



Title	Stable homotopy types of stunted lens spaces mod 4
Author(s)	Kôno, Susumu
Citation	Osaka Journal of Mathematics. 1992, 29(4), p. 697-717
Version Type	VoR
URL	https://doi.org/10.18910/9267
rights	
Note	

The University of Osaka Institutional Knowledge Archive : OUKA

<https://ir.library.osaka-u.ac.jp/>

The University of Osaka

STABLE HOMOTOPY TYPES OF STUNTED LENS SPACES MOD 4

Dedicated to Professor Hideki Ozeki on his 60th birthday

SUSUMU KÔNO

(Received December 26, 1991)

1. Introduction

Let $L^n(q) = S^{2n+1}/(\mathbb{Z}/q)$ be the $(2n+1)$ -dimensional standard lens space mod q . As defined in [8], we set

$$(1.1) \quad \begin{aligned} L_q^{2n+1} &= L^n(q), \\ L_q^{2n} &= \{[z_0, \dots, z_n] \in L^n(q) \mid z_n \text{ is real} \geq 0\}. \end{aligned}$$

The stable homotopy types (S -types) of stunted lens spaces L_q^m/L_q^n have been studied by several authors (e.g. [7], [8], [9], [10], [11] and [12]). For the case $q=2$, D.M. Davis and M. Mahowald have completed the classification of the stable homotopy types of stunted real projective spaces in [7]. Their result shows that we can use structures of J -groups of suspensions of stunted real projective spaces to obtain the necessary conditions for stunted real projective spaces $RP(m)/RP(n)$ and $RP(m+t)/RP(n+t)$ to have the same stable homotopy type as follows: if $RP(m)/RP(n)$ and $RP(m+t)/RP(n+t)$ have the same stable homotopy type, then there exists a non-negative integer N such that

$$\tilde{J}(S^j(RP(m)/RP(n))) \cong \tilde{J}(S^{j-t}(RP(m+t)/RP(n+t)))$$

for each integer j with $j \geq N$ (see [13]). For the case where q is an odd prime, T. Kobayashi has obtained some necessary conditions for stunted lens spaces L_q^m/L_q^n and L_q^{m+t}/L_q^{n+t} to have the same stable homotopy type (cf. [10]). The conditions are also sufficient if $k = [m/2] - [(n+1)/2] \not\equiv 0 \pmod{(q-1)}$ or $n+1 \equiv 0 \pmod{2q^{[k/(q-1)]}}$. We can use structures of J -groups of suspensions of stunted lens spaces mod q to obtain the conditions (see [14]). The object of this paper is to study the stable homotopy types of stunted lens spaces L_q^m/L_q^n for $q=4$ or 8 .

In order to state our results, we prepare functions $h_1, h_2, \alpha, \beta_1, \beta_2$ and γ_1 defined by

$$(1.2) \quad h_1(k) = \begin{cases} [k/4] + [(k+7)/8] + [(k+4)/8] & (k \geq 2) \\ 0 & (1 \geq k \geq 0). \end{cases}$$

$$(1.3) \quad h_2(k) = \begin{cases} [k/4] + [(k+7)/8] + [k/8] + 1 & (k \geq 4) \\ h_1(k) & (3 \geq k \geq 0) . \end{cases}$$

$$(1.4) \quad \alpha(k, n) = \begin{cases} 1 & (n \equiv 0 \pmod{2} \text{ and } k \equiv 1 \pmod{8}, \\ & \text{or } k = 2([n/2] - [(n-1)/2])) \\ 0 & (\text{otherwise}) . \end{cases}$$

(1.5) $\beta_1(k, n)$ is equal to the corresponding integer in the following table:

$k \pmod{8} \backslash n \pmod{4}$	0	1	2	3	4	5	6	7
0	0	0	0	0	0	0	0	0
1	0	0	1	1	0	0	1	1
2	0	0	0	1	0	0	1	1
3	0	0	0	0	0	0	0	0

(1.6) $\beta_2(k, n)$ is equal to the corresponding integer in the following table:

$k \pmod{8} \backslash n \pmod{8}$	0	1	2	3	4	5	6	7
0	0	0	0	0	0	0	0	0
1	0	0	1	1	0	0	1	1
2	0	0	1	1	0	0	1	1
3	0	0	0	0	1	1	0	0
4	0	0	0	0	1	1	0	0
5	0	0	2	2	1	1	1	1
6	0	0	1	2	1	1	1	1
7	0	0	0	0	0	0	0	0

(1.7) $\gamma_1(m, n)$ is equal to the corresponding integer in the following table:

$m-n$ $n \pmod{8}$	1	2	3	4	5	6
0	0	0	1	0	1	0
1	0	1	1	1	1	2
2	0	0	1	0	2	1
3	0	1	1	1	1	1
4	0	0	1	0	1	0
5	0	1	1	1	1	1
6	0	0	1	0	1	1
7	0	1	1	1	2	2

Let $\nu_p(s)$ denote the exponent of the prime p in the prime power decomposition of s .

Theorem 1. If $\nu_2(t) \geq h_1(m-2[(n+1)/2]) + 1 - \alpha(m-2[(n+1)/2], n)$, then L_4^m/L_4^n and L_4^{m+t}/L_4^{n+t} have the same stable homotopy type.

Theorem 2. (1) If L_4^m/L_4^n and L_4^{m+t}/L_4^{n+t} have the same stable homotopy type, then

$$\nu_2(t) \geq [m/2] - [(n+1)/2] + \beta_1(m-2[(n+1)/2], n).$$

(2) Suppose $h_1(m-2[(n+1)/2]) - \alpha(m-2[(n+1)/2], n) \geq i \geq 3$ and $\max\{\nu_2(n+1), \nu_2(m+1)\} \geq i$. If L_4^m/L_4^n and L_4^{m+t}/L_4^{n+t} have the same stable homotopy type, then $\nu_2(t) \geq i+1$.

(3) Suppose $n < m \leq n+6$. If L_4^m/L_4^n and L_4^{m+t}/L_4^{n+t} have the same stable homotopy type, then

$$\nu_2(t) \geq [m/2] - [(n+1)/2] + \gamma_1(m, n).$$

REMARK. It follows from Theorems 1 and 2, that we have obtained necessary and sufficient conditions for spaces L_4^m/L_4^n and L_4^{m+t}/L_4^{n+t} to have the same stable homotopy type if one of the following conditions is satisfied:

- (1) $n < m \leq 2[(n+1)/2] + 3$,
- (2) $\beta_1(m-2[(n+1)/2], n) = 1$,
- (3) $\max\{\nu_2(n+1), \nu_2(m+1)\} \geq h_1(m-2[(n+1)/2]) - \alpha(m-2[(n+1)/2], n)$ and $m \geq n+5$.

Theorem 3. If $\nu_2(t) \geq h_2(m-2[(n+1)/2]) + 1 - \alpha(m-2[(n+1)/2], n)$, then

L_8^m/L_8^n and L_8^{m+t}/L_8^{n+t} have the same stable homotopy type.

Theorem 4. (1) Suppose $m \geq n+5$. If L_8^m/L_8^n and L_8^{m+t}/L_8^{n+t} have the same stable homotopy type, then

$$\nu_2(t) \geq [m/2] - [(n+1)/2] + \beta_2(m-2[(n+1)/2], n).$$

(2) Suppose $h_2(m-2[(n+1)/2]) - \alpha(m-2[(n+1)/2], n) \geq i \geq 3$ and $\max\{\nu_2(n+1), \nu_2(m+1)\} \geq i$. If L_8^m/L_8^n and L_8^{m+t}/L_8^{n+t} have the same stable homotopy type, then $\nu_2(t) \geq i+1$.

(3) Suppose $n < m \leq n+6$. If L_8^m/L_8^n and L_8^{m+t}/L_8^{n+t} have the same stable homotopy type, then

$$\nu_2(t) \geq [m/2] - [(n+1)/2] + \gamma_2(m, n),$$

where $\gamma_2(m, n)$ is the integer defined by

$$\gamma_1(m, n) = \begin{cases} 1 & (n \equiv 0 \pmod{8} \text{ and } m = n+6) \\ \gamma_1(m, n) & (\text{otherwise}). \end{cases}$$

REMARK. It follows from Theorems 3 and 4, that we have obtained necessary and sufficient conditions for spaces L_8^m/L_8^n and L_8^{m+t}/L_8^{n+t} to have the same stable homotopy type if one of the following conditions is satisfied:

- (1) $n < m \leq 2[(n+1)/2] + 3$,
- (2) $n \equiv 0 \pmod{8}$ and $m = n+6$,
- (3) $\beta_2(m-2[(n+1)/2], n) = 2$ or $2[(n+1)/2] \equiv [m/2] - [(n+3)/2] \equiv 2 \pmod{4}$,
- (4) $\max\{\nu_2(n+1), \nu_2(m+1)\} \geq h_2(m-2[(n+1)/2]) - \alpha(m-2[(n+1)/2], n)$ and $m \geq n+5$.

