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# AN OBSTRUCTION TO ASYMPTOTIC SEMISTABILITY AND APPROXIMATE CRITICAL METRICS

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## 1. Introduction

For a polarized algebraic manifold  $(M, L)$  with a Kähler metric of constant scalar curvature in the class  $c_1(L)_{\mathbb{R}}$ , we consider the Kodaira embedding

$$\Phi_{|L^m|} : M \hookrightarrow \mathbb{P}(V_m), \quad m \gg 1,$$

where  $V_m := H^0(M, \mathcal{O}(L^m))^*$ . Even when a linear algebraic group of positive dimension acts nontrivially and holomorphically on  $M$ , we shall show that the vanishing of an obstruction to asymptotic Chow-semistability allows us to generalize Donaldson's construction [3] of approximate solutions for equations of critical metrics<sup>1</sup> of Zhang [20]. This generalization plays a crucial role in our forthcoming paper [14], in which the asymptotic Chow-stability for  $(M, L)$  above will be shown under the vanishing of the obstruction, even when  $M$  admits a group action as above.

## 2. Statement of results

Throughout this paper, we assume that  $L$  is an ample holomorphic line bundle over a connected projective algebraic manifold  $M$ . Let  $n$  and  $d$  be respectively the dimension of  $M$  and the degree of the image  $M_m := \Phi_{|L^m|}(M)$  in the projective space  $\mathbb{P}(V_m)$  with  $m \gg 1$ . Then to this image  $M_m$ , we can associate a nonzero element  $\hat{M}_m$  of  $W_m := \{\text{Sym}^d(V_m)\}^{\otimes n+1}$  such that its natural image  $[\hat{M}_m]$  in  $\mathbb{P}(W_m)$  is the Chow point associated to the irreducible reduced algebraic cycle  $M_m$  on  $\mathbb{P}(V_m)$ . For the natural action of  $H_m := \text{SL}(V_m)$  on  $W_m$  and also on  $\mathbb{P}(W_m)$ , the subvariety  $M_m$  of  $\mathbb{P}(V_m)$  is said to be *Chow-stable* or *Chow-semistable*, according as the orbit  $H_m \cdot \hat{M}$  is closed in  $W_m$  or the origin of  $W_m$  is not in the closure of  $H_m \cdot \hat{M}$  in  $W_m$ . Fix an increasing sequence

$$(2.1) \quad m(1) < m(2) < m(3) < \cdots < m(k) < \cdots$$

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<sup>1</sup>In (2.6) below,  $\omega = c_1(L; h)$  is called a *critical metric* if  $K(q, h)$  is a constant function on  $M$ . The same concept was later re-discovered by Luo [12] (see [14]).

of positive integers  $m(k)$ . For this sequence, we say that  $(M, L)$  is *asymptotically Chow-stable* or *asymptotically Chow-semistable*, according as for some  $k_0 \gg 1$ , the subvariety  $M_{m(k)}$  of  $\mathbb{P}(V_{m(k)})$  is Chow-stable or Chow-semistable for all  $k \geq k_0$ .

Let  $\text{Aut}^0(M)$  denote the identity component of the group of all holomorphic automorphisms of  $M$ . Then the maximal connected linear algebraic subgroup  $G$  of  $\text{Aut}^0(M)$  is the identity component of the kernel of the Jacobi homomorphism

$$\alpha_M: \text{Aut}^0(M) \rightarrow \text{Aut}^0(\text{Alb}(M)), \quad (\text{cf. [4]}).$$

