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CHARACTERISTIC INITIAL BOUNDARY VALUE PROBLEMS FOR SYMMETRIC HYPERBOLIC SYSTEMS

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1. Introduction

Let Ω be a bounded open subset of $\mathbf{R}^n$ with smooth boundary $\partial\Omega$. In $\mathbf{R}\times\Omega$ we consider a first order symmetric hyperbolic system:

$$Lu=\sum_{j=0}^{n}A_j(t,x)\partial_j u+B(t,x)u,$$
$$A_j(t,x),B(t,x)\in C^\infty(\mathbf{R}\times\overline{\Omega}),\quad A_j^*(t,x)=A_j(t,x)$$

with $\partial_0=\partial/\partial t$, $\partial_j=\partial/\partial x_j$, $j=1,\dots,n$ and $u=(u_1,\dots,u_N)$ where $A_0(t,x)$ is positive definite on $\mathbf{R}\times\Omega$. We assume that $A_j(t,x)$ and $B(t,x)$ are independent of t outside a compact subset of $\mathbf{R}\times\overline{\Omega}$. Recall that the boundary matrix is given by

$$A_b(t,x)=\sum_{j=1}^{n}\nu_j(x)A_j(t,x)\quad\text{for } (t,x)\in\mathbf{R}\times\partial\Omega$$

where $\nu(x)=(\nu_1(x),\dots,\nu_n(x))$ is the unit outward normal to Ω at $x\in\partial\Omega$. In this paper we study the initial boundary value problems for L assuming that the boundary space is maximal positive.

A general theory of initial boundary value problem for non singular A_b with maximal positive boundary space was developed by Friedrichs [2], Lax-Phillips [4], Rauch-Massey III [13] and so on. The case of the characteristic boundary has been studied by Lax-Phillips [4], Majda-Osher [6], Rauch [12] and so on. In particular, when dim KerA_b is constant on the boundary, in [11] we find a detailed study on the initial boundary value problem where the regularity was measured by conormal Sobolev spaces. In the characteristic case, one can not expect full regularity even if $f\in H^s(\Omega)$ (see [6], [17]). In [9], [14], in a similar situation, the initial boundary value problems were studied in usual Sobolev space setting aimed to study non linear perturbations. For a concrete problem of this type see [18] which motivated our study.

When dim KerA_b is not constant it is well known (see [5], [10]) that one does not in general get a well posed boundary value problem by merely taking maximal positive boundary conditions, while in [11] we can find some positive results. In [7] we proved the existence of regular solutions in the case that A_b is definite apart from an embedded $n-2$ dimensional submanifold of $\partial\Omega$ on which A_b vanishes under the same conditions assumed also in this paper. In [15] the same question is studied in a similar situation.

In this paper we continue studying the same problem when A_b is non singular outside a set, assumed to be an open set with smooth boundary on which A_b is definite.

Let us set

$$O^+(O^-) \coloneqq \{(t,x) \in \mathbf{R} \times \partial\Omega; A_b(t,x) \text{ is positive (negative) definite}\} \tag{1.1}$$

and denote by $\gamma^\pm$ the boundaries of $O^\pm$ in $\mathbf{R}\times\partial\Omega$. Letting $\gamma = \gamma^+ \cup \gamma^-$ we assume that γ is a smooth embedded $n-1$ dimensional submanifold of $\mathbf{R}\times\partial\Omega$, the boundary matrix $A_b(t,x)$ is non singular on $(\mathbf{R}\times\partial\Omega)\backslash\gamma$ and that $\mathrm{Ker}A_b(t,x)$ is a smooth vector bundle over γ.

The boundary condition takes the form:

$$u(t,x) \in M(t,x) \quad \text{for } (t,x) \in \mathbf{R}\times\partial\Omega$$

where $M(t,x)$ is a linear subspace of $\mathbf{C}^N$. We assume that the boundary space $M(t,x)$ is maximal positive in the sense that

$$\langle A_b(t,x)v, v\rangle \geq 0 \quad \text{for all } v \in M(t,x),$$

$$\dim M(t,x) = \#\{\text{non negative eigenvalues of } A_b(t,x) \text{ counting multiplicity}\}.$$

In particular, (1.1) implies that

$$M(t,x) = \begin{cases} \mathbf{C}^N & \text{on } O^+ \\ \{0\} & \text{on } O^-. \end{cases} \tag{1.2}$$

We also assume that $M(t,x)$ is smooth on each component of $(\mathbf{R}\times\partial\Omega)\setminus\gamma$ up to the boundary and independent of t outside a compact subset of $\mathbf{R}\times\partial\Omega$.

We study the following initial boundary value problem:

$$\begin{cases} Lu = f & \text{in } I\times\Omega \\ u \in M & \text{at } I\times\partial\Omega \\ u(0,\cdot) = u_0 & \text{on } \Omega \end{cases} \tag{IBVP}$$

where $I = (0,T)$. In what follows, we introduce the notation $\mathcal{O} = I\times\Omega$, $\Gamma = I\times\partial\Omega$, $\mathcal{R} = \mathbf{R}\times\Omega$ and $\Delta = \mathbf{R}\times\partial\Omega$.

We make our assumptions precise. Let $(\bar{t},\bar{x}) \in \gamma$ and we work in a neighborhood U of $(\bar{t},\bar{x})$. Let $\{v_1(t,x),\dots,v_p(t,x)\} \subset C^\infty(U)$ be a basis for $\mathrm{Ker}A_b(t,x)$ on $\gamma\cap U$

and set $V(t,x) = (v_1(t,x), \dots, v_p(t,x))$. Take $h(t,x) \in C^\infty(U)$ so that $\gamma \cap U = (\mathbf{R} \times \partial\Omega) \cap \{h(t,x) = 0\}$ where $dh(t,x)$ and $\nu(x)$ are linearly independent on $\gamma \cap U$. Since $(V^* A_b V)(t,x)$ vanishes on $\gamma \cap U$ we can factor out $h(t,x)$;

$$(V^* A_b V)(t,x) = h(t,x) A_{b,\gamma}(t,x) \quad \text{on } (\mathbf{R} \times \partial\Omega) \cap U. \tag{1.3}$$

Moreover we set

$$A_{\gamma/b}(t,x) = V^*(t,x) \left(\sum_{j=0}^{n} (\partial_j h) A_j \right) (t,x) V(t,x). \tag{1.4}$$

For more intrinsic definitions of $A_{b,\gamma}$ and $A_{\gamma/b}$, see [8]. Our assumption is stated as follows:

(1.5) $A_{b,\gamma}(t,x)$ and $A_{\gamma/b}(t,x)$ have the same definiteness on $\gamma \cap U$.

Clearly this condition does not depend on the choice of $v_j(t,x)$ and $h(t,x)$.

Under the conditions (1.5) we discuss the existence and regularity of solutions to (IBVP). We also study asymptotic behavior of solutions near γ.

2. Results for zero initial data

In what follows, if $u = u(t,x)$ is a function of t and x then we denote by $u(t)$ the function of x obtained by freezing t; $u(t)(x) = u(t,x)$.

We denote the formal adjoint of L by L^*:

$$L^* u = -\sum_{j=0}^{n} \partial_j A_j(t,x) u + B^*(t,x) u.$$

For u, $v \in C^{0,1}(\overline{\mathcal{O}})$, Green's identity yields

$$\begin{aligned}(Lu, v)_{L^2(\mathcal{O})} = (u, L^* v)_{L^2(\mathcal{O})} + \iint_\Gamma \langle A_b u, v \rangle dt d\sigma \\ + (A_0(T) u(T), v(T))_{L^2(\Omega)} - (A_0(0) u(0), v(0))_{L^2(\Omega)}.\end{aligned}$$

The adjoint boundary space $M^*(t,x)$ is defined by

$$M^*(t,x) = [A_b(t,x) M(t,x)]^\perp \quad \text{for } (t,x) \in \Delta.$$

In particular, (1.2) implies that

$$M^*(t,x) = \begin{cases} \{0\} & \text{on } O^+ \\ \mathbf{C}^N & \text{on } O^-. \end{cases} \tag{2.1}$$

We recall the following definition (see [1], [2]).

DEFINITION. For $f \in L^2(\mathcal{O})$ and $u_0 \in L^2(\Omega)$ we say $u \in L^2(\mathcal{O})$ is a weak solution to (IBVP) if and only if the identity

$$(u, L^*\psi)_{L^2(\mathcal{O})} = (f, \psi)_{L^2(\mathcal{O})} + (A_0(0)u_0, \psi(0))_{L^2(\Omega)}$$

holds for all $\psi \in C^{0,1}(\overline{\mathcal{O}})$ with $\psi \in M^*$ at Γ and $\psi(T) = 0$.

Take $r(x) \in C^\infty(\overline{\Omega})$ with $dr(x) \neq 0$ on $\partial\Omega$ so that $\Omega = \{r(x) > 0\}$ and $h_\pm(t,x) \in C^\infty(\overline{\mathcal{R}})$ such that $O^\pm = \Delta \cap \{h_\pm(t,x) > 0\}$ where $dh_\pm(t,x)$ and $\nu(x)$ are linearly independent on $\gamma^\pm$. Similarly, take $h(t,x) \in C^\infty(\overline{\mathcal{R}})$ such that $\gamma = \Delta \cap \{h(t,x) = 0\}$ where $dh(t,x)$ and $\nu(x)$ are linearly independent on γ. We assume that $h_\pm(t,x)$ and $h(t,x)$ are independent of t outside a compact subset of $\overline{\mathcal{R}}$. Let us set

$$m_\pm(t,x) = \{r(x)^2 + h_\pm(t,x)^2\}^{1/2}, \quad m(t,x) = \{r(x)^2 + h(t,x)^2\}^{1/2},$$
$$\phi_\pm(t,x) = m_\pm(t,x) - h_\pm(t,x).$$

Note that $\phi_\pm(t,x) > 0$ if $(t,x) \in \overline{\mathcal{R}} \setminus (O^\pm \cup \gamma^\pm)$ and that $\phi_\pm(t,x) = 0$ if $(t,x) \in O^\pm \cup \gamma^\pm$.

We first get the following two propositions.

Proposition 2.1. *If $f \in \phi_-^\tau L^2(\mathcal{O})$ and $u_0 \in \phi_-^\tau(0)L^2(\Omega)$ for some $\tau \geq 1$ then there exists a weak solution $u \in \phi_-^\tau L^2(\mathcal{O})$ to* (IBVP) *satisfying*

$$||\phi_-^{-\tau}u||^2_{L^2(\mathcal{O})} \leq C\{||\phi_-^{-\tau}f||^2_{L^2(\mathcal{O})} + ||\phi_-^{-\tau}(0)u_0||^2_{L^2(\Omega)}\}$$

where $C = C(\tau) > 0$ is independent of f, u_0 and u.

Proposition 2.2. *If $f \in L^2(\mathcal{O})$ and $u_0 \in L^2(\Omega)$ then a weak solution $u \in m_-L^2(\mathcal{O})$ to* (IBVP) *is unique.*

An immediate corollary to Proposition 2.1 and Proposition 2.2 is

Corollary 2.3. *If $f \in \phi_-^\tau L^2(\mathcal{O})$ and $u_0 \in \phi_-^\tau(0)L^2(\Omega)$ for some $\tau \geq 1$ and if $u \in m_-L^2(\mathcal{O})$ is a weak solution to* (IBVP) *then we have $u \in \phi_-^\tau L^2(\mathcal{O})$ and it follows that*

$$||\phi_-^{-\tau}u||^2_{L^2(\mathcal{O})} \leq C\{||\phi_-^{-\tau}f||^2_{L^2(\mathcal{O})} + ||\phi_-^{-\tau}(0)u_0||^2_{L^2(\Omega)}\}$$

where $C = C(\tau) > 0$ is independent of f, u_0 and u.

Our main concern is the regularity of solutions u to (IBVP). Hence we introduce the following spaces: For $q \in \mathbf{Z}_+$ and σ, $\tau \in \mathbf{R}$ we set

(2.2) $$X^q_{(\sigma,\tau)}(\mathcal{O}) = \bigcap_{j=0}^{q} \phi_+^{\sigma+q-j}\phi_-^{\tau+q-j}H^j(\mathcal{O}),$$

(2.3) $$X^q_{0(\sigma,\tau)}(\Omega) = \bigcap_{j=0}^{q} (\phi_+^{\sigma+q-j}\phi_-^{\tau+q-j})(0)H^j(\Omega)$$

where $H^j(\mathcal{O})$ and $H^j(\Omega)$ are usual Sobolev spaces of order j. We define $X^q_{(\sigma,\tau)}(\mathcal{O};\Gamma)$ by (2.2) with $H^j(\mathcal{O};\Gamma)$, the conormal Sobolev space of order j with respect to Γ, instead of $H^j(\mathcal{O})$. The space $X^q_{0(\sigma,\tau)}(\Omega;\partial\Omega)$ is defined similarly (see also [8]). Note that if $f \in X^q_{(\sigma,\tau)}(\mathcal{O})$ (resp. $X^q_{(\sigma,\tau)}(\mathcal{O};\Gamma)$) for some $q \in \mathbf{Z}_+$, $q \geq 1$ and σ, $\tau \in \mathbf{R}$ then $(\partial_0^k f)(0) \in X^{q-1-k}_{0(\sigma,\tau)}(\Omega)$ (resp. $X^{q-1-k}_{0(\sigma,\tau)}(\Omega;\partial\Omega)$) for $k = 0,\ldots,q-1$.

We can now obtain regular solutions to (IBVP) with zero initial data (results for the general case is described in Theorem 5.4 and Theorem 5.5 in Section 5).

