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Osaka University
EQUIVARIANT ALGEBRAIC VECTOR BUNDLES
OVER ADJOINT REPRESENTATIONS

Dedicated to Professor Seiya Sasao on his 60th birthday

MIKIYA MASUDA and TERUKO NAGASE

(Received January 26, 1994)

0. Introduction

Let $G$ be a reductive complex algebraic group and let $B, F$ be $G$-modules over $C$. Let $\text{Vec}_G(B, F)$ denote the set of complex algebraic $G$-vector bundles over $B$ whose fiber at $0 \in B$ is $F$, and let $\text{VEC}_G(B, F)$ denote the set of the $G$-isomorphism classes in $\text{Vec}_G(B, F)$. The set $\text{VEC}_G(B, F)$ has the trivial class represented by the product bundle $B \times F \to B$.

The solution of the Serre conjecture by Quillen [9] and Suslin [11] says that $\text{VEC}_G(B, F)$ is trivial for any $B$ and $F$ when $G$ is trivial. In contrast to this Schwarz [10] discovered that $\text{VEC}_G(B, F)$ is nontrivial for some $B$ and $F$ when $G$ belongs to a class of noncommutative groups that includes all classical groups (see also [5]) ; this depends upon an analysis of $\text{VEC}_G(B, F)$ when the ring $\mathcal{O}(B)^G$ of invariants on $B$ is a polynomial ring in one variable. Subsequently Knop [6] used the result of Schwarz for $G =$ SL$_2$ to show that $\text{VEC}_G(\mathfrak{g}, F)$ is nontrivial for many irreducible $G$-modules $F$ if $G$ is connected and noncommutative, where $\mathfrak{g}$ denotes the adjoint representation of $G$. Note that $\mathcal{O}(\mathfrak{g})^G$ is a polynomial ring in $n$ variables where $n$ is the rank of $G$. We refer the reader to [7] and [8] for further results, where $\text{VEC}_G(B, F)$ is studied from a different point of view.

In this paper we closely look at the result of Schwarz on the SL$_2$ case together with the argument of Knop to prove

**Theorem A.** If $G$ is semisimple, then $\text{VEC}_G(\mathfrak{g}, F)$ is nontrivial for all but finitely many isomorphism classes of irreducible $G$-modules $F$.

**Remark.** If $G$ is commutative, then $\text{VEC}_G(\mathfrak{g}, F)$ is trivial for any $G$-module $F$ because the action of $G$ on $\mathfrak{g}$ is trivial ([3, §2]).

Theorem A is a corollary of Theorem B stated below. Let $R_n$ be the
SL₂-module of homogeneous polynomials of degree \( n \) in two variables. According to \([10]\) \( \text{VEC}_{\text{SL}_2}(R_2, R_m) \) forms an abelian group isomorphic to \( \mathbb{C}^p \) where \( p = \lceil \frac{(m - 1)^2}{4} \rceil \). Suppose \( G \) is connected and noncommutative. We fix a system \( \Sigma \) of simple roots of \( G \). Associated to a simple root \( \alpha \in \Sigma \), Knop defined a map

\[
\Phi^\alpha : \text{VEC}_{G}(\mathfrak{g}, F) \to \text{VEC}_{\text{SL}_2}(R_2, R_m)
\]

where \( m = \langle \chi, \alpha \rangle \) and \( \chi \) is the highest weight of the irreducible \( G \)-module \( F \). He proved that \( \Phi^\alpha \) is surjective if \( \chi \) is regular, i.e. unless \( \chi \) is contained in a reflecting hyperplane \( P_\beta \) for some \( \beta \in \Sigma \).

**Definition.** We call the \( \alpha \)-string \((\chi, \chi - \alpha, \cdots, \chi - m\alpha)\) of \( \chi \) singular if it is contained in some \( P_\beta \) and regular otherwise.

Clearly if \( \chi \) is regular, then the \( \alpha \)-string of \( \chi \) is regular for any \( \alpha \in \Sigma \). But an \( \alpha \)-string happens to be regular even if \( \chi \) is singular, e.g. if \( G \) is semisimple and of rank two, then any dominant weight has a regular \( \alpha \)-string. Hence the following theorem extends the result of Knop mentioned above.