For the maximal algebraic torus  $Z$  in the center of  $G$ , we consider the Lie subalgebra  $\mathfrak{z}$  of  $H^0(M, \mathcal{O}(T^{1,0}M))$  associated to the Lie subgroup  $Z$  of  $\text{Aut}^0(M)$ . For the isotropy subgroup, denoted by  $\tilde{S}_m$ , of  $H_m$  at the point  $[\hat{M}_m] \in \mathbb{P}(W_m)$ , we have a natural isogeny

$$\iota_m: \tilde{S}_m \rightarrow S_m,$$

where  $S_m$  is an algebraic subgroup of  $G$ . For  $Z_m := \iota_m^{-1}(Z)$ , we have a  $Z_m$ -action on  $M$  naturally induced by the  $Z$ -action on  $M$ . Since the  $Z$ -action on  $M$  is liftable to a holomorphic bundle action on  $L$  (see for instance [7]), the restriction of  $\iota_m$  to  $Z_m$  defines an isogeny of  $Z_m$  onto  $Z$ . The vector space  $V_m$  is viewed as the line bundle  $\mathcal{O}_{\mathbb{P}(V_m)}(-1)$  with the zero section blown-down to a point, while the line bundle  $\mathcal{O}_{\mathbb{P}(V_m)}(-1)$  coincides with  $L^{-m}$  when restricted to  $M$ . Hence, the natural  $\tilde{S}_m$ -action on  $V_m$  induces a bundle action of  $Z_m$  on  $L^m$  which covers the  $Z_m$ -action on  $M$ . Infinitesimally, each  $X \in \mathfrak{z}$  induces a holomorphic vector field  $X' \in H^0(L^m, \mathcal{O}(T^{1,0}L^m))$  on  $L^m$ . Since the  $\mathbb{C}^*$ -bundle  $L \setminus \{0\}$  associated to  $L$  is an  $m$ -fold unramified covering of the  $\mathbb{C}^*$ -bundle  $L^m \setminus \{0\}$ , the restriction of  $X'$  to  $L^m \setminus \{0\}$  naturally induces a holomorphic vector field  $X''$  on  $L \setminus \{0\}$ . Since  $X''$  extends to a holomorphic vector field on  $L$ , the mapping  $X \mapsto X''$  defines inclusions

$$(2.2) \quad \rho_m: \mathfrak{z} \hookrightarrow H^0(L, \mathcal{O}(T^{1,0}L)), \quad m = 1, 2, \dots,$$

inducing lifts, from  $M$  to  $L$ , of vector fields in  $\mathfrak{z}$ . For a sequence as in (2.1), we say that *the isotropy actions for  $(M, L)$  are stable* if there exists an integer  $k_0 \gg 1$  such that

$$(2.3) \quad \rho_{m(k)} = \rho_{m(k_0)}, \quad \text{for all } k \geq k_0.$$

For the maximal compact subgroup  $(Z_m)_c$  of  $Z_m$ , take a  $(Z_m)_c$ -invariant Hermitian metric  $\lambda$  for  $L^m$ . By the theory of equivariant cohomology ([1], [8]), we define (see [15], [13]):

$$(2.4) \quad \mathcal{C}\{c_1^{n+1}; L^m\}(X) := \frac{\sqrt{-1}}{2\pi} (n+1) \int_M \lambda^{-1}(X\lambda) c_1(L^m; \lambda)^n, \quad X \in \mathfrak{z},$$

where  $X\lambda$  is as in [13], (1.4.1). Then the  $\mathbb{C}$ -linear map  $\mathcal{C}\{c_1^{n+1}; L^m\}: \mathfrak{z} \rightarrow \mathbb{C}$  which sends each  $X \in \mathfrak{z}$  to  $\mathcal{C}\{c_1^{n+1}; L^m\}(X) \in \mathbb{C}$  is independent of the choice of  $h$ . The following gives an obstruction to asymptotic Chow-semistability (see [5], [15], [16] for related results):

**Theorem A.** *For a sequence as in (2.1), assume that  $(M, L)$  is asymptotically Chow-semistable. Then for some  $k_0 \gg 1$ , the equality  $\mathcal{C}\{c_1^{n+1}; L^{m(k)}\} = 0$  holds for all  $k \geq k_0$ . In particular, for this sequence, the isotropy actions for  $(M, L)$  are stable.*

The following modification of a result in [7] shows that, as an obstruction, the stability condition (2.3) is essential, since the vanishing of (2.4) is straightforward from (2.3).

