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ON THE STRUCTURE OF A BOUNDED DOMAIN WITH A SPECIAL BOUNDARY POINT

Dedicated to Professor Tadashi Kuroda on his 60th birthday

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Introduction. In this paper we study the structure of a bounded domain D in $\boldsymbol{C}^n$ $(n>1)$ with a boundary point $p\in\partial D$ satisfying the following conditions: There exist an open neighborhood U of p and real-valued C^2-functions $\rho_1, \cdots, \rho_k$ $(1\leqq k\leqq n)$ defined on U such that

(C.1) $\rho_1(p)=\cdots=\rho_k(p)=0$;

(C.2) $D\cap U=\{z\in U: \rho_i(z)<0, \quad i=1, \cdots, k\}$;

(C.3) the differential form $\bar\partial\rho_1\wedge\cdots\wedge\bar\partial\rho_k(z)\neq 0$ for all $z\in U$;

(C.4) $\displaystyle\sum_{\alpha,\beta=1}^{n}\frac{\partial^2\rho_i}{\partial z_\alpha\partial\bar z_\beta}(p)\xi_\alpha\bar\xi_\beta\geqq 0, \quad \xi=(\xi_\alpha)\in T \qquad \text{for} \quad i=1, \cdots, k,$

where

$$T=\left\{\xi=(\xi_\alpha)\in\boldsymbol{C}^n: \sum_{\alpha=1}^{n}\frac{\partial\rho_i}{\partial z_\alpha}(p)\xi_\alpha=0, \qquad i=1, \cdots, k\right\};$$

(C.5) for some constant $A\geqq 0$, the function $\rho=\sum_{i=1}^{k}\rho_i+A\sum_{i=1}^{k}\rho_i^2$ is strictly plurisubharmonic on U.

As a typical example of such domains, we have of course a strictly pseudoconvex domain with C^2-smooth boundary (in fact, in this case any boundary point satisfies the above conditions). Furthermore, in a recent paper [8], Pinčuk proved that any bounded pseudoconvex domain D with piecewise C^2-smooth boundary also admits a boundary point $p\in\partial D$ satisfying the conditions (C.1)~(C.5). After that, he used efficiently this fact to show the following interesting

Theorem (Pinčuk [9]). *Let D be a homogeneous bounded domain in $\boldsymbol{C}^n$ $(n>1)$ with piecewise C^2-smooth boundary. Then D is biholomorphically equivalent to a direct product of the open unit balls $\mathcal{B}^{n_i}$ in $\boldsymbol{C}^{n_i}$ $(1\leqq i\leqq k)$: $D\simeq\mathcal{B}^{n_1}\times\cdots\times\mathcal{B}^{n_k}$.*

Here it should be remarked that any homogeneous bounded domain in C^n is pseudoconvex [5] and that $\mathcal{B}^{n_1}\times\cdots\times\mathcal{B}^{n_k}$ is biholomorphically equivalent to the so-called Siegel domain

$$\mathcal{E}(n_1, \cdots, n_k) = \{(z_1, \cdots, z_k, \omega_1, \cdots, \omega_k)\in C^k\times C^{n-k}:\\ \operatorname{Im} z_i - |\omega_i|^2>0, \qquad i=1, \cdots, k\}$$

in $C^k\times C^{n-k}=C^k\times C^{n_1-1}\times\cdots\times C^{n_k-1}$, where $|\cdot|$ denotes the Euclidean norm on C^{n_i-1}.

Now, in order to state our results, let us introduce some notations. For a domain D in C^N, we always denote by $\operatorname{Aut}(D)$ the group consisting of all biholomorphic automorphisms of D. For the open convex cone

$$R_+^k = \{(y_1, \cdots, y_k)\in R^k:\ y_i>0, \qquad i=1, \cdots, k\}$$

in R^k $(1\leqq k\leqq n)$ and an R_+^k-hermitian form H: $C^{n-k}\times C^{n-k}\to C^k$, let $\mathcal{D}(R_+^k, H)$ denote the Siegel domain in $C^k\times C^{n-k}$ associated to R_+^k and H. (For the definition of a Siegel domain, see section 1.) Our main purpose in this paper is to establish the following extension of the Pinčuk's theorem:

Theorem I. *Let D be a bounded domain in C^n $(n>1)$ with a boundary point $p\in\partial D$ satisfying the conditions from* (C.1) *through* (C.5). *Assume that*:

(*) *There exist a compact set K in D, a sequence $\{k_\nu\}$ in K and a sequence $\{f_\nu\}$ in $Aut(D)$ such that $\lim_{\nu\to\infty} f_\nu(k_\nu)=p$.*

Then D is biholomorphically equivalent to a Siegel domain $\mathcal{D}(R_+^k, H)$ in $C^k\times C^{n-k}$. Conversely, every Siegel domain $\mathcal{D}(R_+^k, H)$ in $C^k\times C^{n-k}$ is biholomorphically equivalent to a bounded domain D in C^n satisfying all the conditions (C.1)$\sim$(C.5) *and* (*).

Corollary 1. *Let D be a bounded domain in C^n $(n>1)$ with a boundary point p satisfying the conditions from* (C.1) *through* (C.5). *Assume that there exists a compact subset K of D such that $Aut(D)\cdot K=D$. Then D is biholomorphically equivalent to the Siegel domain $\mathcal{E}(n_1, \cdots, n_k)$ in $C^k\times C^{n-k}$.*

Let D be a domain in C^n. A point $p\in\partial D$ is said to be a *strictly pseudoconvex boundary point of D* if there exist an open neighborhood U of p and a strictly plurisubharmonic function ρ: $U\to R$ such that $D\cap U=\{z\in U: \rho(z)<0\}$ and $d\rho(z)\neq 0$ for all $z\in\partial D\cap U$. Consequently, the conditions from (C.1) through (C.5) are automatically satisfied for a strictly pseudoconvex boundary point p of D. On the other hand, it is easy to see that a Siegel domain $\mathcal{D}(R_+^1, H)$ in $C\times C^{n-1}$ is biholomorphically equivalent to the open unit ball $\mathcal{B}^n$ in C^n. Therefore, as a corollary of Theorem I, we also obtain the following well-known fact due to Wong [12] and Rosay [11]:

Corollary 2. *Let D be a bounded domain in $\boldsymbol{C}^n$ with a strictly pseudoconvex boundary point $p\in\partial D$. Assume that the condition $(*)$ in Theorem I is satisfied. Then D is biholomorphically equivalent to the open unit ball $\mathcal{B}^n$ in $\boldsymbol{C}^n$.*

Next we wish to consider a problem as follows. Let M be a complex manifold of complex dimension n which can be exhausted by biholomorphic images of a fixed complex manifold D, that is, for any compact subset K of M there exists a biholomorphic mapping f_K from D into M such that $K\subset f_K(D)$. Then, how can we describe M using the data of D? We can see many articles related closely to this problem, for instance, Fornaess-Sibony [2], Fornaess-Stout [3] and Fridman [1]. Our second purpose of this paper is to prove the following theorems. (For the precise definitions of terminologies, see section 1.)

Theorem II. *Let M be a connected hyperbolic manifold of complex dimension n in the sense of Kobayashi* [6] *and let D be a bounded domain with piecewise C^2-smooth boundary of special type. Assume that M can be exhausted by biholomorphic images of D. Then M is biholomorphically equivalent either to D or to a Siegel domain $\mathcal{D}(\boldsymbol{R}^k_+, H)$ in $\boldsymbol{C}^k\times\boldsymbol{C}^{n-k}$ $(1\leqq k\leqq n)$.*

Theorem III. *Let D_1 and D_2 be bounded domains with piecewise C^2-smooth boundaries of special type. Then D_1 and D_2 are biholomorphically equivalent if and only if each of them can be exhausted by biholomorphic images of the other.*

In particular, considering the case where M is a connected complete hyperbolic manifold and D is a bounded strictly pseudoconvex domain with C^3-smooth boundary in Theorem II, and also the case where D_1 and D_2 are bounded strictly pseudoconvex domains with C^3-smooth boundaries in Theorem III, we obtain the main results of Fridman [1].

This paper is organized as follows. In section 1 we recall some definitions and a well-known fact on Siegel domains. Section 2 is devoted to proving Theorem I and Corollary 1. And Theorems II and III will be shown in the final section 3.

1. Preliminaries

Let M and N be complex manifolds and $\mathrm{Hol}(N, M)$ the family of all holomorphic mappings from N into M. A sequence $\{f_\nu\}$ in $\mathrm{Hol}(N, M)$ is said to be *compactly divergent on* N if, for any compact sets L, K in N, M respectively, there exists an integer ν_0 such that $f_\nu(L)\cap K=\emptyset$ for all $\nu\geqq\nu_0$. According to Wu [13], we shall define the tautness of complex manifolds as follows:

DEFINITION 1. A complex manifold M is said to be *taut* if $\mathrm{Hol}(N, M)$ is a normal family for any complex manifold N, i.e., any sequence in $\mathrm{Hol}(N, M)$ contains a subsequence which is either uniformly convergent on every compact subset of N or compactly divergent on N.

Let d_M, d_N be the Kobayashi pseudodistances of M, N respectively [6]. The following distance-decreasing property will play an important role in the proofs of our theorems: *Let f: $N \to M$ be a holomorphic mapping. Then*

$$(1.1) \qquad d_M(f(p), f(q)) \leqq d_N(p, q) \qquad \textit{for all} \quad p, q \in N.$$

Consequently, every biholomorphic mapping f from N onto M is an isometry with respect to d_N and d_M; and if N is a complex submanifold of M, then $d_M(p, q) \leqq d_N(p, q)$ for all $p, q \in N$.

DEFINITION 2. A bounded domain D in $\boldsymbol{C}^n$ is said to *have a piecewise C^r-smooth boundary* ($r \geqq 1$) if there exist a finite open covering $\{U_j\}_{j=1}^N$ of an open neighborhood V of ∂D, the boundary of D, and C^r-functions ρ_j: $U_j \to \boldsymbol{R}$, $j=1, \cdots, N$, such that

(i) $D \cap V = \{z \in V$: for $j=1, \cdots, N$, either $z \notin U_j$ or $z \in U_j$, $\rho_j(z) < 0\}$;

(ii) for every set $\{j_1, \cdots, j_i\}$, $1 \leqq j_1 < \cdots < j_i \leqq N$, the differential form

$$d\rho_{j_1} \wedge \cdots \wedge d\rho_{j_i}(z) \neq 0 \qquad \text{for all} \quad z \in \bigcap_{l=1}^{i} U_{j_l}.$$

We call $\{U_j, \rho_j\}_{j=1}^N$ a defining system for D.

Note that, by the condition (ii) the set $S_j = \{z \in U_j$: $\rho_j(z) = 0\}$ is a closed C^r-smooth real hypersurface of U_j for $j=1, \cdots, N$. Without loss of generality, we may assume in Definition 2 that: If $p \in \partial D \cap S_{j_0}$ and $p \notin S_j$ for $j \neq j_0$, then there is an open neighborhood U of p such that $D \cap U = \{z \in U$: $\rho_{j_0}(z) < 0\}$. And, in an arbitrary small neighborhood of any point $p \in \partial D \cap S_j$, the boundary ∂D contains a non-empty open subset of S_j.

DEFINITION 3. Let D be a bounded domain in $\boldsymbol{C}^n$ with piecewise C^2-smooth boundary and let $\{U_j, \rho_j\}_{j=1}^N$ be its defining system. Then ∂D is said to be *of special type* if, for an arbitrary given point $p \in \partial D$, one can find a subset $\{j_1, \cdots, j_k\}$ ($1 \leqq k \leqq n$) of $\{1, \cdots, N\}$ and an open neighborhood U of p, $U \subset \bigcap_{l=1}^{k} U_{j_l}$, such that the system ($p$; U; $\rho_{j_1}, \cdots, \rho_{j_k}$) satisfies all the conditions from (C.1) through (C.5) in the introduction. We call (U; $\rho_{j_1}, \cdots, \rho_{j_k}$) a defining system for D in the neighborhood U of p.

Obviously, any bounded strictly pseudoconvex domain with C^2-smooth boundary is also a domain with piecewise C^2-smooth boundary of special type. We present here a simple example of bounded domains with piecewise C^2-smooth, but not smooth, boundaries of special type.

EXAMPLE 1. Take two arbitrary constants a, $b > 0$, $a \neq b$ and consider the domain

$$D = \{(z_1, z_2) \in \boldsymbol{C}^2 \colon a|z_1|^2 + b|z_2|^2 < 1,\ b|z_1|^2 + a|z_2|^2 < 1\}.$$

Then it is easily checked that D is a bounded domain with piecewise C^2-smooth boundary of special type.