**Theorem B.** Suppose \( G \) is connected and noncommutative. Then

1. \( \Phi^\alpha \) is surjective if the \( \alpha \)-string of \( \chi \) is regular,
2. the image of \( \Phi^\alpha \) contains a subspace of dimension \( \lceil \frac{m}{2} \rceil \lceil \frac{m}{2} - 1 \rceil / 2 \) if the \( \alpha \)-string of \( \chi \) is singular.

Theorem B implies that \( \text{VEC}_{G}(\mathfrak{g}, F) \) is nontrivial provided \( m \geq 4 \). If \( G \) is semisimple, then there are only finitely many irreducible \( G \)-modules \( F \) such that \( \langle \chi, \alpha \rangle \leq 3 \) for all \( \alpha \in \Sigma \). Therefore Theorem A follows from Theorem B.

1. **The \( \text{SL}_2 \) case**

In this section we translate the result of Schwarz on the \( \text{SL}_2 \) case into an explicit form. Let \( G = \text{SL}_2 \) and \( T \) be its maximal torus consisting of diagonal matrices. Remember that \( R_n \) is the \( G \)-module of homogeneous polynomials of degree \( n \) in two variables, say \( x \) and \( y \). Since the \( G \)-orbit of \( R_2^T = \{ bxy | b \in \mathbb{C} \} \) is dense in \( R_2 \), the inclusion map \( i : R_2^T \to R_2 \) induces an injective homomorphism

\[
i^\ast : \text{Mor}(R_2, \text{End}(R_m))^G \to \text{Mor}(R_2^T, (\text{End}(R_m)^T)^W)
\]

where \( W \) denotes the Weyl group \( N_G(T)/T \), which is of order two. Note that the one dimensional subspaces of \( R_m \) spanned by \( x^{m-n}y^n \) are mutually non-isomorphic \( T \)-modules.

**Lemma 1.1.** Any element \( \sigma \in \text{Mor}(R_2^T, (\text{End}(R_m)^T)^W) \) is of the form

\[
(\sigma(bxy))(x^{m-n}y^n) = f_n(b)x^{m-n}y^n
\]
with polynomials $f_n(b)$ such that $f_n(-b) = f_{m-n}(b)$ for $n = 0, 1, \cdots, m$.

Proof. It follows from Schur's lemma that $\sigma$ is of the form

$$(\sigma(bxy))(x^{m-n}y^n) = f_n(b)x^{m-n}y^n$$

with polynomials $f_n(b)$ for any $n$. The element of $G$ mapping $x$ to $y$ and $y$ to $-x$ is a representative of the nontrivial element of $W$. It acts on $\mathbb{R}_2^2$ as multiplication by $-1$ and on $(\text{End}(R_m))^G$ by conjugation of the element of $\text{End}(R_m)$ mapping $x^{m-n}y^n$ to $(-1)^n x^n y^{m-n}$. Hence it follows from the equivariance with respect to the action of $W$ that $f_n(-b) = f_{m-n}(b)$. This proves the lemma. \(\square\)

It is well-known (and easy to prove) that $\mathcal{O}(\mathbb{R}_2)^G$ is a polynomial ring $\mathbb{C}[\Delta]$ where $\Delta$ is the discriminant defined by $\Delta(ax^2 + bxy + cy^2) = b^2 - 4ac$. We note that $\text{Mor}(\mathbb{R}_2, \text{End}(R_m))^G$ is an algebra over $\mathcal{O}(\mathbb{R}_2)^G = \mathbb{C}[\Delta]$. The following lemma describes the algebra structure.

**Lemma 1.2.** $\text{Mor}(\mathbb{R}_2, \text{End}(R_m))^G = (\mathbb{C}[\Delta])[\gamma]/\prod_{n=0}^{m} (\gamma - (m-2n)\sqrt{\Delta})$ where $\gamma$ is homogeneous of degree one with respect to the coordinates of $\mathbb{R}_2$ and expressed on $\mathbb{R}_2$ as

$$(\gamma(bxy))(x^{m-n}y^n) = (m-2n)bx^{m-n}y^n.$$ (\*)

**Remark.** Since $(\gamma - (m-2k)\sqrt{\Delta})(\gamma - (m-2(m-k))\sqrt{\Delta}) = \gamma^2 - (m-2k)^2\Delta$, the product $\prod_{n=0}^{\infty} (\gamma - (m-2n)\sqrt{\Delta})$ is actually a polynomial of $\gamma$ and $\Delta$.