**Theorem B.** *For sufficiently large  $(n+2)$  distinct integers  $m_k$ ,  $k = 0, 1, \dots, n+1$ , suppose that  $\rho_{m_0} = \rho_{m_1} = \dots = \rho_{m_{n+1}}$ . Then  $\mathcal{C}\{c_1^{n+1}; L^{m_k}\} = 0$  for all  $k$ .*

If  $\dim Z = 0$ , by setting  $m(k) = k$  in (2.1) for all  $k > 0$ , we see that  $\rho_m$  are trivial for all  $m \gg 1$ , and consequently (2.3) holds. Note also that Donaldson's result [3] treating the case  $\dim G = 0$  depends on his construction of approximate solutions for equations of critical metrics of Zhang [20]. In Theorem C down below, assuming (2.3), we generalize Donaldson's construction to the case  $\dim G > 0$ .

Put  $N_m := \dim_{\mathbb{C}} V_m - 1$ . Let  $h$  be a Hermitian metric for  $L$  such that  $\omega = c_1(L; h)$  is a Kähler metric on  $M$ . By the inner product

$$(2.5) \quad (\sigma, \sigma')_h := \int_M \langle \sigma, \sigma' \rangle_h \omega^n, \quad \sigma, \sigma' \in V_m^*,$$

on  $V_m^* = H^0(M, \mathcal{O}(L^m))$ , we choose a unitary basis  $\{\sigma_0^{(m)}, \sigma_1^{(m)}, \dots, \sigma_{N_m}^{(m)}\}$  for  $V_m^*$ . Here,  $\langle \sigma, \sigma' \rangle_h$  denotes the function on  $M$  obtained as the pointwise inner product of the sections  $\sigma, \sigma'$  by the Hermitian metric  $h^m$  on  $L^m$ . Put

$$(2.6) \quad K(q, h) := \frac{n!}{m^n} \sum_{i=0}^{N_m} \|\sigma_i^{(m)}\|_h^2,$$

where  $\|\sigma\|_h^2 := \langle \sigma, \sigma \rangle_h$  for all  $\sigma \in V_m^*$ , and we set  $q := 1/m$ . We then have the asymptotic expansion of Tian-Zelditch (cf. [18], [19]) for  $m \gg 1$ :

$$(2.7) \quad K(q, h) = 1 + a_1(\omega)q + a_2(\omega)q^2 + a_3(\omega)q^3 + \dots,$$

where  $a_i(\omega)$ ,  $i = 1, 2, \dots$ , are smooth functions on  $M$ . Then  $a_1(\omega) = \sigma_\omega/2$  (cf. [11]) for the scalar curvature  $\sigma_\omega$  of  $\omega$ . Put  $C_q := \{m^n c_1(L)^n [M]/n!\}^{-1} (N_m + 1)$ . Then

**Theorem C.** *For a Kähler metric  $\omega_0$  in the class  $c_1(L)_{\mathbb{R}}$  of constant scalar curvature, choose a Hermitian metric  $h_0$  for  $L$  such that  $\omega_0 = c_1(L; h_0)$ . For a sequence as in (2.1), assume that the isotropy actions for  $(M, L)$  are stable, i.e., (2.3) holds. Put  $q = 1/m(k)$ . Then there exists a sequence of real-valued smooth functions  $\varphi_k$ ,  $k = 1, 2, \dots$ , on  $M$  such that  $h(l) := h_0 \exp(-\sum_{k=1}^l q^k \varphi_k)$  satisfies  $K(q, h(l)) - C_q = O(q^{l+2})$  for each nonnegative integer  $l$ .*

The last equality  $K(q, h(l)) - C_q = O(q^{l+2})$  means that there exist a positive real constant  $A = A_l$  independent of  $q$  such that  $\|K(q, h(l)) - C_q\|_{C^0(M)} \leq A_l q^{l+2}$  for all  $0 \leq q \leq 1$  on  $M$ . By [19], for every nonnegative integer  $j$ , a choice of a larger constant  $A = A_{j,l} > 0$  keeps Theorem C still valid even if  $C^0(M)$ -norm is replaced by  $C^j(M)$ -norm.