We fix a coordinate system $(z, w)=(z_1, \cdots, z_n, w_1, \cdots, w_m)$ in $\boldsymbol{C}^n \times \boldsymbol{C}^m$. For a given open convex cone Ω in $\boldsymbol{R}^n$ not containing any full straight line, a mapping H: $\boldsymbol{C}^m \times \boldsymbol{C}^m \to \boldsymbol{C}^n$ is called an *Ω-hermitian form* if

(i) H is complex linear with respect to the first variable;

(ii) $\overline{H(u, v)}=H(v, u)$ for all $u, v \in \boldsymbol{C}^m$;

(iii) $H(u, u) \in \bar{\Omega}$ for all $u \in \boldsymbol{C}^m$, where $\bar{\Omega}$ denotes the topological closure of Ω in $\boldsymbol{R}^n$;

(iv) $H(u, u)=0$ if and only if $u=0$.

According to Pjateckiĭ-Šapiro [10], we define a Siegel domain as follows:

DEFINITION 4. For a given open convex cone Ω in $\boldsymbol{R}^n$ not containing any full straight line and an Ω-hermitian form H: $\boldsymbol{C}^m \times \boldsymbol{C}^m \to \boldsymbol{C}^n$, the domain

$$\mathcal{D}(\Omega, H)=\{(z, w) \in \boldsymbol{C}^n \times \boldsymbol{C}^m : \operatorname{Im} z-H(w, w) \in \Omega\}$$

in $\boldsymbol{C}^n \times \boldsymbol{C}^m$ is called a *Siegel domain of the second kind associated to Ω and H.* In the case $m=0$, $\mathcal{D}(\Omega, H)$ reduces to the domain

$$\mathcal{D}(\Omega)=\{z \in \boldsymbol{C}^n : \operatorname{Im} z \in \Omega\}.$$

This is called a *Siegel domain of the first kind.*

In this paper we regard a Siegel domain of the first kind as the special case of the second kind and by a *Siegel domain* we mean a Siegel domain of the first or the second kind.

Let (z', z'') be a coordinate system in $\boldsymbol{C}^k \times \boldsymbol{C}^{n-k}=\boldsymbol{C}^n$ $(1 \leqq k \leqq n)$ with $z'=(z_1, \cdots, z_k)$, $z''=(z_{k+1}, \cdots, z_n)$ and consider a Siegel domain

$$\mathcal{D}(\boldsymbol{R}_+^k, H)=\{(z', z'') \in \boldsymbol{C}^k \times \boldsymbol{C}^{n-k} : \operatorname{Im} z'-H(z'', z'') \in \boldsymbol{R}_+^k\}$$

in $\boldsymbol{C}^n$ associated to $\boldsymbol{R}_+^k$ and an $\boldsymbol{R}_+^k$-hermitian form H as in the introduction. Then we have the following

Lemma (Pjateckiĭ-Šapiro [10]). *There exists a biholomorphic mapping C from $\mathcal{D}(\boldsymbol{R}_+^k, H)$ onto a subdomain D of the direct product of the open unit balls $\mathcal{B}^{n_1} \times \cdots \times \mathcal{B}^{n_k}$ $(n_1+\cdots+n_k=n)$.*

For later use of the concrete description of the biholomorphic mapping C: $\mathcal{D}(\boldsymbol{R}_+^k, H) \to D$, we shall recall here the proof.

Proof of Lemma. With respect to the coordinate system $z'=(z_1, \cdots, z_k)$ in $\boldsymbol{C}^k$, H can be written as $H=(H_1, \cdots, H_k)$, where every H_i is a positive semi-definite hermitian form on $\boldsymbol{C}^{n-k}$. Hence we can express

$$H_i(z_1'', z_2'') = \sum_{j=1}^{m_i} L_i^j(z_1'')\overline{L_i^j(z_2'')}\,, \qquad i = 1, \cdots, k$$

for $z_1'', z_2'' \in \boldsymbol{C}^{n-k}$ with complex linear forms L_i^j on $\boldsymbol{C}^{n-k}$. Thanks to the positive definiteness of the hermitian form $\sum_{i=1}^{k} H_i$, we can now select $n-k$ linearly independent forms among the set $\{L_i^j\colon j=1, \cdots, m_i,\ i=1, \cdots, k\}$, say $L_1^1, \cdots, L_1^{n_1-1}, \cdots, L_k^1, \cdots, L_k^{n_k-1}$ with $n_1+\cdots+n_k=n$ and $n_i \geqq 1$, $i=1, \cdots, k$. Define the hermitian form $\tilde{H}_i\colon \boldsymbol{C}^{n-k}\times\boldsymbol{C}^{n-k}\to\boldsymbol{C}$ by putting, for $i=1, \ldots, k$,

$$\tilde{H}_i(z_1'', z_2'') = \begin{cases} \sum_{j=1}^{n_i-1} L_i^j(z_1'')\overline{L_i^j(z_2'')} & \text{if} \quad n_i \geqq 2\,, \\ 0 & \text{if} \quad n_i = 1 \end{cases}$$

for $z_1'', z_2'' \in \boldsymbol{C}^{n-k}$ and set $\tilde{H}=(\tilde{H}_1, \cdots, \tilde{H}_k)$. Then $\tilde{H}\colon \boldsymbol{C}^{n-k}\times\boldsymbol{C}^{n-k}\to\boldsymbol{C}^k$ is an $\boldsymbol{R}_+^k$-hermitian form and $\mathcal{D}(\boldsymbol{R}_+^k, H)\subset\mathcal{D}(\boldsymbol{R}_+^k, \tilde{H})$. Set

$$\omega_i = (L_i^1(z''), \cdots, L_i^{n_i-1}(z''))\,, \qquad i = 1, \cdots, k\,.$$

Then $(z_1, \cdots, z_k, \omega_1, \cdots, \omega_k)$ defines a linear coordinate system in $\boldsymbol{C}^k\times\boldsymbol{C}^{n_1-1}\times\cdots\times\boldsymbol{C}^{n_k-1}=\boldsymbol{C}^k\times\boldsymbol{C}^{n-k}$ and, with respect to this coordinate system, the domain $\mathcal{D}(\boldsymbol{R}_+^k, \tilde{H})$ can be written in the form

$$\mathcal{D}(\boldsymbol{R}_+^k, \tilde{H}) = \{(z_1, \cdots, z_k, \omega_1, \cdots, \omega_k)\in\boldsymbol{C}^k\times\boldsymbol{C}^{n-k}\colon \\ \operatorname{Im} z_i - |\omega_i|^2>0\,,\ i = 1, \cdots, k\}\,.$$

It is now an easy matter to check that $\mathcal{D}(\boldsymbol{R}_+^k, \tilde{H})$ is biholomorphically equivalent to the direct product

$$\mathcal{B}^{n_1}\times\cdots\times\mathcal{B}^{n_k} = \{(z_1', z_{k+1}', \cdots, z_k', z_{2k}')\in\boldsymbol{C}\times\boldsymbol{C}^{n_1-1}\times\cdots\times\boldsymbol{C}\times\boldsymbol{C}^{n_k-1}\colon \\ |z_i'|^2+|z_{k+i}'|^2<1\,,\ i = 1, \cdots, k\}$$

via the mapping

$$\tilde{C}\colon z_i' = \frac{z_i-\sqrt{-1}}{z_i+\sqrt{-1}}\,, \quad z_{k+i}' = \frac{2\omega_i}{z_i+\sqrt{-1}} \qquad \text{for} \quad i = 1, \cdots, k\,.$$

So, if we define the non-singular linear mapping $L\colon \boldsymbol{C}^n\to\boldsymbol{C}^n$ by

$$L(z', z'') = (z', L_1^1(z''), \cdots, L_1^{n_1-1}(z''), \cdots, L_k^1(z''), \cdots, L_k^{n_k-1}(z''))\,,$$

then the composition $C=\tilde{C}\circ L$ gives rise to a biholomorphic mapping from $\mathcal{D}(\boldsymbol{R}_+^k, H)$ into $\mathcal{B}^{n_1}\times\cdots\times\mathcal{B}^{n_k}$. Putting $D=C(\mathcal{D}(\boldsymbol{R}_+^k, H))$, we therefore obtain a desired biholomorphic mapping $C\colon \mathcal{D}(\boldsymbol{R}_+^k, H)\to D$. Q.E.D.

From the construction, it is obvious that C can be extended to a biholomorphic mapping from an open neighborhood V of the origin $0=(0, \cdots, 0)\in$

$\partial\mathcal{D}(\boldsymbol{R}^k_+, H)$ onto an open neighborhood U of the point $p=(-1, 0, \cdots, -1, 0)\in \partial D\subset \boldsymbol{C}\times \boldsymbol{C}^{n_1-1}\times\cdots\times\boldsymbol{C}\times\boldsymbol{C}^{n_k-1}$.

REMARK. The above lemma holds for any Siegel domain $\mathcal{D}(\Omega, H)$ in $\boldsymbol{C}^n\times\boldsymbol{C}^m$ [10].

2. Proofs of Theorem I and Corollary 1

Throughout this section, the following notation will be used for $i=1, \cdots, k$:

$$\nabla\rho_i(z) = 2\left(\frac{\partial\rho_i}{\partial\bar{z}_1}(z), \cdots, \frac{\partial\rho_i}{\partial\bar{z}_n}(z)\right), \qquad z\in U;$$
$$S_i = \{z\in U\colon \rho_i(z) = 0\},$$

where $\rho_i\colon U\to\boldsymbol{R}$ is the function given in (C.1)$\sim$(C.5) in the introduction. We note that, for every $i=1, \cdots, k$, the vector $\nabla\rho_i(\zeta)$ is perpendicular to the closed C^2-smooth real hypersurface S_i at each point $\zeta\in S_i$ with respect to the Euclidean structure on $\boldsymbol{C}^n=\boldsymbol{R}^{2n}$.

Proof of Theorem I. Generalizing the idea of "stretched coordinate system" due to Pinčuk [8], [9], we first prove that a bounded domain D satisfying the conditions from (C.1) through (C.5) and (*) is biholomorphically equivalent to some Siegel domain $\mathcal{D}(\boldsymbol{R}^k_+, H)$. By the compactness of K and (C.3), we may assume without loss of generality that

$$\lim_{\nu\to\infty} k_\nu = k_0 \qquad \text{for some point } k_0 \text{ of } K \tag{2.1}$$

and

$$\nabla\rho_i(p) = (0, \cdots, 0, \overset{\overset{i}{\vee}}{2}, 0, \cdots, 0), \qquad i = 1, \cdots, k, \tag{2.2}$$

where $\underset{\vee}{i}$ means that the number 2 appears at the i-th position. Now we will proceed in steps.

1) Some subsequence $\{f_{\nu_j}\}$ of $\{f_\nu\}$ converges uniformly on compact subsets of D to the constant mapping $C_p\colon D\to\boldsymbol{C}^n$ defined by $C_p(z)=p$, $z\in D$. In fact, owing to the boundedness of D, we can select a subsequence $\{f_{\nu_j}\}$ of $\{f_\nu\}$ which converges uniformly on compact subsets of D to a holomorphic mapping $f\colon D\to\boldsymbol{C}^n$. Clearly $f(k_0)=p$ and $f(D)\subset\bar{D}$. Choose an open neighborhood V of k_0 such that $f(V)\subset U$ and consider the plurisubharmonic function $\rho\circ f\colon V\to\boldsymbol{R}$, where $\rho\colon U\to\boldsymbol{R}$ is the strictly plurisubharmonic function given in (C.5). Then

$$\rho\circ f(k_0) = 0 \quad \text{and} \quad \rho\circ f(z)\leqq 0, \qquad z\in V$$

and hence $\rho\circ f(z)=0$ for all $z\in V$. In view of strict plurisubharmonicity of ρ, this means that $f(z)=p$ on V. Hence $f=C_p$ by the theorem of identity, as desired.