Proof. This may be known, but for the sake of completeness we shall give the proof.

First we claim that $\text{Mor}(\mathbb{R}_2, \text{End}(R_m))^G$ is free and of rank $m+1$ as a $\mathbb{C}[\Delta]$-module; more precisely, the degrees of the generators are 0, 1, 2, $\cdots$, $m$. This can be seen as follows. By the self-duality of $R_m$ and the Clebsch-Gordan formula ([4, p. 170]) we have

$$\text{End}(R_m) \cong R_m \otimes R_m \cong \bigoplus_{k=0}^{m} R_{2k}.$$ Hence $\text{Mor}(\mathbb{R}_2, \text{End}(R_m))^G \cong \bigoplus_{k=0}^{m} \text{Mor}(\mathbb{R}_2, R_{2k})^G$. Here it is easy to see that $\text{Mor}(\mathbb{R}_2, R_{2k})^G$ is free and of rank one as a $\mathbb{C}[\Delta]$-module, in fact, the generator is given by the $k$th power map. This implies the claim.

Suppose $\gamma \in \text{Mor}(\mathbb{R}_2, \text{End}(R_m))^G$ is homogeneous and of degree one. Then it follows from Lemma 1.1 that

$$(\gamma(bxy))(x^{m-n}y^n) = c_n bx^{m-n}y^n$$

with constants $c_n$ such that $c_n = -c_{m-n}$. Let $g \in G$ be the unipotent matrix with 1 in the upper right hand corner. Since $gx = x$ and $gy = x + y$ (hence $g(xy) = x^2$
+ xy), it follows from equivariance that
\[ \gamma(bx^2 + bxy) = g\gamma(bxy)g^{-1}. \]

We view elements in End(R^m) as matrices by taking a basis \( \{x^n, x^{n-1}y, \ldots, y^n\} \) of \( R_m \). Since \( \gamma \) is homogeneous and of degree one, the entries of the matrix \( \gamma(ax^2 + bxy + cy^2) \) are linear combinations of \( a, b, \) and \( c \). The equivariance of \( \gamma \) with respect to the action of \( T \) implies that the \((i, j)\) entries of \( \gamma(ax^2 + bxy + cy^2) \) vanish whenever \( |i-j| \geq 2 \). (In fact, the diagonal entries are scalar multiples of \( b \), the \((i, i+1)\) entries are those of \( a \), and the \((i+1, i)\) entries are those of \( c \).) In particular, the \((1, j)\) entries of \( \gamma(bx^2 + bxy) \) are zero for \( j \geq 3 \). The vanishing of the \((1, j)\) entries \( (3 \leq j \leq m+1) \) of the matrix at the right hand side of the identity above yields \( m-1 \) equations among the constants \( c_n \). An elementary computation shows that
\[ c_n = (1-n)c_0 + nc_1. \]

This together with the relation \( c_n = -c_{m-n} \) shows \( c_n = (m-2n)c_0/m \). The identities (*) are then obtained by setting \( c_0 = m \).

The identities (*) imply that \( \gamma'(0 \leq j \leq m) \) are linearly independent over \( C[\Delta] \) when restricted to \( R^2 \). Since the \( G \)-orbit of \( R^2 \) is dense in \( R_2 \), the \( \gamma'(0 \leq j \leq m) \) are linearly independent over \( C[\Delta] \) as elements of \( \text{Mor}(R_2, \text{End}(R_m))^G \). Moreover the identities (*) show that the element is \( \prod_{n=0}^{m}(\gamma-(m-2n)/\Delta) \) is zero when restricted to \( R^2 \), and hence zero actually as an element of \( \text{Mor}(R_2, \text{End}(R_m))^G \). As claimed above \( \text{Mor}(R_2, \text{End}(R_m))^G \) is free and of rank \( m+1 \) as a \( C[\Delta] \)-module. This shows that the identity \( \prod_{n=0}^{m}(\gamma-(m-2n)/\Delta) = 0 \) is the only relation in \( \text{Mor}(R_2, \text{End}(R_m))^G \). This completes the proof. \( \square \)