Theorem 2.4. *For $q \in \mathbf{Z}_+$, $q \geq 1$ there is a $\Sigma(q) > 0$ such that if $f \in X^q_{(-\sigma,\tau)}(\mathcal{O};\Gamma) \cap \phi_- L^2(\mathcal{O})$, for some σ, $\tau > \Sigma(q)$, satisfies $(\partial_0^k f)(0) = 0$ for $k = 0,\ldots,q-1$ then there exists a weak solution $u \in X^q_{(-\sigma,\tau)}(\mathcal{O};\Gamma) \cap \phi_- L^2(\mathcal{O})$ to* (IBVP) *with zero initial data which satisfies*

$$\|u\|^2_{X^q_{(-\sigma,\tau)}(\mathcal{O};\Gamma)} + \|\phi_-^{-1}u\|^2_{L^2(\mathcal{O})} \leq C\{\|f\|^2_{X^q_{(-\sigma,\tau)}(\mathcal{O};\Gamma)} + \|\phi_-^{-1}f\|^2_{L^2(\mathcal{O})}\}$$

where $C = C(q,\sigma,\tau) > 0$ is independent of f and u.

We can get a rough estimate of asymptotic behavior of weak solutions near γ.

Theorem 2.5. *For $q \in \mathbf{Z}_+$ there is a $\Sigma(q) > 0$ such that if $f \in X^{q+[n/2]+1}_{(-\sigma,\tau)}(\mathcal{O}) \cap \phi_- L^2(\mathcal{O})$, for some σ, $\tau > \Sigma(q)$, satisfies $(\partial_0^k f)(0) = 0$ for $k = 0,\ldots,q+[n/2]$ and if $u \in m_- L^2(\mathcal{O})$ is a weak solution to* (IBVP) *with zero initial data then we have $u \in m^{-(q+[n/2]+1)}\phi_+^{-\sigma}\phi_-^{\tau}C^q(\overline{\mathcal{O}})$.*

3. Existence and uniqueness of solutions to (IBVP)

Let us set

$$m_\pm(t,x;\kappa,\mu) = \{\kappa r(x)^2 + (\mu r(x) - h_\pm(t,x))^2\}^{1/2},$$
$$\phi_\pm(t,x;\kappa,\mu) = m_\pm(t,x;\kappa,\mu) + \mu r(x) - h_\pm(t,x)$$

for $\kappa > 0$ and $\mu \in \mathbf{R}$. Then we can choose a $C = C(\kappa,\mu) > 0$ satisfying

$$C^{-1}m_\pm(t,x) \leq m_\pm(t,x;\kappa,\mu) \leq Cm_\pm(t,x),$$

$$C^{-1}\phi_\pm(t,x) \le \phi_\pm(t,x;\kappa,\mu) \le C\phi_\pm(t,x) \quad \text{for } (t,x) \in \mathcal{R}.$$

Thus it suffices to prove the results in Section 2 with $m_\pm(t,x;\kappa,\mu)$ and $\phi_\pm(t,x;\kappa,\mu)$ instead of $m_\pm(t,x)$ and $\phi_\pm(t,x)$. In what follows, we simply write $m_\pm(t,x)$ and $\phi_\pm(t,x)$ for $m_\pm(t,x;\kappa,\mu)$ and $\phi_\pm(t,x;\kappa,\mu)$ respectively.

We denote by $\|\cdot\|_{\mathcal{O}}$, $\|\cdot\|_{\mathcal{R}}$ and $\|\cdot\|_\Omega$ the norm in $L^2(\mathcal{O})$, $L^2(\mathcal{R})$ and in $L^2(\Omega)$ respectively. The following a priori estimate is obtained by much the same way as in [8]. (for details see Lemma 5.4 in [8]).

Lemma 3.1. *There are c_0, $C_1 > 0$ such that for $\tau \ge 0$ we can take a $\Lambda(\tau) \in$ $\mathbf{R}$ having the following properties: If $\mathrm{Re}\lambda > \Lambda(\tau)$, $-\infty < T_1 < T_2 < \infty$ and if $u \in C_0^{0,1}(\overline{\mathcal{R}})$ with $u \in M^*$ at $(T_1,T_2)\times\partial\Omega$ then it follows that*

$$\begin{aligned}(\mathrm{Re}\lambda - \Lambda(\tau))\|u\|^2_{(T_1,T_2)\times\Omega} + c_0\|u(T_1)\|^2_\Omega \\ \le C_1\{\|\phi_-^\tau(L^* + \overline{\lambda}A_0)\phi_-^{-\tau}u\|^2_{(T_1,T_2)\times\Omega} + \|u(T_2)\|^2_\Omega\}.\end{aligned}$$

Applying this we can prove Proposition 2.1.

Proof of Proposition 2.1. Let us set

$$E = \{\phi_-^\tau(L^* + \overline{\lambda}A_0)e^{\overline{\lambda}t}\psi;\ \psi \in C^{0,1}(\overline{\mathcal{O}}) \text{ with } \psi \in M^* \text{ at } \Gamma \text{ and } \psi(T) = 0\}$$

and we study the map

$$T : E \ni \phi_-^\tau(L^* + \overline{\lambda}A_0)e^{\overline{\lambda}t}\psi \mapsto (f,\psi)_{\mathcal{O}} + (A_0(0)u_0, \psi(0))_\Omega \in \mathbf{C}.$$

From Lemma 3.1 with $u = e^{\overline{\lambda}t}\phi_-^\tau\psi$ and $T_1 = 0$, $T_2 = T$ we obtain that

$$\begin{aligned}&|T\phi_-^\tau(L^* + \overline{\lambda}A_0)e^{\overline{\lambda}t}\psi|^2 \\ &\quad\le C\{\|e^{\overline{\lambda}t}\phi_-^\tau\psi\|^2_{\mathcal{O}}\|e^{-\overline{\lambda}t}\phi_-^{-\tau}f\|^2_{\mathcal{O}} + \|(\phi_-^\tau\psi)(0)\|^2_\Omega\|\phi_-^{-\tau}(0)u_0\|^2_\Omega\} \\ &\quad\le C'(\lambda)\|\phi_-^\tau(L^* + \overline{\lambda}A_0)e^{\overline{\lambda}t}\psi\|^2_{\mathcal{O}}\{\|e^{-\overline{\lambda}t}\phi_-^{-\tau}f\|^2_{\mathcal{O}} + \|\phi_-^{-\tau}(0)u_0\|^2_\Omega\}.\end{aligned}$$

By Hahn-Banach theorem there is a $w \in L^2(\mathcal{O})$ such that

$$\begin{aligned}&\|w\|^2_{\mathcal{O}} \le C(\lambda)\{\|e^{-\overline{\lambda}t}\phi_-^{-\tau}f\|^2_{\mathcal{O}} + \|\phi_-^{-\tau}(0)u_0\|^2_\Omega\}, \\ &(w, \phi_-^\tau(L^* + \overline{\lambda}A_0)e^{\overline{\lambda}t}\psi)_{\mathcal{O}} = (f,\psi)_{\mathcal{O}} + (A_0(0)u_0, \psi(0))_\Omega\end{aligned}$$

for every $\psi \in C^{0,1}(\overline{\mathcal{O}})$ with $\psi \in M^*$ at Γ and $\psi(T) = 0$. Then $u = e^{\lambda t}\phi_-^\tau w$ is a desired weak solution. □

For the proof of Proposition 2.2 and Theorem 2.4 we study the following boundary value problem:

$$\text{(BVP)} \qquad \begin{cases} (L+\lambda A_0)u = f & \text{in } \mathbf{R}\times\Omega = \mathcal{R} \\ u \in M & \text{at } \mathbf{R}\times\partial\Omega = \Delta \end{cases}$$

where $\lambda \in \mathbf{C}$ is a parameter.

DEFINITION. For $f \in L^2(\mathcal{R})$ we say $u \in L^2(\mathcal{R})$ is a weak solution to (BVP) if and only if the identity

$$(u, (L^* + \overline{\lambda}A_0)\psi)_{L^2(\mathcal{R})} = (f, \psi)_{L^2(\mathcal{R})}$$

holds for all $\psi \in C_0^{0,1}(\overline{\mathcal{R}})$ with $\psi \in M^*$ at Δ.

We now set $\phi_{\pm,\eta}(t,x) = \phi_\pm(t,x) - \eta$ and $\mathcal{R}_{\pm,\eta} = \mathcal{R} \cap \{\phi_{\pm,\eta} > 0\}$ for $\eta \geq 0$. The following proposition is a key result to prove Proposition 2.2 and Theorem 2.4.

Proposition 3.2. *There is a* $\Lambda \in \mathbf{R}$ *such that if* $\mathrm{Re}\lambda > \Lambda$ *and if* $f \in L^2(\mathcal{R})$ *with* $\mathrm{supp} f \subset \overline{\mathcal{R}}_{-,\eta} \cap \{t \geq T_0\}$ *for some* $\eta > 0$ *and* $T_0 \in \mathbf{R}$ *then there exists a weak solution* $u \in L^2(\mathcal{R})$ *to* (BVP) *with* $\mathrm{supp} u \subset \overline{\mathcal{R}}_{-,\eta} \cap \{t \geq T_0\}$.

To prove this we shall need a few lemmas which are proved by repeating the same arguments as in [8] (see Lemma 5.6, Proposition 5.2 and Corollary 7.8 in [8]).

Lemma 3.3. *There is a* $\eta_0 > 0$ *such that for* $\tau \geq 0$ *we can take a* $\Lambda(\tau) \in \mathbf{R}$ *verifying the following properties: If* $0 < \eta < \eta_0$, $\mathrm{Re}\lambda > \Lambda(\tau)$ *and if* $u \in C_0^{0,1}(\overline{\mathcal{R}})$ *with* $\mathrm{supp} u \cap \{\phi_{-,\eta} = 0\} = \emptyset$ *and* $u \in M^*$ *at* $\Delta \cap \{\phi_{-,\eta} > 0\}$ *then it follows that*

$$\begin{aligned}(\mathrm{Re}\lambda - \Lambda(\tau))\|u\|^2_{\mathcal{R}_{-,\eta}} &+ c_0(\tau - 1/4)\|\phi_{-,\eta}^{-1/2}u\|^2_{\mathcal{R}_{-,\eta}} \\ &\leq C_1\|\phi_{-,\eta}^{\tau+1/2}(L^* + \overline{\lambda}A_0)\phi_{-,\eta}^{-\tau}u\|^2_{\mathcal{R}_{-,\eta}}\end{aligned}$$

where c_0, $C_1 > 0$ *depend only on* η.

Lemma 3.4. *There are* c_0, C_1, $\Sigma_0 > 0$ *such that for* σ, $\tau > \Sigma_0$ *we can take a* $\Lambda(\sigma,\tau) \in \mathbf{R}$ *verifying the following properties: If* $\mathrm{Re}\lambda > \Lambda(\sigma,\tau)$, $-\infty < T_1 < T_2 < \infty$ *and if* $u \in C_0^{0,1}(\overline{\mathcal{R}})$ *with* $u \in M$ *at* $(T_1,T_2)\times\partial\Omega$ *then it follows that*

$$\begin{aligned}&(\mathrm{Re}\lambda - \Lambda(\sigma,\tau))\|m^{1/2}u\|^2_{(T_1,T_2)\times\Omega} \\ &\quad + c_0(\min(\sigma,\tau) - \Sigma_0)\|u\|^2_{(T_1,T_2)\times\Omega} + c_0\|(m^{1/2}u)(T_2)\|^2_\Omega \\ &\quad \leq C_1\{\|m\phi_+^\sigma\phi_-^{-\tau}(L+\lambda A_0)\phi_+^{-\sigma}\phi_-^\tau u\|^2_{(T_1,T_2)\times\Omega} + \|(m^{1/2}u)(T_1)\|^2_\Omega\}.\end{aligned}$$

Lemma 3.5. *Let $u \in L^2(\mathcal{R})$, with* $\operatorname{supp} u \subset \overline{\mathcal{R}}_{-,\eta}$ *for some $\eta > 0$, be a weak solution to* (BVP). *Then there is a $\{u_\epsilon\} \subset C_0^\infty(\overline{\mathcal{R}})$ with* $\operatorname{supp} u_\epsilon \subset \overline{\mathcal{R}}_{-,\eta_0}$ *and $u_\epsilon \in M$ at Δ such that if $\sigma \geq 4$ and $\tau \in \mathbf{R}$ then $\phi_+^\sigma \phi_-^\tau u_\epsilon$ is also a weak solution to* (BVP). *Moreover we have*

$$u_\epsilon \to u, \quad \phi_+^\sigma \phi_-^\tau (L + \lambda A_0) u_\epsilon \to \phi_+^\sigma \phi_-^\tau (L + \lambda A_0) u \quad \textit{in } L^2(\mathcal{R}) \quad \textit{as } \epsilon \to 0$$

where $\eta_0 > 0$ depends only on η.

Proof of Proposition 3.2. Using Lemma 3.3 and repeating arguments similar to those in Proposition 3.2 in [8] we can find a $u \in L^2(\mathcal{R})$ with $\operatorname{supp} u \subset \overline{\mathcal{R}}_{-,\eta}$ which is a weak solution to (BVP). We choose a $\{u_\epsilon\}$ as in Lemma 3.5. Then Lemma 3.4 shows that

$$\begin{aligned}&(\min(\sigma,\tau) - \Sigma_0)\|\phi_+^\sigma \phi_-^{-\tau} u_\epsilon\|^2_{(S_0,T_0)\times\Omega} \\ &\quad \leq C_1\{\|m\phi_+^\sigma \phi_-^{-\tau}(L + \lambda A_0)u_\epsilon\|^2_{(S_0,T_0)\times\Omega} + \|(m^{1/2}\phi_+^\sigma \phi_-^{-\tau} u_\epsilon)(S_0)\|^2_\Omega\}.\end{aligned}$$

Letting $S_0 \to -\infty$ and $\epsilon \to 0$ we have

$$\begin{aligned}&(\min(\sigma,\tau) - \Sigma_0)\|\phi_+^\sigma \phi_-^{-\tau} u\|^2_{(-\infty,T_0)\times\Omega} \\ &\quad \leq C_1\|m\phi_+^\sigma \phi_-^{-\tau}(L + \lambda A_0)u\|^2_{(-\infty,T_0)\times\Omega} = \|m\phi_+^\sigma \phi_-^{-\tau} f\|^2_{(-\infty,T_0)\times\Omega} = 0.\end{aligned}$$

This implies $\operatorname{supp} u \subset \{t \geq T_0\}$ which proves the assertion. □

We now give the proof of Proposition 2.2.