2) For later purpose, we wish to construct a family of biholomorphic mappings $h^{\zeta}\colon \boldsymbol{C}^n \to \boldsymbol{C}^n$, depending continuously on $\zeta \in \partial D \cap U$, which has good properties. First we notice by (2.2) that the square matrix $\left(\dfrac{\partial \rho_i}{\partial z_j}(z)\right)_{1 \leqq i, j \leqq k}$ has non-zero determinant for each point z belonging to a sufficiently small neighborhood of p. So, shrinking U if necessary, we can assume that the affine mapping $\varphi^{\zeta}\colon \boldsymbol{C}^n \to \boldsymbol{C}^n$ defined by

$$\varphi^{\zeta}\colon \begin{cases} u_i = \sum_{\alpha=1}^{n} \dfrac{\partial \rho_i}{\partial z_\alpha}(\zeta)(z_\alpha - \zeta_\alpha), & i = 1, \cdots, k; \\ u_j = z_j - \zeta_j, & j = k+1, \cdots, n \end{cases}$$

is non-singular for each point $\zeta \in \partial D \cap U$. Setting $\tilde{\rho}_i^{\zeta} = \rho_i \circ (\varphi^{\zeta})^{-1}$ for $i = 1, \cdots, k$, we have then by Taylor's formula

$$\tilde{\rho}_i^{\zeta}(u) = 2 \operatorname{Re} [u_i + \sum_{\alpha,\beta=1}^{n} a^i_{\alpha\beta}(\zeta) u_\alpha u_\beta] + \tilde{H}_i^{\zeta}(u) + \tilde{\alpha}_i^{\zeta}(u)$$

in a neighborhood of the origin, where $\tilde{\alpha}_i^{\zeta}(u) = o(|u|^2)$ and

$$a^i_{\alpha\beta}(\zeta) = \frac{1}{2} \frac{\partial^2 \tilde{\rho}_i^{\zeta}}{\partial u_\alpha \partial u_\beta}(0), \quad \tilde{H}_i^{\zeta}(u) = \sum_{\alpha,\beta=1}^{n} \frac{\partial^2 \tilde{\rho}_i^{\zeta}}{\partial u_\alpha \partial \bar{u}_\beta}(0) u_\alpha \bar{u}_\beta .$$

Define

$$\psi^{\zeta}\colon \begin{cases} w_i = u_i + \sum_{\alpha,\beta=k+1}^{n} a^i_{\alpha\beta}(\zeta) u_\alpha u_\beta , & i = 1, \cdots, k; \\ w_j = u_j , & j = k+1, \cdots, n \end{cases}$$

for $\zeta \in \partial D \cap U$. It is clear that $\psi^{\zeta}\colon \boldsymbol{C}^n \to \boldsymbol{C}^n$ is a biholomorphic mapping with the inverse mapping

$$(\psi^{\zeta})^{-1}\colon \begin{cases} u_i = w_i - \sum_{\alpha,\beta=k+1}^{n} a^i_{\alpha\beta}(\zeta) w_\alpha w_\beta , & i = 1, \cdots, k; \\ u_j = w_j , & j = k+1, \cdots, n . \end{cases}$$

Therefore the composition $h^{\zeta} = \psi^{\zeta} \circ \varphi^{\zeta}\colon \boldsymbol{C}^n \to \boldsymbol{C}^n$ is a biholomorphic mapping from $\boldsymbol{C}^n$ onto $\boldsymbol{C}^n$ for each $\zeta \in \partial D \cap U$. From the construction of h^{ζ}, it is obvious that

$$h^{\zeta} \to h^{p} \quad \text{as} \quad \zeta \to p \quad \text{locally uniformly on } \boldsymbol{C}^n;$$
$$h^{\zeta}(\zeta) = 0 \qquad \text{for all} \quad \zeta \in \partial D \cap U .$$

Set $\rho_i^{\zeta} = \tilde{\rho}_i^{\zeta} \circ (\psi^{\zeta})^{-1} = \rho_i \circ (h^{\zeta})^{-1}$ for $i = 1, \cdots, k$. Then each ρ_i^{ζ} can be expressed in the form

(2.3) $$\rho_i^\zeta(w) = 2\,\mathrm{Re}\,(w_i + K_i^\zeta(w)) + H_i^\zeta(w) + \alpha_i^\zeta(w)$$

in a neighborhood of the origin, where

(2.4) $$K_i^\zeta(w) = \sum_{\alpha,\beta=1}^{k} a_{\alpha\beta}^i(\zeta) w_\alpha w_\beta + 2\sum_{\alpha=1}^{k}\sum_{\beta=k+1}^{n} a_{\alpha\beta}^i(\zeta) w_\alpha w_\beta\,,$$
$$H_i^\zeta(w) = \sum_{\alpha,\beta=1}^{n} \frac{\partial^2 \rho_i^\zeta}{\partial w_\alpha \partial \bar{w}_\beta}(0) w_\alpha \bar{w}_\beta \quad \text{and} \quad \alpha_i^\zeta(w) = o(|w|^2)\,.$$

(Here we have used the fact that $\dfrac{\partial^2 \tilde{\rho}_i^\zeta}{\partial u_\alpha \partial \bar{u}_\beta}(0) = \dfrac{\partial^2 \rho_i^\zeta}{\partial w_\alpha \partial \bar{w}_\beta}(0)$, $1 \leqq \alpha, \beta \leqq n$, in our case.)

3) This step is a preparation of the next one. We set

$$N = \frac{1}{2}(\nabla\rho_1(p) + \cdots + \nabla\rho_k(p)) = (\underbrace{1, \cdots, 1}_{k \text{ times}}, 0, \cdots, 0)\,.$$

Then the vector N is not tangent to every hypersurface S_i at the point $p \in \partial D \cap S_1 \cap \cdots \cap S_k$, because $\nabla\rho_i(p)$ is perpendicular to S_i at p for every $i=1, \cdots, k$. This guarantees us the existences of positive numbers ε_1, ε_2 and ε_3 such that

(2.5) $$B_p(\varepsilon_1) \subset \{z = \zeta + \lambda N : \zeta \in \partial D \cap B_p(\varepsilon_2),\ |\lambda| < \varepsilon_3\} \subset U\,,$$

where $B_p(\varepsilon_i)$ stands for the open Euclidean ε_i-ball with center at p. Now, we put

$$p^\nu = f_\nu(k_\nu) \qquad \text{for} \quad \nu = 1, 2, \cdots.$$

Then, by virtue of the first step 1) and (2.5) we may assume (by passing to a subsequence if necessary) that $\{f_\nu\}$ converges on compact subsets of D to the constant mapping $C_p(z) \equiv p$ and that every point p^ν, $\nu=1, 2, \cdots$, has the following form:

(2.6) $$p^\nu = \zeta^\nu + \lambda^\nu N \qquad \text{for some} \quad \zeta^\nu \in \partial D \cap U\,,\ \lambda^\nu < 0\,.$$

(The negativity of λ^ν is a direct consequence of (C.2).) It is clear that $\zeta^\nu \to p$, $\lambda^\nu \to 0$ as $\nu \to \infty$. For the sake of simplicity, we shall set

$$h^\nu = h^{\zeta^\nu} \quad \text{and} \quad \rho_i^\nu = \rho_i^{\zeta^\nu}\,, \qquad i = 1, \cdots, k$$

for $\nu=1, 2, \cdots$, where h^{ζ^ν} and $\rho_i^{\zeta^\nu}$ are the mappings and functions defined in 2). Then

$$h^\nu(p^\nu) = (\delta_1^\nu, \cdots, \delta_k^\nu, 0, \cdots, 0)\,, \qquad \nu = 1, 2, \cdots$$

with

$$\delta_i^\nu = \lambda^\nu \sum_{\alpha=1}^{k} \frac{\partial \rho_i}{\partial z_\alpha}(\zeta^\nu)\,, \qquad i = 1, \cdots, k;\ \nu = 1, 2, \cdots.$$

Therefore, putting

$$(2.7) \qquad r_\nu = (|\delta_1^\nu|^2 + \cdots + |\delta_k^\nu|^2)^{1/2} \quad \text{and} \quad s_\nu = \sqrt{r_\nu}$$

for $\nu = 1, 2, \cdots$, we obtain by (2.2) that

$$(2.8) \qquad \lim_{\nu\to\infty} \delta_i^\nu / r_\nu = -1/\sqrt{k} \qquad \text{for} \quad i = 1, \cdots, k\,.$$

In particular, we can assume that

$$0 < |\delta_i^\nu| < 1 \qquad \text{for} \quad i = 1, \cdots, k; \; \nu = 1, 2, \cdots.$$

Now let us fix a family $\{D_j\}_{j=1}^\infty$ of relatively compact subdomains of D such that

$$D = \bigcup_{j=1}^\infty D_j \supset \cdots \supset D_{j+1} \supset D_j \supset \cdots \supset D_1 \supset K\,,$$

where K is the compact subset of D as in the theorem. Taking D_j arbitrarily, we set $D' = D_j$ for simplicity. Since $f_\nu(z) \to p$ uniformly on D', there exists an integer $\nu(D')$ such that

$$f_\nu(D') \subset D \cap U \qquad \text{for all} \quad \nu \geqq \nu(D')\,.$$

We define the mappings L^ν: $\boldsymbol{C}^n \to \boldsymbol{C}^n$ and F^ν: $D' \to \boldsymbol{C}^n$ for $\nu = 1, 2, \cdots$ by setting

$$L^\nu(w) = \left(-\frac{w_1}{\delta_1^\nu}, \cdots, -\frac{w_k}{\delta_k^\nu}, \frac{w_{k+1}}{s_\nu}, \cdots, \frac{w_n}{s_\nu} \right), \qquad w = (w_\alpha) \in \boldsymbol{C}^n$$

and

$$F^\nu(z) = L^\nu \circ h^\nu \circ f_\nu(z)\,, \qquad z \in D'\,,$$

where δ_i^ν, s_ν are the numbers given by (2.7). Then L^ν are non-singular linear transformations of $\boldsymbol{C}^n$ and F^ν are biholomorphic mappings from D' into $\boldsymbol{C}^n$. Moreover it follows from (C.2) and the construction of F^ν that

$$(2.9) \qquad F^\nu(k_\nu) = (\underbrace{-1, \cdots, -1}_{k \text{ times}}, 0, \cdots, 0)\,, \qquad F^\nu(D') \subset W_\nu$$

for all $\nu \geqq \nu(D')$, where

$$(2.10) \quad W_\nu = \{w \in \boldsymbol{C}^n\colon (L^\nu)^{-1}(w) \in h^\nu(U),\ \rho_i^\nu \circ (L^\nu)^{-1}(w) < 0,\ i = 1, \cdots, k\}$$

for $\nu = 1, 2, \cdots$.

4) Set $\rho^\nu = \sum_{i=1}^k \rho_i^\nu + A \sum_{i=1}^k (\rho_i^\nu)^2$ for $\nu = 1, 2, \cdots$. By virtue of (C.5) every ρ^ν is a strictly plurisubharmonic function on $h^\nu(U)$. Since $\dfrac{\partial \rho_i^\nu}{\partial w_\alpha}(0) = \delta_{i\alpha}$ (the usual Kronecker's symbol), we see that every ρ^ν has the form

$$\rho^{\nu}(w) = 2\,\mathrm{Re}\left[\sum_{i=1}^{k}(1+2A\rho_i(\zeta^{\nu}))w_i+\frac{1}{2}K^{\nu}(w)\right]+H^{\nu}(w)+\alpha^{\nu}(w) \tag{2.11}$$

in a neighborhood of the origin, where $\alpha^{\nu}(w)=o(|w|^2)$ and

$$K^{\nu}(w) = \sum_{\alpha,\beta=1}^{n}\frac{\partial^2\rho^{\nu}}{\partial w_{\alpha}\partial w_{\beta}}(0)w_{\alpha}w_{\beta}\,, \quad H^{\nu}(w) = \sum_{\alpha,\beta=1}^{n}\frac{\partial^2\rho^{\nu}}{\partial w_{\alpha}\partial \overline{w}_{\beta}}(0)w_{\alpha}\overline{w}_{\beta}\,.$$

Since ρ^{ν} is strictly plurisubharmonic on $h^{\nu}(U)$ and $\alpha^{\nu}(w)=o(|w|^2)$ for all $\nu=1, 2, \cdots$ and since $h^{\nu}\to h^{p}$, $H^{\nu}\to H^{p}$ and $\alpha^{\nu}\to\alpha^{p}$ locally uniformly, we can find an open neighborhood W^* of the origin 0 with $W^*\subset h^{\nu}(U)$ for all $\nu=1, 2, \cdots$ and a positive constant C, which depends neither on ν nor on $w\in W^*$, such that

$$H^{\nu}(w)+\alpha^{\nu}(w)\geqq C|w|^2 \qquad \text{on} \quad W^* \tag{2.12}$$

for all $\nu=1, 2, \cdots$.