This paper is organized as follows. In section 2 we prepare some lemmas and recall known results. We prove Theorems 1, 2, 3 and 4 in the final section.

The author would like to express his gratitude to Professor Akie Tamamura and Professor Kensô Fujii for helpful suggestions.

2. Preliminaries

In this section we prepare some lemmas and recall known results which are needed to prove Theorems 1, 2, 3 and 4.

Lemma 2.1. Let m, n, k and s be non-negative integers with $2^s \leq k < 2^{s+1}$. Assume that $\binom{m}{i} \equiv \binom{n}{i} \pmod{2}$ for $1 \leq i \leq k$. Then $m \equiv n \pmod{2^{s+1}}$.

Proof. Suppose that $m = \sum_{j=0}^N a_j 2^j$, $n = \sum_{j=0}^N b_j 2^j$ and $i = \sum_{j=0}^N c_{i,j} 2^j$ ($1 \leq i \leq k$), where a_j, b_j and $c_{i,j}$ are non-negative integers with $a_j \leq 1, b_j \leq 1$ and $c_{i,j} \leq 1$ ($1 \leq i \leq k, 1 \leq j \leq N$). Then we have $\binom{m}{i} \equiv \prod_{j=0}^N \binom{a_j}{c_{i,j}} \pmod{2}$ and $\binom{n}{i} \equiv \prod_{j=0}^N$

$\binom{b_j}{c_{i,j}} \pmod{2}$ for $1 \leq i \leq k$. It follows from the hypothesis that we have $a_j = b_j$ for $0 \leq j \leq s$; that is, $m \equiv n \pmod{2^{s+1}}$. q.e.d.

Let $q \geq 2$ be an integer and q_0, q_1, \dots, q_n be integers relatively prime to q . Consider the (\mathbf{Z}/q) -action on the unit sphere $S^{2n+1} \subset \mathbf{C}^{n+1}$ given by

$$\exp(2\pi\sqrt{-1}/q)(z_0, \dots, z_n) = (z_0 \cdot \exp(2q_0\pi\sqrt{-1}/q), \dots, z_n \cdot \exp(2q_n\pi\sqrt{-1}/q)).$$

Then $S^{2n+1}(q; q_0, \dots, q_n)$ denotes the space S^{2n+1} with this action,

$$L_q^{2n+1}(q_0, \dots, q_n) = S^{2n+1}(q; q_0, \dots, q_n)/(\mathbf{Z}/q)$$

and $L_q^{2n}(q_0, \dots, q_n)$ is the subspace of $L_q^{2n+1}(q_0, \dots, q_n)$ defined by

$$L_q^{2n}(q_0, \dots, q_n) = \{[z_0, \dots, z_n] \in L_q^{2n+1}(q_0, \dots, q_n) \mid z_n \text{ is real} \geq 0\}.$$

For $0 \leq n < m \leq 2l+1$, we set

$$L_q^m/L_q^n(q_0, \dots, q_l) = L_q^m(q_0, \dots, q_l)/L_q^n(q_0, \dots, q_l),$$

which is called a stunted lens space mod q . Then we have

$$(2.2) \quad L_q^m/L_q^n(1, \dots, 1) = L_q^m/L_q^n.$$

Considering the (\mathbf{Z}/q) -action on $S^{2l+1}(q; q_0, \dots, q_l) \times \mathbf{C}^k$ given by

$$\begin{aligned} & \exp(2\pi\sqrt{-1}/q)(z, w_1, \dots, w_k) \\ &= (\exp(2\pi\sqrt{-1}/q) \cdot z, w_1 \cdot \exp(2a_1\pi\sqrt{-1}/q), \dots, w_k \cdot \exp(2a_k\pi\sqrt{-1}/q)) \end{aligned}$$

for $(z, w_1, \dots, w_k) \in S^{2l+1}(q; q_0, \dots, q_l) \times \mathbf{C}^k$, we have a complex k -dimensional vector bundle

$$\eta(a_1, \dots, a_k): (S^{2l+1}(q; q_0, \dots, q_l) \times \mathbf{C}^k)/(\mathbf{Z}/q) \rightarrow L_q^{2l+1}(q_0, \dots, q_l).$$

We use same symbol for the restriction of $\eta(a_1, \dots, a_k)$ to $L_q^n(q_0, \dots, q_l)$ ($n \leq 2l+1$) and denote the complex line bundle $\eta(1)$ by η . Then we have

$$(2.3) \quad \eta(a_1, \dots, a_k) \cong \eta^{a_1} \oplus \dots \oplus \eta^{a_k}.$$

Let $\xi: E(\xi) \rightarrow X$ be a real vector bundle over a finite CW -complex X with disk bundle $B(\xi)$ and sphere bundle $S(\xi)$. Then the Thom complex X^ξ of ξ is defined as the quotient space $B(\xi)/S(\xi)$. Define $f: S^{2n+1} \times D^{2k} \rightarrow S^{2k+2n+1}$ by

$$f(z_0, \dots, z_n, w) = (w, (1 - \|w\|^2)^{1/2} z_0, \dots, (1 - \|w\|^2)^{1/2} z_n)$$

for $(z_0, \dots, z_n, w) \in S^{2n+1} \times D^{2k}$, where $D^{2k} \subset \mathbf{C}^k$ is the unit disk. Then we have

(2.4) *Let a_1, \dots, a_k be integers relatively prime to q . Then f induces the following homeomorphisms.*

- (1) $(L_q^{2n+1}(q_0, \dots, q_n))^{r(\eta(a_1, \dots, a_k))} \approx L_q^{2k+2n+1}/L_q^{2k-1}(a_1, \dots, a_k, q_0, \dots, q_n).$
- (2) $(L_q^{2n}(q_0, \dots, q_n))^{r(\eta(a_1, \dots, a_k))} \approx L_q^{2k+2n}/L_q^{2k-1}(a_1, \dots, a_k, q_0, \dots, q_n).$
- (3) $(L_q^{2n+1}(q_0, \dots, q_n))^{r(\eta(a_1, \dots, a_k))}/S^{2k} \approx L_q^{2k+2n+1}/L_q^{2k}(a_1, \dots, a_k, q_0, \dots, q_n).$
- (4) $(L_q^{2n}(q_0, \dots, q_n))^{r(\eta(a_1, \dots, a_k))}/S^{2k} \approx L_q^{2k+2n}/L_q^{2k}(a_1, \dots, a_k, q_0, \dots, q_n).$

We define the function $h(q, k)$ by setting

$$(2.5) \quad h(q, k) = \text{ord} \langle J(r(\eta) - 2) \rangle,$$

where $J(r(\eta) - 2)$ is the image of $r(\eta) - 2 \in \widetilde{KO}(L_q^k)$ by the J -homomorphism $J: \widetilde{KO}(L_q^k) \rightarrow \widetilde{J}(L_q^k)$.

REMARK. The function $h(q, k)$ have been determined completely (cf. [8]).

Spaces X and Y are said to have the same stable homotopy type ($X \underset{s}{\approx} Y$) if there exist non-negative integers a and b such that $S^a X$ and $S^b Y$ have the same homotopy type. For stunted generalized lens spaces, we have

(2.6) (1) If $mn \equiv 0 \pmod{2}$, then $L_q^m/L_q^n(q_0, \dots, q_{[m/2]})$ and L_q^m/L_q^n have the same stable homotopy type. In particular, $L_q^m(q_0, \dots, q_{[m/2]})$ and L_q^m have the same stable homotopy type.

(2) Let $a_1, \dots, a_k, b_1, \dots, b_k, q_0, \dots, q_n, r_0, \dots, r_n$ and a be integers relatively prime to q .

i) Assume $q_0 \cdots q_n \equiv \pm a^{n+1} r_0 \cdots r_n \pmod{q}$. Then

$$L_q^{2n+2k+1}/L_q^{2k-1}(a_1, \dots, a_k, q_0, \dots, q_n)$$

and $L_q^{2n+2k+1}/L_q^{2k-1}(b_1, \dots, b_k, r_0, \dots, r_n)$ have the same stable homotopy type.

ii) If $k \equiv 0 \pmod{h(q, 2n+1)}$, then the spaces

$$L_q^{2n+2k+1}/L_q^{2k-1}(a_1, \dots, a_k, q_0, \dots, q_n)$$

and $L_q^{2n+2k+1}/L_q^{2k-1}$ have the same stable homotopy type.