Proof of Proposition 2.2. Assuming that $u \in m_- L^2(\mathcal{O})$ is a weak solution to (BVP) with $f = 0$ and $u_0 = 0$ we wish to show $u = 0$. Let $g \in C_0^\infty(\mathcal{O})$. Repeating the same arguments as in Proposition 3.2 we can find a $v \in L^2(\mathcal{R})$, with $\operatorname{supp} v \subset \overline{\mathcal{R}}_{+,\eta} \cap \{t \leq T - \eta\}$ for some $\eta > 0$, which is a weak solution to the following adjoint boundary value problem

$$\text{(BVP*)} \qquad \begin{cases} (L + \overline{\lambda} A_0)v = g & \text{in } \mathbf{R} \times \Omega = \mathcal{R} \\ v \in M^* & \text{at } \mathbf{R} \times \partial\Omega = \Delta. \end{cases}$$

Let us choose $\chi \in C_0^\infty(\mathbf{R})$ so that $\chi = 1$ near 0 and set

$$v_k = \chi(k^{-1}t)(1 - \chi(km_-))v, \quad g_k = (L^* + \overline{\lambda} A_0)v_k$$

for $k > 0$ large enough. Then v_k is also a weak solution to (BVP*) replaced g by g_k. Since $\operatorname{supp} v_k$ is compact and $\operatorname{supp} v_k \cap \gamma = \emptyset$ then Theorem 4 in [12] gives a $\{v_{k,\epsilon}\} \subset C_0^1(\overline{\mathcal{R}})$ with $v_{k,\epsilon} \in M^*$ at Δ such that

$$v_{k,\epsilon} \to v_k, \quad (L^* + \overline{\lambda} A_0)v_{k,\epsilon} \to g_k \quad \text{in } L^2(\mathcal{R}) \quad \text{as } \epsilon \to 0.$$

Noticing $e^{-\overline{\lambda}t}v_{k,\epsilon} \in C^1(\overline{\mathcal{O}})$ with $e^{-\overline{\lambda}t}v_{k,\epsilon} \in M^*$ at Γ and $(e^{-\overline{\lambda}t}v_{k,\epsilon})(T) = 0$ and recalling that u is a weak solution to (BVP) with $f = 0$ and $u_0 = 0$ we obtain $(u, L^*e^{-\overline{\lambda}t}v_{k,\epsilon})_{\mathcal{O}} = 0$. Letting $\epsilon \to 0$ we have $(e^{-\lambda t}u, g_k)_{\mathcal{O}} = 0$. We note that $(e^{-\lambda t}u, g_k)_{\mathcal{O}} \to (e^{-\lambda t}u, g)_{\mathcal{O}}$ as $k \to \infty$. Indeed we can write

$$\begin{aligned}(e^{-\lambda t}u, g_k)_{\mathcal{O}} &= (e^{-\lambda t}u, \chi(k^{-1}t)(1-\chi(km_-))g)_{\mathcal{O}} \\ &\quad - (e^{-\lambda t}u, k^{-1}\chi'(k^{-1}t)\chi(km_-)A_0v)_{\mathcal{O}} \\ &\quad + (e^{-\lambda t}u, \chi(k^{-1}t)k\chi'(km_-)M_-v)_{\mathcal{O}} \\ &= I_1 + I_2 + I_3\end{aligned}$$

with $M_- = \sum_{j=0}^{n}(\partial_j m_-)A_j$. The dominated convergence theorem shows that $I_1 \to (e^{-\lambda t}u, g)_{\mathcal{O}}$, $I_2 \to 0$ as $\epsilon \to 0$. We turn to I_3. Since $u = m_-w$ for some $w \in L^2(\mathcal{O})$ and $|\theta\chi'(\theta)| \leq C$ for some $C > 0$ the dominated convergence theorem again proves that $I_3 = (e^{-\lambda t}w, \chi(k^{-1}t)km_-\chi'(km_-)M_-v)_{\Omega} \to 0$ as $\epsilon \to 0$. Thus we have $(e^{-\lambda t}u, g)_{\Omega} = 0$. Noticing that $C_0^\infty(\mathcal{O})$ is dense in $L^2(\mathcal{O})$ we conclude $u = 0$. $\square$

4. Proofs of results for zero initial data

We start with the proof of Theorem 2.4. Recalling that $A_j(t,x)$, $B(t,x)$ and $h_\pm(t,x)$ are independent of t outside a compact subset of $\overline{\mathcal{R}}$ and repeating the same arguments as in [8] we can prove the following two propositions (see Proposition 3.1 and Proposition 11.3 in [8]).

Proposition 4.1. *For $q \in \mathbf{Z}_+$ there is a $\Sigma(q) > 0$ such that for σ, $\tau > \Sigma(q)$ we can take a $\Lambda(q,\sigma,\tau) \in \mathbf{R}$ verifying the following properties: If* $\mathrm{Re}\lambda > \Lambda(q,\sigma,\tau)$ *and $f \in X^q_{(-\sigma,\tau)}(\mathcal{R};\Delta) \cap L^2(\mathcal{R})$ and if $u \in L^2(\mathcal{R})$, with* $\mathrm{supp}\, u \subset \overline{\mathcal{R}}_{-,\eta}$ *for some $\eta > 0$, is a weak solution to* (BVP) *then it follows that $u \in X^q_{(-\sigma,\tau)}(\mathcal{R};\Delta)$.*

Proposition 4.2. *For $q \in \mathbf{Z}_+$ there is a $\Sigma(q) > 0$ such that for σ, $\tau > \Sigma(q)$ we can take a $\Lambda(q,\sigma,\tau) \in \mathbf{R}$ verifying the following properties: If* $\mathrm{Re}\lambda > \Lambda(q,\sigma,\tau)$ *and if $u \in X^{q+[n/2]+2}_{(-\sigma,\tau)}(\mathcal{R};\Delta) \cap L^2(\mathcal{R})$ is a weak solution to* (BVP) *with $f \in C_0^\infty(\mathcal{R})$ then it follows that*

$$||u||^2_{X^q_{(-\sigma,\tau)}(\mathcal{R};\Delta)} \leq C_1||(L+\lambda A_0)u||^2_{X^q_{(-\sigma,\tau)}(\mathcal{R};\Delta)}$$

where $C_1 = C_1(q,\sigma,\tau,\lambda) > 0$ is independent of u and f.

An immediate corollary to these propositions is

Corollary 4.3. *For $q \in \mathbf{Z}_+$ there is a $\Sigma(q) > 0$ such that for σ, $\tau > \Sigma(q)$ we can take a $\Lambda(q,\sigma,\tau) \in \mathbf{R}$ verifying the following properties: If* $\mathrm{Re}\lambda > \Lambda(q,\sigma,\tau)$ *and* *if* $u \in L^2(\mathcal{R})$, *with* $\mathrm{supp} u \subset \overline{\mathcal{R}}_{-,\eta}$ *for some* $\eta > 0$, *is a weak solution to* (BVP) *with* $f \in C_0^\infty(\mathcal{R})$ *then it follows that* $u \in X^q_{(-\sigma,\tau)}(\mathcal{R};\Delta)$ *and*

$$\|u\|^2_{X^q_{(-\sigma,\tau)}(\mathcal{R};\Delta)} \leq C_1\|(L+\lambda A_0)u\|^2_{X^q_{(-\sigma,\tau)}(\mathcal{R};\Delta)}$$

where $C_1 = C_1(q,\sigma,\tau,\lambda) > 0$ *is independent of* u *and* f.

Proof of Theorem 2.4. We can take a $\tilde{f} \in X^q_{(-\sigma,\tau)}(\mathcal{R};\Delta)$ with $\mathrm{supp}\tilde{f} \subset \{0 \leq t \leq \tilde{T}\}$ such that $\tilde{f} = f$ on $\mathcal{O}$ and

$$\|\tilde{f}\|_{X^q_{(-\sigma,\tau)}(\mathcal{R};\Delta)} \leq C\|f\|_{X^q_{(-\sigma,\tau)}(\mathcal{O};\Gamma)} \tag{4.1}$$

where $C > 0$ and $\tilde{T} > T$ are independent of f and $\tilde{f}$ (we give the proof of this fact in Corollary 7.11 below). Let us choose $\chi \in C_0^\infty(\mathbf{R})$ so that $\chi = 1$ near 0 and $\rho \in C_0^\infty(\mathbf{R}^{n+1})$ with $\mathrm{supp}\rho \subset \{(t,x);\ 0 < t < 1, |x| < 1\}$ such that $\rho \geq 0$ and $\iint \rho dt dx = 1$ and set

$$f_{k,\epsilon}(t,x) = (((1-\chi(kr))\tilde{f}) * \rho_\epsilon)(t,x), \quad \rho_\epsilon(t,x) = \epsilon^{-(n+1)}\rho(\epsilon^{-1}t,\epsilon^{-1}x)$$

for $k > 0$ large enough and $0 < \epsilon < 1$ small enough where $r = r(x)$ is a defining function of Ω. Then we have $f_{k,\epsilon} \in C_0^\infty(\mathcal{R})$ with $\mathrm{supp} f_{k,\epsilon} \subset \{0 \leq t \leq \tilde{T}+1\}$ for $\epsilon > 0$ small enough. Moreover it follows from the proof of Lemma 6.4 in [8] that

$$\phi_-^{-1}f_{k,\epsilon} \to \phi_-^{-1}f \quad \text{in } L^2(\mathcal{O}), \quad f_{k,\epsilon} \to \tilde{f} \quad \text{in } X^q_{(-\sigma,\tau)}(\mathcal{R};\Delta)$$

as $\epsilon \to 0$ and $k \to \infty$. Let $\lambda \in \mathbf{C}$ be $\mathrm{Re}\lambda > 0$ large enough and set $F_{k,\epsilon} = e^{-\lambda t}f_{k,\epsilon}$. Then Proposition 3.2 gives a weak solution $U_{k,\epsilon} \in L^2(\mathcal{R})$ to (BVP) with f replaced by $F_{k,\epsilon}$ with $\mathrm{supp}U_{k,\epsilon} \subset \overline{\mathcal{R}}_{-,\eta} \cap \{t \geq 0\}$ for some $\eta > 0$. From Corollary 4.3 it follows that $U_{k,\epsilon} \in X^q_{(-\sigma,\tau)}(\mathcal{R};\Delta)$ and

$$\|U_{k,\epsilon}\|^2_{X^q_{(-\sigma,\tau)}(\mathcal{R};\Delta)} \leq C_1\|F_{k,\epsilon}\|^2_{X^q_{(-\sigma,\tau)}(\mathcal{R};\Delta)}.$$

Now if we write $u_{k,\epsilon} = e^{\lambda t}U_{k,\epsilon}$ then we have $u_{k,\epsilon} \in X^q_{(-\sigma,\tau)}(\mathcal{O};\Gamma) \cap \phi_- L^2(\mathcal{O})$ and

$$\|u_{k,\epsilon}\|^2_{X^q_{(-\sigma,\tau)}(\mathcal{O};\Gamma)} \leq C_1\|f_{k,\epsilon}\|^2_{X^q_{(-\sigma,\tau)}(\mathcal{R};\Delta)}. \tag{4.2}$$

We first show that $u_{k,\epsilon}$ is a weak solution to (IBVP) replaced f and u_0 by $f_{k,\epsilon}$ and 0. Let $\psi \in C^{0,1}(\overline{\mathcal{O}})$ with $\psi \in M^*$ at Γ and $\psi(T) = 0$. We choose a $\tilde{\psi} \in C_0^{0,1}(\overline{\mathcal{R}})$ with $\mathrm{supp}\tilde{\psi} \subset \{t \leq T\}$ such that $\tilde{\psi} \in M^*$ at Δ and $\tilde{\psi}(T) = 0$. Since $U_{k,\epsilon}$ is a weak solution to (BVP) it follows that

$$(U_{k,\epsilon}, (L^* + \overline{\lambda}A_0)e^{-\overline{\lambda}t}\tilde{\psi})_{\mathcal{R}} = (F_{k,\epsilon}, e^{-\overline{\lambda}t}\tilde{\psi})_{\mathcal{R}}.$$

Noticing $\mathrm{supp} U_{k,\epsilon} \cap \mathrm{supp}\tilde{\psi} \subset \overline{\mathcal{O}}$ we get $(u_{k,\epsilon}, L^*\psi)_{\mathcal{O}} = (f_{k,\epsilon}, \psi)_{\mathcal{O}}$, and hence $u_{k,\epsilon}$ is a weak solution to (IBVP). Therefore it follows from Corollary 2.3 that

$$\|\phi_-^{-1} u_{k,\epsilon}\|_{\mathcal{O}}^2 \le C\|\phi_-^{-1} f_{k,\epsilon}\|_{\mathcal{O}}^2. \tag{4.3}$$

Combining (4.2) and (4.3) we obtain

$$\|u_{k,\epsilon}\|^2_{X^q_{(-\sigma,\tau)}(\mathcal{O};\Gamma)} + \|\phi_-^{-1} u_{k,\epsilon}\|^2_{\mathcal{O}} \le C_1\{\|f_{k,\epsilon}\|^2_{X^q_{(-\sigma,\tau)}(\mathcal{R};\Delta)} + \|\phi_-^{-1} f_{k,\epsilon}\|^2_{\mathcal{O}}\}. \tag{4.4}$$

Since $\{f_{k,\epsilon}\}$ is a Cauchy sequence in $X^q_{(-\sigma,\tau)}(\mathcal{R};\Delta) \cap \phi_- L^2(\mathcal{O})$ then $\{u_{k,\epsilon}\}$ has a limit point u in $X^q_{(-\sigma,\tau)}(\mathcal{O};\Gamma) \cap \phi_- L^2(\mathcal{O})$. Then u is a desired weak solution to (IBVP) with zero initial data and the desired estimate follows from (4.4) and (4.1). □

We turn to the proof of Theorem 2.5. The proof easily follows from

Proposition 4.4. *Let $u \in X^q_{(\sigma,\tau)}(\mathcal{O};\Gamma)$ and $Lu \in X^q_{(\sigma,\tau)}(\mathcal{O})$ for some $q \in \mathbf{Z}_+$ and σ, $\tau \in \mathbf{R}$. Then it follows that $u \in m^{-q} X^q_{(\sigma,\tau)}(\mathcal{O})$ and*

$$\|m^q u\|_{X^q_{(\sigma,\tau)}(\mathcal{O})} \le C\{\|u\|_{X^q_{(\sigma,\tau)}(\mathcal{O};\Gamma)} + \|Lu\|_{X^q_{(\sigma,\tau)}(\mathcal{O})}\}$$

where $C = C(q,\sigma,\tau) > 0$ is independent of u.

