1. Introduction

Let $L^s(q)=S^{2n+1}/(Z/q)$ be the $(2n+1)$-dimensional standard lens space mod $q$. As defined in [8], we set

$$L^s_{q+1}=L^s(q),$$

$$L^s_q=\{[z_0, \ldots, z_n]|\in L^s(q)|z_n \text{ is real} \geq 0\}.$$

The stable homotopy types (S-types) of stunted lens spaces $L^s_m/L^s_q$ have been studied by several authors (e.g. [7], [8], [9], [10], [11] and [12]). For the case $q=2$, D.M. Davis and M. Mahowald have completed the classification of the stable homotopy types of stunted real projective spaces in [7]. Their result shows that we can use structures of $J$-groups of suspensions of stunted real projective spaces $RP(m)/RP(n)$ and $RP(m+t)/RP(n+t)$ to have the same stable homotopy type as follows: if $RP(m)/RP(n)$ and $RP(m+t)/RP(n+t)$ have the same stable homotopy type, then there exists a non-negative integer $N$ such that

$$J(S^j(RP(m)/RP(n)))=J(S^j-t(RP(m+t)/RP(n+t)))$$

for each integer $j$ with $j \geq N$ (see [13]). For the case where $q$ is an odd prime, T. Kobayashi has obtained some necessary conditions for stunted lens spaces $L^s_m/L^s_q$ and $L^s_{m+t}/L^s_q$ to have the same stable homotopy type (cf. [10]). The conditions are also sufficient if $k=m/2-[(n+1)/2]=0 \mod (q-1)$ or $n+1=0 \mod 2q^{(k-1)/2}$. We can use structures of $J$-groups of suspensions of stunted lens spaces mod $q$ to obtain the conditions (see [14]). The object of this paper is to study the stable homotopy types of stunted lens spaces $L^s_m/L^s_q$ for $q=4$ or 8.

In order to state our results, we prepare functions $h_1$, $h_2$, $\alpha$, $\beta_1$, $\beta_2$, and $\gamma_1$ defined by

$$h_1(k)=\begin{cases} [k/4]+[(k+7)/8]+[(k+4)/8] & (k \geq 2) \\ 0 & (1 \geq k \geq 0) \end{cases}.$$
\[ h_2(k) = \begin{cases} \lfloor k/4 \rfloor + \lfloor (k+7)/8 \rfloor + \lfloor k/8 \rfloor + 1 & (k \geq 4) \\ \alpha(k, n) & (3 \leq k \leq 0) \end{cases} \]

\[ \alpha(k, n) = \begin{cases} 1 & (n \equiv 0 \pmod{2} \text{ and } k \equiv 1 \pmod{8}), \\ 0 & \text{otherwise}. \end{cases} \]

\[ \beta_1(k, n) \] is equal to the corresponding integer in the following table:

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<th>( k \pmod{8} )</th>
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\[ \beta_2(k, n) \] is equal to the corresponding integer in the following table:

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\( \gamma_1(m, n) \) is equal to the corresponding integer in the following table:
Let $v_p(s)$ denote the exponent of the prime $p$ in the prime power decomposition of $s$.

**Theorem 1.** If $v_2(t) \geq h_1(m-2 \cdot [(n+1)/2]) + 1 - \alpha(m-2 \cdot [(n+1)/2], n)$, then $L_i^p/L_i$ and $L_i^{p+t}/L_i^{p+t}$ have the same stable homotopy type.

**Theorem 2.**

1. If $L_i^p/L_i$ and $L_i^{p+t}/L_i^{p+t}$ have the same stable homotopy type, then

2. Suppose $h_1(m-2 \cdot [(n+1)/2]) - \alpha(m-2 \cdot [(n+1)/2], n) \geq i \geq 3$ and $\max \{v_2(n+1), v_2(m+1)\} \geq i$. If $L_i^p/L_i$ and $L_i^{p+t}/L_i^{p+t}$ have the same stable homotopy type, then $v_2(t) \geq i+1$.

3. Suppose $n < m \leq n+6$. If $L_i^p/L_i$ and $L_i^{p+t}/L_i^{p+t}$ have the same stable homotopy type, then

$$v_2(t) \geq [m/2] - [(n+1)/2] + \gamma_1(m, n).$$

**Remark.** It follows from Theorems 1 and 2, that we have obtained necessary and sufficient conditions for spaces $L_i^p/L_i$ and $L_i^{p+t}/L_i^{p+t}$ to have the same stable homotopy type if one of the following conditions is satisfied:

1. $n < m \leq 2 \cdot [(n+1)/2] + 3$,
2. $\beta_1(m-2 \cdot [(n+1)/2], n) = 1$,
3. $\max \{v_2(n+1), v_2(m+1)\} \geq h_1(m-2 \cdot [(n+1)/2]) - \alpha(m-2 \cdot [(n+1)/2], n)$ and $m \geq n+5$.