Proof. Set $X = L_q^{2n+1}(q_0, \dots, q_n)$. According to [17], we have

$$K(X) \cong \mathbf{Z}[\eta]/(\eta^q - 1, (\eta - 1)^{n+1})$$

and $\text{ord } \tilde{K}(X) = q^n$. If a is an integer relatively prime to q , then

$$\begin{aligned} \bigcap_j (\sum_j j^{f(j)} (\psi^j - 1) \tilde{K}(X)) &= \sum_j (\bigcap_e j^e (\psi^j - 1) \tilde{K}(X)) \\ &\supset \bigcap_e a^e (\psi^a - 1) \tilde{K}(X) = (\psi^a - 1) \tilde{K}(X) \\ &\ni (\psi^a - 1)(\eta - 1) = \eta^a - \eta. \end{aligned}$$

Since the Adams operations are compatible with the real restriction $r: K \rightarrow KO$ [4],

$$r(\eta^a - \eta) = r(\eta^a) - r(\eta) \in \bigcap_f (\sum_j j^{f(j)} (\psi^j - 1) \widetilde{KO}(X)).$$

By [2], [3] and [19], this implies that

$$J(r(\eta^a) - 2) = J(r(\eta) - 2)$$

in $\mathcal{J}(X)$. If a_1, \dots, a_k are integers relatively prime to q ,

$$J(r(\eta(a_1, \dots, a_k)) - 2k) = J(k(r(\eta)) - 2k)$$

in $\mathcal{J}(X)$. Suppose that $a_1, \dots, a_k, b_1, \dots, b_k, q_0, \dots, q_n, r_0, \dots, r_n$ and a be integers relatively prime to q . According to [5, Proposition (2.6)], we have

$$L_q^{2n+2k+1}/L_q^{2k-1}(a_1, \dots, a_k, q_0, \dots, q_n) \cong_s L_q^{2n+2k+1}/L_q^{2k-1}(a, \dots, a, q_0, \dots, q_n)$$

and

$$L_q^{2n+2k+1}/L_q^{2k-1}(b_1, \dots, b_k, r_0, \dots, r_n) \cong_s L_q^{2n+2k+1}/L_q^{2k-1}(1, \dots, 1, r_0, \dots, r_n).$$

Suppose that $q_0 \cdots q_n \equiv \pm a^{n+1} r_0 \cdots r_n \pmod{q}$. Identify $S^{2n+2k+1}$ with the iterated join

$$S^1 * \cdots * S^1 = \{\lambda_0 z_0 + \cdots + \lambda_{n+k} z_{n+k} \mid \sum_j \lambda_j = 1, \lambda_j \geq 0, z_j \in S^1\}.$$

Choose integers \bar{q}_i ($0 \leq i \leq n$) with $q_i \bar{q}_i \equiv 1 \pmod{q}$. Denote the generator $\exp(2\pi\sqrt{-1}/q)$ of \mathbb{Z}/q by g . Then, the map

$$f: S^{2n+2k+1}(q; a, \dots, a, q_0, \dots, q_n) \rightarrow S^{2n+2k+1}(q; 1, \dots, 1, r_0, \dots, r_n)$$

defined by

$$f = 1 * \cdots * 1 * (ar_0 \bar{q}_0) * \cdots * (ar_n \bar{q}_n),$$

is a map of degree $a^{n+1} r_0 \cdots r_n \bar{q}_0 \cdots \bar{q}_n \equiv \pm 1 \pmod{q}$ with $f \circ g = g^a \circ f$. Modify f to get a map h of degree ± 1 with $h \circ g = g^a \circ h$ and

$$h|S^{2k-1}(q; a, \dots, a) = f|S^{2k-1}(q; a, \dots, a)$$

(see the proof of [6, (29.4)] for the detail). Then h induces a homotopy equivalence

$$\bar{h}: L_q^{2n+2k+1}/L_q^{2k-1}(a, \dots, a, q_0, \dots, q_n) \rightarrow L_q^{2n+2k+1}/L_q^{2k-1}(1, \dots, 1, r_0, \dots, r_n).$$

This completes the proof of (2) i).

Now we turn to the proof of (1). Since $\mathcal{J}(X)$ has a finite order, we may assume that $k > 1$. Let a be an integer with $aq_0 \cdots q_n \equiv 1 \pmod{q}$. Then

$$L_q^{2n+2k+1}/L_q^{2k}(a_1, \dots, a_k, q_0, \dots, q_n) \underset{\cong}{\simeq} L_q^{2n+2k+1}/L_q^{2k}(1, \dots, 1, a, q_0, \dots, q_n).$$

By the proof above, there exists an equivariant map

$$h: S^{2n+2k+1}(q; 1, \dots, 1) \rightarrow S^{2n+2k+1}(q; 1, \dots, 1, a, q_0, \dots, q_n)$$

of degree 1, which induces homotopy equivalences

$$\bar{h}: L_q^{2n+2k+1}/L_q^{2k} \rightarrow L_q^{2n+2k+1}/L_q^{2k}(1, \dots, 1, a, q_0, \dots, q_n)$$

and $\bar{h}|L_q^{2n+2k}/L_q^{2k}: L_q^{2n+2k}/L_q^{2k} \rightarrow L_q^{2n+2k}/L_q^{2k}(1, \dots, 1, a, q_0, \dots, q_n)$. Thus $L_q^{2n+2k+1}/L_q^{2k}(a_1, \dots, a_k, q_0, \dots, q_n)$ is stably homotopically equivalent to $L_q^{2n+2k+1}/L_q^{2k}$ and $L_q^{2n+2k}/L_q^{2k}(a_1, \dots, a_k, q_0, \dots, q_n)$ is stably homotopically equivalent to L_q^{2n+2k}/L_q^{2k} . The equivariant map

$$\begin{aligned} f &= 1 * \dots * 1 * q_0 * \dots * q_n: S^{2n+2k+1}(q; 1, \dots, 1) \\ &\rightarrow S^{2n+2k+1}(q; 1, \dots, 1, q_0, \dots, q_n) \end{aligned}$$

induces a homotopy equivalence

$$f|L_q^{2n+2k}/L_q^{2k-1}: L_q^{2n+2k}/L_q^{2k-1} \rightarrow L_q^{2n+2k}/L_q^{2k-1}(1, \dots, 1, q_0, \dots, q_n).$$

Thus $L_q^{2n+2k}/L_q^{2k-1}(a_1, \dots, a_k, q_0, \dots, q_n)$ is stably homotopically equivalent to L_q^{2n+2k}/L_q^{2k-1} . This completes the proof of (1).

Finally, we prove the part ii) of (2). Suppose that $k \equiv 0 \pmod{h(q, 2n+1)}$. Since the order of $J(r(\eta)-2) \in \tilde{J}(X)$ coincides with $h(q, 2n+1)$, we have

$$\begin{aligned} &L_q^{2n+2k+1}/L_q^{2k-1}(a_1, \dots, a_k, q_0, \dots, q_n) \\ &\underset{\cong}{\simeq} L_q^{2n+2k+1}/L_q^{2k-1}(1, \dots, 1, q_0, \dots, q_n) \\ &\approx (L_q^{2n+1}(q_0, \dots, q_n))^{r(k\eta)} \\ &\underset{\cong}{\simeq} (L_q^{2n+1}(q_0, \dots, q_n))^{2k} \\ &\simeq S^{2k} L_q^{2n+1}(q_0, \dots, q_n) \vee S^{2k} \\ &\underset{\cong}{\simeq} S^{2k} L_q^{2n+1} \vee S^{2k} \\ &\simeq (L_q^{2n+1})^{2k} \underset{\cong}{\simeq} (L_q^{2n+1})^{r(k\eta)} \approx L_q^{2n+2k+1}/L_q^{2k-1}, \end{aligned}$$

by [5, Proposition (2.6)], (2.4) and (2.6) (1). This completes the proof of the part ii) of (2). q.e.d.

Let X and Y be pointed CW -complexes. The stable homotopy group from X to Y is defined by

$$(2.7) \quad \{X, Y\} = \lim_{n \rightarrow \infty} [S^n X, S^n Y]_0.$$

The following assertion is proved by making use of Puppe exact sequences.

$$(2.8) \quad (1) \quad \text{If } k > m, \text{ then } \{L_q^m/L_q^n(q_0, \dots, q_{[m/2]}), S^k\} \cong 0.$$

(2) If $k \leq n$, then $\{S^k, L_q^m/L_q^n(q_0, \dots, q_{[m/2]})\} \cong 0$.