Admitting for the moment that Proposition 4.4 holds we shall prove Theorem 2.5.

Proof of Theorem 2.5. Let $q' = q + [n/2] + 1$. Theorem 2.4 gives a weak solution $v \in X^{q'}_{(-\sigma,\tau)}(\mathcal{O};\Gamma) \cap \phi_- L^2(\mathcal{O})$ to (IBVP) with zero initial data, and hence it follows from Proposition 2.2 that $u = v \in X^{q'}_{(-\sigma,\tau)}(\mathcal{O};\Gamma)$. Therefore Proposition 4.4 implies that

$$u \in m^{-q'} X^{q'}_{(-\sigma,\tau)}(\mathcal{O}) \hookrightarrow m^{-q'} \phi_+^{-\sigma} \phi_-^{\tau} H^{q'}(\mathcal{O}) \hookrightarrow m^{-q'} \phi_+^{-\sigma} \phi_-^{\tau} C^q(\overline{\mathcal{O}})$$

which shows the assertion. □

To prove Proposition 4.4 we localize the problem. Let us take a covering $\{U_i\}_{i=0}^l$ of $\mathcal{O}$ as follows: First we cover Γ by coordinate patches U_i, $i = 1, \dots, l$, with coordinate systems $\chi_i : U_i \cap \mathcal{O} \to \{(\tau,\xi);\ a_i < \tau < b_i,\ |\xi| < 1,\ \xi_1 > 0\}$ such that $\tau = t \circ \chi_i^{-1}$ and $\xi_1 = r \circ \chi_i^{-1}$ where $0 \le a_i < b_i \le T$. Next we cover $\mathcal{O} \setminus \bigcup_{i=1}^l U_i$ by $U_0 \subset\subset \mathbf{R} \times \Omega$. Choose a partition of unity $\{\psi_i\}_{i=0}^l$ subordinate to this covering $\{U_i\}_{i=0}^l$ and set $u_i = \psi_i u$. If $U_i \cap \Gamma = \emptyset$ then Proposition 4.4 with u_i instead of u is easily checked.

Now we suppose that $U_i \cap \Gamma \ne \emptyset$. Performing a change of independent variables we may assume that $r = x_1$ and $\mathrm{supp} u_i \subset \overline{I}_i \times \{|x| < 1, x_1 \ge 0\}$ where $I_i = (a_i, b_i)$.

In what follows, we write

$$\partial = (\partial_0, \partial_1, \partial_2, \dots, \partial_n), \qquad \partial_x = (\partial_1, \partial_2, \dots, \partial_n),$$
$$Z = (\partial_0, x_1\partial_1, \partial_2, \dots, \partial_n), \quad Z_x = (x_1\partial_1, \partial_2, \dots, \partial_n).$$

Proposition 4.4 is an immediate consequence of Lemma 4.5 below.

Lemma 4.5. *Let $p \in \mathbf{Z}_+$, $\alpha \in \mathbf{Z}_+^{n+1}$ and assume that $p + |\alpha| \le q$. Then we have*

$$\begin{aligned}(4.5) \qquad &\|m^p \phi_+^{-\sigma-q+p+|\alpha|} \phi_-^{-\tau-q+p+|\alpha|} \partial_1^p Z^\alpha u_i\|_{I_i \times \mathbf{R}_+^n} \\ &\le C\{\|u\|_{X^q_{(\sigma,\tau)}(\mathcal{O};\Gamma)} + \|Lu\|_{X^q_{(\sigma,\tau)}(\mathcal{O})}\}\end{aligned}$$

where $C = C(q, \sigma, \tau, p, \alpha) > 0$.

Proof of Proposition 4.4. Let $\alpha \in \mathbf{Z}_+^{n+1}$ and assume that $|\alpha| \le q$. If we write $\alpha' = (\alpha_0, 0, \alpha_2, \dots, \alpha_n)$ then it follows that

$$\begin{aligned}&\|m^q \phi_+^{-\sigma-q+|\alpha|} \phi_-^{-\tau-q+|\alpha|} \partial^\alpha u_i\|_{I_i \times \mathbf{R}_+^n} \\ &\le C\|m^{\alpha_1} \phi_+^{-\sigma-q+\alpha_1+|\alpha'|} \phi_-^{-\tau-q+\alpha_1+|\alpha'|} \partial_1^{\alpha_1} Z^{\alpha'} u_i\|_{I_i \times \mathbf{R}_+^n} \\ &\le C'\{\|u\|_{X^q_{(\sigma,\tau)}(\mathcal{O};\Gamma)} + \|Lu\|_{X^q_{(\sigma,\tau)}(\mathcal{O})}\}.\end{aligned}$$

Arguments similar to those in Lemma 6.1 in [8] imply that

$$\|m^q u_i\|_{X^q_{(\sigma,\tau)}(\mathcal{O})} \le C \sum_{|\alpha| \le q} \|m^q \phi_+^{-\sigma-q+|\alpha|} \phi_-^{-\tau-q+|\alpha|} \partial^\alpha u_i\|_{I_i \times \mathbf{R}_+^n}$$

which shows that

$$(4.6) \qquad \|m^q u_i\|_{X^q_{(\sigma,\tau)}(\mathcal{O})} \le C\{\|u\|_{X^q_{(\sigma,\tau)}(\mathcal{O};\Gamma)} + \|Lu\|_{X^q_{(\sigma,\tau)}(\mathcal{O})}\}.$$

Summing (4.6) from $i = 0$ to $i = l$ we get the desired estimate. □

We shall prove Lemma 4.5. The interesting patches are at γ. Note that $h_\pm \partial_1$ is written as a sum of $a(t,x)Z^\beta$, $|\beta| \le 1$ and $a(t,x)L$ where $a(t,x) \in \mathcal{B}^\infty(\mathbf{R}^{n+1})$, the set of all smooth functions on $\mathbf{R}^{n+1}$ with bounded derivatives of all order, which may differ from line to line.

Lemma 4.6. *$(h_\pm \partial_1)^p Z^\alpha$, $p \ge 1$ and $|\alpha| \le q$, is written as a sum of the following terms:*

$$a(t,x) m^{-k} x_1^i h_\pm^j (h_\pm \partial_1)^l Z^\beta, \qquad 0 \le l \le p-1,\ 0 \le k \le 2(q+p-1-l),$$

$$\begin{aligned} & && 0 \le k-i-j \le q,\ |\beta| \le q-k+i+j+1, \\ & a(t,x)m^{-k}x_1^i h_\pm^j (h_\pm \partial_1)^l Z^\beta L, && 0 \le l \le p-1,\ 0 \le k \le 2(q+p-1-l), \\ & && 0 \le k-i-j \le q,\ |\beta| \le q-k+i+j. \end{aligned}$$

Proof. We first consider the case $p = 1$. Note that $(h_\pm \partial_1)Z^\alpha$ is written as a sum of $a(t,x)Z^\beta Z^\alpha$, $|\beta| \le 1$ and $a(t,x)LZ^\alpha$. Here $a(t,x)Z^\beta Z^\alpha$ can be written as a desired sum. We turn to $a(t,x)LZ^\alpha$. Since $LZ^\alpha = Z^\alpha L + [L, Z^\alpha]$, Lemma 10.5 in [7] shows that LZ^α can be also written as a desired sum.

We next consider the case $p \ge 2$. Using that $(h_\pm\partial_1)^p Z^\alpha = (h_\pm\partial_1)^{p-1}(h_\pm\partial_1)Z^\alpha$ and the results for the case $p = 1$ we conclude the assertion. □

Lemma 4.7. $\|m^p \phi_+^{-\sigma-q+p+|\alpha|}\phi_-^{-\tau-q+p+|\alpha|}\partial_1^p Z^\alpha u_i\|_{I_i \times \mathbf{R}^n_+}$, $p+|\alpha| \le q$ *and* $p \ge 1$, *is bounded from above by a sum of the following terms*:

$$\begin{aligned} & \|m^l \phi_+^{-\sigma-q+l+|\beta|}\phi_-^{-\tau-q+l+|\beta|}\partial_1^l Z^\beta u_i\|_{I_i\times\mathbf{R}^n_+}, && 0 \le l \le p-1,\ l+|\beta| \le q, \\ & \|m^l \phi_+^{-\sigma-q+l+|\beta|}\phi_-^{-\tau-q+l+|\beta|}\partial_1^l Z^\beta L u_i\|_{I_i\times\mathbf{R}^n_+}, && 0 \le l \le p-1,\ l+|\beta| \le q. \end{aligned}$$

Proof. Since $m \le C(|x_1| + |h_\pm|)$ for some $C > 0$ it follows that

$$\begin{aligned} & \|m^p \phi_+^{-\sigma-q+p+|\alpha|}\phi_-^{-\tau-q+p+|\alpha|}\partial_1^p Z^\alpha u_i\|_{I_i\times\mathbf{R}^n_+} \\ & \le C\{\|x_1^p \phi_+^{-\sigma-q+p+|\alpha|}\phi_-^{-\tau-q+p+|\alpha|}\partial_1^p Z^\alpha u_i\|_{I_i\times\mathbf{R}^n_+} \\ & \qquad + \|h_\pm^p \phi_+^{-\sigma-q+p+|\alpha|}\phi_-^{-\tau-q+p+|\alpha|}\partial_1^p Z^\alpha u_i\|_{I_i\times\mathbf{R}^n_+}\} \\ & = C\{I_1 + I_2\}. \end{aligned}$$

Noticing that $x_1^p \partial_1^p$ can be written as a sum of Z_1^l, $0 \le l \le p$, we have

$$I_1 \le C\sum_{l=0}^{p} \|\phi_+^{-\sigma-q+(l+|\alpha|)}\phi_-^{-\tau-q+(l+|\alpha|)} Z_1^l Z^\alpha u_i\|_{I_i\times\mathbf{R}^n_+}.$$

Thus I_1 is bounded from above by a desired sum. Moreover Lemma 4.6 implies that I_2 can be also bounded from above by a desired sum. □

Proof of Lemma 4.5. We proceed by induction on p. From Lemma 6.1 in [8] the case $p = 0$ is trivial. Inductively assume that the statement is true up to $p-1$. Lemma 4.7 shows that $\|m^p \phi_+^{-\sigma-q+p+|\alpha|}\phi_-^{-\tau-q+p+|\alpha|}\partial_1^p Z^\alpha u_i\|_{I_i\times\mathbf{R}^n_+}$ is bounded from above by a sum of the following terms:

$$\begin{aligned} & \|m^l \phi_+^{-\sigma-q+l+|\beta|}\phi_-^{-\tau-q+l+|\beta|}\partial_1^l Z^\beta u_i\|_{I_i\times\mathbf{R}^n_+} = I_1, && 0 \le l \le p-1,\ l+|\beta| \le q, \\ & \|m^l \phi_+^{-\sigma-q+l+|\beta|}\phi_-^{-\tau-q+l+|\beta|}\partial_1^l Z^\beta L u_i\|_{I_i\times\mathbf{R}^n_+} = I_2, && 0 \le l \le p-1,\ l+|\beta| \le q. \end{aligned}$$

By the inductive hypothesis I_1 can be bounded from above by the right-hand side of (4.5). We turn to I_2. Since $Lu_i = \psi_i Lu + \sum_{j,k}(\partial_j\psi_i)A_j u_k$ the inductive hypothesis implies that I_2 can also be bounded from above by the right-hand side of (4.5). This proves the assertion for p. $\square$

5. Results for general initial data

Now we shall extend the definition of a weak solution to (IBVP). Let us set

$$\mathcal{L}^2(\mathcal{O}) = \bigcup_{\sigma,\tau\geq 0} \phi_+^{-\sigma}\phi_-^{\tau} L^2(\mathcal{O}), \quad \mathcal{L}_0^2(\Omega) = \bigcup_{\sigma,\tau\geq 0} (\phi_+^{-\sigma}\phi_-^{\tau})(0)L^2(\Omega).$$

Noticing (2.1) we introduce the following definition.

Definition. For $f \in \mathcal{L}^2(\mathcal{O})$ and $u_0 \in \mathcal{L}_0^2(\Omega)$ we say $u \in \mathcal{L}^2(\mathcal{O})$ is a weak solution to (IBVP) if and only if the identity

$$(u, L^*\psi)_{L^2(\mathcal{O})} = (f, \psi)_{L^2(\mathcal{O})} + (A_0(0)u_0, \psi(0))_{L^2(\Omega)}$$

holds for all $\psi \in C^{0,1}(\overline{\mathcal{O}})$ with $\psi \in M^*$ at Γ, $\psi(T) = 0$ and $\psi = 0$ on a neighborhood of O^+.