5) In this step we shall show that some subsequence of $\{F^{\nu}\}$ converges uniformly on every compact subset of D to a holomorphic mapping F: $D\to\boldsymbol{C}^n$. To see this, consider first the domains

$$V_{\nu} = \{w\in\boldsymbol{C}^n\colon (L^{\nu})^{-1}(w)\in h^{\nu}(U),\ \rho^{\nu}\circ(L^{\nu})^{-1}(w)<0\}$$

for $\nu=1, 2, \cdots$. Shrinking U if necessary, we may assume without loss of generality that $D\cap U\subset\{z\in U\colon \rho(z)<0\}$, where ρ: $U\to\boldsymbol{R}$ is the function defined in (C.5). Therefore

$$W_{\nu}\subset V_{\nu} \qquad \text{for all} \quad \nu = 1, 2, \cdots, \tag{2.13}$$

where W_{ν} are the domains given by (2.10). Let us put for a while

$$w^{\nu} = F^{\nu}(z)\,, \quad z\in D' \qquad \text{for} \quad \nu\geqq\nu(D')\,.$$

Since $(L^{\nu})^{-1}(w^{\nu})=h^{\nu}\circ f_{\nu}(z)\to 0$ uniformly on D', we may assume, by taking a subsequence if necessary, that

$$(L^{\nu})^{-1}(F^{\nu}(D'))\subset W^* \qquad \text{for all} \quad \nu=1, 2, \cdots,$$

where W^* denotes the open neighborhood of the origin as in (2.12). Hence it follows from the relations (2.9) and (2.11)$\sim$(2.13) that

$$\begin{aligned} 0&>\rho^{\nu}\circ(L^{\nu})^{-1}(w^{\nu}) \\ &\geqq 2\,\mathrm{Re}\left[-\sum_{i=1}^{k}(1+2A\rho_i(\zeta^{\nu}))\delta_i^{\nu}w_i^{\nu}+\frac{1}{2}K^{\nu}\circ(L^{\nu})^{-1}(w^{\nu})\right] \\ &\quad +C[|\delta_1^{\nu}w_1^{\nu}|^2+\cdots+|\delta_k^{\nu}w_k^{\nu}|^2+r_{\nu}(|w_{k+1}^{\nu}|^2+\cdots+|w_n^{\nu}|^2)] \end{aligned}$$

and so

$$0>2\,\mathrm{Re}\left[-\sum_{i=1}^{k}(1+2A\rho_i(\zeta^\nu))\frac{\delta_i^\nu}{r_\nu}w_i^\nu+\frac{1}{2r_\nu}K^\nu\circ(L^\nu)^{-1}(w^\nu)\right]$$
$$+C(|w_{k+1}^\nu|^2+\cdots+|w_n^\nu|^2)$$

for all $w^\nu\in F^\nu(D')$ and all $\nu=1, 2, \cdots$. This implies that if we define the domain $\mathscr{B}$ in $\boldsymbol{C}^{n-k+1}$ and the holomorphic mappings Φ^ν: $D'\to\boldsymbol{C}^{n-k+1}$, $\nu=1, 2, \cdots$, by

$$\mathscr{B}=\{(u, v)\in\boldsymbol{C}\times\boldsymbol{C}^{n-k}: 2\,\mathrm{Re}\,u+C\,|v|^2<0\}$$

and

$$\Phi^\nu=\left(-\sum_{i=1}^{k}(1+2A\rho_i(\zeta^\nu))\frac{\delta_i^\nu}{r_\nu}F_i^\nu+\frac{1}{2r_\nu}K^\nu\circ(L^\nu)^{-1}\circ F^\nu,\ F_{k+1}^\nu, \cdots, F_n^\nu\right),$$

then every Φ^ν gives rise to a holomorphic mapping from D' into $\mathscr{B}$. On the other hand, it is easy to see that $\mathscr{B}$ is biholomorphically equivalent to the open unit ball

$$\mathscr{B}^{n-k+1}=\{(z', z'')\in\boldsymbol{C}\times\boldsymbol{C}^{n-k}: |z'|^2+|z''|^2<1\}$$

in $\boldsymbol{C}^{n-k+1}$ via the correspondence

$$(u, v)\mapsto(z', z'')=\left(\frac{u+1}{u-1}, \frac{\sqrt{2C}\,v}{u-1}\right).$$

In particular, $\mathscr{B}$ is a taut domain, so that $\{\Phi^\nu\}$ forms a normal family. Moreover, using the facts (2.8), (2.9) and $\lim_{\nu\to\infty}\rho_i(\zeta^\nu)=\rho_i(p)=0$ for every $i=1, \cdots k$, we can show by direct computations that

$$\Phi^\nu(k_\nu)=\left(\sum_{i=1}^{k}(1+2A\rho_i(\zeta^\nu))\frac{\delta_i^\nu}{r_\nu}+\frac{1}{2r_\nu}K^\nu\circ(L^\nu)^{-1}\circ F^\nu(k_\nu), 0, \cdots, 0\right)$$
$$\to(-\sqrt{k}, 0, \cdots, 0)\in\mathscr{B}\quad\text{as}\quad\nu\to\infty,$$

which says that $\{\Phi^\nu\}$ is not compactly divergent on D'. Therefore we can assume that $\{\Phi^\nu\}$ converges uniformly on compact subsets of D' to a holomorphic mapping $\Phi=(\Phi_1, \cdots, \Phi_{n-k+1})$: $D'\to\mathscr{B}\subset\boldsymbol{C}^{n-k+1}$, that is,

$$-\sum_{i=1}^{k}(1+2A\rho_i(\zeta^\nu))\frac{\delta_i^\nu}{r_\nu}F_i^\nu+\frac{1}{2r_\nu}K^\nu\circ(L^\nu)^{-1}\circ F^\nu\to\Phi_1 \tag{2.14}$$

and

$$F_j^\nu\to\Phi_{j-k+1}, \qquad j=k+1, \cdots, n \tag{2.15}$$

as $\nu\to\infty$ uniformly on compact subsets of D'.

It remains to show that, for every i_0, $1\leqq i_0\leqq k$, the sequence $\{F_{i_0}^\nu\}$ also contains a convergent subsequence. To do this, choose an $\varepsilon>0$ so small that the function $\rho_\varepsilon=\rho+\varepsilon\rho_{i_0}$ is still strictly plurisubharmonic on U. Clearly

$D\cap U\subset\{z\in U:\ \rho_\varepsilon(z)<0\}$. Set $\rho_\varepsilon^\nu=\rho_\varepsilon\circ(h^\nu)^{-1}$ for $\nu=1, 2, \cdots$. Then, by (2.3) and (2.11) ρ_ε^ν can be written in the form

$$\rho_\varepsilon^\nu(w)=2\,\mathrm{Re}\Big[\sum_{i=1}^{k}(1+2A\rho_i(\zeta^\nu))w_i+\frac{1}{2}K^\nu(w)+\varepsilon w_{i_0}\Big]$$
$$+H^\nu(w)+\varepsilon H_{i_0}^\nu(w)+\alpha^\nu(w)+\varepsilon A_{i_0}^\nu(w)$$

in a neighborhood of the origin, where

$$A_{i_0}^\nu(w)=2\,\mathrm{Re}\,[K_{i_0}^\nu(w)]+\alpha_{i_0}^\nu(w)\,.$$

Setting

$$C_1=\min_{|w|=1}.\,H^p(w)>0\,,\quad C_2=\max_{|w|=1}.\,|H_{i_0}^p(w)|\geqq 0$$

and choosing an $\varepsilon_0>0$ so small that $C_1-\varepsilon_0C_2>0$, we obtain that, for every ε, $0<\varepsilon\leqq\varepsilon_0$,

$$H^p(w)+\varepsilon H_{i_0}^p(w)\geqq 4C|w|^2\,,\qquad w\in\boldsymbol{C}^n$$

with $C=(C_1-\varepsilon_0C_2)/4$. So, recalling the facts that $h^\nu\to h^p$, $H^\nu+\varepsilon H_{i_0}^\nu\to H^p+\varepsilon H_{i_0}^p$ and $\alpha^\nu\to\alpha^p$ local uniformly and $\alpha^\nu(w)=o(|w|^2)$, $\alpha_{i_0}^\nu(w)=o(|w|^2)$ for all $\nu=1, 2, \cdots$, we can find an open neighborhood W^* of the origin, $W^*\subset h^\nu(U)$ for $\nu=1, 2, \cdots$, and a constant $C_3>0$, which depends neither on ν nor on $w\in W^*$, such that

$$|A_{i_0}^\nu(w)|\leqq C_3|w|^2\qquad\text{on}\quad W^*;$$
$$H^\nu(w)+\varepsilon H_{i_0}^\nu(w)+\alpha^\nu(w)\geqq 2C\,|w|^2\qquad\text{on}\quad W^*$$

for all sufficiently large ν. Here, as we have already seen above, $\varepsilon>0$ may be chosen as small as we wish. Thus we can assume without loss of generality that

$$H^\nu(w)+\varepsilon H_{i_0}^\nu(w)+\alpha^\nu(w)+\varepsilon A_{i_0}^\nu(w)\geqq C\,|w|^2\qquad\text{on}\quad W^*$$

for all large ν, and hence by the same reasoning as in the first half of this step 5), we can assume, by taking a subsequence if necessary, that the sequence

$$\Big\{-\sum_{i=1}^{k}(1+2A\rho_i(\zeta^\nu))\frac{\delta_i^\nu}{r_\nu}F_i^\nu+\frac{1}{2r_\nu}K^\nu\circ(L^\nu)^{-1}\circ F^\nu-\varepsilon\frac{\delta_{i_0}^\nu}{r_\nu}F_{i_0}^\nu\Big\}_{\nu=1}^{\infty}$$

converges uniformly on compact subsets of D' to a holomorphic function $\tilde\Phi_1$ on D'. This combined with (2.8) and (2.14) yields that

$$F_{i_0}^\nu\to\frac{\sqrt{k}}{\varepsilon}(\tilde\Phi_1-\Phi_1)\qquad\text{as}\quad\nu\to\infty$$

uniformly on compact subsets of D'. Finally we have shown that $\{F^\nu\}$ converges uniformly on compact subsets of D' to a holomorphic mapping $F(j)$: $D'=D_j\to \boldsymbol{C}^n$. Since j was arbitrary, we can extract, by the usual diagonal arguments, a sequence $\{F^{\nu_i}\}$ of $\{F^\nu\}$ which converges uniformly on every compact subset of D_j to the holomorphic mapping $F(j)$: $D_j\to \boldsymbol{C}^n$ for all $j=1, 2, \cdots$. Therefore we can define a holomorphic mapping F: $D\to \boldsymbol{C}^n$ by setting $F(z)=F(j)(z)$ for all $z\in D_j$, $j=1, 2, \cdots$.

6) This step is devoted to proving that the range of F: $D\to \boldsymbol{C}^n$ lies in the closure of a domain $\mathcal{W}$, which is biholomorphically equivalent to some Siegel domain $\mathcal{D}(\boldsymbol{R}^k_+, H)$. We first define the functions H_i: $\boldsymbol{C}^{n-k}\times \boldsymbol{C}^{n-k}\to \boldsymbol{C}$, $i=1, \cdots, k$, by

$$(2.16)\qquad H_i(u, v) = \frac{\sqrt{k}}{2}\sum_{\alpha,\beta=k+1}^{n}\frac{\partial^2\rho_i^p}{\partial w_\alpha \partial \overline{w}_\beta}(0)u_\alpha \overline{v}_\beta\,, \qquad u=(u_\alpha),\ v=(v_\beta)\in \boldsymbol{C}^{n-k}$$

and consider the domain

$$(2.17)\qquad \mathcal{W} = \{(w_1, \cdots, w_k, w'')\in \boldsymbol{C}^k\times \boldsymbol{C}^{n-k}\colon \operatorname{Re} w_i + H_i(w'', w'')<0\,,\ i=1, \cdots, k\}\,.$$

It is easily seen from the construction of h^p and (2.2) that the differential $(dh^p)_p$ of h^p: $\boldsymbol{C}^n\to \boldsymbol{C}^n$ at the point p is the identity mapping and

$$\begin{aligned} T &= \left\{\xi=(\xi_\alpha)\in \boldsymbol{C}^n\colon \sum_{\alpha=1}^{n}\frac{\partial \rho_i}{\partial z_\alpha}(p)\xi_\alpha = 0\,, \qquad i=1, \cdots, k\right\} \\ &= \{(0, \cdots, 0, \xi'')\in \boldsymbol{C}^k\times \boldsymbol{C}^{n-k}\colon \xi''\in \boldsymbol{C}^{n-k}\}\,. \end{aligned}$$

Hence

$$H_i(u, u)\geqq 0 \qquad \text{for all} \quad u\in \boldsymbol{C}^{n-k},\ i=1, \cdots, k$$

by (C.4) and

$$\sum_{i=1}^{k}\left[\sum_{\alpha,\beta=k+1}^{n}\frac{\partial^2\rho_i^p}{\partial w_\alpha \partial \overline{w}_\beta}(0)\,u_\alpha \overline{u}_\beta\right] = \sum_{\alpha,\beta=k+1}^{n}\frac{\partial^2\rho^p}{\partial w_\alpha \partial \overline{w}_\beta}(0)u_\alpha \overline{u}_\beta \quad \text{for all} \quad u=(u_\alpha)\in \boldsymbol{C}^{n-k}\,.$$

This last equality combined with our assumption (C.5) guarantees that the hermitian form $\sum_{i=1}^{k} H_i$ is positive definite on $\boldsymbol{C}^{n-k}$. Consequently, the mapping $H=(H_1, \cdots, H_k)$: $\boldsymbol{C}^{n-k}\times \boldsymbol{C}^{n-k}\to \boldsymbol{C}^k$ is an $\boldsymbol{R}^k_+$-hermitian form. Let $\mathcal{D}(\boldsymbol{R}^k_+, H)$ be the Siegel domain in $\boldsymbol{C}^k\times \boldsymbol{C}^{n-k}$ associated to the convex cone $\boldsymbol{R}^k_+$ and this $\boldsymbol{R}^k_+$-hermitian form H and let L: $\boldsymbol{C}^n\to \boldsymbol{C}^n$ be the linear mapping defined by

$$L(w', w'') = (-\sqrt{-1}\,w', w''), \qquad (w', w'')\in \boldsymbol{C}^k\times \boldsymbol{C}^{n-k} = \boldsymbol{C}^n\,.$$

Then L gives rise to a biholomorphic mapping from $\mathcal{W}$ onto $\mathcal{D}(\boldsymbol{R}^k_+, H)$. In particular, we see that $\mathcal{W}$ is a complete hyperbolic domain and hence it is taut [4], [6].