### 3. An obstruction to asymptotic semistability

The purpose of this section is to prove Theorems A and B. Fix a sequence as in (2.1), and in this section, any kind of stability is considered with respect to this sequence.

**Proof of Theorem A.** Assume that  $(M, L)$  is asymptotically Chow-semistable, i.e., for some  $k_0 \gg 1$ , the subvariety  $M_{m(k)}$  of  $\mathbb{P}(V_{m(k)})$  is Chow-semistable for all  $k \geq k_0$ . Then the isotropy representation of  $Z_{m(k)}$  on the line  $\mathbb{C} \cdot \hat{M}_{m(k)}$  is trivial (cf. [5], [15]) for  $k \geq k_0$ , and hence by [15], (3.5) (cf. [16]; [20], (1.5)), we obtain the required equality

$$(3.1) \quad \mathcal{C}\{c_1^{n+1}; L^{m(k)}\}(X) = 0, \quad X \in \mathfrak{z},$$

for all  $k \geq k_0$ . For  $\lambda$  in (2.4), by setting  $h := \lambda^{1/m}$ , we have a Hermitian metric  $h$  for  $L$ . Put  $\chi_m := \mathcal{C}\{c_1^{n+1}, L^m\}/m^{n+1}$  for positive integers  $m$ . Then by the Leibniz rule,

$$(3.2) \quad \chi_m(X) = \frac{\sqrt{-1}}{2\pi} (n+1) \int_M h^{-1}(Xh)_{\rho_m} c_1(L; h)^n, \quad X \in \mathfrak{z},$$

where the complexified action  $(Xh)_{\rho_m}$  of  $X$  on  $h$  as in [13], (1.4.1), is taken via the lifting  $\rho_m$  in (2.2). Then by (3.1),

$$\chi_{m(k_0)} = \chi_{m(k_0+1)} = \dots = \chi_{m(k)} = \dots,$$

and since lifts in (2.2), from  $M$  to  $L$ , of holomorphic vector fields in  $\mathfrak{z}$  are completely characterized by  $\chi_m$  (cf. [7]), we obtain (2.3), as required.  $\square$

**Proof of Theorem B.** For  $q := \text{l.c.m}\{m_k; k = 0, 1, \dots, n+1\}$ , we take a  $q$ -fold unramified cover  $\nu: \tilde{Z} \rightarrow Z$  between algebraic tori. Then the  $Z$ -action on  $M$  naturally

induces a  $\tilde{Z}$ -action on  $M$  via this covering. Since  $\nu$  factors through  $Z_{m_k}$ , the lift, from  $M$  to  $L^{m_k}$ , of the  $Z_{m_k}$ -action naturally induces a lift, from  $M$  to  $L^{m_k}$ , of the  $\tilde{Z}$ -action. The assumption

$$(3.3) \quad \rho_{m_0} = \rho_{m_1} = \cdots = \rho_{m_{n+1}}$$

shows that the lifts, from  $M$  to  $L^{m_k}$ ,  $k = 0, 1, \dots, n+1$ , of the  $\tilde{Z}$ -action come from the same infinitesimal action of  $\mathfrak{z}$  as vector fields on  $L$ . For brevity, the common  $\rho_{m_k}$  in (3.3) will be denoted just by  $\rho$ . Then the proof of [6], Theorem 5.1, is valid also in our case, and the formula in the theorem holds. By  $Z_{m_k} \subset \mathrm{SL}(V_{m_k})$  and by its contragredient representation, the  $\tilde{Z}$ -action on  $V_{m_k}^* = H^0(M, \mathcal{O}(L^{m_k}))$  comes from an algebraic group homomorphism:  $\tilde{Z} \rightarrow \mathrm{SL}(V_{m_k}^*)$ . Hence, by the notation in (3.2) above,  $\int_M h^{-1}(Xh) \rho c_1(L; h)^n = 0$  for all  $X \in \mathfrak{z}$ , i.e.,  $\mathcal{C}\{c_1^{n+1}; L^{m_k}\} = 0$  for all  $k$ , as required.  $\square$