(3) Suppose that $[m/2] > [(n+1)/2]$. Then we have

$$i) \quad \{S^{n+1}, L_q^m/L_q^n(q_0, \dots, q_{[m/2]})\} = \langle \{i_{n+1}\} \rangle \cong \begin{cases} \mathbf{Z} & (n: \text{odd}) \\ \mathbf{Z}/q & (n: \text{even}), \end{cases}$$

where i_{n+1} is the composition

$$S^{n+1} \approx L_q^{n+1}/L_q^n(q_0, \dots, q_{[m/2]}) \subset L_q^m/L_q^n(q_0, \dots, q_{[m/2]}) .$$

$$ii) \quad \{L_q^m/L_q^n(q_0, \dots, q_{[m/2]}), S^m\} = \langle \{p_m\} \rangle \cong \begin{cases} \mathbf{Z} & (m: \text{odd}) \\ \mathbf{Z}/q & (m: \text{even}), \end{cases}$$

where p_m is the composition

$$L_q^m/L_q^n(q_0, \dots, q_{[m/2]}) \rightarrow L_q^m/L_q^{m-1}(q_0, \dots, q_{[m/2]}) \approx S^m .$$

The following lemma implies that the family of stunted lens spaces is closed under S -duality.

Lemma 2.9. Suppose that $k=2[m/2]+1-2[(n+1)/2] \geq 3$, $N \equiv 0 \pmod{2h(q, k)}$ and $N > m+1$. Then the S -dual of

$$L_q^m/L_q^n(a_1, \dots, a_{[(n+1)/2]}, q_0, \dots, q_{[k/2]})$$

$$\text{is } L_q^{N-n-2}/L_q^{N-m-2}(b_1, \dots, b_{[(N-m-1)/2]}, q_0, \dots, q_{[k/2]}) .$$

Proof. If $nm \equiv 1 \pmod{2}$, then the S -dual of

$$X = L_q^m/L_q^n(a_1, \dots, a_{[(n+1)/2]}, q_0, \dots, q_{[k/2]}) \\ \cong (L_q^k(q_0, \dots, q_{(k-1)/2}))^{((n+1)/2)r(\eta)}$$

is $(L_q^k(q_0, \dots, q_{(k-1)/2}))^{-\tau - ((n+1)/2)r(\eta)}$, where τ denotes the tangent bundle of $L_q^k(q_0, \dots, q_{(k-1)/2})$ (cf. [5, Theorem (3.5)]). Since

$$\tau \oplus 1 \cong r(\eta(q_0, \dots, q_{(k-1)/2})),$$

it follows from [5, Proposition (2.6)] and (2.6) that

$$(L_q^k(q_0, \dots, q_{(k-1)/2}))^{-\tau - ((n+1)/2)r(\eta)} \\ \cong (L_q^k(q_0, \dots, q_{(k-1)/2}))^{(N/2)r(\eta) - ((k+1)/2)r(\eta) - ((n+1)/2)r(\eta)} \\ = (L_q^k(q_0, \dots, q_{(k-1)/2}))^{((N-m-1)/2)r(\eta)} \\ \cong L_q^{N-n-2}/L_q^{N-m-2}(b_1, \dots, b_{[(N-m-1)/2]}, q_0, \dots, q_{[k/2]}) = Y .$$

This implies that there are positive integers a, b and M such that the M -dual of $S^a X$ is $S^b Y$. Let

$$u: S^a X \wedge S^b Y \rightarrow S^M$$

be a M -duality pairing. Then the homomorphism

$$\Gamma_u^i: \{S^i, S^a X\} \rightarrow \{S^{i+b} Y, S^M\}$$

defined by $\Gamma_u^i(\{f\}) = \{S^i u \circ (f \wedge 1)\}$ for $f: S^{l+i} \rightarrow S^{l+a} X$, is an isomorphism. It follows from (2.8) that $M = a + b + N - 1$ and

$$\{S^{n+1}, X\} = \langle \{i_{n+1}\} \rangle \cong \{Y, S^{N-n-2}\} = \langle \{p_{N-n-2}\} \rangle \cong \mathbb{Z}.$$

Hence, by the isomorphism

$$\{S^{n+1}, X\} \cong \{S^{a+n+1}, S^a X\} \xrightarrow{\Gamma_u^{a+n+1}} \{S^{a+b+n+1} Y, S^{a+b+N-1}\} \cong \{Y, S^{N-n-2}\},$$

$\{i_{n+1}\}$ corresponds to either $\{p_{N-n-2}\}$ or $-\{p_{N-n-2}\}$. This implies that there exists a homotopy commutative diagram

$$\begin{array}{ccc} S^{l+a+n+1} \wedge S^b Y & \xrightarrow{1 \wedge S^b(p_{N-n-2})} & S^{l+a+n+1} \wedge S^{b+N-n-2} \\ \downarrow S^{l+a}(i_{n+1}) \wedge 1 & & \approx \downarrow v \\ S^{l+a} X \wedge S^b Y & \xrightarrow{S^l u} & S^{l+a+b+N-1}, \end{array}$$

where v and $S^l u$ are $(l+a+b+N-1)$ -duality pairings. It follows that $C_{S^{l+a}(i_{n+1})}$ is the $(l+a+b+N)$ -dual of $C_{S^b(p_{N-n-2})}$. Since

$$C_{S^{l+a}(i_{n+1})} \cong S^{l+a}(L_q^m/L_q^{n+1}(a_1, \dots, a_{(n+1)/2}, q_0, \dots, q_{[k/2]}))$$

and

$$C_{S^b(p_{N-n-2})} \cong S^{b+1}(L^{N-n-3}/L^{N-m-2}(b_1, \dots, b_{(N-m-1)/2}, q_0, \dots, q_{[k/2]})),$$

this implies that L_q^m/L_q^{n+1} is the S -dual of L^{N-n-3}/L^{N-m-2} . In the similar way it is shown that L_q^{N-n-2}/L_q^{N-m-1} is the S -dual of L_q^{m-1}/L_q^n . Using this fact, in the similar way it is shown that L_q^{m-1}/L_q^{n+1} is the S -dual of L_q^{N-n-3}/L_q^{N-m-1} . q.e.d.

REMARK. The partial results for the case where q is a prime of this lemma have been obtained in [18].

It follows from (2.6) and Lemma 2.9 that, in the following cases,

$$L_q^m/L_q^n(a_1, \dots, a_{[(n+1)/2]}, q_0, \dots, q_{[m/2]-[(n+1)/2]})$$

and L_q^m/L_q^n have the same stable homotopy type:

- (1) $q=2, 3, 4$ or 6 ,
- (2) $mn \equiv 0 \pmod{2}$,
- (3) $q_0 \cdots q_{[m/2]-[(n+1)/2]} \equiv \pm a^{[m/2]-[(n-1)/2]} \pmod{q}$,

- (4) $n+1 \equiv 0 \pmod{2h(q, m-n-1)}$,
 (5) $m+1 \equiv 0 \pmod{2h(q, m-n-1)}$.

QUESTION. Is it true that $L_q^m/L_q^n(q_0, \dots, q_{[m/2]})$ and L_q^m/L_q^n have the same stable homotopy type for any case? If it is not, then how many stable homotopy types are there for fixed m, n and q ?

From now on, we restrict ourselves to standard stunted lens spaces. According to [5, Propositions (2.6) and (2.9)], (2.4) and Lemma 2.9, we obtain the following.

(2.10) Set $k=m-2[(n+1)/2]$ and $l=2[m/2]-n$.

- (1) If $t \equiv 0 \pmod{2h(q, k)}$, then L_q^m/L_q^n and L_q^{m+t}/L_q^{n+t} have the same stable homotopy type.
 (2) If $k \geq 2$ and $n+1 \equiv 0 \pmod{2h(q, k)}$, then $t \equiv 0 \pmod{2h(q, k)}$ if and only if L_q^m/L_q^n and L_q^{m+t}/L_q^{n+t} have the same stable homotopy type.
 (3) If $t \equiv 0 \pmod{2h(q, l)}$, then L_q^m/L_q^n and L_q^{m+t}/L_q^{n+t} have the same stable homotopy type.
 (4) If $l \geq 2$ and $m+1 \equiv 0 \pmod{2h(q, l)}$, then $t \equiv 0 \pmod{2h(q, l)}$ if and only if L_q^m/L_q^n and L_q^{m+t}/L_q^{n+t} have the same stable homotopy type.

(2.11) (1) ([12, I; Theorem 1.1]) Let p be a prime and r a positive integer with $p^r > 2$. Suppose that $k=m-2[(n+1)/2] \geq 2$. Then $t \equiv 0 \pmod{2p^{[(k-2)/2(p-1)]}}$ if $L_{p^r}^m/L_{p^r}^n$ and $L_{p^r}^{m+t}/L_{p^r}^{n+t}$ have the same stable homotopy type.

(2) Let $r \geq 2$ be a positive integer and set $k=m-2[(n+1)/2]$. Then $v_2(t) \geq [k/2] + \beta_1(k, n)$ if $L_{2^r}^m/L_{2^r}^n$ and $L_{2^r}^{m+t}/L_{2^r}^{n+t}$ have the same stable homotopy type, where β_1 is the function defined by (1.5).