We shall study here the functions ρ_i^ν more closely. By virtue of (2.3) we can express

$$\rho_i^\nu(w) = 2\,\mathrm{Re}\, w_i + H_i^\nu(w) + A_i^\nu(w)$$

in a neighborhood of the origin, where

$$A_i^\nu(w) = 2\,\mathrm{Re}\,[K_i^\nu(w)] + \alpha_i^\nu(w)\,.$$

Since $K_i^\nu \to K_i^p$, $\alpha_i^\nu \to \alpha_i^p$ for $i=1, \cdots, k$ and $K_i^\nu(0, \cdots, 0, w'')=0$, $w'' \in \boldsymbol{C}^{n-k}$ for $i=1, \cdots, k$; $\nu=1, 2, \cdots$ by (2.4), we can show that there exist a constant $C>0$ and a positive function $\eta(t)$, which are independent on ν, such that

$$|2\,\mathrm{Re}\,[K_i^\nu(w)]| \leqq C \sum_{\alpha=1}^{k} |w_\alpha|\,|w| \qquad \text{for all} \quad w = (w_\alpha) \in \boldsymbol{C}^n\,,$$

$$\lim_{t \to +0} \eta(t) = 0 \quad \text{and} \quad |\alpha_i^\nu(w)| \leqq \eta(|w|^2)|w|^2$$

in a neighborhood of the origin. Hence we have that, for every $i=1, \cdots, k$,

$$\begin{aligned}\frac{|A_i^\nu \circ (L^\nu)^{-1}(w)|}{r_\nu} &\leqq C \sum_{\alpha=1}^{k} \frac{|\delta_\alpha^\nu|}{r_\nu} |w_\alpha|\,|(L^\nu)^{-1}(w)| \\ &\quad + \eta(|(L^\nu)^{-1}(w)|^2)\Big[\sum_{i=1}^{k} \frac{|\delta_i^\nu w_i|^2}{r_\nu} + \sum_{j=k+1}^{n} |w_j|^2\Big] \\ &\quad \to 0 \quad \text{as} \quad \nu \to \infty\end{aligned}$$

uniformly on every compact subset of $\boldsymbol{C}^n$. By a routine calculation, we can also prove that, for $i=1, \cdots, k$,

$$\frac{H_i^\nu \circ (L^\nu)^{-1}(w)}{r_\nu} \to H_i^p(0, \cdots, 0, w'') \qquad \text{as} \quad \nu \to \infty$$

uniformly on every compact subset of $\boldsymbol{C}^n$.

Now, we change the notation and may assume that $\{F^\nu\}$ itself converges uniformly on every compact set in D to the holomorphic mapping $F\colon D \to \boldsymbol{C}^n$. Choose a point $z \in D$ arbitrarily and put again

$$w^\nu = F^\nu(z)$$

(starting with some index $\nu=\nu(z)$). Then it follows from (2.9) and (2.10) that, for every $i=1, \cdots, k$,

$$\begin{aligned} 0 &> \frac{\rho_i^\nu \circ (L^\nu)^{-1}(w^\nu)}{r_\nu} \\ &= 2\,\mathrm{Re}\Big(-\frac{\delta_i^\nu}{r_\nu} w_i^\nu\Big) + \frac{1}{r_\nu}\,[H_i^\nu \circ (L^\nu)^{-1}(w^\nu) + A_i^\nu \circ (L^\nu)^{-1}(w^\nu)]\end{aligned}$$

for all sufficiently large ν, and so letting ν tend to infinity, we have

$$0 \geqq 2\,\mathrm{Re}\Big(\frac{1}{\sqrt{k}} F_i(z)\Big) + H_i^p(0, \cdots, 0, F_{k+1}(z), \cdots, F_n(z))$$

for every $i=1, \cdots, k$. Clearly this means that $F(z) \in \overline{\mathcal{W}}$ and accordingly $F(D) \subset \overline{\mathcal{W}}$.

7) We claim that $F(D) \subset \mathcal{W}$. To see this, observe that the interior of the closure $\overline{\mathcal{W}}$ coincides with $\mathcal{W}$ itself in our case. Hence the problem reduces to showing that $F\colon D \to \boldsymbol{C}^n$ is an open mapping. Now we define the biholomorphic mappings $g^\nu\colon W_\nu \to D$, $\nu=1, 2, \cdots$, as follows:

$$g^\nu(w) = f_\nu^{-1} \circ (h^\nu)^{-1} \circ (L^\nu)^{-1}(w)\,, \qquad w \in W_\nu\,,$$

where W_ν are the domains given by (2.10). It is clear that

$$g^\nu \circ F^\nu_{|D_j} = id_{D_j} \quad \text{and} \quad F^\nu \circ g^\nu_{|F^\nu(D_j)} = id_{F^\nu(D_j)} \tag{2.18}$$

for $\nu \geqq \nu(D_j)$, $j=1, 2, \cdots$. Let W' be an arbitrary subdomain of $\mathcal{W}$ with compact closure. Then

$$h^\nu(U) \to h^p(U)\,, \quad (L^\nu)^{-1}(w) \to 0$$

and, for every $i=1, \cdots, k$,

$$\frac{\rho_i^\nu \circ (L^\nu)^{-1}(w)}{r_\nu} \to 2\,\mathrm{Re}\Big(\frac{1}{\sqrt{k}} w_i\Big) + H_i^p(0, \cdots, 0, w'') < 0$$

uniformly on W'. This assures us the existence of an integer $\nu(W')$ such that

$$W' \subset W_\nu \qquad \text{for all} \quad \nu \geqq \nu(W')\,. \tag{2.19}$$

Now, keeping the fact

$$F(k_0) = \lim_{\nu \to \infty} F^\nu(k_\nu) = (-1, \cdots, -1, 0, \cdots, 0) \in \mathcal{W}$$

in mind, we choose open neighborhoods W', D' of the points $(-1, \cdots, -1, 0, \cdots, 0)$, k_0 with compact closures in $\mathcal{W}$, D, respectively, in such a way that $F(\bar{D}') \subset W'$. Then there is a large integer $\nu(D', W')$ such that

$$F^\nu(D') \subset W' \qquad \text{for all} \quad \nu \geqq \nu(D', W')\,. \tag{2.20}$$

We here assert that $F\colon D \to \boldsymbol{C}^n$ is injective on D', so that $F(D')$ is a non-empty open subset of $\boldsymbol{C}^n$. To verify this assertion, assume that $F(z')=F(z'')=w$ for $z', z'' \in D'$. It follows then from (1.1) and (2.18)~(2.20) that

$$\begin{aligned} d_{W'}(F^\nu(z'), F^\nu(z'')) &= d_{g^\nu(W')}(g^\nu(F^\nu(z')), g^\nu(F^\nu(z''))) \\ &= d_{g^\nu(W')}(z', z'') \geqq d_D(z', z'') \end{aligned}$$

for all $\nu \geqq \max.(\nu(W'), \nu(D', W'))$, and so

$$0 \leqq d_D(z', z'') \leqq \lim_{\nu\to\infty} d_{W'}(F^\nu(z'), F^\nu(z'')) = d_{W'}(w, w) = 0 .$$

Clearly this means that $z'=z''$, as asserted. On the other hand, being the local uniform limit of regular holomorphic mappings F^ν, the mapping F: $D\to \boldsymbol{C}^n$ is either regular on D or the Jacobian determinant of F vanishes identically on D [7; p. 80]. But, as we have already seen above, $F(D)$ contains a non-empty open subset of $\boldsymbol{C}^n$. Thus F: $D\to\boldsymbol{C}^n$ must be regular on D and $F(D)\subset \mathcal{W}$.

8) As the final step, we show that F: $D\to\mathcal{W}$ is, in fact, a biholomorphic mapping from D onto $\mathcal{W}$. First let us fix a family $\{\tilde{W}_j\}_{j=1}^\infty$ of relatively compact subdomains $\tilde{W}_j$ of $\mathcal{W}$ such that

$$\mathcal{W} = \bigcup_{j=1}^{\infty} \tilde{W}_j \supset \cdots \supset \tilde{W}_{j+1} \supset \tilde{W}_j \supset \cdots \supset \tilde{W}_1 \ni (\underbrace{-1, \cdots, -1}_{k \text{ times}}, 0, \cdots, 0) .$$

Choosing a $\tilde{W}_j$ arbitrarily, we put $W'=\tilde{W}_j$ for simplicity. By (2.19) there exists an integer $\nu(W')$ such that $W'\subset W_\nu$ for all $\nu\geqq\nu(W')$. So that the restriction $G^\nu=g^\nu_{|W'}$ defines a biholomorphic mapping from W' into D for every $\nu\geqq\nu(W')$. By the Montel's theorem, some subsequence of $\{G^\nu\}$ converges uniformly on compact subsets of W' to a holomorphic mapping $G(j)$: $W'=\tilde{W}_j\to\bar{D}\subset\boldsymbol{C}^n$. Hence, in exactly the same way as in the construction of F: $D\to\mathcal{W}$, we can define a holomorphic mapping G: $\mathcal{W}\to\bar{D}\subset\boldsymbol{C}^n$. Once it is shown that $G(\mathcal{W})\subset D$, our proof can be completed, because in such a case (2.18) implies that

$$G\circ F = id_D \quad \text{and} \quad F\circ G = id_{\mathcal{W}} .$$

But, since $G(-1, \cdots, -1, 0, \cdots, 0)=k_0$ by (2.9), (2.18) and since $F(D)\subset\mathcal{W}$ as above, by interchanging the roles of D' and W', F and G in the preceding step 7), we can prove that G: $\mathcal{W}\to\boldsymbol{C}^n$ is a regular mapping on $\mathcal{W}$. Then it follows at once from [3; Lemma 0] or [7; p. 79] that $G(\mathcal{W})\subset D$. We have shown eventually that the composition $L\circ F$: $D\to\mathcal{D}(\boldsymbol{R}^k_+, H)$ gives a biholomorphic equivalence between D and $\mathcal{D}(\boldsymbol{R}^k_+, H)$, which was the first half of the theorem.