Then by using arguments similar to those in the proof of Proposition 2.1, Proposition 2.2 and Corollary 2.3 we obtain

Proposition 5.1. *If $f \in \phi_+^{-\sigma}\phi_-^{\tau} L^2(\mathcal{O})$ and $u_0 \in (\phi_+^{-\sigma}\phi_-^{\tau})(0)L^2(\Omega)$ for some $\sigma \geq 0$ and $\tau \geq 1$ then there exists a weak solution $u \in \phi_+^{-\sigma}\phi_-^{\tau} L^2(\mathcal{O})$ to* (IBVP) *satisfying*

$$\|\phi_+^{\sigma}\phi_-^{-\tau}u\|_{L^2(\mathcal{O})}^2 \leq C\{\|\phi_+^{\sigma}\phi_-^{-\tau}f\|_{L^2(\mathcal{O})}^2 + \|(\phi_+^{\sigma}\phi_-^{-\tau})(0)u_0\|_{L^2(\Omega)}^2\}$$

where $C = C(\sigma, \tau) > 0$ is independent of f, u_0 and u.

Proposition 5.2. *If $f \in \mathcal{L}^2(\mathcal{O})$ and $u_0 \in \mathcal{L}_0^2(\Omega)$ then a weak solution $u \in m_- L^2(\mathcal{O})$ to* (IBVP) *is unique.*

Corollary 5.3. *If $f \in \phi_+^{-\sigma}\phi_-^{\tau} L^2(\mathcal{O})$ and $u_0 \in (\phi_+^{-\sigma}\phi_-^{\tau})(0)L^2(\Omega)$ for some $\sigma \geq 0$ and $\tau \geq 1$ and if $u \in m_- L^2(\mathcal{O})$ is a weak solution to* (IBVP) *then we have $u \in \phi_+^{-\sigma}\phi_-^{\tau} L^2(\mathcal{O})$ and it follows that*

$$\|\phi_+^{\sigma}\phi_-^{-\tau}u\|_{L^2(\mathcal{O})}^2 \leq C\{\|\phi_+^{\sigma}\phi_-^{-\tau}f\|_{L^2(\mathcal{O})}^2 + \|(\phi_+^{\sigma}\phi_-^{-\tau})(0)u_0\|_{L^2(\Omega)}^2\}$$

where $C = C(\sigma, \tau) > 0$ is independent of f, u_0 and u.

In order to get regularity results we introduce "compatibility conditions". Let $f \in X^q_{(-\sigma,\tau)}(I \times \Omega)$ and $u_0 \in X^q_{0(-\sigma,\tau)}(\Omega)$ for $q \in \mathbf{Z}_+$, $q \geq 1$ and σ, $\tau \geq 0$. Then we define $u^{(k)}$, $k = 0, \dots, q-1$, as follows:

$$u^{(0)} = u_0, \quad u^{(k)} = (\partial_0^{k-1} A_0^{-1} f)(0) - \sum_{i=0}^{k-1} \binom{k-1}{i} K_i u^{(k-1-i)} \quad \text{for} \quad k \geq 1$$

where $K_i = \sum_{j=1}^{n} (\partial_0^i A_0^{-1} A_j)(0)\partial_j + (\partial_0^i A_0^{-1} B)(0)$. Note that

$$u^{(k)} \in X^{q-k}_{0(-\sigma,\tau)}(\Omega) \hookrightarrow X^1_{0(-\sigma,\tau)}(\Omega) \hookrightarrow (\phi_+^{-\sigma}\phi_-^{\tau})(0)H^1(\Omega),$$

and hence $(\phi_+^{\sigma}\phi_-^{-\tau})(0)u^{(k)} \in L^2(\partial\Omega)$. We write $T_k(f, u_0) = u^{(k)}$ for $k = 0, \dots, q-1$.

Let $\delta > 0$ be small enough and choose $P(t,x) \in C^\infty((-\delta,\delta) \times \partial\Omega; M_N(\mathbf{C}))$ such that $v \in M(t,x)$ if and only if $P(t,x)v = 0$ for every $(t,x) \in (-\delta,\delta) \times \partial\Omega$.

DEFINITION. For $q \in \mathbf{Z}_+$, $q \geq 1$ and σ, $\tau \geq 0$ we say $f \in X^q_{(-\sigma,\tau)}(\mathcal{O})$ and $u_0 \in X^q_{0(-\sigma,\tau)}(\Omega)$ satisfy the compatibility conditions up to order $q-1$ if and only if the following identities hold:

$$\sum_{i=0}^{k} \binom{k}{i} (\partial_0^i P)(0)(\phi_+^{\sigma}\phi_-^{-\tau})(0)u^{(k-i)} = 0 \quad \text{on } \partial\Omega \setminus \gamma_0 \quad \text{for } k = 0, \dots, q-1.$$

Here $u^{(k)} = T_k(f, u_0)$, $k = 0, \dots, q-1$, and $\gamma_0 = \{x \in \partial\Omega;\ (0,x) \in \gamma\}$.

Theorem 5.4. *For $q \in \mathbf{Z}_+$, $q \geq 1$ there is a $\Sigma(q) > 0$ such that if $f \in X^q_{(-\sigma,\tau)}(\mathcal{O})$ and $u_0 \in X^q_{0(-\sigma,\tau)}(\Omega)$, for some σ, $\tau > \Sigma(q)$, satisfy the compatibility conditions up to order $q-1$ then there exists a weak solution $u \in X^q_{(-\sigma,\tau)}(\mathcal{O};\Gamma)$ to* (IBVP) *which satisfies*

$$\|u\|^2_{X^q_{(-\sigma,\tau)}(\mathcal{O};\Gamma)} \leq C\{\|f\|^2_{X^q_{(-\sigma,\tau)}(\mathcal{O})} + \|u_0\|^2_{X^q_{0(-\sigma,\tau)}(\Omega)}\}$$

where $C = C(q,\sigma,\tau) > 0$ is independent of f, u_0 and u.

From Theorem 5.4 and Proposition 4.4 we can derive a rough estimate of asymptotic behavior of weak solutions near γ.

Theorem 5.5. *For $q \in \mathbf{Z}_+$ there is a $\Sigma(q) > 0$ such that if $f \in X^{q+[n/2]+1}_{(-\sigma,\tau)}(\mathcal{O})$ and $u_0 \in X^{q+[n/2]+1}_{0(-\sigma,\tau)}(\Omega)$, for some σ, $\tau > \Sigma(q)$, satisfy the compatibility conditions up to order $q+[n/2]$ and if $u \in m_- L^2(\mathcal{O})$ is a weak solution to* (IBVP) *then we have $u \in m^{-(q+[n/2]+1)}\phi_+^{-\sigma}\phi_-^{\tau} C^q(\overline{\mathcal{O}})$.*

6. Proofs of results for general initial data

In this section we give the proof of Theorem 5.4. From Lemma 3.4 we recall that

Lemma 6.1. *There are $C, \Sigma_0 > 0$ such that for σ, $\tau > \Sigma_0$ we can take a $\Lambda(\sigma, \tau) \in \mathbf{R}$ verifying the following properties: If $\mathrm{Re}\lambda > \Lambda(\sigma, \tau)$ and if $u \in C_0^{0,1}(\overline{\mathcal{R}})$, with $\mathrm{supp} u \cap (O^- \cup \gamma^-) = \emptyset$, is a weak solution to (IBVP) then it follows that*

$$
\begin{aligned}
&(\min(\sigma, \tau) - \Sigma_0)\|e^{-\lambda t}\phi_+^{\sigma}\phi_-^{-\tau}u\|_{\mathcal{O}}^2 \\
&\quad \le C\{\|me^{-\lambda t}\phi_+^{\sigma}\phi_-^{-\tau}Lu\|_{\mathcal{O}}^2 + \|(\phi_+^{\sigma}\phi_-^{-\tau}u)(0)\|_{\Omega}^2\}.
\end{aligned}
$$

This implies the following a priori estimate.

Proposition 6.2. *For $q \in \mathbf{Z}_+$, $q \ge 1$ there is a $\Sigma(q) > 0$ such that if σ, $\tau > \Sigma(q)$ and if $u \in C^{q+1}(\overline{\mathcal{O}})$, with $\mathrm{supp} u \cap (O^- \cup \gamma^-) = \emptyset$, is a weak solution to (IBVP) then it follows that*

$$
\begin{aligned}
&\|u\|_{X^q_{(-\sigma,\tau)}(\mathcal{O};\Gamma)}^2 \\
&\quad \le C\left\{\|mLu\|_{X^q_{(-\sigma,\tau)}(\mathcal{O})}^2 + \sum_{k=0}^{q-1}\|(\partial_0^k Lu)(0)\|_{X^{q-1-k}_{0(-\sigma,\tau)}(\Omega)}^2 + \|u_0\|_{X^q_{0(-\sigma,\tau)}(\Omega)}^2\right\}
\end{aligned}
$$

where $C = C(q, \sigma, \tau) > 0$ is independent of u.

Proof. Localizing the problem as in Proposition 4.4 and repeating the same arguments as in Proposition 10.1 in [8] we can obtain that

$$
\|u\|_{X^q_{(-\sigma,\tau)}(\mathcal{O};\Gamma)}^2 \le C\left\{\|mLu\|_{X^q_{(-\sigma,\tau)}(\mathcal{O};\Gamma)}^2 + \sum_{k=0}^{q}\|(\partial_0^k u)(0)\|_{X^{q-k}_{0(-\sigma,\tau)}(\Omega)}^2\right\}.
$$

Since $\partial_0 = A_0^{-1}(L - \sum_{j=1}^n A_j\partial_j - B)$ we note that $(\partial_0^k u)(0)$, $k \ge 1$, is written as a sum of $(\partial_x^\alpha u)(0)$, $|\alpha| \le k$ and $(\partial_x^\beta \partial_0^l Lu)(0)$, $l + |\beta| \le k - 1$. This completes the proof. □

Let us set

$$
\gamma_0^{\pm} = \{x \in \partial\Omega;\ (0, x) \in \gamma^{\pm}\}, \quad \gamma_0 = \{x \in \partial\Omega;\ (0, x) \in \gamma\},
$$
$$
O_0^{\pm} = \{x \in \partial\Omega;\ (0, x) \in O^{\pm}\}.
$$

For the proof of Theorem 5.4 we shall extract from technical details and sum up in the following two lemmas.

Lemma 6.3. *Let $q \in \mathbf{Z}_+$ and set $\tilde{q} = q+[n/2]+2$. Suppose that $f \in H^{\tilde{q}}(\mathcal{O})$ and $u_0 \in H^{\tilde{q}}(\Omega)$ with* $\mathrm{supp} f \cap (O^+ \cup O^- \cup \gamma) = \emptyset$ *and* $\mathrm{supp} u_0 \cap (O_0^+ \cup O_0^- \cup \gamma_0) = \emptyset$ *and that $u \in X^{\tilde{q}}_{(-\sigma+\tilde{q},\tau+\tilde{q})}(\mathcal{O};\Gamma)$, for some σ, $\tau \geq \tilde{q}$, is a weak solution to* (IBVP) (*we remark that $X^{\tilde{q}}_{(-\sigma+\tilde{q},\tau+\tilde{q})}(\mathcal{O};\Gamma) \hookrightarrow X^{q}_{(-\sigma,\tau)}(\mathcal{O};\Gamma)$*). *Then there exists a $\{u_l\} \subset C^{q+1}(\overline{\mathcal{O}})$ with* $\mathrm{supp} u_l \cap (O^+ \cup O^- \cup \gamma) = \emptyset$ *and $u_l \in M$ at Γ which satisfies that*

$$
\begin{aligned}
&mLu_l && \to mf && \text{in } X^{q}_{(-\sigma,\tau)}(\mathcal{O}), \\
&(\partial_0^k Lu_l)(0) && \to (\partial_0^k f)(0) && \text{in } X^{q-1-k}_{0(-\sigma,\tau)}(\Omega), \qquad k = 0,\dots q-1, \\
&u_l(0) && \to u_0 && \text{in } X^{q}_{0(-\sigma,\tau)}(\Omega), \\
&u_l && \to u && \text{in } X^{q}_{(-\sigma,\tau)}(\mathcal{O};\Gamma),
\end{aligned}
$$

as $l \to \infty$.

Lemma 6.4. *Let $f \in X^{q}_{(-\sigma,\tau)}(\mathcal{O})$ and $u_0 \in X^{q}_{0(-\sigma,\tau)}(\Omega)$, for $q \in \mathbf{Z}_+$ and $q \geq 1$, satisfy the compatibility conditions up to order $q-1$. Then for $q' \in \mathbf{Z}_+$, $q' \geq q$ there exist $\{f_l\} \subset H^{q'}(\mathcal{O})$ and $\{u_{0l}\} \subset H^{q'}(\Omega)$ with* $\mathrm{supp} f_l \cap (O^+ \cup O^- \cup \gamma) = \emptyset$ *and* $\mathrm{supp} u_{0l} \cap (O_0^+ \cup O_0^- \cup \gamma_0) = \emptyset$ *such that f_l and u_{0l} satisfy the compatibility conditions up to order $q'-1$ and moreover*

$$mf_l \to mf \quad \text{in } X^{q}_{(-\sigma,\tau)}(\mathcal{O}), \tag{6.1}$$

$$(\partial_0^k f_l)(0) \to (\partial_0^k f)(0) \quad \text{in } X^{q-1-k}_{0(-\sigma,\tau)}(\Omega), \quad k = 0,\dots q-1, \tag{6.2}$$

$$u_{0l} \to u_0 \quad \text{in } X^{q}_{0(-\sigma,\tau)}(\Omega), \tag{6.3}$$

as $l \to \infty$.

Admitting these lemmas we give the proof of Theorem 5.4.

Proof of Theorem 5.4. Let us set $\tilde{q} = q + [n/2] + 2$ and $q' = 2\tilde{q} + 1$. First we suppose that $f \in H^{q'}(\mathcal{O})$ and $u_0 \in H^{q'}(\Omega)$, with $\mathrm{supp} f \cap (O^+ \cup O^- \cup \gamma) = \emptyset$ and $\mathrm{supp} u_0 \cap (O_0^+ \cup O_0^- \cup \gamma_0) = \emptyset$, satisfy the compatibility conditions up to order $q'-1$. Noticing that the rank of $M(t,x)$ is constant on each component of $\Gamma \setminus \mathrm{supp} f$ and repeating the same arguments as in Lemma 3.1 in [13] we can find a $w \in H^{\tilde{q}+1}(\mathcal{O})$ with $\mathrm{supp} w \cap (O^+ \cup O^- \cup \gamma) = \emptyset$ such that $w(t,x) \in M(t,x)$ for $(t,x) \in \Gamma$, $w(0) = u_0$ and $(\partial_0^k(Lw - f))(0) = 0$ for $k = 0,\dots,\tilde{q}-1$.