(3) Suppose that $q \equiv 0 \pmod{2}$ and $m \geq n+2$. Then $v_2(t) \geq [\log_2 2(m-n-1)]$ if L_q^m/L_q^n and L_q^{m+t}/L_q^{n+t} have the same stable homotopy type.

Proof. Suppose that $q \equiv 0 \pmod{2}$ and $m \geq n+2$. It is well known that

$$H^*(L_q^m; \mathbf{Z}/2) \cong \begin{cases} (\mathbf{Z}/2)[u]/(u^{m+1}) & (q \equiv 2 \pmod{4}) \\ (\mathbf{Z}/2)[u, v]/(u^2, v^{[m/2]+1}, uv^{m-[m/2]}) & (q \equiv 0 \pmod{4}), \end{cases}$$

where $\deg u=1$ and $\deg v=2$. The action of the Steenrod squares is given by

$$\begin{cases} Sq^i(u^j) = \binom{j}{i} u^{j+i} & (q \equiv 2 \pmod{4}) \\ Sq^{2i}(v^j) = \binom{j}{i} v^{j+i} & (q \equiv 0 \pmod{4}) \\ Sq^{2i}(uv^j) = \binom{j}{i} uv^{j+i} & (q \equiv 0 \pmod{4}) \\ Sq^{2i+1} = 0 & (q \equiv 0 \pmod{4}). \end{cases}$$

Assume that L_q^m/L_q^n and L_q^{m+t}/L_q^{n+t} have the same stabel homotopy type. Then $t \equiv 0 \pmod{2}$. It follows from the naturality of the Steenrod squares that we have

$$(2.12) \quad \begin{cases} \binom{n+1}{i} \equiv \binom{n+t+1}{i} \pmod{2} & (1 \leq i \leq m-n-1, q \equiv 2 \pmod{4}) \\ \binom{[(n+1)/2]}{i} \equiv \binom{[(n+t+1)/2]}{i} \pmod{2} & (2 \leq 2i \leq m-n-1, q \equiv 0 \pmod{4}). \end{cases}$$

Let s be the integer with $2^s \leq m-n-1 < 2^{s+1}$. By Lemma 2.1, (2.12) implies that $\nu_2(t) \geq s+1 = [\log_2 2(m-n-1)]$. This completes the proof of (3).

(2) It follows from [11, Theorem 1.1] that

$$\nu_2(t) \geq [m/2] - [(n+1)/2] + C(m, n)$$

if $L_{2^r}^m/L_{2^r}^n$ and $L_{2^r}^{m+t}/L_{2^r}^{n+t}$ have the same stable homotopy type, where $C(m, n)$ is the function defined by

$$C(m, n) = \begin{cases} 1 & (n \equiv 1, 5 \text{ or } 6 \pmod{8} \text{ and } m \equiv 1, 4 \text{ or } 5 \pmod{8}) \\ 0 & (\text{otherwise}). \end{cases}$$

Then (2) is obtained by making use of the S -duality (Lemma 2.9). q.e.d.

In order to state the next proposition, we set

$$(2.13) \quad \begin{cases} a_0(m, n) = [m/4] + [(m+7)/8] + [(m+4)/8] \\ \quad - [(n+1)/4] - [(n+1)/8] - [(n+6)/8] \\ b_0(m, n) = [m/8] + [(m+6)/8] - [(n+7)/8] - [(n+5)/8]. \end{cases}$$

$$(2.14) \quad \begin{cases} a(j, m, n) = \min \{ \nu+1, a_0(m+j, n+j) \} \\ b(j, m, n) = \min \{ \nu+1, b_0(m+j, n+j) \} \end{cases},$$

where ν is the integer defined by

$$\nu = \begin{cases} \nu_2(j) & (j \neq 0) \\ m & (j = 0). \end{cases}$$

Let $m(s)$ denote the function defined on positive integers as follows (cf. [3]):

$$\nu_p(m(s)) = \begin{cases} 0 & (p \neq 2 \text{ and } s \not\equiv 0 \pmod{p-1}) \\ 1 + \nu_p(s) & (p \neq 2 \text{ and } s \equiv 0 \pmod{p-1}) \\ 1 & (p = 2 \text{ and } s \not\equiv 0 \pmod{2}) \\ 2 + \nu_2(s) & (p = 2 \text{ and } s \equiv 0 \pmod{2}). \end{cases}$$

Proposition 2.15 ([15, Theorem 3]). *Let j , m and n be non-negative in-*

tegers with $m > n$ and $j \equiv n+1 \equiv 0 \pmod{4}$. Then we have

$$\tilde{J}(S^j(L_4^m/L_4^n)) \cong \begin{cases} \mathbf{Z}/m((n+j+1)/2) \cdot 2^c \oplus \mathbf{Z}/2^{d+i} \oplus \mathbf{Z}/2^k & (b(j, m, n) \geq 0) \\ \mathbf{Z}/m((n+j+1)/2) & (b(j, m, n) < 0), \end{cases}$$

where i, k, c and d are integers defined by

$$(2.16) \quad \begin{cases} i = \begin{cases} \min \{v_2(n+1)-1, a(j, m, n)\} & (n+j \equiv 7 \pmod{8}) \\ \min \{v_2(n+1), a(j, m, n)\} & (n+j \equiv 3 \pmod{8}) \end{cases} \\ k = \min \{v_2(n+1)-1, b(j, m, n)\} \\ c = \max \{a(j, m, n)-i, b(j, m, n)-k\} \\ d = \min \{a(j, m, n)-i, b(j, m, n)-k\}. \end{cases}$$

In order to state the next proposition, we set

$$(2.17) \quad \begin{cases} a_3(m, n) = [(m-2)/8] - [(n+5)/8] \\ a_4(m, n) = [m/8] - [(n+7)/8] \\ a_5(m, n) = a_0(m, n) + [m/8] - [(m-4)/8] \\ a_6(m, n) = [(m+4)/8] + [(m-2)/8] - [(n+1)/4]. \end{cases}$$

$$(2.18) \quad \begin{cases} b_1(j, m, n) = \min \{v - a_3(n, n+4), a_5(m, n)\} \\ b_2(j, m, n) = \min \{v - a_3(n-5, n), a_6(m, n)\} \\ b_3(j, m, n) = \min \{v+1, a_4(m, n)\}, \end{cases}$$

where v is the integer defined in (2.14).

Proposition 2.19 ([16, Theorem 2]). *Let j, m and n be non-negative integers with $m > n$ and $j \equiv 0 \pmod{8}$.*

(1) *If $n \not\equiv 3 \pmod{4}$ and $m \geq 2[(n+6)/8] + 2[n/8] + 4[(n+15)/8]$, then we have*

$$\tilde{J}(S^j(L_8^m/L_8^n)) \cong \begin{cases} (\oplus_{i=1}^3 \mathbf{Z}/2^{b_i(j, m, n)}) \oplus \mathbf{Z}/2 & (n \equiv 2 \pmod{8}) \\ \oplus_{i=1}^3 \mathbf{Z}/2^{b_i(j, m, n)} & (\text{otherwise}). \end{cases}$$

(2) *If $n \equiv 3 \pmod{4}$ and $2[(n+6)/8] + 2[n/8] + 4[(n+15)/8] > m > n$, then we have*

$$\tilde{J}(S^j(L_8^m/L_8^n)) \cong \begin{cases} \mathbf{Z}/2^{b_1(j, m, n)} \oplus \mathbf{Z}/4 & (n \equiv 2 \pmod{8} \text{ and } m \geq n+6) \\ \mathbf{Z}/8 & (n \equiv 1 \pmod{8} \text{ and } m \geq n+3) \\ K\tilde{O}(S^j(L_8^m/L_8^n)) & (\text{otherwise}). \end{cases}$$

(3) *If $n \equiv 3 \pmod{4}$ and $m \geq n+5$, then we have*

$$\tilde{J}(S^j(L_8^m/L_8^n)) \cong \mathbf{Z}/m((n+j+1)/2) \cdot 2^{c_1} \oplus \mathbf{Z}/2^{c_2+i_1} \oplus \mathbf{Z}/2^{i_2} \oplus \mathbf{Z}/2^{b_3(j, m, n)},$$

where i_1, i_2, c_1 and c_2 are integers defined by

$$(2.20) \quad \begin{cases} i_1 = \begin{cases} \min \{b_1(j, m, n), v_2(n+1)-1\} & (n \equiv 7 \pmod{8}) \\ \min \{b_1(j, m, n), v_2(n+1)\} & (n \equiv 3 \pmod{8}) \end{cases} \\ i_2 = \begin{cases} \min \{b_2(j, m, n), v_2(n+1)-2\} & (n \equiv 7 \pmod{8}) \\ \min \{b_2(j, m, n), v_2(n+1)-1\} & (n \equiv 3 \pmod{8}) \end{cases} \\ c_1 = \max \{b_1(j, m, n)-i_1, b_2(j, m, n)-i_2\} \\ c_2 = \min \{b_1(j, m, n)-i_1, b_2(j, m, n)-i_2\}. \end{cases}$$