In order to prove the converse, let us take an arbitrary Siegel domain

$$\mathcal{D}(\boldsymbol{R}^k_+, H) = \{(z', z'')\in\boldsymbol{C}^k\times\boldsymbol{C}^{n-k}\colon \operatorname{Im} z' - H(z'', z'')\in\boldsymbol{R}^k_+\}$$

in $\boldsymbol{C}^k\times\boldsymbol{C}^{n-k}$ and consider the biholomorphic mapping

$$C\colon \mathcal{D}(\boldsymbol{R}^k_+, H) \to D = C(\mathcal{D}(\boldsymbol{R}^k_+, H))\subset\mathcal{B}^{n_1}\times\cdots\times\mathcal{B}^{n_k}$$

constructed in section 1. We want to show that this bounded domain D satisfies all the conditions from (C.1) through (C.5) and $(*)$ at the point $p=(-1, 0, \cdots, -1, 0)\in\partial D\subset\boldsymbol{C}\times\boldsymbol{C}^{n_1-1}\times\cdots\times\boldsymbol{C}\times\boldsymbol{C}^{n_k-1}$. As was remarked at the end of section 1, C can be extended to a biholomorphic mapping from an open neighborhood V of the origin $0 \in \partial \mathcal{D}(\boldsymbol{R}_+^k, H)$ onto an open neighborhood U of the point $p=(-1, 0, \cdots, -1, 0) \in \partial D$. We shall denote this extended mapping by the same letter C. Now we set, for $i=1, \cdots, k$,

$$\tilde{\rho}_i(z) = \frac{-1}{|z_i+\sqrt{-1}|^2} \cdot (\operatorname{Im} z_i - H_i(z'', z'')),$$

where H_i is the i-th component function of the $\boldsymbol{R}_+^k$-hermitian form H: $\boldsymbol{C}^{n-k} \times \boldsymbol{C}^{n-k} \to \boldsymbol{C}^k$. Shrinking V if necessary, we may assume that every $\tilde{\rho}_i$ is a real analytic function on V. And it is clear that

$$\mathcal{D}(\boldsymbol{R}_+^k, H) \cap V = \{z \in V : \tilde{\rho}_i(z) < 0, \qquad i=1, \cdots, k\}.$$

Here we introduce the following notations:

E_k : The $k \times k$ unit matrix.

$E(i)$: The $k \times k$ matrix having 1 in the (i, i)-position and 0 elsewhere $(1 \leqq i \leqq k)$.

0 : The zero matrix with suitable size.

$\tilde{H}_i = (h^i_{\alpha\bar{\beta}})_{k+1 \leqq \alpha, \beta \leqq n}$: The hermitian matrix defined by

$$H_i(u, v) = \sum_{\alpha, \beta = k+1}^{n} h^i_{\alpha\bar{\beta}} u_\alpha \bar{v}_\beta, \quad u = (u_\alpha), \quad v = (v_\beta) \in \boldsymbol{C}^{n-k}; \ i = 1, \cdots, k.$$

And $X > 0$ $(\geqq 0)$ means that X is positive (semi-) definite, where X is a hermitian matrix. With these notations, we can show by direct computations that

$$\left(\frac{\partial \tilde{\rho}_i}{\partial z_\alpha}(0)\right)_{1 \leqq i \leqq k, 1 \leqq \alpha \leqq n} = \left(\frac{\sqrt{-1}}{2} E_k \,\Big|\, 0\right);$$

$$\left(\frac{\partial^2 \tilde{\rho}_i}{\partial z_\alpha \partial \bar{z}_\beta}(0)\right)_{1 \leqq \alpha, \beta \leqq n} = \left(\begin{array}{c|c} E(i) & 0 \\ \hline 0 & \tilde{H}_i \end{array}\right), \qquad i = 1, \cdots, k.$$

Hence we have

$$\bar{\partial} \tilde{\rho}_1 \wedge \cdots \wedge \bar{\partial} \tilde{\rho}_k(0) \neq 0;$$

$$\left(\frac{\partial^2 \tilde{\rho}_i}{\partial z_\alpha \partial \bar{z}_\beta}(0)\right)_{1 \leqq \alpha, \beta \leqq n} \geqq 0, \qquad i = 1, \cdots, k;$$

$$\left(\frac{\partial^2}{\partial z_\alpha \partial \bar{z}_\beta}\Big(\sum_{i=1}^{k} \tilde{\rho}_i\Big)(0)\right)_{1 \leqq \alpha, \beta \leqq n} = \left(\begin{array}{c|c} E_k & 0 \\ \hline 0 & \sum_{i=1}^{k} \tilde{H}_i \end{array}\right) > 0.$$

Therefore, if we define the functions ρ_i: $U \to \boldsymbol{R}$ by

$$\rho_i = \tilde{\rho}_i \circ C^{-1} \qquad \text{for} \quad i = 1, \cdots, k$$

using the biholomorphic mapping C: $V \to U$, the conditions (C.1)$\sim$(C.5) are satisfied for the system $(p;\ U;\ \rho_1, \cdots, \rho_k)$. Furthermore, considering the one-parameter subgroup

$$\varphi_t\colon (z', z'') \mapsto (e^t z', e^{(1/2)t} z''), \qquad t \in \boldsymbol{R}$$

of $\mathrm{Aut}(\mathcal{D}(\boldsymbol{R}_+^k, H))$, we can see that

$$\begin{aligned} &\{C\circ\varphi_t\circ C^{-1}\}_{t\in\boldsymbol{R}} \subset \mathrm{Aut}(D);\\ &(C\circ\varphi_t\circ C^{-1})(0) = \left(\frac{e^t-1}{e^t+1}, 0, \cdots, \frac{e^t-1}{e^t+1}, 0\right)\\ &\qquad\qquad \to (-1, 0, \cdots, -1, 0) = p \quad \text{as} \quad t\to -\infty. \end{aligned}$$

(Note that $0 = C(\sqrt{-1}, \cdots, \sqrt{-1}, 0, \cdots, 0) \in D$.) Obviously this guarantees us that the condition $(*)$ is also satisfied. Thus the proof of Theorem I has been completed. Q.E.D.

Proof of Corollary 1. By our assumption there exist a sequence $\{k_\nu\}$ in K and a sequence $\{f_\nu\}$ in $\mathrm{Aut}(D)$ such that $\lim_{\nu\to\infty} f_\nu(k_\nu) = p$. As an immediate consequence of Theorem I, it follows that D is biholomorphically equivalent to a Siegel domain $\mathcal{D}(\boldsymbol{R}_+^k\ H)$. The proof is thus reduced to the following general

Proposition. *Let $\mathcal{D} = \mathcal{D}(\boldsymbol{R}_+^n, H)$ be a Siegel domain in $\boldsymbol{C}^n \times \boldsymbol{C}^m$ associated to $\boldsymbol{R}_+^n$ and H. Assume that there exists a compact subset K of $\mathcal{D}$ such that $Aut(\mathcal{D})\cdot K = \mathcal{D}$. Then $\mathcal{D}$ is biholomorphically equivalent to some Siegel domain $\mathcal{E}(m_1, \cdots, m_n)$ in $\boldsymbol{C}^n \times \boldsymbol{C}^m$ as in the introduction.*

Proof. We shall prove this fact along the same line as in the homogeneous case by Pinčuk [9]. With respect to the given coordinate system $(z, w) = (z_1, \cdots, z_n, w_1, \cdots, w_m)$ in $\boldsymbol{C}^n \times \boldsymbol{C}^m$, the $\boldsymbol{R}_+^n$-hermitian form H can be expressed as $H = (H_1, \cdots, H_n)$. According to [9], we may assume without loss of generality that there exists a direct product decomposition $\boldsymbol{C}^m = \boldsymbol{C}^{m_1-1} \times \cdots \times \boldsymbol{C}^{m_n-1}$ $(m_1 + \cdots + m_n = n+m$ and $m_i \geqq 1$ for $i = 1, \cdots, n)$ satisfying the following conditions i) and ii): Set, for $i = 1, \cdots, n$,

$$\begin{aligned} \omega_i &= (w_{m_1+\cdots+m_{i-1}-(i-2)}, \cdots, w_{m_1+\cdots+m_i-i});\\ H_i(w, w) &= H_i(\omega_1, \cdots, \omega_n), \end{aligned}$$

where $m_0 = 0$. Then, for every $i = 1, \cdots, n$ we have:

i) $H_i(\omega_1, \cdots, \omega_i, \cdots, \omega_n) = H_i(\omega_1, \cdots, \omega_i, 0, \cdots, 0)$;

ii) $H_i(0, \cdots, 0, \omega_i, 0, \cdots, 0)$ is positive definite on $\boldsymbol{C}^{m_i-1}$ whenever $m_i > 1$.

Now, choosing a sequence of positive numbers δ_ν such that $\delta_\nu \downarrow 0$, we put

$$p^\nu = (\sqrt{-1}\,\delta_\nu^{2(2n-1)}, \sqrt{-1}\,\delta_\nu^{2(2n-2)}, \cdots, \sqrt{-1}\,\delta_\nu^{2n}, 0, \cdots, 0)$$

for $\nu=1, 2, \cdots$. Then $p^{\nu}\in\mathcal{D}$ and $\lim_{\nu\to\infty} p^{\nu}=0\in\partial\mathcal{D}$. Especially, there exist sequences $\{k_{\nu}\}\subset K$ and $\{f_{\nu}\}\subset\mathrm{Aut}(\mathcal{D})$ such that

$$p^{\nu}=f_{\nu}(k_{\nu}) \qquad \text{for} \quad \nu=1, 2, \cdots.$$

As we have already seen as in the proof of Theorem I, there exists a strictly plurisubharmonic function ρ defined on an open neighborhood U of the origin $0\in\partial\mathcal{D}$ such that $\rho(0)=0$ and $\mathcal{D}\cap U\subset\{(z, w)\in U\colon \rho(z, w)<0\}$. Therefore, by the same reasoning as in the step 1) of the proof of Theorem I, we can assume that $\{f_{\nu}\}$ converges uniformly on compact subsets of $\mathcal{D}$ to the constant mapping $C_0(z, w)\equiv 0$. Now define the mappings $L^{\nu}\colon \boldsymbol{C}^n\times\boldsymbol{C}^m\to\boldsymbol{C}^n\times\boldsymbol{C}^m$ and $F^{\nu}\colon \mathcal{D}\to\boldsymbol{C}^n\times\boldsymbol{C}^m$ for $\nu=1, 2, \cdots$ by

$$L^{\nu}: \tilde{z}_i=\frac{z_i}{\delta_{\nu}^{2(2n-i)}}, \quad \tilde{\omega}_i=\frac{\omega_i}{\delta_{\nu}^{2n-i}}, \qquad i=1, \cdots, n$$

for $(z_1, \cdots, z_n, \omega_1, \cdots, \omega_n)=(z, w)\in\boldsymbol{C}^n\times\boldsymbol{C}^m$ and

$$F^{\nu}(z, w)=L^{\nu}\circ f_{\nu}(z, w), \qquad (z, w)\in\mathcal{D}.$$

Then L^{ν} are non-singular linear transformations of $\boldsymbol{C}^n\times\boldsymbol{C}^m$ and F^{ν} are biholomorphic mappings from $\mathcal{D}$ into $\boldsymbol{C}^n\times\boldsymbol{C}^m$ such that

$$F^{\nu}(k_{\nu})=(\underbrace{\sqrt{-1}, \cdots, \sqrt{-1}}_{n \text{ times}}, 0, \cdots, 0) \tag{2.21}$$

for all $\nu=1, 2, \cdots$. Let us consider the image domains $W_{\nu}=F^{\nu}(\mathcal{D})$, $\nu=1, 2, \cdots$. Set

$$\rho_i^{\nu}(\tilde{z}, \tilde{\omega})=\mathrm{Im}\,\tilde{z}_i-H_i(\delta_{\nu}^{i-1}\tilde{\omega}_1, \delta_{\nu}^{i-2}\tilde{\omega}_2, \cdots, \delta_{\nu}\tilde{\omega}_{i-1}, \tilde{\omega}_i, 0, \cdots, 0) \tag{2.22}$$

for $i=1, \cdots, n$; $\nu=1, 2, \cdots$. Then it is easily seen that

$$W_{\nu}=\{(\tilde{z}, \tilde{\omega})\in\boldsymbol{C}^n\times\boldsymbol{C}^m\colon \rho_i^{\nu}(\tilde{z}, \tilde{\omega})>0, \qquad i=1, \cdots, n\}$$

for $\nu=1, 2, \cdots$. Hence every $\Phi^{\nu}=(F_1^{\nu}, \cdots, F_n^{\nu})$, $\nu=1, 2, \cdots$, is a holomorphic mapping from $\mathcal{D}$ into the Siegel domain $\mathcal{D}(\boldsymbol{R}_+^n)=\{(\tilde{z}_1, \cdots, \tilde{z}_n)\in\boldsymbol{C}^n\colon \mathrm{Im}\,\tilde{z}_i>0, i=1, \cdots, n\}$ (which is of course a taut domain) such that $\Phi^{\nu}(k_{\nu})=(\sqrt{-1}, \cdots, \sqrt{-1})\in\mathcal{D}(\boldsymbol{R}_+^n)$ by (2.21). Thus, passing to a subsequence if necessary, we can assume that $\{\Phi^{\nu}\}$ converges uniformly on compact subsets of $\mathcal{D}$ to a holomorphic mapping $\Phi\colon \mathcal{D}\to\mathcal{D}(\boldsymbol{R}_+^n)$. Then, from ii) and the inequality $\rho_1^{\nu}\circ F^{\nu}(z, w)>0, (z, w)\in\mathcal{D}$, for $\nu=1, 2, \cdots$, it follows that the sequence $\{\Psi_1^{\nu}=(F_{n+1}^{\nu}, \cdots, F_{n+m_1-1}^{\nu})\}$ is bounded on every compact set in $\mathcal{D}$, so that we may assume that $\{\Psi_1^{\nu}\}$ converges uniformly on compact subsets of $\mathcal{D}$ to a holomorphic mapping $\Psi_1\colon \mathcal{D}\to\boldsymbol{C}^{m_1-1}$. In this case we see that the sequence $\{\Psi_2^{\nu}=(F_{n+m_1}^{\nu}, \cdots, F_{n+m_1+m_2-2}^{\nu})\}$ must be also bounded on every compact set in $\mathcal{D}$.