#### 4. Proof of Theorem C

Throughout this section, we assume that the first Chern class  $c_1(L)_{\mathbb{R}}$  admits a Kähler metric of constant scalar curvature. Then a result of Lichnérowicz [10] (see also [9]) shows that  $G$  is a reductive algebraic group, and consequently the identity component of the center of  $G$  coincides with  $Z$  in the introduction. Let  $K$  be a maximal compact subgroup of  $G$ . Then the maximal compact subgroup  $Z_c$  of  $Z$  satisfies

$$(4.1) \quad Z_c \subset K.$$

For an arbitrary  $K$ -invariant Kähler metric  $\omega$  on  $M$  in the class  $c_1(L)_{\mathbb{R}}$ , we write  $\omega$  as the Chern form  $c_1(L; h)$  for some Hermitian metric  $h$  for  $L$ . Let  $\Psi(q, \omega)$  denote the power series in  $q$  given by the right-hand side of (2.7). Then

$$(4.2) \quad \int_M \{\Psi(q, \omega) - C_q\} \omega^n = \int_M \left\{ -C_q + \frac{n!}{m^n} \sum_{i=0}^{N_m} \|\sigma_i^{(m)}\|_h^2 \right\} \omega^n = 0.$$

Let  $h_0$  be a Hermitian metric for  $L$  such that  $\omega_0 := c_1(L; h_0)$  is a Kähler metric of constant scalar curvature on  $M$ . We write

$$\omega_0 = \frac{\sqrt{-1}}{2\pi} \sum_{\alpha, \beta} g_{\alpha\bar{\beta}} dz^\alpha \wedge d\bar{z}^\beta,$$

for a system  $(z^1, z^2, \dots, z^n)$  of holomorphic local coordinates on  $M$ . In view of [10] (see also [9]), replacing  $\omega_0$  by  $g^* \omega_0$  for some  $g \in G$  if necessary, we may assume that  $\omega_0$  is  $K$ -invariant. Let  $D_0$  be the Lichnérowicz operator, as defined in [2], (2.1), for the Kähler manifold  $(M, \omega_0)$ . Since  $\omega_0$  has a constant scalar curvature,  $D_0$  is a real operator. Let  $\mathcal{F}$  denote the space of all real-valued smooth  $K$ -invariant functions

$\varphi$  such that  $\int_M \varphi \omega_0^n = 0$ . Since the operator  $D_0$  preserves the space  $\mathcal{F}$ , we write  $D_0$  as an operator  $D_0: \mathcal{F} \rightarrow \mathcal{F}$ , and the kernel in  $\mathcal{F}$  of this operator will be denoted by  $\text{Ker } D_0$ . Let  $\mathfrak{z}_c$  denote the Lie subalgebra of  $\mathfrak{z}$  corresponding to the maximal compact subgroup  $Z_c$  of  $Z$ . Then

$$(4.3) \quad \gamma: \text{Ker } D_0 \cong \mathfrak{z}_c, \quad \eta \leftrightarrow \gamma(\eta) := \text{grad}_{\omega_0}^{\mathbb{C}} \eta,$$

where  $\text{grad}_{\omega_0}^{\mathbb{C}} \eta := (1/\sqrt{-1}) \sum g^{\bar{\beta}\alpha} \eta_{\bar{\beta}} \partial/\partial z^{\alpha}$  denotes the complex gradient of  $\eta$  with respect to  $\omega_0$ . We then consider the orthogonal projection

$$P: \mathcal{F} (= \text{Ker } D_0 \oplus \text{Ker } D_0^{\perp}) \rightarrow \text{Ker } D_0.$$