Now we set $g = Lw$ and consider the following initial boundary value problem:

$$
\text{(IBVP}'\text{)} \qquad
\begin{cases}
Lv = f - g & \text{in } I \times \Omega = \mathcal{O} \\
v \in M & \text{at } I \times \partial\Omega = \Gamma \\
v(0) = 0 & \text{on } \Omega.
\end{cases}
$$

Since $f-g \in X^{\tilde{q}}_{(-\sigma+\tilde{q},\tau+\tilde{q})}(\mathcal{O};\Gamma) \cap \phi_- L^2(\mathcal{O})$ and $(\partial_0^k(f-g))(0)=0$ for $k=0,\dots,\tilde{q}-1$ Theorem 2.4 gives a weak solution $v \in X^{\tilde{q}}_{(-\sigma+\tilde{q},\tau+\tilde{q})}(\mathcal{O};\Gamma)$ to (IBVP′). If we set $u=v+w$ then it follows that $u \in X^{\tilde{q}}_{(-\sigma+\tilde{q},\tau+\tilde{q})}(\mathcal{O};\Gamma)$ and u is a weak solution to (IBVP). Moreover combining Proposition 6.2 and Lemma 6.3 we have

$$\begin{aligned}
&\|u\|^2_{X^q_{(-\sigma,\tau)}(\mathcal{O};\Gamma)}\\
&\le C\left\{\|mLu\|^2_{X^q_{(-\sigma,\tau)}(\mathcal{O})}+\sum_{k=0}^{q-1}\|(\partial_0^k Lu)(0)\|^2_{X^{q-1-k}_{0(-\sigma,\tau)}(\Omega)}+\|u_0\|^2_{X^q_{0(-\sigma,\tau)}(\Omega)}\right\}\\
&\le C'\{\|f\|^2_{X^q_{(-\sigma,\tau)}(\mathcal{O})}+\|u_0\|^2_{X^q_{0(-\sigma,\tau)}(\Omega)}\}.
\end{aligned}$$

Next we suppose that $f \in X^q_{(-\sigma,\tau)}(\mathcal{O})$ and $u_0 \in X^q_{0(-\sigma,\tau)}(\Omega)$ satisfy the compatibility conditions up to order $q-1$. Then by using Lemma 6.4 and standard limiting argument we conclude the assertion. □

Proof of Lemma 6.3. Proposition 4.4 implies that $u \in m^{-\tilde{q}} X^{\tilde{q}}_{(-\sigma+\tilde{q},\tau+\tilde{q})}(\mathcal{O}) \hookrightarrow X^{\tilde{q}}_{(-\sigma,\tau)}(\mathcal{O})$. Let us choose $\chi \in C_0^\infty(\mathbf{R})$ so that $\chi=1$ near 0 and set

$$\alpha_l(t,x)=1-\chi(l(\phi_+\phi_-)(t,x)),\quad u_l(t,x)=(\alpha_l u)(t,x)$$

for $l>0$ large enough. Then $\{u_l\}$ is a desired sequence. Indeed since $\alpha_l \in C_0^\infty(\overline{\mathcal{R}})$ and $\alpha_l=0$ on a neighborhood of $O^+ \cup O^- \cup \gamma$ it follows that $u_l \in H^{\tilde{q}}(\mathcal{O}) \hookrightarrow C^{q+1}(\overline{\mathcal{O}})$. Thus it is easily checked that $u_l \in M$ at Γ. For the proof of the desired convergences it suffices to show that

$$Lu_l \to f \quad \text{in } X^q_{(-\sigma,\tau)}(\mathcal{O}),\quad u_l \to u \quad \text{in } X^{q+1}_{(-\sigma,\tau)}(\mathcal{O}) \quad \text{as } l\to\infty. \tag{6.4}$$

Note that $f \in X^q_{(-\sigma,\tau)}(\mathcal{O})$, $\phi_+^{-1}\phi_-^{-1}u \in X^q_{(-\sigma,\tau)}(\mathcal{O})$, $u \in X^{q+1}_{(-\sigma,\tau)}(\mathcal{O})$ and

$$Lu_l-f=-\chi(l\phi_+\phi_-)f-\tilde{\chi}(l\phi_+\phi_-)\sum_{j=0}^n(\partial_j\phi_+\phi_-)A_j\phi_+^{-1}\phi_-^{-1}u,$$

$$u_l-u=-\chi(l\phi_+\phi_-)u$$

where $\tilde{\chi}(\theta)=\theta\chi'(\theta)$. Thus using arguments similar to those in Lemma 6.5 in [8] we can prove (6.4). □

Proof of Lemma 6.4. The proof of Lemma 6.4 proceeds in three steps.

FIRST STEP: If we write $u^{(k)}=T_k(f,u_0)$ for $k=0,\dots,q-1$ then we can find a $u \in X^q_{(-\sigma,\tau)}(\mathcal{O})$ such that $(\partial_0^k u)(0)=u^{(k)}$ for $k=0,\dots,q-1$ (we give the proof of this fact in Proposition 7.1 below). Now let us choose $\chi \in C_0^\infty(\mathbf{R})$ so that $\chi = 1$ near 0 and set with $\alpha_l(t,x) = 1 - \chi(lm(t,x))$

$$(6.5)\quad f_l(t,x) = (\alpha_l f)(t,x) - \sum_{j=0}^{n}((\partial_j \alpha_l)A_j u)(t,x), \quad u_{0l}(x) = (\alpha_l(0)u_0)(x)$$

for $l > 0$ large enough. Then we remark that $f_l \in X^q_{(-\sigma,\tau)}(\mathcal{O})$ and $u_{0l} \in X^q_{0(-\sigma,\tau)}(\Omega)$ with $\mathrm{supp} f_l \cap \gamma = \emptyset$ and $\mathrm{supp} u_{0l} \cap \gamma_0 = \emptyset$.

Lemma 6.5. *Let f_l and u_{0l} be given by* (6.5). *Then*
(i) *f_l and u_{0l} satisfy the compatibility conditions up to order $q-1$.*
(ii) *f_l and u_{0l} satisfy* (6.1), (6.2) *and* (6.3).

Proof. We first consider the assertion (i). We note that

$$(\phi_+^\sigma \phi_-^{-\tau})(0)(\partial_0^k Pu)(0) = \sum_{i=0}^{k}\binom{k}{i}(\partial_0^i P)(0)(\phi_+^\sigma \phi_-^{-\tau})(0)u^{(k-i)} = 0 \quad \text{on } \partial\Omega \setminus \gamma_0$$

for $k = 0, \ldots, q-1$. If we write $u_l^{(k)} = T_k(f_l, u_{0l})$ for $k = 0, \ldots, q-1$ then we have $u_l^{(k)} = (\partial_0^k \alpha_l u)(0)$, and hence it follows that

$$\begin{aligned}\sum_{i=0}^{k}\binom{k}{i}(\partial_0^i P)(0)(\phi_+^\sigma \phi_-^{-\tau})(0)u_l^{(k-i)} &= (\phi_+^\sigma \phi_-^{-\tau})(0)(\partial_0^k P\alpha_l u)(0)\\ &= \sum_{i=0}^{k}\binom{k}{i}(\partial_0^i \alpha_l)(0)(\phi_+^\sigma \phi_-^{-\tau})(0)(\partial_0^{k-i}Pu)(0) = 0 \quad \text{on } \partial\Omega \setminus \gamma_0.\end{aligned}$$

We turn to the assertion (ii). With $\tilde{\chi}(\theta) = \theta\chi'(\theta)$ we can write

$$f_l - f = -\chi(lm)f - m^{-1}\tilde{\chi}(lm)\sum_{j=0}^{n}(\partial_j m)A_j u, \quad u_{0l} - u_0 = -\chi(lm(0))u_0.$$

Therefore (6.1) and (6.3) are easily checked and since $(\partial_0^k f)(0) \in X^{q-1-k}_{0(-\sigma,\tau)}(\Omega)$ and $(\partial_0^k u)(0) = u^{(k)} \in X^{q-k}_{0(-\sigma,\tau)}(\Omega)$ for $k = 0, \ldots, q-1$ we can prove (6.2). □

In what follows, we may assume that $\mathrm{supp} f \cap \gamma = \emptyset$ and $\mathrm{supp} u_0 \cap \gamma_0 = \emptyset$.

SECOND STEP: Let $\chi \in C_0^\infty(\mathbf{R})$ be as in First step and set with $\alpha_l(t,x) = 1 - \chi(l(\phi_+\phi_-)(t,x))$

$$(6.6)\qquad f_l(t,x) = (\alpha_l f)(t,x), \quad u_{0l}(x) = (\alpha_l(0)u_0)(x)$$

for $l > 0$ large enough. Then we remark that $f_l \in X^q_{(-\sigma,\tau)}(\mathcal{O})$ and $u_{0l} \in X^q_{0(-\sigma,\tau)}(\Omega)$ with $\mathrm{supp} f_l \cap (O^+ \cup O^- \cup \gamma) = \emptyset$ and $\mathrm{supp} u_{0l} \cap (O_0^+ \cup O_0^- \cup \gamma_0) = \emptyset$. In particular, this implies that $f_l \in H^q(\mathcal{O})$ and $u_{0l} \in H^q(\Omega)$.

Lemma 6.6. *Let f_l and u_{0l} be given by* (6.6). *Then the same conclusion as in* Lemma 6.5 *holds.*

Proof. We first consider the assertion (i). Noticing that $\alpha_l = 1$ near $\Gamma \setminus (O^+ \cup O^- \cup \operatorname{supp} f)$ and $\alpha_l = 0$ near $(O^+ \cup O^-) \setminus \operatorname{supp} f$ and that $f_l = f = 0$ on $\operatorname{supp} f$ we obtain that

$$f_l = f \quad \text{near } \Gamma \setminus (O^+ \cup O^-), \quad f_l = 0 \quad \text{near } O^+ \cup O^- \cup \gamma.$$

Similarly we have

$$u_{0l} = u_0 \quad \text{near } \partial\Omega \setminus (O_0^+ \cup O_0^-), \quad u_{0l} = 0 \quad \text{near } O_0^+ \cup O_0^- \cup \gamma_0.$$

Therefore if we write $u^{(k)} = T_k(f, u_0)$ and $u_l^{(k)} = T_k(f_l, u_{0l})$ for $k = 0, \ldots, q-1$ then it follows that

$$u_l^{(k)} = u^{(k)} \quad \text{near } \partial\Omega \setminus (O_0^+ \cup O_0^-), \quad u_l^{(k)} = 0 \quad \text{near } O_0^+ \cup O_0^- \cup \gamma_0.$$

This proves the assertion (i). The assertion (ii) is easily checked. □

In what follows, we may assume that $f \in H^q(\mathcal{O})$ and $u_0 \in H^q(\Omega)$ with $\operatorname{supp} f \subset \overline{\mathcal{O}} \cap \{\phi_+ > \eta,\ \phi_- > \eta\}$ and $\operatorname{supp} u_0 \subset \overline{\Omega} \cap \{\phi_+(0) > \eta,\ \phi_-(0) > \eta\}$ for some $\eta > 0$.

THIRD STEP: Recalling that $A_b(t,x)$ is non singular on $\Gamma \cap \{\phi_+ > \eta,\ \phi_- > \eta\}$ and using the same arguments as in Lemma 3.3 in [13] we can find $\{f_l\} \subset H^{q'}(\mathcal{O})$ and $\{u_{0l}\} \subset H^{q'}(\Omega)$, with $\operatorname{supp} f_l \subset \overline{\mathcal{O}} \cap \{\phi_+ > \delta,\ \phi_- > \delta\}$ and $\operatorname{supp} u_{0l} \subset \overline{\Omega} \cap \{\phi_+(0) > \delta,\ \phi_-(0) > \delta\}$ for some $\delta = \delta(\eta) > 0$, such that f_l and u_{0l} satisfy the compatibility conditions up to order $q' - 1$ and it follows that

$$f_l \to f \quad \text{in } H^q(\mathcal{O}), \quad u_{0l} \to u_0 \quad \text{in } H^q(\Omega) \quad \text{as } l \to \infty.$$

In particular, this implies that

$$f_l \to f \quad \text{in } X^q_{(-\sigma,\tau)}(\mathcal{O}), \quad u_{0l} \to u_0 \quad \text{in } X^q_{0(-\sigma,\tau)}(\Omega) \quad \text{as } l \to \infty$$

which shows (6.1), (6.2) and (6.3). Therefore $\{f_l\}$ and $\{u_{0l}\}$ are desired sequences. Thus we conclude the proof of Lemma 6.4. □

7. Auxiliary lemmas

In this section we first show the following proposition which is used in the proof of Lemma 6.4.

Proposition 7.1. *Let $q \in \mathbf{Z}_+$, $q \geq 1$ and σ, $\tau \in \mathbf{R}$. If $u^{(k)} \in X^{q-k}_{0(\sigma,\tau)}(\Omega)$ for $k = 0,\dots,q-1$ then there exists a $u \in X^q_{(\sigma,\tau)}(\mathbf{R}\times\Omega)$ with $(\partial_0^k u)(0) = u^{(k)}$, $k = 0,\dots,q-1$, such that*

$$\|u\|_{X^q_{(\sigma,\tau)}(\mathbf{R}_+\times\Omega)} \leq C\sum_{k=0}^{q-1}\|u^{(k)}\|_{X^{q-k}_{0(\sigma,\tau)}(\Omega)}$$

where $C = C(q,\sigma,\tau) > 0$ is independent of $u^{(k)}$ and u.