Indeed, assuming the contrary, we can find a sequence $\{(z_i, w_i)\}_{i=1}^{\infty}$ in $\mathcal{D}$ such that $(z_i, w_i) \to (z_0, w_0) \in \mathcal{D}$ and $r_i = |\Psi_2^{\nu_i}(z_i, w_i)| \to \infty$. Without loss of generality, we can assume that

$$\frac{1}{r_i}\Psi_2^{\nu_i}(z_i, w_i) \to \tilde{\omega}_2^0 \in \boldsymbol{C}^{m_2-1} \qquad \text{with} \quad |\tilde{\omega}_2^0| = 1 .$$

On the other hand, from the inequalities $\rho_2^{\nu_i} \circ F^{\nu_i}(z_i, w_i) > 0$ we obtain that

$$\begin{aligned} 0 \leqq H_2 &\left(\frac{\delta_{\nu_i}}{r_i}\Psi_1^{\nu_i}(z_i, w_i), \frac{1}{r_i}\Psi_2^{\nu_i}(z_i, w_i), 0, \cdots, 0\right) \\ &< \frac{\operatorname{Im} F_2^{\nu_i}(z_i, w_i)}{r_i^2} \to 0 \qquad \text{as} \quad i \to \infty \end{aligned}$$

and so

$$H_2(0, \tilde{\omega}_2^0, 0, \cdots, 0) = 0 .$$

This contradicts the fact ii). Therefore we can select a convergent subsequence of $\{\Psi_2^{\nu}\}$. Repeating this process, we obtain eventually a subsequence of $\{F^{\nu}\}$ converging uniformly on every compact set in $\mathcal{D}$ to a holomorphic mapping $F: \mathcal{D} \to \boldsymbol{C}^n \times \boldsymbol{C}^m$.

Now we set

$$\tilde{H}_i(\tilde{\omega}_i) = H_i(0, \cdots, 0, \tilde{\omega}_i, 0, \cdots, 0) , \qquad i = 1, \cdots, n$$

and define the Siegel domain $\tilde{\mathcal{D}}$ in $\boldsymbol{C}^n \times \boldsymbol{C}^m$ by

$$\tilde{\mathcal{D}} = \{(\tilde{z}, \tilde{\omega}) \in \boldsymbol{C}^n \times \boldsymbol{C}^m \colon \operatorname{Im} \tilde{z}_i - \tilde{H}_i(\tilde{\omega}_i) > 0 , \qquad i = 1, \cdots, n\} .$$

We also define the holomorphic mappings $G^{\nu}: W_{\nu} \to \mathcal{D}$ by

$$G^{\nu}(\tilde{z}, \tilde{\omega}) = f_{\nu}^{-1} \circ (L^{\nu})^{-1}(\tilde{z}, \tilde{\omega}) , \qquad (\tilde{z}, \tilde{\omega}) \in W_{\nu}$$

for $\nu = 1, 2, \cdots$. Then, in exactly the same way as in the proof of Theorem I we can show that $F(\mathcal{D}) \subset \tilde{\mathcal{D}}$ and $\{G^{\nu}\}$ contains a subsequence which converges uniformly on every compact set in $\tilde{\mathcal{D}}$ to a holomorphic mapping $G: \tilde{\mathcal{D}} \to \mathcal{D}$ such that $G \circ F = id_{\mathcal{D}}$ and $F \circ G = id_{\tilde{\mathcal{D}}}$. Thus $\tilde{\mathcal{D}}$ is biholomorphically equivalent to $\mathcal{D}$. Since $\tilde{\mathcal{D}}$ is obviously biholomorphically equivalent to the Siegel domain $\mathcal{E}(m_1, \cdots, m_n)$ in $\boldsymbol{C}^n \times \boldsymbol{C}^m$, our proof has been completed. Q.E.D.

3. Proofs of Theorems II and III

Using exactly the same technique as in the proof of Theorem I, we shall show the theorems.

Proof of Theorem II. To begin with, we fix a family $\{M_j\}_{j=1}^{\infty}$ of relatively compact subdomains of M such that

$$M=\bigcup_{j=1}^{\infty} M_j \supset \cdots \supset M_{j+1} \supset M_j \supset \cdots \supset M_1 . \tag{3.1}$$

Since M can be exhausted by biholomorphic images of D, there exists a sequence $\{\varphi_\nu\}_{\nu=1}^{\infty}$ of biholomorphic mappings from D into M such that

$$M_\nu \subset \varphi_\nu(D), \qquad \nu=1, 2, \cdots .$$

We set

$$\psi_\nu=\varphi_\nu^{-1}: \varphi_\nu(D) \rightarrow D, \qquad \nu=1, 2, \cdots .$$

Since D is bounded, for each $j=1, 2, \cdots$ some subsequence of $\{\psi_{\nu|M_j}\}_{\nu \geqq j}$ converges uniformly on compact subsets of M_j to a holomorphic mapping $\psi(j)$: $M_j \rightarrow \bar{D} \subset \boldsymbol{C}^n$. Thus, after taking a subsequence and relabelling if necessary, we can assume that $\{\psi_\nu\}$ converges uniformly on every compact set K in M to a holomorphic mapping ψ: $M \rightarrow \boldsymbol{C}^n$, starting with some index $\nu=\nu(K)$. Clearly $\psi(M) \subset \bar{D}$. Since D is a bounded domain with piecewise C^2-smooth boundary of special type, the same reasoning as in the step 1) of the proof of Theorem I yields that, if $\psi(x_0) \in \partial D$ for some point $x_0 \in M$, then $\psi(x)=\psi(x_0)$ for all $x \in M$. Therefore the proof is now divided into the following two cases.

Case 1: $\psi(M) \subset D$.

We shall prove that M is biholomorphically equivalent to D in this case. We first claim that ψ: $M \rightarrow D$ is injective, so that it defines a biholomorphic mapping from M onto the image domain $\psi(M) \subset D$. Assume that $\psi(x')=\psi(x'')=z$ for $x', x'' \in M$. It follows then from (1.1) that

$$\begin{aligned} d_D(\psi_\nu(x'), \psi_\nu(x'')) &= d_{\varphi_\nu(D)}(\varphi_\nu(\psi_\nu(x'), \varphi_\nu(\psi_\nu(x''))) \\ &= d_{\varphi_\nu(D)}(x', x'') \geqq d_M(x', x'') \end{aligned}$$

for all sufficiently large ν. Consequently we have $x'=x''$, because M is hyperbolic and $d_D(\psi_\nu(x'), \psi_\nu(x'')) \rightarrow d_D(z, z)=0$. Thus ψ: $M \rightarrow D$ is injective.

We next claim that ψ: $M \rightarrow D$ is surjective. By the argument above, we can identify M with the bounded domain $\psi(M)$ in $\boldsymbol{C}^n$, and hence some subsequence $\{\varphi_{\nu_j}\}$ of $\{\varphi_\nu\}$ converges uniformly on compact subsets of D to a holomorphic mapping φ: $D \rightarrow \bar{M} \subset \boldsymbol{C}^n$. Once it is shown that $\varphi(D) \subset M$, the sequence $\{\varphi_{\nu_j}(z)\}$ lies in a compact subset of M for any $z \in D$. Hence

$$z=\lim_{j \rightarrow \infty} \psi_{\nu_j}(\varphi_{\nu_j}(z))=\psi(\varphi(z)) \in \psi(M),$$

which means the surjectivity of ψ. Therefore, it is enough to show that $\varphi(D) \subset M$. Changing the notation, we may assume that $\{\varphi_\nu\}$ converges uniformly on compact subsets of D to φ. Choose an open neighborhood D' of $\psi(\bar{M}_1)$ with compact closure in D and fix an integer ν_0 so large that $\psi_\nu(\bar{M}_1) \subset D'$ for all $\nu \geqq \nu_0$, where M_1 is the subdomain of M appeared in (3.1). Then, for any point $x \in M_1$ there exists a sequence of points z_ν of D' such that $\varphi_\nu(z_\nu)=x$ for all $\nu \geqq \nu_0$. We can assume that $z_\nu \to z$ for some point z of $\bar{D}'$. Hence $x=\lim_{\nu\to\infty} \varphi_\nu(z_\nu)=\varphi(z)\in\varphi(D)$, and accordingly $M_1 \subset \varphi(D)$. On the other hand, being the local uniform limit of regular holomorphic mappings $\varphi_\nu: D \to M \subset \boldsymbol{C}^n$, φ is either regular on D or the Jacobian determinant of φ vanishes identically on D. But, since $\varphi(D)$ contains the non-empty open set M_1 as above, we conclude that $\varphi: D \to \boldsymbol{C}^n$ is regular on D. Then it follows immediately that $\varphi(D) \subset M$, completing the proof.

Case 2. $$\psi(M) = \{p\} \subset \partial D .$$

Since D is a bounded domain with piecewise C^2-smooth boundary of special type, there exist an open neighborhood U of p and real-valued C^2-functions $\rho_1, \cdots, \rho_k$ $(1 \leqq k \leqq n)$ defined on U satisfying the conditions from (C.1) through (C.5) in the introduction. Let us fix a point $x_0 \in M_1$ arbitrarily and put

$$p^\nu = \psi_\nu(x_0) \qquad \text{for} \quad \nu = 1, 2, \cdots .$$

Then $p^\nu=\psi_\nu(x_0) \to \psi(x_0)=p$ as $\nu \to \infty$. Changing the coordinate system and shrinking U if necessary, we may assume that

$$\nabla\rho_i(p) = (0, \cdots, 0, \overset{\overset{i}{\vee}}{2}, 0, \cdots, 0) , \qquad i = 1, \cdots, k$$

as in (2.2) and every point p^ν, $\nu=1, 2, \cdots$, can be written in the form

$$p^\nu = \zeta^\nu + \lambda^\nu N \qquad \text{for some} \quad \zeta^\nu \in \partial D \cap U , \qquad \lambda^\nu < 0 ,$$

where

$$N = \frac{1}{2}(\nabla\rho_1(p)+\cdots+\nabla\rho_k(p)) = (\underbrace{1, \cdots, 1}_{k \text{ times}}, 0, \cdots, 0) .$$

We define two families of biholomorphic mappings $\{h^\nu\}$ and $\{L^\nu\}$ by the same manner as in the proof of Theorem I. Let $\{M_j\}_{j=1}^\infty$ be the monotone increasing sequence of relatively compact subdomains of M as in (3.1). Taking an M_j arbitrarily, we set $M'=M_j$ for simplicity. Since $\psi_\nu(x) \to p$ uniformly on M', there exists an integer $\nu(M')$ such that

$$\psi_\nu(M') \subset D \cap U \qquad \text{for all} \quad \nu \geqq \nu(M') .$$

Define now the biholomorphic mappings $F^\nu: M' \to \boldsymbol{C}^n$ $\nu \geqq \nu(M')$ by

$$F^\nu(x) = L^\nu \circ h^\nu \circ \psi_\nu(x) , \qquad x \in M' .$$

Then, repeating exactly the same arguments as in the proof of Theorem I, we can show that some subsequence of $\{F^\nu\}$ converges uniformly on compact subsets of M' to a holomorphic mapping $F(j)$: $M'=M_j \to \overline{\mathcal{W}} \subset \boldsymbol{C}^n$, where $\mathcal{W}$ is the domain in $\boldsymbol{C}^n$ given by (2.17). Since j was arbitrary, we obtain a holomorphic mapping F: $M \to \overline{\mathcal{W}} \subset \boldsymbol{C}^n$ such that

$$F(x_0) = (\underbrace{-1, \cdots, -1}_{k \text{ times}}, 0, \cdots, 0) \in \mathcal{W}\,.$$