Starting from  $h(0) = h_0$  and  $\omega(0) := \omega_0$ , we inductively define a Hermitian metric  $h(k)$  for  $L$ , and a Kähler metric  $\omega(k) := c_1(L; h(k))$ , called the *k-approximate solution*, by

$$\begin{aligned} h(k) &= h(k-1) \exp(-q^k \varphi_k), & k = 1, 2, \dots, \\ \omega(k) &= \omega(k-1) + \frac{\sqrt{-1}}{2\pi} q^k \partial \bar{\partial} \varphi_k, & k = 1, 2, \dots, \end{aligned}$$

for a suitable function  $\varphi_k \in \text{Ker } D_0^{\perp}$ , where we require  $h(k)$  to satisfy  $K(q, h(k)) - C_q = O(q^{k+2})$ . In other words, by (4.2), each  $\omega(k)$  is required to satisfy the following conditions:

$$(4.4) \quad (1 - P)\{\Psi(q, \omega(k)) - C_q\} \equiv 0, \quad \text{modulo } q^{k+2},$$

$$(4.5) \quad P\{\Psi(q, \omega(k)) - C_q\} \equiv 0, \quad \text{modulo } q^{k+2}.$$

If  $k = 0$ , then  $\omega(0) = \omega_0$ , and by [11], both (4.4) and (4.5) hold for  $k = 0$ . Hence, let  $l \geq 1$  and assume (4.4) and (4.5) for  $k = l-1$ . It then suffices to find  $\varphi_l \in \text{Ker } D_0^{\perp}$  satisfying both (4.4) and (4.5) for  $k = l$ . Put

$$\Phi(q, \varphi) := (1 - P) \left\{ \Psi \left( q, \omega(l-1) + \frac{\sqrt{-1}}{2\pi} q^l \partial \bar{\partial} \varphi \right) - C_q \right\}, \quad \varphi \in \text{Ker } D_0^{\perp}.$$

Then by (4.4) applied to  $k = l-1$ , we have  $\Phi(q, 0) \equiv u_l q^{l+1}$  modulo  $q^{l+2}$ , where  $u_l$  is a function in  $\text{Ker } D_0^{\perp}$ . Since  $2\pi\omega(l-1) = 2\pi\omega_0 + \sqrt{-1} \sum_{k=1}^{l-1} q^k \partial \bar{\partial} \varphi_k$ , we have  $\omega(l-1) = \omega_0$  at  $q = 0$ . Since the scalar curvature of  $\omega_0$  is constant, the variation formula for the scalar curvature (see for instance [2], (2.5); [3]) shows that

$$\Phi(q, \varphi_l) \equiv \Phi(q, 0) - q^{l+1} \frac{D_0 \varphi_l}{2} \equiv (2u_l - D_0 \varphi_l) \frac{q^{l+1}}{2},$$

modulo  $q^{l+2}$ . Since  $u_l$  is in  $\text{Ker } D_0^{\perp}$ , there exists a unique  $\varphi_l \in \text{Ker } D_0^{\perp}$  such that  $2u_l = D_0 \varphi_l$  on  $M$ . Fixing such  $\varphi_l$ , we obtain  $h(l)$  and  $\omega(l)$ . Thus (4.4) is true for  $k = l$ .

Now, we have only to show that (4.5) is true for  $k = l$ . Before checking this, we give some preliminary remarks. Note that  $C_q = 1 + O(q)$ . Moreover, by (2.7),  $\Psi(q, \omega) = 1 + q\{a_1(\omega) + a_2(\omega)q + \dots\}$ , and hence

$$\begin{aligned}\Psi(q, \omega(l)) - C_q &= \Psi\left(q, \omega(l-1) + \frac{\sqrt{-1}}{2\pi}q^l \partial\bar{\partial}\varphi_l\right) - C_q \\ &\equiv \Psi(q, \omega(l-1)) - C_q \equiv 0, \quad \text{modulo } q^{l+1}.\end{aligned}$$