For the proof of Proposition 7.1 it suffices to prove the assertion for $\sigma = \tau = -q$. Indeed assume that the statement for $\sigma = \tau = -q$ is true. We consider the general case. Let $u^{(k)} \in X^{q-k}_{0(\sigma,\tau)}(\Omega)$ for $k = 0,\dots,q-1$. We define $v^{(k)}$, $k = 0,\dots,q-1$, as follows:

$$v^{(0)} = (\phi_+^{-\sigma-q}\phi_-^{-\tau-q})(0)u^{(0)},$$

$$v^{(k)} = (\phi_+^{-\sigma-q}\phi_-^{-\tau-q})(0)\left\{u^{(k)} - \sum_{i=0}^{k-1}\binom{k}{i}(\partial_0^{k-i}\phi_+^{\sigma+q}\phi_-^{\tau+q})(0)v^{(i)}\right\} \quad \text{for } k \geq 1.$$

Then for each $v^{(k)}$ it follows that $v^{(k)} \in X^{q-k}_{0(-q,-q)}(\Omega)$ and

$$\|v^{(k)}\|_{X^{q-k}_{0(-q,-q)}(\Omega)} \leq C\sum_{i=0}^{k}\|u^{(i)}\|_{X^{q-i}_{0(\sigma,\tau)}(\Omega)}.$$

By the hypothesis we can find a $v \in X^q_{(-q,-q)}(\mathcal{O})$, with $(\partial_0^k v)(0) = v^{(k)}$ for $k = 0,\dots,q-1$, such that

$$\|v\|_{X^q_{(-q,-q)}(\mathcal{O})} \leq C\sum_{k=0}^{q-1}\|v^{(k)}\|_{X^{q-k}_{0(-q,-q)}(\Omega)}.$$

If we set $u = \phi_+^{\sigma+q}\phi_-^{\tau+q}v$ then we have $u \in X^q_{(\sigma,\tau)}(\mathcal{O})$. Noticing that $\partial_0^k u = \sum_{i=0}^k\binom{k}{i}(\partial_0^{k-i}\phi_+^{\sigma+q}\phi_-^{\tau+q})\partial_0^i v$ we obtain $(\partial_0^k u)(0) = u^{(k)}$ for $k = 0,\dots,q-1$. Moreover it follows that

$$\begin{aligned}\|u\|_{X^q_{(\sigma,\tau)}(\mathcal{O})} &= \|v\|_{X^q_{(-q,-q)}(\mathcal{O})}\\ &\leq C\sum_{k=0}^{q-1}\|v^{(k)}\|_{X^{q-k}_{0(-q,-q)}(\Omega)} \leq C'\sum_{k=0}^{q-1}\|u^{(k)}\|_{X^{q-k}_{0(\sigma,\tau)}(\Omega)}.\end{aligned}$$

Therefore u is a desired function, and hence we conclude the assertion for σ, $\tau \in \mathbf{R}$.

To prove Proposition 7.1 for $\sigma = \tau = -q$ we shall localize the problem. Let us take a covering $\{U_i\}_{i=0}^l$ of $\{t = 0\}\times\Omega$ as follows: First we cover $\{t = 0\}\times\partial\Omega$

by coordinate patches U_i, $i = 1,\dots,l$, with coordinate systems $\chi_i : U_i \cap (\mathbf{R}\times\Omega) \to \{(\tau,\xi);\ |\tau| < \delta,\ |\xi| < 1,\ \xi_1 > 0\}$ such that $\tau = t\circ\chi_i^{-1}$ and $\xi_1 = r\circ\chi_i^{-1}$ where $\delta > 0$ is small enough. Next we cover $(\{t = 0\}\times\Omega)\setminus\bigcup_{i=1}^l U_i$ by $U_0 \subset\subset \mathbf{R}\times\Omega$. Choose a partition of unity $\{\psi_i\}_{i=0}^l$ subordinate to this covering $\{U_i\}_{i=0}^l$ and set $u_i^{(k)} = \psi_i u^{(k)}$. It suffices to show Proposition 7.1 for $\sigma = \tau = -q$ with $u_i^{(k)}$ instead of $u^{(k)}$. Performing a change of independent variables we are led to the case where

$$U = \{(t,x);\ |t| < \delta,\ |x| < 1\},\quad \Omega = \mathbf{R}_+^n = \{x;\ x_1 > 0\},\quad r = x_1,$$
$$\mathrm{supp}u^{(k)} \subset \{x;\ |x| < 1-\epsilon_0,\ x_1 \geq 0\}\quad \text{for } k = 0,\dots,q-1$$

with $\epsilon_0 > 0$ small enough.

Now suppose that $q \in \mathbf{Z}_+$ $(q \geq 1)$ is given. For a fixed $k \in \mathbf{Z}_+$ $(0 \leq k \leq q-1)$ and a fixed $v \in X^{q-k}_{0(-q,-q)}(\mathbf{R}^n_+)$ we consider the following functions:

$$w(t,x) = \psi(t)t^k\Phi(t,x)V(t,x),\quad \Phi(t,x) = \chi(t(\phi_+^{-1}\phi_-^{-1})(t,x)),$$
$$V(t,x) = \int_{\mathbf{R}^n} v(x+ty)\rho(y)dy \quad \text{for } (t,x) \in \overline{\mathbf{R}}_+\times\mathbf{R}^n_+$$

where

(7.1) $\psi \in C_0^\infty(\mathbf{R})$ with $\mathrm{supp}\psi \subset \{t;\ |t| < \delta\}$,

(7.2) $\chi \in C_0^\infty(\mathbf{R})$ with $\mathrm{supp}\chi \subset \{\theta;\ |\theta| < 1\}$,

(7.3) $\rho \in C_0^\infty(\mathbf{R}^n)$ with $\mathrm{supp}\rho \subset \{y;\ |y| < \epsilon_0,\ y_1 > \epsilon_0/2,\ y_2 < 0\}$

and they satisfy $\psi = 1$ near 0, $\chi(0) = 1$ and $\int\rho(y)dy = 1$. Then we obtain the following two results.

Lemma 7.2. *It follows that* $w \in X^q_{(-q,-q)}(\mathbf{R}_+\times\mathbf{R}^n_+)$ *and*

$$\|w\|_{X^q_{(-q,-q)}(\mathbf{R}_+\times\mathbf{R}^n_+)} \leq C\|v\|_{X^{q-k}_{0(-q,-q)}(\mathbf{R}^n_+)}$$

where $C = C(q,k) > 0$.

Lemma 7.3. $(\partial_0^i w)(0,x)$, $i = 0,\dots,q-1$, *has the following properties*:

(i) $(\partial_0^i w)(0,x) = 0$ *for* $i = 0,\dots,k-1$ *and* $(\partial_0^k w)(0,x) = v(x)$.

(ii) $(\partial_0^i w)(0) \in X^{q-i}_{0(-q,-q)}(\mathbf{R}^n_+)$ *with* $\mathrm{supp}(\partial_0^i w)(0) \subset \{x;\ |x| < 1-\epsilon_0,\ x_1 \geq 0\}$ *and*

$$\|(\partial_0^i w)(0)\|_{X^{q-i}_{0(-q,-q)}(\mathbf{R}^n_+)} \leq C\|v\|_{X^{q-k}_{0(-q,-q)}(\mathbf{R}^n_+)}$$

where $C = C(q,k,i) > 0$.

Admitting that these results hold we shall complete the proof of Proposition 7.1.

Proof of Proposition 7.1. Let $u^{(k)} \in X^{q-k}_{0(-q,-q)}(\mathbf{R}^n_+)$ with $\operatorname{supp} u^{(k)} \subset \{x;\ |x| < 1-\epsilon_0,\ x_1 \geq 0\}$ for $k = 0, \ldots, q-1$. Let us set

$$u(t,x) = \sum_{k=0}^{q-1} w_k(t,x), \quad w_k(t,x) = \psi(t)t^k\Phi(t,x)\int v^{(k)}(x+ty)\rho(y)dy$$

where ψ, ρ and Φ are as above. Here we define $v^{(i)}$, $i = 0, \ldots, q-1$, as follows:

$$v^{(0)} = u^{(0)}, \quad v^{(i)} = u^{(i)} - \sum_{k=0}^{i-1}(\partial_0^i w_k)(0) \quad \text{for } i \geq 1.$$

Then for each $v^{(i)}$ it follows from Lemma 7.3 that $v^{(i)} \in X^{q-i}_{0(-q,-q)}(\mathbf{R}^n_+)$ with $\operatorname{supp} v^{(i)} \subset \{x;\ |x| < 1-\epsilon_0,\ x_1 \geq 0\}$ and

$$\|v^{(i)}\|_{X^{q-i}_{0(-q,-q)}(\mathbf{R}^n_+)} \leq C\|u^{(k)}\|_{X^{q-k}_{0(-q,-q)}(\mathbf{R}^n_+)},$$

and hence Lemma 7.2 shows that $u \in X^q_{(-q,-q)}(\mathbf{R}_+ \times \mathbf{R}^n_+)$ and

$$\begin{aligned}\|u\|_{X^q_{(-q,-q)}(\mathbf{R}_+\times\mathbf{R}^n_+)} &\leq \sum_{k=0}^{q-1}\|w_k\|_{X^q_{(-q,-q)}(\mathbf{R}_+\times\mathbf{R}^n_+)} \\ &\leq C\sum_{k=0}^{q-1}\|v^{(k)}\|_{X^{q-k}_{0(-q,-q)}(\mathbf{R}^n_+)} \leq C'\sum_{k=0}^{q-1}\|u^{(k)}\|_{X^{q-k}_{0(-q,-q)}(\mathbf{R}^n_+)}.\end{aligned}$$

Moreover we have $(\partial_0^k u)(0) = u^{(k)}$ for $k = 0, \ldots, q-1$. Therefore u is a desired function. □

We shall show Lemma 7.2 and Lemma 7.3. For the proofs we prepare several lemmas.

Lemma 7.4. *$t^i(\partial^\alpha V)(t,x)$, $0 \leq i \leq |\alpha|$, is written as a sum of the following terms:*

$$\int(\partial_x^\beta v)(x+ty)\tilde{\rho}(y)dy, \quad |\beta| = |\alpha| - i$$

where $\tilde{\rho}$ is of type (7.3).

Proof. We first consider the case $i = 0$. Since $\partial_0\{v(x+ty)\} = \sum_{j=1}^n y_j(\partial_j v)(x+ty)$ the assertion for $i = 0$ is clear. We turn to the case $i \geq 1$. For $j = 1, \ldots, n$ it follows from $t(\partial_j v)(x+ty) = \partial_{y_j}\{v(x+ty)\}$ that

$$t\int(\partial_j v)(x+ty)\tilde{\rho}(y)dy = \int v(x+ty)(-\partial_j\tilde{\rho})(y)dy.$$

Thus the assertion is proved. □

From this we obtain the following lemma which is easily checked.

Lemma 7.5. *$(\partial^\alpha w)(t,x)$, $|\alpha| \le q$, is written as a sum of the following terms*:

$$\tilde{\psi}(t)t^{k-j}(\partial^\beta\Phi)(t,x)\int(\partial_x^\gamma v)(x+ty)\tilde{\rho}(y)dy,$$
$$j+|\beta|+|\gamma| \le |\alpha|,\ 0\le j\le k,\ |\gamma|\le q-k$$

where $\tilde{\psi}$ and $\tilde{\rho}$ are of type (7.1) *and* (7.3) *respectively.*

To get the estimate for w the following lemmas will be used.

Lemma 7.6. *For $i \in \mathbf{Z}_+$ and $\alpha \in \mathbf{Z}_+^{n+1}$ there is a $C>0$ such that*

$$|t^i(\partial^\alpha\Phi)(t,x)| \le C(\phi_+^{i-|\alpha|}\phi_-^{i-|\alpha|})(t,x) \quad \textit{for } 0\le t\le\delta,\ |x|\le 1,\ x_1\ge 0.$$

Proof. We first consider the case $|\alpha|=0$. If we write $\tilde{\chi}(\theta) = \theta^i\chi(\theta)$ then it follows that $t^i\Phi(t,x) = (\phi_+^i\phi_-^i)(t,x)\tilde{\chi}(t(\phi_+^{-1}\phi_-^{-1})(t,x))$ which proves the assertion for $|\alpha|=0$. We turn to the case $|\alpha|\ge 1$. Note that $t^i(\partial^\alpha\Phi)(t,x)$ is written as a sum of the following terms:

$$t^{i+j}(\partial^{\beta_1}\phi_+^{-1}\phi_-^{-1})(t,x)\cdots(\partial^{\beta_l}\phi_+^{-1}\phi_-^{-1})(t,x)\chi^{(l)}(t(\phi_+^{-1}\phi_-^{-1})(t,x)),$$
$$1\le l\le|\alpha|,\ 0\le j\le l,\ |\beta_1|+\cdots+|\beta_l|+l = |\alpha|+j$$

with $\chi^{(l)}(\theta) = d^l\chi(\theta)/d\theta^l$. Using $|(\partial^\beta\phi_+^{-1}\phi_-^{-1})(t,x)| \le C(\phi_+^{-|\beta|-1}\phi_-^{-|\beta|-1})(t,x)$ and repeating the same arguments as above we can conclude the proof. □

Lemma 7.7. *Taking $\mu>0$ large enough we have*

$$\phi_\pm(t,x)\le\phi_\pm(0,x+ty) \quad \textit{for} \quad 0\le t\le\delta,\ |x|\le 1,\ x_1\ge 0,\ |y|<\epsilon_0,\ y_1>\epsilon_0/2.$$

Proof. If we set $f(\xi,\eta) = (\kappa\xi^2+\eta^2)^{1/2}+\eta$ for $(\xi,\eta)\in\mathbf{R}_+\times\mathbf{R}$ then we can write $\phi_\pm(t,x) = f(x_1,\mu x_1 - h_\pm(t,x))$. Since $(\partial_\xi f)(\xi,\eta)\ge 0$ and $(\partial_\eta f)(\xi,\eta)\ge 0$ it suffices to show that $\mu x_1 - h_\pm(t,x) \le \mu(x_1+ty_1) - h_\pm(0,x+ty)$ because $x_1 \le x_1+ty_1$. Since $|h_\pm(0,x+ty) - h_\pm(t,x)| \le Ct$ for some $C>0$ it follows that

$$\begin{aligned}&\{\mu(x_1+ty_1) - h_\pm(0,x+ty)\} - \{\mu x_1 - h_\pm(t,x)\}\\ &\quad = \mu t y_1 - (h_\pm(0,x+ty) - h_\pm(t,x)) \ge (\mu\epsilon_0/2 - C)t.\end{aligned}$$

Therefore taking $\mu > 0$ large enough we can prove the assertion. □

The following lemma is easily checked.