It remains to prove that: (i) $F(M) \subset \mathcal{W}$ and (ii) F: $M \to \mathcal{W}$ is, in fact, a biholomorphic mapping from M onto $\mathcal{W}$. But, the assertion (i) can be shown by considering the biholomorphic mappings g^ν: $W_\nu \to M$ given by

$$g^\nu(w) = \varphi_\nu \circ (h^\nu)^{-1} \circ (L^\nu)^{-1}(w)\,, \qquad w \in W_\nu$$

for $\nu=1, 2, \cdots$, where W_ν are the domains in $\boldsymbol{C}^n$ given as in (2.10), and by repeating exactly the same arguments as in the step 7) of the proof of Theorem I. To prove (ii), we first assert that F: $M \to \mathcal{W}$ is injective, so that M is biholomorphically equivalent to the image domain $F(M) \subset \mathcal{W}$. In fact, assume that $F(x')=F(x'')=w$ for x', $x'' \in M$. Let W' be an open neighborhood of w with compact closure in $\mathcal{W}$. Let $\nu_0 \in \boldsymbol{N}$ be so large that $F^\nu(x')$, $F^\nu(x'') \in W'$ and $W' \subset W_\nu$ for all $\nu \geqq \nu_0$. Then

$$\begin{aligned} d_{W'}(F^\nu(x'), F^\nu(x'')) &= d_{g^\nu(W')}(g^\nu(F^\nu(x')), g^\nu(F^\nu(x''))) \\ &= d_{g^\nu(W')}(x', x'') \geqq d_M(x', x'') \end{aligned}$$

for all $\nu \geqq \nu_0$. Letting ν tend to infinity, we obtain that $d_M(x', x'')=0$ and hence $x'=x''$ by the hyperbolicity of M. Therefore F: $M \to \mathcal{W}$ is injective. On the other hand, being biholomorphically equivalent to a Siegel domain $\mathcal{D}(\boldsymbol{R}^k_+, H)$, $\mathcal{W}$ is biholomorphically equivalent to a bounded domain in $\boldsymbol{C}^n$ by the lemma in Sect. 1. Thus we may regard M as a bounded domain in $\boldsymbol{C}^n$. Repeating the same arguments as in 8) of the proof of Theorem I, we can now verify the assertion (ii). As a result, we have shown that M is biholomorphically equivalent to a Siegel domain $\mathcal{D}(\boldsymbol{R}^k_+, H)$ in the Case 2. Q.E.D.

The following example tells us that both the cases in Theorem II actually occur.

EXAMPLE 2. Consider the following two domains in $\boldsymbol{C}^2$:

$$\begin{aligned} \mathcal{E}(2) &= \{(z, w) \in \boldsymbol{C}^2 \colon \operatorname{Im} z - |w|^2 > 0\}\,; \\ D &= \{(z_1, z_2) \in \boldsymbol{C}^2 \colon |z_1|^2 + |z_2|^2 < 1\,, \qquad |z_1|^2 + 16|z_2|^2 < 4\}\,. \end{aligned}$$

Then $\mathcal{E}(2)$ is a Siegel domain biholomorphically equivalent to the open unit ball $\mathcal{B}^2$ in $\boldsymbol{C}^2$ and D is a subdomain of $\mathcal{B}^2$ with piecewise C^2-smooth, but not smooth, boundary of special type. Moreover we have:

i) $\mathcal{E}(2)$ can be exhausted by biholomorphic images of D.

ii) $\mathcal{E}(2)$ is not biholomorphically equivalent to D.

To see these facts, we put $p_0=(1, 0)\in\partial\mathcal{B}^2$ and choose a small open neighborhood U of p_0 in $\boldsymbol{C}^2$ in such a way that $D\cap U=\mathcal{B}^2\cap U$. (The existence of such a neighborhood U is obvious, because p_0 is an interior point of the domain $\{(z_1, z_2)\in\boldsymbol{C}^2\colon |z_1|^2+16|z_2|^2<4\}$.) By the homogeneity of $\mathcal{B}^2$, there exists a sequence $\{f_\nu\}$ in $\mathrm{Aut}(\mathcal{B}^2)$ such that $f_\nu(0)\to p_0$, where 0 denotes the origin of $\boldsymbol{C}^2$. Without loss of generality, we may assume that $f_\nu(z)\to p_0$ uniformly on every compact subset of $\mathcal{B}^2$. So, for any compact set K in $\mathcal{B}^2$ there exists an integer ν_0 such that $f_{\nu_0}(K)\subset\mathcal{B}^2\cap U=D\cap U$. Setting $F_K=f_{\nu_0}^{-1}|_D$, we obtain a biholomorphic mapping F_K from D into $\mathcal{B}^2$ such that $K\subset F_K(D)$. This implies that $\mathcal{B}^2$, and hence $\mathcal{E}(2)$, can be exhausted by biholomorphic images of D, which was the assertion i). Since $\mathcal{B}^2$ is not biholomorphically equivalent to D by [8; Theorem 1.1], we have also the assertion ii).

Proof of Theorem III. If D_1 and D_2 are biholomorphically equivalent, it is trivial that each of them can be exhausted by biholomorphic images of the other. Therefore we have only to prove the converse.

Assume that each of D_1 and D_2 can be exhausted by biholomorphic images of the other and that D_1 and D_2 are not biholomorphically equivalent. Let $\{D_1^j\}_{j=1}^\infty$ (resp. $\{D_2^j\}_{j=1}^\infty$) be an increasing sequence of relatively compact subdomains of D_1 (resp. of D_2) such that $D_1=\bigcup_{j=1}^\infty D_1^j$ (resp. $D_2=\bigcup_{j=1}^\infty D_2^j$). Then the proof of Theorem II guarantees the existences of biholomorphic imbeddings $\varphi_\nu\colon D_1\to D_2$, $\Phi_\nu\colon D_2\to D_1$ for $\nu=1, 2, \cdots$ and boundary points $p_1\in\partial D_1$, $p_2\in\partial D_2$ satisfying the following conditions i), ii) and iii): Let

$$\psi_\nu=\varphi_\nu^{-1}\colon \varphi_\nu(D_1)\to D_1\,,\quad \Psi_\nu=\Phi_\nu^{-1}\colon \Phi_\nu(D_2)\to D_2$$

be the inverse mappings of φ_ν, Φ_ν respectively, and let $(U_1; \rho_1^1, \cdots, \rho_{k_1}^1)$, $(U_2; \rho_1^2, \cdots, \rho_{k_2}^2)$ be defining systems for D_1, D_2 in the neighborhoods U_1, U_2 of p_1, p_2 respectively. Then we have

i) $\bar{D}_2^\nu\subset\varphi_\nu(D_1)$, $\bar{D}_1^\nu\subset\Phi_\nu(D_2)$ for $\nu=1, 2, \cdots$;

ii) $\psi_\nu(z)\to p_1$, $\Psi_\nu(z)\to p_2$ uniformly on compact subsets of D_1, D_2 respectively;

iii) D_1 (resp. D_2) is biholomorphically equivalent to a Siegel domain $\mathcal{D}(\boldsymbol{R}_+^{k_2}, H_2)$ (resp. $\mathcal{D}(\boldsymbol{R}_+^{k_1}, H_1)$), where H_2 (resp. H_1) is the $\boldsymbol{R}_+^{k_2}$ (resp. $\boldsymbol{R}_+^{k_1}$)-hermitian form as in (2.16) defined by the Levi-forms of $\rho_1^2, \cdots, \rho_{k_2}^2$ (resp. of $\rho_1^1, \cdots, \rho_{k_1}^1$).

Fix now two points $x_0^1\in D_1^1$ and $x_0^2\in D_2^1$ arbitrarily. By virtue of i) and ii) we can find a number $n(\nu_0)$ for each $\nu_0=1, 2, \cdots$ such that

$$\psi_{\nu_0}(D_2^{\nu_0})\subset\Phi_\nu(D_2)\qquad\text{for all}\quad \nu\geqq n(\nu_0);\tag{3.2}$$

$$\Psi_{n(\nu_0)}(\psi_{\nu_0}(x_0^2)) \in B_{p_2}\left(\frac{1}{\nu_0}\right), \tag{3.3}$$

where $B_{p_2}\left(\frac{1}{\nu_0}\right)$ denotes the open Euclidean $\left(\frac{1}{\nu_0}\right)$-ball with center at p_2. We set, for $\nu=1, 2, \cdots$,

$$f_\nu = \Psi_{n(\nu)} \circ \psi_{\nu | D_2^\nu} : D_2^\nu \to D_2 .$$

Owing to the boundedness of D_2, we can assume that $\{f_\nu\}$ converges uniformly on every compact set in D_2 to a holomorphic mapping f: $D_2 \to \bar{D}_2 \subset \boldsymbol{C}^n$. But, in view of (3.3) and the fact that ∂D_2 is a piecewise C^2-smooth boundary of special type, we can see that $f(z)=p_2$ for all $z \in D_2$. So, if we put

$$p_2^\nu = f_\nu(x_0^2) \qquad \text{for} \quad \nu=1, 2, \cdots ,$$

then $p_2^\nu = f_\nu(x_0^2) \to f(x_0^2) = p_2$. Changing the coordinate system and shrinking U_2 if necessary, we may assume that

$$\nabla \rho_i^2(p_2) = (0, \cdots, 0, \overset{\overset{i}{\vee}}{2}, 0, \cdots, 0) , \qquad i = 1, \cdots, k_2$$

and every point p_2^ν, $\nu=1, 2, \cdots$, has the form

$$p_2^\nu = \zeta_2^\nu + \lambda_2^\nu N_2 \qquad \text{for some} \quad \zeta_2^\nu \in \partial D_2 \cap U_2,\ \lambda_2^\nu < 0 ,$$

where

$$N_2 = \frac{1}{2}(\nabla \rho_1^2(p_2) + \cdots + \nabla \rho_{k_2}^2(p_2)) = (\underbrace{1, \cdots, 1}_{k_2 \text{ times}}, 0, \cdots, 0)$$

Let us define the families $\{h_2^\nu\}$, $\{(\rho_i^2)^\nu\}$ and $\{L_2^\nu\}$ by the same manner as in the proof of Theorem I. Set $D_2' = D_2^j$ as before. Then, since $f_\nu(z) \to p_2$ uniformly on D_2', there exists an integer $\nu(D_2')$ such that

$$f_\nu(D_2') \subset D_2 \cap U_2 \qquad \text{for all} \quad \nu \geqq \nu(D_2') .$$

Therefore we can define the biholomorphic mappings F_2^ν: $D_2' \to \boldsymbol{C}^n$ for $\nu \geqq \nu(D_2')$ by

$$F_2^\nu(z) = L_2^\nu \circ h_2^\nu \circ f_\nu(z) , \qquad z \in D_2' .$$

Now it is not difficult to check (along the same line as in the proof of Theorem I) that some subsequence of $\{F_2^\nu\}$ converges uniformly on compact subsets of D_2' to a holomorphic mapping $F_2(j)$: $D_2' = D_2^j \to \overline{\mathcal{W}}_2 \subset \boldsymbol{C}^n$, where $\mathcal{W}_2$ is the domain in $\boldsymbol{C}^n$ as in (2.17) defined by the Levi-forms of $\rho_1^2, \cdots, \rho_{k_2}^2$. Since j was arbitrary, we obtain a holomorphic mapping F_2: $D_2 \to \overline{\mathcal{W}}_2 \subset \boldsymbol{C}^n$ such that

$$F_2(x_0^2) = (\underbrace{-1, \cdots, -1}_{k_2 \text{ times}}, 0, \cdots, 0) \in \mathcal{W}_2 .$$

It remains to prove that $F_2(D_2) \subset \mathcal{W}_2$ and F_2 is biholomorphic mapping from D_2 onto $\mathcal{W}_2$. But this can be done with exactly the same arguments as in the proof of Theorem I, if we consider the biholomorphic mappings $g_2^\nu \colon W_2^\nu \to D_2$, $\nu = 1, 2, \cdots$, defined by

$$g_2^\nu(w) = \varphi_\nu \circ \Phi_{n(\nu)} \circ (h_2^\nu)^{-1} \circ (L_2^\nu)^{-1}(w) , \qquad w \in W_2^\nu ,$$

where W_2^ν are the domains as in (2.10) defined by h_2^ν, $(\rho_i^2)^\nu$ and L_2^ν. Recalling the fact that $\mathcal{W}_2$ is biholomorphically equivalent to the Siegel domain $\mathcal{D}(\boldsymbol{R}_+^{k_2}, H_2)$, we conclude by iii) that D_1 and D_2 are biholomorphically equivalent. This contradicts our assumption. Therefore, if each of D_1 and D_2 can be exhausted by biholomorphic images of the other, they must be biholomorphically equivalent.

Q.E.D.

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