Denote by \( M_k^m \) (resp., \( N_k^m \)) the linear space consisting of homomogeneous elements of degree \( k \) in \( i^* \text{Mor}(R_2, \text{End}(R_m))^G \) (resp., \( \text{Mor}(R_2, \text{End}(R_m))^G \)) and set \( M^m = \prod_{k=1}^m M_k^m \), \( N^m = \prod_{k=1}^m N_k^m \). An elementary calculation together with Lemmas 1.1 and 1.2 shows that
\[ \dim N_k^m = \begin{cases} (m+1)/2, & \text{if } m \text{ is odd} \\ (m+1+(-1)^k)/2, & \text{if } m \text{ is even} \end{cases} \]
\[ \dim M_k^m = \begin{cases} \lceil k/2 \rceil + 1, & \text{if } k \leq m-2 \\ \dim N_k^m, & \text{if } k \geq m-1 \end{cases} \]
and
\[ \dim N^m/M^m = [(m-1)^2/4]. \]

Remember that \( \Delta : R_2 \to C \) is an invariant polynomial. It is known that any element of \( \text{Vec}_{\text{SL}_2}(R_2, R_m) \) is trivial over \( \Delta^{-1}(C-0) \) ([5, VII.2.6]). Moreover, given \( E \in \text{Vec}_{\text{SL}_2}(R_2, R_m) \), there is a finite subset \( S \) of \( C-0 \) such that \( E \) is trivial.
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over $\Delta^{-1}(C-S)$ ([3, 6.2]). Hence one can find a transition function $\psi_E$ of $E$ in $\text{Mor}(\Delta^{-1}(C-(SU(0))), \text{End}(R_m))^G$. The choice of $\psi_E$ is not unique and one can always arrange $\psi_E$ such that the restriction $\psi_E|R_m^T$ is defined at 0 with value the identity. By the $T$-equivariance $\psi_E|R_m^T$ is a diagonal matrix with rational functions as entries with respect to the basis $\{x^m, x^{m-1}y, \cdots, y^m\}$ of $R_m$. We expand those rational functions into formal power series of the coordinate $b$ of $R_m$. This gives a correspondence

$$\phi : \text{Vec}_{L^2}(R_2, R_m) \rightarrow 1+N^m$$

defined by $\phi(E) = \phi_E|R_m^T$. A general result of Schwarz [10] or Kraft-Schwarz [5, VII.3.4] applied to the $SL_2$ case implies

**Theorem 1.3** ([5], [10]). The map $\phi$ induces a bijection

$$\Psi : \text{VEC}_{SL_2}(R_2, R_m) \cong 1+N^m/M^m.$$

2. The map $\Phi^a$

In this section $G$ is connected and noncommutative. We recall the definition of the map $\Phi^a : \text{VEC}_G(g, F) \rightarrow \text{VEC}_{SL_2}(R_2, R_m)$ mentioned in the introduction. Let $T$ be a maximal torus of $G$. Denote the Lie algebra of $T$ by $t$. Let $L$ be the subgroup of $G$ generated by $T$ and the root groups $U_a$ and $U_{-a}$ (see [2, 26.3]). Let $L'$ be the commutator subgroup of $L$ and $Z$ be the identity component of the center of $L$. Then $L'$ is isomorphic to $SL_2$ or $SO_3$, the subgroup $Z$ is a codimension one torus in $T$ and $L=ZL'$. We choose and fix an element $\xi_0 \in t$ whose centralizer is exactly $L$. This is equivalent to saying that $\xi_0 \in P_a$ but $\xi_0 \notin P_{\beta}$ for any $\beta \neq \alpha \in \Sigma$. Denote by $a$ the affine space $\xi_0 + \text{Lie } L' \subset g$, which is $L$ invariant. The action of $Z$ on $a$ is trivial and $a$ is isomorphic to $R_2$ as $L'$-varieties because $\text{Lie } L'$ and $R_2$ are isomorphic representations.

Given $E \in \text{Vec}_G(g, F)$, we restrict it to $a$. Since $a$ is fixed under the action of $Z$, the restricted bundle $E|a$ decomposes into eigenbundles according to the weights of $F$ viewed as a $Z$-module. Let $(E|a)^{\chi}$ denote the eigenbundle of $E|a$ corresponding to the highest weight $\chi$ restricted to $Z$. Since $Z$ commutes with $L'$, $(E|a)^{\chi}$ is an $L'$-vector bundle. The correspondence $E \rightarrow (E|a)^{\chi}$ induces the desired map $\Phi^a$.