(4) If $n \equiv 3 \pmod{4}$ and $n+5 > m > n$, then we have

$$J(S^j(L_8^m/L_8^n)) \cong \begin{cases} \mathbf{Z}/m((n+j+1)/2) \oplus \widetilde{KO}(S^j(L_8^m/L_8^{n+1})) & (m \geq n+2) \\ \mathbf{Z}/m((n+j+1)/2) & (m = n+1). \end{cases}$$

In order to state the next proposition, we set

$$(2.21) \quad (1) \quad \begin{cases} \sigma_{2^i} = \eta^{2^i} - 1 & (0 \leq i \leq 2) \\ \sigma_6 = \sigma_4 \sigma_2 \\ \sigma_{2i+1} = \sigma_{2i} \sigma_1 & (1 \leq i \leq 3). \end{cases}$$

(2) Let $F(x)$ denote the free abelian group generated by $x_1, x_2, x_3, x_4, x_5, x_6$ and x_7 . Then X_i and $X_i(n)$ ($1 \leq i \leq 7, n \geq 0$) denote the elements of $F(x)$ defined by $X_1 = 4x_1 + 2x_3 + 2x_5 + x_7$, $X_2 = 2x_2 + x_6$, $X_3 = 2x_3 + x_7$, $X_6 = x_6 + x_7$, $X_i = x_i$ ($i = 4, 5$ or 7), $X_1(n) = 2^{[n/2]} X_1$,

$$\begin{aligned} X_2(n) &= 2^{[n/4]} X_2 - 2^{2[n/4]} X_1, \\ X_3(n) &= 2^{[(n-2)/4]} X_3 + (2^{[n/2]} - 2^{2[(n-2)/4]+1}) X_1, \\ X_4(n) &= 2^{[n/8]} X_4 + (2^{[n/4]} - 2^{2[n/8]}) X_2 + 2^{[n/4]+2[n/8]} X_1, \\ X_5(n) &= 2^{[(n-2)/8]} X_5 + (2^{[(n-2)/4]} - 2^{2[(n-2)/8]}) X_3 \\ &\quad - 2^{[(n+2)/4]+2[(n-2)/8]} X_1, \\ X_6(n) &= 2^{[(n-4)/8]} X_6 + (2^{[n/4]} - 2^{2[(n-4)/8]+1}) X_2 \\ &\quad - 2^{[n/4]+2[(n-4)/8]+1} X_1 \end{aligned}$$

and

$$\begin{aligned} X_7(n) &= 2^{[(n-6)/8]} X_7 - (2^{[(n-2)/4]} - 2^{2[(n-6)/8]+1}) (X_3 - 2X_2) \\ &\quad + 2^{2[(n+2)/8]+[(n-2)/4]} X_1. \end{aligned}$$

(3) Let $\varphi: F(x) \rightarrow \tilde{K}(L_8^m)$ be the homomorphism defined by setting $\varphi(x_i) = \sigma_i$ ($1 \leq i \leq 7$).

Proposition 2.22 (Kobayashi and Sugawara [12]). *The homomorphism φ is an epimorphism, and the kernel of φ coincides with the subgroup of $F(x)$ generated by $\{X_i(m) \mid 1 \leq i \leq 7\}$.*

For each integer n with $0 \leq n < m$, we denote the inclusion map of L_8^n into

L_8^m by i_n^m , and denote the kernel of the homomorphism

$$(i_n^m)^!: \tilde{K}(L_8^m) \rightarrow \tilde{K}(L_8^n)$$

by V_n . Set $u = [(n+1)/2]$ and $S_i = \varphi(X_i(2u))$ ($1 \leq i \leq 7$). Then V_{2u} is the subgroup of $\tilde{K}(L_8^n)$ generated by S_i ($1 \leq i \leq 7$), and we have

$$\tilde{K}(L_8^m/L_8^n) \cong \begin{cases} V_{2u} & (n \equiv 0 \pmod{2}) \\ \mathbf{Z} \oplus V_{2u} & (n \equiv 1 \pmod{2}) \end{cases}.$$

According to [1], we have the following lemma.

Lemma 2.23. *The Adams operations are given by the following formulae, where $s_i = \varphi(X_i)$ ($1 \leq i \leq 7$) and $k \equiv 1 \pmod{2}$.*

- (1) $\psi^k(s_i) = s_i$ ($i = 1, 2$ or 4).
- (2) $\psi^k(s_3) = \begin{cases} s_3 & (k \equiv 1 \pmod{4}) \\ -s_3 - 2s_2 & (k \equiv 3 \pmod{4}) \end{cases}.$
- (3) $\psi^k(s_5) = \begin{cases} s_5 & (k \equiv 1 \pmod{8}) \\ s_5 + s_6 & (k \equiv 3 \pmod{8}) \\ -s_5 - 2s_4 & (k \equiv 5 \pmod{8}) \\ -s_5 - 2s_4 - s_6 & (k \equiv 7 \pmod{8}) \end{cases}.$
- (4) $\psi^k(s_6) = \begin{cases} s_6 & (k \equiv \pm 1 \pmod{8}) \\ -s_6 & (k \equiv \pm 3 \pmod{8}) \end{cases}.$
- (5) $\psi^k(s_7) = \begin{cases} s_7 & (k \equiv 1 \pmod{8}) \\ -s_7 + 2s_4 & (k \equiv 3 \pmod{8}) \\ s_7 - 2s_6 & (k \equiv 5 \pmod{8}) \\ -s_7 + 2s_4 + 2s_6 & (k \equiv 7 \pmod{8}) \end{cases}.$

3. Proofs of Theorems

In this section we prove Theorems 1, 2, 3 and 4.

3.1. Proof of Theorems 1 and 3. According to [8], we have

$$h(q, k) = \begin{cases} 2^{h_1(k)} & (q = 4) \\ 2^{h_2(k)} & (q = 8) \end{cases}.$$

Then Theorems 1 and 3 follows from (2.10) for the case $m \geq n+3$. Note that we have $L_q^{n+1}/L_q^n \approx S^{n+1}$ and

$$L_q^{n+2}/L_q^n \cong \begin{cases} S^{n+1} \vee S^{n+2} & (n \equiv 1 \pmod{2}) \\ S^n L_q^2 & (n \equiv 0 \pmod{2}) \end{cases}.$$

This completes the proof of Theorems 1 and 3.

3.2. Proof of Theorem 2. The part (1) is obtained by (2.11) (2).

Suppose that the spaces L_4^m/L_4^n and L_4^{m+t}/L_4^{n+t} have the same stable homotopy type, where $\nu_2(n+1) \geq i$ and $h_1(m-2[(n+1)/2]) \geq i \geq 3$. Then $m \geq n+5$ and there exists a homotopy equivalence

$$f: S^j(L_4^m/L_4^n) \rightarrow S^{j-t}(L_4^{m+t}/L_4^{n+t}),$$

which induces an isomorphism

$$(3.1) \quad J(f^!): \tilde{J}(S^{j-t}(L_4^{m+t}/L_4^{n+t})) \rightarrow \tilde{J}(S^j(L_4^m/L_4^n)).$$

We can assume that $\nu_2(j) \geq h_1(m-2[(n+1)/2]) = h_1(m-n-1)$. By (2.11) (3), $t \equiv 0 \pmod{8}$. It follows from (2.11) (1) and Proposition 2.15 that we have $\nu_2(t) \geq [(m-n-1)/2]$,

$$\tilde{J}(S^j(L_4^m/L_4^n)) \cong \mathbf{Z}/m((n+j+1)/2) \cdot 2^{c_1} \oplus \mathbf{Z}/2^{d_1+i_1} \oplus \mathbf{Z}/2^{k_1}$$

and

$$\tilde{J}(S^{j-t}(L_4^{m+t}/L_4^{n+t})) \cong \mathbf{Z}/m((n+j+1)/2) \cdot 2^{c_2} \oplus \mathbf{Z}/2^{d_2+i_2} \oplus \mathbf{Z}/2^{k_2},$$

where $i_l = \min \{\nu_2(n+1)-1, h_1(m-n-1)\}$,

$$\begin{aligned} i_2 &= \min \{\nu_2(n+t+1)-1, h_1(m-n-1)\}, \\ k_1 &= \min \{\nu_2(n+1)-1, [(m-n-1)/8] + [(m-n+5)/8]\}, \\ k_2 &= \min \{\nu_2(n+t+1)-1, [(m-n-1)/8] + [(m-n+5)/8]\}, \\ c_l &= h_1(m-n-1) - i_l \quad (l=1 \text{ or } 2) \end{aligned}$$

and $d_l = [(m-n-1)/8] + [(m-n+5)/8] - k_l$ ($l=1$ or 2). Since $c_l \geq d_l \geq 0$ and $\nu_2(m((n+j+1)/2)) \geq i_l \geq k_l$ ($l=1$ or 2), the isomorphism (3.1) implies that $c_1 = c_2$, and hence

$$\min \{\nu_2(n+1)-1, h_1(m-n-1)\} = \min \{\nu_2(n+t+1)-1, h_1(m-n-1)\}.$$

Since $\nu_2(n+1) \geq i$, this implies that we have $\nu_2(n+t+1) > i$ if $\nu_2(n+1) > i$ and $\nu_2(n+t+1) = i$ if $\nu_2(n+1) = i$. Thus we have $\nu_2(t) \geq i+1$. The proof of the part (2) of Theorem 2 is completed by making use of the S -duality (Lemma 2.9).