Lemma 7.8. *Let $u \in L^2(\mathbf{R}^n_+)$ with* $\operatorname{supp} u \subset \{|x| \le 1,\ x_1 \ge 1\}$ *and let $\tilde\rho$ be of type* (7.3). *Suppose that $a(t,x,y)$ with* $\operatorname{supp} a \subset \{0 \le t \le \delta\}$ *satisfies $|a(t,x,y)| \le C$ for $t \in \mathbf{R}_+$, $|x| \le 1$, $x_1 \ge 1$, $y \in \operatorname{supp}\tilde\rho$ where $C>0$ is independent of t, x and y. If we set*

$$U(t,x) = \int a(t,x,y)u(x+ty)\tilde\rho(y)dy$$

then it follows that $U(t,x) \in L^2(\mathbf{R}_+ \times \mathbf{R}^n_+)$ and

$$\|U\|_{\mathbf{R}_+\times\mathbf{R}^n_+} \le C'\|u\|_{\mathbf{R}^n_+}$$

where $C' > 0$ is independent of u and U.

Now we give the proofs of Lemma 7.2 and Lemma 7.3

Proof of Lemma 7.2. By using a reasoning similar to that in Lemma 6.1 in [8] it suffices to show that

$$\|\phi_+^{|\alpha|}\phi_-^{|\alpha|}\partial^\alpha w\|_{\mathbf{R}_+\times\mathbf{R}^n_+} \le C \sum_{|\beta|\le q-k} \|(\phi_+^{k+|\beta|}\phi_-^{k+|\beta|})(0)\partial_x^\beta v\|_{\mathbf{R}^n_+}$$

for $|\alpha| \le q$. From Lemma 7.5 we recall that $\phi_+^{|\alpha|}\phi_-^{|\alpha|}\partial^\alpha w$ is written as a sum of the following terms:

$$\int a(t,x,y)((\phi_+^{k+|\gamma|}\phi_-^{k+|\gamma|})(0)\partial_x^\gamma v)(x+ty)\tilde\rho(y)dy$$

where

$$\begin{aligned}a(t,x,y) = \tilde\psi(t)(\phi_+^{|\alpha|}\phi_-^{|\alpha|})(t,x)t^{k-j}(\partial^\beta\Phi)(t,x)(\phi_+^{-k-|\gamma|}\phi_-^{-k-|\gamma|})(0,x+ty),\\ j+|\beta|+|\gamma| \le |\alpha|,\ 0 \le j \le k,\ |\gamma| \le q-k.\end{aligned}$$

From Lemma 7.6 and Lemma 7.7 it follows that $|a(t,x,y)| \le C$ for some $C > 0$, and hence using Lemma 7.8 we conclude the proof. □

Proof of Lemma 7.3. The assertion (i) is clear. We consider the assertion (ii). We may assume $k \le i \le q-1$. For the proof it suffices to show that

$$\|(\phi_+^{i+|\alpha|}\phi_-^{i+|\alpha|})(0)\partial_x^\alpha(\partial_0^i w)(0)\|_{\mathbf{R}^n_+} \le C \sum_{|\gamma|\le q-k} \|(\phi_+^{k+|\gamma|}\phi_-^{k+|\gamma|})(0)\partial_x^\gamma v\|_{\mathbf{R}^n_+} \tag{7.4}$$

for $|\alpha| \le q-i$. Lemma 7.5 shows that $\partial_x^\alpha(\partial_0^i w)(0,x)$ is written as a sum of $(\partial^\beta\Phi)(0,x)(\partial_x^\gamma v)(x)$, $|\beta|+|\gamma| \le i+|\alpha|-k$. Thus the left-hand side of (7.4) is bounded from above by a sum of the following terms:

$$\|a_{\beta,\gamma}(\cdot)(\phi_+^{k+|\gamma|}\phi_-^{k+|\gamma|})(0)\partial_x^\gamma v\|_{\mathbf{R}^n_+}$$

where $a_{\beta,\gamma}(x) = (\phi_+^{i+|\alpha|-k-|\gamma|}\phi_-^{i+|\alpha|-k-|\gamma|}\partial^\beta\Phi)(0,x)$. From Lemma 7.6 it follows that $|a_{\beta,\gamma}(x)| \le C$ for some $C > 0$, and hence we can prove the assertion (ii). □

An immediate consequence of Lemma 7.1 is

Corollary 7.9. *Let $q \in \mathbf{Z}_+$ and σ, $\tau \in \mathbf{R}$. For $f \in X^q_{(\sigma,\tau)}(\mathbf{R}_- \times \Omega)$ there exists a $\tilde f \in X^q_{(\sigma,\tau)}(\mathbf{R} \times \Omega)$ with $\tilde f = f$ on $\mathbf{R}_- \times \Omega$ such that*

$$\|\tilde f\|_{X^q_{(\sigma,\tau)}(\mathbf{R}\times\Omega)} \le C\|f\|_{X^q_{(\sigma,\tau)}(\mathbf{R}_-\times\Omega)}$$

where $C = C(q,\sigma,\tau) > 0$ is independent of $\tilde f$ and f.

We can also obtain the following proposition.

Proposition 7.10. *Let $q \in \mathbf{Z}_+$, $q \ge 1$ and σ, $\tau \in \mathbf{R}$. If $u^{(k)} \in X^{q-k}_{0(\sigma,\tau)}(\Omega;\partial\Omega)$ for $k = 0,\dots,q-1$ there exists a $u \in X^q_{(\sigma,\tau)}(\mathbf{R}\times\Omega;\mathbf{R}\times\partial\Omega)$ with $(\partial_0^k u)(0) = u^{(k)}$, $k = 0,\dots,q-1$, such that*

$$\|u\|_{X^q_{(\sigma,\tau)}(\mathcal{O};\Gamma)} \le C\sum_{k=0}^{q-1} \|u^{(k)}\|_{X^{q-k}_{0(\sigma,\tau)}(\Omega;\partial\Omega)}$$

where $C = C(q,\sigma,\tau) > 0$ is independent of $u^{(k)}$ and u.

An immediate corollary to this proposition is

Corollary 7.11. *Let $q \in \mathbf{Z}_+$ and σ, $\tau \in \mathbf{R}$. For $f \in X^q_{(\sigma,\tau)}(\mathbf{R}_-\times\Omega;\mathbf{R}_-\times\partial\Omega)$ there exists a $\tilde f \in X^q_{(\sigma,\tau)}(\mathbf{R}\times\Omega;\mathbf{R}\times\partial\Omega)$ with $\tilde f = f$ on $\mathbf{R}_-\times\Omega$ such that*

$$\|\tilde f\|_{X^q_{(\sigma,\tau)}(\mathbf{R}\times\Omega;\mathbf{R}\times\partial\Omega)} \le C\|f\|_{X^q_{(\sigma,\tau)}(\mathbf{R}_-\times\Omega;\mathbf{R}_-\times\partial\Omega)}$$

where $C = C(q,\sigma,\tau) > 0$ is independent of $\tilde{f}$ and f.

Proof of Proposition 7.10. By the same arguments as in Proposition 7.1 it suffices to prove Proposition 7.10 for $\sigma = -q$ and $\tau = q$. By localization we may assume that $u^{(k)} \in X^{q-k}_{0(-q,q)}(\mathbf{R}^n_+)$ with $\operatorname{supp} u^{(k)} \subset \{x;\ |x| < 1-\epsilon_0,\ x_1 \geq 0\}$ for $k = 0,\dots,q-1$. Now let us set

$$u(t,x) = \sum_{k=0}^{q-1} w_k(t,x), \quad w_k(t,x) = \psi(t)t^k\Phi(t,x)\int v^{(k)}(x_1e^{ty_1}, x'+ty')\rho(y)dy$$

with $x = (x_1,x') = (x_1,x_2,\dots,x_n)$ where ψ, ρ, Φ and $v^{(i)}$, $i = 0,\dots,q-1$, are as in the proof of Proposition 7.1. Then u is shown to be a desired function using the following lemma instead of Lemma 7.7. □

Lemma 7.12. *Taking coordinate patches U small enough and coordinate systems χ appropriate, if necessary, we obtain that*

$$\phi_+(t,x) \leq C\phi_+(0, x_1e^{ty_1}, x'+ty'), \tag{7.5}$$

$$\phi_-^{-1}(t,x) \leq C\phi_-^{-1}(0, x_1e^{ty_1}, x'+ty') \tag{7.6}$$

for $0 \leq t \leq \delta$, $|x| \leq 1$, $x_1 \geq 0$, $|y| < \epsilon_0$, $y_2 < 0$ where $C > 0$ is independent of t, x and y.

Proof of Lemma 7.12. Let us set $U_0 = \{x;\ (0,x) \in U\}$. We shall prove the case $U_0 \cap \gamma_0^\pm \neq \emptyset$. Otherwise the proof is easier. There are two cases as follows:
(I) $(\partial_2 h_\pm,\dots,\partial_n h_\pm)(0,x) \neq (0,\dots,0)$ for any $x \in U_0 \cap \gamma_0^\pm$.
(II) $(\partial_2 h_\pm,\dots,\partial_n h_\pm)(0,\bar{x}) = (0,\dots,0)$ for some $\bar{x} \in U_0 \cap \gamma_0^\pm$.

We first consider the case (I). Then we may assume that χ satisfies not only $\tau = t\circ\chi^{-1}$ and $\xi_1 = r\circ\chi^{-1}$ but also $\xi_2 = \pm h_\pm\circ\chi^{-1}$. Performing a change of independent variables we can write

$$\phi_\pm(t,x) = \{\kappa x_1^2 + (\mu x_1 \mp x_2)^2\}^{1/2} + \mu x_1 \mp x_2.$$

Since $(\partial_1\phi_-)(t,x) \geq 0$ and $(\partial_2\phi_-)(t,x) \geq 0$ it follows from $x_1e^{ty_1} \leq e^{\delta\epsilon_0}x_1$ and $x_2 + ty_2 \leq x_2$ that

$$\begin{aligned}
&\phi_-(0, x_1e^{ty_1}, x'+ty') \\
&\leq \phi_-(0, e^{\delta\epsilon_0}x_1, x') = \phi_-(t, e^{\delta\epsilon_0}x_1, x') = \phi_-(t,x;\kappa e^{2\delta\epsilon_0}, \mu e^{\delta\epsilon_0}) \\
&\leq C\phi_-(t,x;\kappa,\mu) = C\phi_-(t,x)
\end{aligned}$$

which shows (7.6). Similarly we can prove (7.5).

We turn to the case (II). We note that $\pm(\partial_0 h_\pm)(0,\bar{x}) > 0$. Indeed if the identity $(\partial_0 h_-)(0,\bar{x}) = 0$ holds then $A_{\gamma/b}(0,\bar{x}) = 0$ would follow from (1.4). This is incompatible with (1.5). Suppose that $(\partial_0 h_-)(0,\bar{x}) > 0$ holds. Then we would have $h_-(t,\bar{x}) > 0$ if $0 < t < \delta$ and $h_-(t,\bar{x}) < 0$ if $-\delta < t < 0$. In particular, we obtain $(t,\bar{x}) \in O^-$ if $0 < t < \delta$, and hence $A_b(t,\bar{x})$ is negative definite there. On the other hand, since it follows from (1.4) that $A_{\gamma/b}(0,\bar{x})$ is positive definite then $A_{b,\gamma}(0,\bar{x})$ is also positive definite. This is incompatible with (1.1) and (1.3). Therefore we have $(\partial_0 h_-)(0,\bar{x}) < 0$. Similarly we can prove $(\partial_0 h_+)(0,\bar{x}) > 0$. Thus taking U small enough we may assume that $\pm(\partial_0 h_\pm)(t,x) \geq c_0$ on U for some $c_0 > 0$.

Now we shall show (7.6). If we set $f(\xi,\eta) = \{\kappa\xi^2 + (\mu\xi - \eta)^2\}^{1/2} + \mu\xi - \eta$ for $(\xi,\eta) \in \mathbf{R}_+ \times \mathbf{R}$ then we can write $\phi_-(t,x) = f(x_1, h_-(t,x))$. Since $(\partial_\xi f)(\xi,\eta) \geq 0$ and $(\partial_\eta f)(\xi,\eta) \leq 0$ it suffices to show that $h_-(0, x_1 e^{ty_1}, x' + ty') \geq h_-(t,x)$ because $x_1 e^{ty_1} \leq e^{\delta\epsilon_0} x_1$. Indeed admitting this assertion we have

$$\phi_-(0, x_1 e^{ty_1}, x' + ty') \leq \phi_-(t, x; \kappa e^{2\delta\epsilon_0}, \mu e^{\delta\epsilon_0}) \leq C\phi_-(t,x)$$

which concludes (7.6). Note that

$$\begin{aligned} &h_-(0, x_1 e^{ty_1}, x' + ty') - h_-(t,x) \\ &= -t\int_0^1 (\partial_0 h_-)(t - \theta t, x_1 + \theta x_1(e^{ty_1} - 1), x' + \theta t y')d\theta \\ &\quad + x_1(e^{ty_1} - 1)\int_0^1 (\partial_1 h_-)(\cdots)d\theta + \sum_{j=2}^n t y_j \int_0^1 (\partial_j h_-)(\cdots)d\theta \\ &\geq c_0 t - C\epsilon_0 t. \end{aligned}$$

Therefore taking $\epsilon_0 > 0$ small enough we can prove the assertion. Similarly we can obtain (7.5). □

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