3. Proof of Theorem B

Let $\Delta \subset C[g]^G$ be the discriminant and put $g_0=\Delta^{-1}(C-(0))$. For a finite subset $S \subset C-(0)$ we set $g_S=\Delta^{-1}(C-S)$. Similarly we set $t_0=t \cap g_0, t_S=t \cap g_S$. Since $g_0$ is the set of regular semisimple elements, we have

$$g_0=G \times N_G(T)t_0.$$

We construct a $G$-vector bundle over $g$ by glueing the product $G$-vector
bundles $g_0 \times F \to g_0$ and $g_s \times F \to g_s$ over $g_{s_0} = g_0 \cap g_s$ using a transition function, where $S_0 = S \cup \{0\}$. The transition function is a $G$-equivariant morphism

$$\varphi : g_{s_0} \to \text{GL}(F)$$

where $G$ acts on $\text{GL}(F)$ by conjugation. It follows from (3.1) that the restriction map

$$\text{Mor}(g_{s_0}, \text{GL}(F))^G \to \text{Mor}(t_{s_0}, \text{GL}(F)^W)$$

is bijective, where $W$ is the Weyl group $N_G(T)/T$. Thus we are led to study $W$-equivariant morphisms from $t_{s_0}$ to $\text{GL}(F)^T$.

Decompose

$$F = \bigoplus_{\eta \in \chi(T)} M(\eta)$$

as $T$-modules

where $\chi(T)$ denotes the set of characters of $T$ and $M(\eta)$ is a (not necessarily one dimensional) $T$-module with character $\eta$. It follows from Schur’s lemma that

$$\text{GL}(F)^T = \prod_{\eta \in \chi(T)} \text{GL}(M(\eta)).$$

Hence an element of $\text{Mor}(t_{s_0}, \text{GL}(F)^T)^W$ is given by a family of morphisms

$$\varphi_\eta : t_{s_0} \to \text{GL}(M(\eta))$$

satisfying

$$(3.2) \quad \varphi_{w\eta}(\xi) = \bar{w} \circ \varphi_\eta(w^{-1}\xi) \circ \bar{w}^{-1} \quad \text{for all } w \in W \text{ and } \xi \in t_{s_0}$$

where $\bar{w} \in N_G(T)$ is a representative of $w$. The action of $\bar{w}$ induces an isomorphism from $M(\eta)$ to $M(w\eta)$ as $T$-modules.

We define $\varphi_\eta \equiv 1$ unless $\eta$ is in the $W$-orbit of the $\alpha$-string of $\chi$, i.e. unless $\eta = w(\chi - n\alpha)$ for some $w \in W$ and $0 \leq n \leq m$. If $\eta$ is in the $W$-orbit of the $\alpha$-string of $\chi$, then $\dim M(\eta) = 1$ ([1, p. 125, Exercise 1]); so $\text{GL}(M(\eta)) = \mathbb{C}^*$. Hence $\varphi_\eta$ is a rational function on $t$ which has neither zero nor a pole on $t_{s_0}$. Moreover in this case (3.2) reduces to

$$\varphi_{w\eta}(\xi) = \varphi_\eta(w^{-1}\xi) \quad \text{for all } w \in W \text{ and } \xi \in t_{s_0}.$$  

In order to choose a family $\{\varphi_\eta\}$ which satisfies (3.3), It suffices to choose a subfamily $\{\varphi_\eta|\eta\}$ in the $\alpha$-string of $\chi$ which satisfies (3.3) whenever $\eta$ and $w\eta$ are in the $\alpha$-string of $\chi$. We note that the reflection $s_\alpha$ relative to the reflecting hyperplane $P_\alpha$ reflects the $\alpha$-string of $\chi$, i.e. $s_\alpha(\chi - n\alpha) = \chi - (m - n)\alpha$ for any $n$

Lemma 3.4. (1) If $w(\chi - k\alpha) = \chi - l\alpha$ for some $0 \leq k, l \leq m$, then $k = l$ or $k = m - l$.

(2) If $w(\chi - k\alpha) = \chi - ka$ and $\chi - ka$ is regular, then $w$ is the identity.
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Proof. (1) First we recall the following general fact ([1, 10.3]). Let \( \lambda, \mu \) be elements in the closure \( \overline{C} \) of the Weyl chamber relative to the simple root system \( \Sigma \). If \( \lambda \) and \( \lambda \) are both in \( \overline{C} \). Then it follows from the above fact that \( \lambda \), \( \mu \) are in \( \overline{C} \). Since \( s_\alpha w(\chi - ka) = \chi - (m - l)\alpha \), we are in the same situation as above, hence \( k = m - l \). The remaining two cases can be treated in the same way.