The part (3) is obtained by (2.11) (3) and the parts (1) and (2) of Theorem 2. This completes the proof of Theorem 2.

3.3. Proof of Theorem 4. Suppose that the spaces L_8^m/L_8^n and L_8^{m+t}/L_8^{n+t} have the same stable homotopy type. Then there exists a homotopy equivalence

$$f: S^j(L_8^m/L_8^n) \rightarrow S^{j-t}(L_8^{m+t}/L_8^{n+t}),$$

which induces isomorphisms

$$(3.2) \quad f^*: \tilde{K}(S^{j-t}(L_8^{m+t}/L_8^{n+t})) \rightarrow \tilde{K}(S^j(L_8^m/L_8^n))$$

and

$$(3.3) \quad J(f^!): \tilde{J}(S^{j-t}(L_8^{m+t}/L_8^{n+t})) \rightarrow \tilde{J}(S^j(L_8^m/L_8^n)).$$

We can assume that $v_2(j) \geq a_0(m, n) + 1$. Suppose $m \geq n + 5$. Then, by (2.11) (3), $t \equiv 0 \pmod{8}$. If $n \not\equiv 3 \pmod{4}$, then Proposition 2.19 asserts that the exponent of the group $\tilde{J}(S^j(L_8^m/L_8^n))$ is equal to $2^{b_1(j, m, n)}$ and the exponent of the group $\tilde{J}(S^{j-t}(L_8^{m+t}/L_8^{n+t}))$ is equal to $2^{b_1(j-t, m+t, n+t)}$, and the isomorphism (3.3) implies that $b_1(j, m, n) = b_1(j-t, m+t, n+t)$. In the case $n \equiv 3 \pmod{4}$, f induces a homotopy equivalence

$$\tilde{f}: S^j(L_8^m/L_8^{n+1}) \rightarrow S^{j-t}(L_8^{m+t}/L_8^{n+t+1}),$$

which induces an isomorphism

$$J(\tilde{f}^!): \tilde{J}(S^{j-t}(L_8^{m+t}/L_8^{n+t+1})) \rightarrow \tilde{J}(S^j(L_8^m/L_8^{n+1})).$$

Since $b_1(j, m, n) = b_1(j, m, n+1)$ and $b_1(j-t, m+t, n+t) = b_1(j-t, m+t, n+t+1)$, the isomorphism $J(\tilde{f}^!)$ implies that $b_1(j, m, n) = b_1(j-t, m+t, n+t)$. In either case, we have $b_1(j, m, n) = b_1(j-t, m+t, n+t)$. Hence $v_2(j-t) - a_3(n, n+4) \geq a_5(m, n)$, and

$$(3.4) \quad v_2(t) \geq \begin{cases} a_5(m, n) - 2 & (n \equiv 0, 1 \text{ or } 7 \pmod{8}) \\ a_5(m, n) - 1 & (\text{otherwise}). \end{cases}$$

By Lemma 2.9, (2.11) (2) and (3.4), the part (1) of Theorem 4 is obtained except for the case $n \equiv m - n \equiv 2 \pmod{8}$. So, assume that $n \equiv m - 2 \equiv 2 \pmod{8}$. Let Y_i be the element of $\tilde{K}(S^j(L_8^m/L_8^n))$, which corresponds to $I^{j/2}(S_i)$ by the isomorphism

$$(p_n^m)^!: \tilde{K}(S^j(L_8^m/L_8^n)) \xrightarrow{\cong} I^{j/2}(V_n),$$

where I denotes the Bott periodicity isomorphism ($1 \leq i \leq 7$). Set $u_1 = Y_6$,

$$\begin{aligned} u_2 &= Y_4 + Y_2 + 2^{(m-n-10)/8}(2Y_2 + Y_1) + 2^{(3m-3n-6)/8} Y_1, \\ u_3 &= Y_7 - Y_2 - Y_1 + 2^{(m-n-10)/8}(2Y_2 + Y_1 - Y_3) + 2^{(3m-3n-14)/8} Y_1 \\ &\quad - 2^{(2m-2n-12)/8} Y_1, \\ u_4 &= 2Y_4 - Y_1 + 2^{(m-n-2)/8}(2Y_2 + Y_1) + 2^{(3m-3n+2)/8} Y_1, \\ u_5 &= Y_5 + Y_1 - 2^{(3m-3n-6)/8} Y_1, \\ u_6 &= 2Y_6 - Y_2 \end{aligned}$$

and $u_7 = 2Y_7 + Y_3 - 2Y_2 - 2Y_1 + 2^{(m-n-2)/8}(2Y_2 + Y_1 - Y_3) + 2^{(3m-3n-6)/8} Y_1$. Then, by Proposition 2.22, we have

$$\tilde{K}(S^j(L_8^m/L_8^n)) \simeq \langle \{u_i \mid 1 \leq i \leq 7\} \rangle / \langle \{2^{a(i)} u_i \mid 1 \leq i \leq 7\} \rangle,$$

where

$$\begin{aligned} a(1) &= (m-n+4)/2, \\ a(2) &= a(3) = (m-n+2)/4, \\ a(5) &= a(6) = (m-n-2)/8 \end{aligned}$$

and $a(4) = a(7) = (m-n-10)/8$. According to Lemma 2.23, the Adams operation ψ^5 is given by the following formulae.

$$(3.5) \quad \begin{cases} (1) & \psi^5(u_1) = 5^{j/2}(u_1 - u_6). \\ (2) & \psi^5(u_2) = 5^{j/2} u_2. \\ (3) & \psi^5(u_3) = 5^{j/2}(u_3 - u_6). \\ (4) & \psi^5(u_4) = 5^{j/2} u_4. \\ (5) & \psi^5(u_5) = 5^{j/2}(2^{(m-n-2)/8}(2u_2 - u_4) - u_4 - u_5). \\ (6) & \psi^5(u_6) = 5^{j/2}(-u_6). \\ (7) & \psi^5(u_7) = 5^{j/2}(u_7 - 2u_6). \end{cases}$$

Choose $v_i \in \tilde{K}(S^{j-t}(L_8^{m+t}/L_8^{n+t}))$ similarly as $u_i \in \tilde{K}(S^j(L_8^m/L_8^n))$ ($1 \leq i \leq 7$), and set

$$f^*(v_i) = \sum_{k=1}^7 a_{ik} u_k \quad (1 \leq i \leq 7).$$

By the equality $\psi^5 \circ f^* = f^* \circ \psi^5$, we have

$$(3.6) \quad 5^{j/2} a_{11} \equiv 5^{(j-t)/2}(a_{11} - a_{61}) \pmod{2^{a(1)}}.$$

$$(3.7) \quad 5^{j/2} a_{31} \equiv 5^{(j-t)/2}(a_{31} - a_{61}) \pmod{2^{a(1)}}.$$

Since $v_2(a_{31}) \geq a(1) - a(3) = (m-n+6)/4$, (3.7) implies that $a_{61} \equiv 0 \pmod{2^{a(1)}}$. It follows from (3.6) that we have $(5^{t/2} - 1)a_{11} \equiv 0 \pmod{2^{a(1)}}$. Note that $a_{11} \equiv 1 \pmod{2}$. According to [15, Lemma 3.1], we see that $v_2(t) + 1 \geq a(1) = (m-n+4)/2$. Hence

$$v_2(t) \geq (m-n+2)/2 = [m/2] - [(n+1)/2] + 1.$$

This completes the proof of the part (1) of Theorem 4.