By [17], p. 35, the  $G$ -action on  $M$  is liftable to a bundle action of  $G$  on the real line bundle  $(L \cdot \bar{L})^{1/2} = (L^m \cdot \bar{L}^m)^{1/2m}$ . Then the induced  $K$ -action on  $(L \cdot \bar{L})^{1/2}$  is unique, because liftings, from  $M$  to  $L^m$ , of the  $G$ -action differ only by scalar multiplications of  $L^m$  by characters of  $Z$ . In this sense,  $h(l)$  is  $K$ -invariant. Put  $r := \dim_{\mathbb{C}} Z$ . Then we can write  $Z_m = \mathbb{G}_m^r = \{t = (t_1, t_2, \dots, t_r) \in (\mathbb{C}^*)^r\}$ . By the natural inclusion

$$\psi_m: Z_m \hookrightarrow H_m = \mathrm{SL}(V_m),$$

we can choose a unitary basis  $\{\tau_0, \tau_1, \dots, \tau_{N_m}\}$  for  $(V_m^*, (\cdot, \cdot)_{h(l)})$  (cf. (2.5)) such that, for some integers  $\alpha_{ij}$  with  $\sum_i \alpha_{ij} = 0$ , the contragredient representation  $\psi_m^*$  of  $\psi_m$  is given by

$$\psi_m^*(t)\tau_i = \left( \prod_{j=1}^r t_j^{\alpha_{ij}} \right) \tau_i, \quad i = 0, 1, \dots, N_m,$$

for all  $t \in (\mathbb{C}^*)^r = Z_m$ . Now by (2.3), for some  $\rho: \mathfrak{z} \hookrightarrow H^0(L, \mathcal{O}(T^{1,0}L))$ , we can write  $\rho_{m(k)} = \rho$  for all  $k \geq k_0$ . Consider the Kähler metric  $\omega_m := c_1(L; h_m)$  on  $M$  in the clasas  $c_1(L)_{\mathbb{R}}$ , where  $h_m := (|\tau_0|^2 + |\tau_1|^2 + \dots + |\tau_{N_m}|^2)^{-1/m}$ . From now on, let  $m = m(k)$ , where  $k$  is running through all integers  $\geq k_0$ . Put  $X_j := t_j \partial/\partial t_j$ . Then  $\{X_1, X_2, \dots, X_r\}$  forms a  $\mathbb{C}$ -basis for the Lie algebra  $\mathfrak{z}$  such that, using the notation as in (3.2), we have

$$(4.6) \quad h_m^{-1}(X_j h_m)_{\rho} = -\frac{\sum_i \alpha_{ij} |\tau_i|^2}{m \sum_i |\tau_i|^2}, \quad 1 \leq j \leq r, \quad \text{for } m = m(k) \text{ with } k \geq k_0,$$

where in the numerator and the denominator, the sum is taken over all integers  $i$  such that  $0 \leq i \leq N_m$ . From (2.3) and Theorem B, using the notation as in (3.2), we obtain

$$(4.7) \quad \int_M h(l)^{-1}(X_j h(l))_{\rho} \omega(l)^n = 0, \quad 1 \leq j \leq r.$$

By  $\int_M h_0^{-1}(X_j h_0)_{\rho} \omega_0^n / \int_M \omega_0^n = 0$ , we have  $\eta_j := h_0^{-1}(X_j h_0)_{\rho} \in \mathrm{Ker} D_0$ . Then  $\gamma(\eta_j) = \sqrt{-1} X_j$ . Hence  $\{\eta_1, \eta_2, \dots, \eta_r\}$  is an  $\mathbb{R}$ -basis for  $\mathrm{Ker} D_0$ . Since  $\Psi(q, \omega(l)) \equiv C_q$

modulo  $q^{l+1}$ , it follows that

$$(4.8) \quad -C_q + \frac{n!}{m^n} \sum_{i=0}^{N_m} \|\tau_i\|_{h(l)}^2 \equiv v_l q^{l+1}$$

modulo  $q^{l+2}$  for some  $v_l \in \text{Ker } D_0$ , because (4.4) is true for  $k = l$ . In view of (4.2), (4.6),  $h_m - h_0 = O(q)$  and  $\omega(l) - \omega_0 = O(q)$ , we see from (4.8) that, modulo  $q^{l+2}$ ,

