every strongly supplemented module M is represented as $M = (\sum_{i \in I} L_i) + K$ where each L_i is local and K is semihollow nonlocal (if $K \neq 0$) and the sum is irredundant.*

Corollary 11. *The followings are equivalent for a module M.*

(1) *M is a finitely generated supplemented module.*

(2) *M is a finitely generated module and every maximal submodule of M has a supplement.*

(3) $M=L_1+L_2+\cdots+L_n$ *where each* L_i *is local.*

Proof. Since a local module is cyclic hollow and the Jacobson radical of finitely generated module is small, we have the corollary by Theorem 8.

References

[1] P. Fleury: *A note on dualizing goldie dimension*, Canad. Math. Bull. **17** (1974), 511–517.

[2] J.S. Golan: *Quasiperfect modules*, Quart. J. Math. Oxford (2) **22** (1971), 173–182.

[3] F. Kasch und E. Mares: *Eine Kennzeichnung Semi-perfekter Moduln*, Nagoya Math. J. **27** (1966), 525–529.

[4] Y. Miyashita: *Quasi-projective modules, perfect modules, and a theorem for modular lattices*, J. Fac. Sci. Hokkaido Univ. (I) **19** (1966), 86–110.

[5] K.M. Rangaswamy: *Modules with finite spanning dimension*, Canad. Math. Bull. **20** (1977), 255–262.

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