Suppose that $v_2(n+1) \geq i$ and $h_2(m-2[(n+1)/2]) \geq i \geq 3$. Then $m \geq n+5$ and $t \equiv 0 \pmod{8}$ by (2.11) (3). It follows from (2.11) (1) and Proposition 2.19 that we have $v_2(t) \geq [(m-n-1)/2]$,

$$\tilde{J}(S^j(L_8^m/L_8^n)) \simeq \mathbf{Z}/m((n+j+1)/2) \cdot 2^{c_1} \oplus \mathbf{Z}/2^{c_2+i_1} \oplus \mathbf{Z}/2^{i_2} \oplus \mathbf{Z}/2^{i_3}$$

and

$$\tilde{J}(S^{j-t}(L_8^{m+t}/L_8^{n+t})) \simeq \mathbf{Z}/m((n+j+1)/2) \cdot 2^{d_1} \oplus \mathbf{Z}/2^{d_2+k_1} \oplus \mathbf{Z}/2^{k_2} \oplus \mathbf{Z}/2^{k_3},$$

where $i_1 = \min \{v_2(n+1) - 1, h_2(m-m-1)\}$,

$$k_1 = \min \{v_2(n+t+1) - 1, h_2(m-n-1)\},$$

$$\begin{aligned}
 i_2 &= \min \{ \nu_2(n+1) - 2, [(m-n+3)/8] + [(m-n-3)/8] \}, \\
 k_2 &= \min \{ \nu_2(n+t+1) - 2, [(m-n+3)/8] + [(m-n-3)/8] \}, \\
 i_3 &= k_3 = [(m-n-1)/8], \\
 c_1 &= h_2(m-n-1) - i_1, \\
 d_1 &= h_2(m-n-1) - k_1, \\
 c_2 &= [(m-n+3)/8] + [(m-n-3)/8] - i_2,
 \end{aligned}$$

and $d_2 = [(m-n+3)/8] + [(m-n-3)/8] - k_2$. Since $c_1 \geq c_2 \geq 0$, $c_2 + i_1 \geq i_3$,

$$\begin{aligned}
 \nu_2(\mathfrak{m}((n+j+1)/2)) &\geq i_1 \geq i_2, \\
 \nu_2(\mathfrak{m}((n+j+1)/2)) + c_1 &\geq h_2(m-n-1) + 1
 \end{aligned}$$

and $\max \{d_2 + k_1, k_2, k_3\} \leq h_2(m-n-1)$, the isomorphism (3.3) implies that $c_1 = d_1$, and hence

$$\min \{ \nu_2(n+1) - 1, h_2(m-n-1) \} = \min \{ \nu_2(n+t+1) - 1, h_2(m-n-1) \}.$$

Since $\nu_2(n+1) \geq i$, this implies that we have $\nu_2(n+t+1) > i$ if $\nu_2(n+1) > i$ and $\nu_2(n+t+1) = i$ if $\nu_2(n+1) = i$. Thus we have $\nu_2(t) \geq i+1$. The proof of the part (2) of Theorem 4 is completed by making use of the S -duality (Lemma 2.9).

By (2.11) (3) and the parts (1) and (2) of Theorem 4, the part (3) of Theorem 4 is obtained except for the case $n \equiv 0 \pmod{8}$ and $m = n+6$. So, assume that $n \equiv 0 \pmod{8}$ and $m = n+6$. Let Y_i be the element of $\tilde{K}(S^j(L_8^m/L_8^n))$, which corresponds to $I^{j/2}(S_i)$ by the isomorphism

$$(p_n^m)^!: \tilde{K}(S^j(L_8^m/L_8^n)) \xrightarrow{\cong} I^{j/2}(V_n)$$

($1 \leq i \leq 7$). Set $u_1 = Y_5$, $u_2 = Y_6 + 2Y_5$ and $u_3 = Y_7 + 2Y_5$. Then, by Proposition 2.22, we have

$$\tilde{K}(S^j(L_8^m/L_8^n)) \cong \langle \{u_i \mid 1 \leq i \leq 3\} \rangle / \langle \{2^{a(i)} u_i \mid 1 \leq i \leq 3\} \rangle,$$

where $a(1) = 32$ and $a(2) = a(3) = 4$. According to Lemma 2.23, we have the following formulae.

$$(3.8) \quad \begin{cases} (1) & \psi^{-1}(u_1) = 5u_1 + u_2. \\ (2) & \psi^5(u_1) = 5^{j/2}(17u_1 + 2u_2). \\ (3) & \psi^{-1}(u_2) = 8u_1 - u_2. \\ (4) & \psi^5(u_2) = 5^{j/2}(16u_1 + u_2). \\ (5) & \psi^{-1}(u_3) = -8u_1 - u_3. \\ (6) & \psi^5(u_3) = 5^{j/2}(16u_1 + u_3). \end{cases}$$

Choose $v_i \in \tilde{K}(S^{j-t}(L_8^{m+t}/L_8^{n+t}))$ similarly as $u_i \in \tilde{K}(S^j(L_8^m/L_8^n))$ ($1 \leq i \leq 3$), and set

$$f^*(v_i) = \sum_{k=1}^3 a_{ik} u_k \quad (1 \leq i \leq 3).$$

By the equality $\psi^{-1} \circ f^* = f^* \circ \psi^{-1}$, we have

$$(3.9) \quad 8a_{12} - 8a_{13} \equiv a_{21} \pmod{32}.$$

By the equality $\psi^5 \circ f^* = f^* \circ \psi^5$, we have

$$(3.10) \quad 5^{j/2}(17a_{11} + 16a_{12} + 16a_{13}) \equiv 5^{(j-t)/2}(17a_{11} + 2a_{21}) \pmod{32}.$$

By (3.9), (3.10) and the fact $5^{t/2} \equiv 1 \pmod{2^{v_2(t)+1}}$, we have

$$(5^{t/2} - 1)a_{11} \equiv 0 \pmod{32}.$$

Note that $a_{11} \equiv 1 \pmod{2}$. According to [15, Lemma 3.1], we see that $v_2(t) + 1 \geq 5$. Hence

$$v_2(t) \geq 4 = [m/2] - [(n+1)/2] + 1.$$

This completes the proof of Theorem 4.

References

- [1] J.F. Adams: *Vector fields on spheres*, Ann. of Math. **75** (1962), 603–632.
- [2] J.F. Adams: *On the groups $J(X)$ -I*, Topology **2** (1963), 181–195.
- [3] J.F. Adams: *On the groups $J(X)$ -II, -III*, Topology **3** (1965), 137–171, 193–222.
- [4] J.F. Adams and G. Walker: *On complex Stiefel manifolds*, Proc. Camb. Phil. Soc. **61** (1965), 81–103.
- [5] M.F. Atiyah: *Thom complexes*, Proc. London Math. Soc. **11** (1961), 291–310.
- [6] M.M. Cohen: *A course in simple-homotopy theory*, Graduate Texts in Mathematics 10, Springer-Verlag, 1973.
- [7] D.M. Davis and M. Mahowald: *Classification of the stable homotopy types of stunted real projective spaces*, Pacific J. Math. **125** (1986), 335–345.
- [8] K. Fujii, T. Kobayashi and M. Sugawara: *Stable homotopy types of stunted lens spaces*, Mem. Fac. Sci. Kochi Univ. (Math.) **3** (1982), 21–27.
- [9] T. Kambe, H. Matsunaga and H. Toda: *A note on stunted lens space*, J. Math. Kyoto Univ. **5** (1966), 143–149.
- [10] T. Kobayashi: *Stable homotopy types of stunted lens spaces mod p* , Mem. Fac. Sci. Kochi Univ. (Math.) **5** (1984), 7–14.
- [11] T. Kobayashi: *Stable homotopy types of stunted lens spaces mod 2^r* , Mem. Fac. Sci. Kochi Univ. (Math.) **11** (1990), 17–22.
- [12] T. Kobayashi and M. Sugawara: *On stable homotopy types of stunted lens spaces*, Hiroshima Math. J. **1** (1971), 287–304; **II**, **7** (1977), 689–705.
- [13] S. Kôno and A. Tamamura: *On J -groups of $S^l(RP(t-l)/RP(n-l))$* , Math. J. Okayama Univ. **24** (1982), 45–51.
- [14] S. Kôno and A. Tamamura: *J -groups of the suspensions of the stunted lens spaces mod p* , Osaka J. Math. **24** (1987), 481–498.

- [15] S. Kôno and A. Tamamura: *J-groups of suspensions of stunted lens spaces mod 4*, Osaka J. Math. **26** (1989), 319–345.
- [16] S. Kôno and A. Tamamura: *J-groups of suspensions of stunted lens spaces mod 8*, to appear.
- [17] N. Mahammed: *A propos de la K-théorie des espaces lenticulaires*, C.R. Acad. Sc. Paris **271** (1970), 639–642.
- [18] M. Mimura, J. Mukai and G. Nishida: *Representing elements of stable homotopy groups by symmetric maps*, Osaka J. Math. **11** (1974), 105–111.
- [19] D. Quillen: *The Adams conjecture*, Topology **10** (1971), 67–80.

Department of Mathematics
Osaka University
Toyonaka, Osaka 560, Japan