$$\begin{aligned} q^{l+1} \int_M \eta_j v_l \omega_0^n &\equiv \int_M \eta_j \left( -C_q + \frac{n!}{m^n} \sum_{i=0}^{N_m} \|\tau_i\|_{h(l)}^2 \right) \{\omega(l)\}^n \\ &\equiv \int_M h_0^{-1}(X_j h_0)_\rho \left( -C_q + \frac{n!}{m^n} \sum_{i=0}^{N_m} \|\tau_i\|_{h(l)}^2 \right) \{\omega(l)\}^n \\ &\equiv \int_M h_m^{-1}(X_j h_m)_\rho \left( -C_q + \frac{n!}{m^n} \sum_{i=0}^{N_m} \|\tau_i\|_{h(l)}^2 \right) \{\omega(l)\}^n \\ &\equiv \int_M \frac{\sum_i \alpha_{ij} \|\tau_i\|_{h(l)}^2}{m \sum_i \|\tau_i\|_{h(l)}^2} \left( C_q - \frac{n!}{m^n} \sum_{i=0}^{N_m} \|\tau_i\|_{h(l)}^2 \right) \{\omega(l)\}^n. \end{aligned}$$

Since  $\sum_i \alpha_{ij} = 0$  for all  $j$ , we obtain, modulo  $q^{l+2}$ ,

$$\begin{aligned} q^{l+1} \int_M \eta_j v_l \omega_0^n &\equiv C_q \int_M \frac{\sum_i \alpha_{ij} \|\tau_i\|_{h(l)}^2}{m \sum_i \|\tau_i\|_{h(l)}^2} \{\omega(l)\}^n \equiv C_q \int_M h_m^{-1}(X_j h_m)_\rho \{\omega(l)\}^n \\ &\equiv C_q \int_M \{h_m^{-1}(X_j h_m)_\rho - h(l)^{-1}(X_j h(l))_\rho\} \{\omega(l)\}^n, \end{aligned}$$

where the equivalence just above follows from (4.7). The last integrand is rewritten as

$$\begin{aligned} h_m^{-1}(X_j h_m)_\rho - h(l)^{-1}(X_j h(l))_\rho &= X_j \log \left( \frac{h_m}{h(l)} \right) = -\frac{1}{m} X_j \log \left( \frac{n!}{m^n} \sum_{i=0}^{N_m} \|\tau_i\|_{h(l)}^2 \right) \\ &\equiv -q X_j \log(C_q + v_l q^{l+1}) \equiv -C_q^{-1}(X_j v_l) q^{l+2} \equiv 0, \quad \text{mod } q^{l+2}. \end{aligned}$$

Therefore,  $\int_M \eta_j v_l \omega_0^n = 0$  for all  $j$ . From  $v_l \in \text{Ker } D_0$ , it now follows that  $v_l = 0$ . This shows that (4.5) is true for  $k = l$ , as required.  $\square$

## 5. Concluding remarks

As in Donaldson's work [3], the construction of approximate solutions in Theorem C is a crucial step to the approach of the stability problem for a polarized algebraic manifold with a Kähler metric of constant scalar curvature. Actually, in a forthcoming paper [14], this construction allows us to prove the following:

**Theorem.** *For a sequence as in (2.1), assume that the isotropy actions for  $(M, L)$  are stable. Assume further that  $c_1(L)_{\mathbb{R}}$  admits a Kähler metric of constant scalar curvature. Then for this sequence,  $(M, L)$  is asymptotically Chow-stable.*

Moreover, if a sequence (2.1) exists in such a way that (2.3) holds, then the same argument as in the case  $\dim G = 0$  (cf. [3]) is applied, and we can also show the uniqueness, modulo the action of  $G$ , of the Kähler metrics of constant scalar curvature in the polarization class  $c_1(L)_{\mathbb{R}}$ . We finally remark that, if  $\dim G = 0$ , the asymptotic Chow-stability implies the asymptotic stability in the sense of Hilbert schemes (cf. [17], p.215). Hence the result of Donaldson [3] follows from the theorem just above.

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