(2) The isotropy subgroup of \( W \) at a regular element in \( t \) is trivial ([1, 10.3]). This implies (2).

We denote \( \varphi_{x^{-n}} \) by \( \varphi_n \). We shall find a family \( \{\varphi_n(0 \leq n \leq m)\} \) satisfying (3.3). Let \( \delta \) be the product of positive roots. It is well known that

\[
\delta(s_\beta \xi) = -\delta(\xi) \quad \text{for any } \beta \in \Sigma
\]

([1, 10.2]). We take a family of polynomials \( \{p_n(0 \leq n \leq m)\} \) in one variable such that

\[
p_0 = p_m = 1 \quad \text{and} \quad p_n(-\delta) = p_{m-n}(\delta)
\]

for any \( n \) and define

\[
\varphi_n(\xi) = p_n(\delta(\xi)).
\]

Suppose the \( \alpha \)-string of \( \chi \) is regular. Since \( \varphi_0 = \varphi_m = 1 \), it follows from (3.3) and Lemma 3.4 that the identity \( \varphi_{m-n}(\xi) = \varphi_n(s_\alpha \xi) \) for each \( n \) is the only condition which the family \( \{\varphi_n\} \) must satisfy. But it is satisfied by (3.5), (3.6) and (3.7).

Suppose the \( \alpha \)-string of \( \chi \) is singular. Then we require one more condition on the family \( \{p_n\} \) that they be all even functions. Since \( \delta(w\xi)^2 = \delta(\xi)^2 \) for any \( w \in W \) by (3.5), it follows from Lemma 3.4, (3.6) and (3.7) that (3.3) is satisfied.

Let \([E_p]\) denote the isomorphism class of the \( G \)-vector bundle \( E_p \in \text{Vec}_G(\theta, F) \) defined by a family of polynomials \( \{p_n\} \) satisfying the conditions mentioned above. We shall observe \( \Phi^a([E_p]) \). As discussed in §1 elements in \( \text{VEC}_{\text{St}}(R^2, \mathbb{R}_m) \) are detected by their transition functions restricted to \( \mathbb{R}_m^3 \). By definition \( \Phi^a([E_p]) \) is \( [(E_p)[\alpha]] \) and \( \alpha \) is the affine space \( \xi_0 + \text{Lie } L' \) which is isomorphic to \( \mathbb{R}_2 \) as \( L' \)-varieties. Then \( \mathbb{R}_m^3 \) corresponds to \( t \cap \alpha = \{\xi_0 + bh \mid b \in C\} \) where \( h \in t \cap \text{Lie } L' \) with \( \alpha(h) = 1 \). Thus \( \Phi^a([E_p]) \) corresponds to the family \( \{p_n(\delta(\xi_0 + bh_\alpha))\} \) through the map \( \Psi \) in Theorem 1.3. Remember that \( \xi_0 \) is chosen in such a way that \( \xi_0 \in P_\alpha \) but \( \xi_0 \notin P_\beta \) for any \( \beta \neq \alpha \in \Sigma \). Since \( \delta \) is the product of positive roots, \( \delta(\xi_0 + bh_\alpha) \) is a polynomial of \( b \) with zero constant term and nonzero degree one term.

In case the \( \alpha \)-string of \( \chi \) is regular, the condition we imposed on \( \{p_n\} \) is only (3.6). Then it is not difficult to see that the composition \( \Psi \circ \Phi^a \) is surjective, hence \( \Phi^a \) is surjective as \( \Psi \) is bijective.

In case the \( \alpha \)-string of \( \chi \) is singular, the conditions we imposed on \( \{p_n\} \) are (3.6) and that \( p_n \) are even functions. Then it is also not difficult to see that \( \Psi \circ \Phi^a \) contains the image of even degree elements of \( 1 + N^m \) in \( 1 + N^m/M^m \). An elemen-
tary calculation shows that the image is a subspace of dimension $\lfloor m/2 \rfloor (\lfloor m/2 \rfloor - 1)/2$. This completes the proof of Theorem B.

References


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