**Theorem 3.** If $v_2(t) \geq h_2(m-2 \cdot [(n+1)/2]) + 1 - \alpha(m-2 \cdot [(n+1)/2], n)$, then
$L^n_6/L^+_6$ and $L^{n+1}_6/L^{n+1}_6$ have the same stable homotopy type.

**Theorem 4.** (1) Suppose $m \geq n+5$. If $L^n_6/L^+_6$ and $L^{n+1}_6/L^{n+1}_6$ have the same stable homotopy type, then
\[ \nu_2(t) \geq [m/2] - [(n+1)/2] + \beta_2(m-2([(n+1)/2], n)) \]

(2) Suppose $h_2(m-2([(n+1)/2]) - \alpha(m-2([(n+1)/2], n)) \geq i \geq 3$ and $\max \{\nu_2(n+1), \nu_2(m+1)\} \geq i$. If $L^n_6/L^+_6$ and $L^{n+1}_6/L^{n+1}_6$ have the same stable homotopy type, then $\nu_2(t) \geq i+1$.

(3) Suppose $n < m \leq n+6$. If $L^n_6/L^+_6$ and $L^{n+1}_6/L^{n+1}_6$ have the same stable homotopy type, then
\[ \nu_2(t) \geq [m/2] - [(n+1)/2] + \gamma_2(m, n), \]

where $\gamma_2(m, n)$ is the integer defined by
\[ \begin{align*}
\gamma_1(m, n) &= \begin{cases} 
1 & (n \equiv 0 \pmod 8 \text{ and } m = n+6) \\
\gamma_1(m, n) & \text{(otherwise)} 
\end{cases} 
\end{align*} \]

**Remark.** It follows from Theorems 3 and 4, that we have obtained necessary and sufficient conditions for spaces $L^n_6/L^+_6$ and $L^{n+1}_6/L^{n+1}_6$ to have the same stable homotopy type if one of the following conditions is satisfied:

(1) $n < m \leq [m/2] - [(n+1)/2] + 3$,
(2) $n \equiv 0 \pmod 8$ and $m = n + 6$,
(3) $\beta_2(m-2([(n+1)/2]), n) = 2$ or $2([(n+1)/2]) - [m/2] - [(n+3)/2] \equiv 2 \pmod 4$,
(4) $\max \{\nu_2(n+1), \nu_2(m+1)\} \geq h_2(m-2([(n+1)/2]) - \alpha(m-2([(n+1)/2], n))$ and $m \geq n+5$.

This paper is organized as follows. In section 2 we prepare some lemmas and recall known results. We prove Theorems 1, 2, 3 and 4 in the final section.

The author would like to express his gratitude to Professor Akie Tamamura and Professor Kenso Fujii for helpful suggestions.

2. Preliminaries

In this section we prepare some lemmas and recall known results which are needed to prove Theorems 1, 2, 3 and 4.

**Lemma 2.1.** Let $m, n, k$ and $s$ be non-negative integers with $2i \leq k < 2i+1$. Assume that \( \binom{m}{i} \equiv \binom{n}{i} \pmod 2 \) for $1 \leq i \leq k$. Then $m \equiv n \pmod {2^{i+1}}$.

**Proof.** Suppose that $m = \sum_{i=0}^{\gamma_0} a_i 2^i$, $n = \sum_{j=0}^{\gamma_0} b_j 2^j$ and $i = \sum_{j=0}^{\gamma_0} c_{i,j} 2^j (1 \leq i \leq k)$, where $a_i, b_j$ and $c_{i,j}$ are non-negative integers with $a_j \leq 1, b_j \leq 1$ and $c_{i,j} \leq 1 (1 \leq i \leq k, 1 \leq j \leq N)$. Then we have \( \binom{m}{i} \equiv \prod_{i=0}^{\gamma_0} \binom{a_i}{c_{i,j}} \pmod 2 \) and \( \binom{n}{i} \equiv \prod_{i=0}^{\gamma_0} \binom{b_j}{c_{i,j}} \pmod 2 \).
(\begin{bmatrix} b_j \\ c_{ij} \end{bmatrix} \pmod{2}) \text{ for } 1 \leq i \leq k. \text{ It follows from the hypothesis that we have } a_j = b_j \text{ for } 0 \leq j \leq s; \text{ that is, } m \equiv n \pmod{2^{s+1}}. \text{ q.e.d.}

Let \( q \geq 2 \) be an integer and \( q_0, q_1, \cdots, q_n \) be integers relatively prime to \( q \). Consider the \(( \mathbb{Z}/q)\)-action on the unit sphere \( S^{2n+1} \subset \mathbb{C}^{n+1} \) given by

\[
\exp(2\pi \sqrt{-1}/q)(x_0, \cdots, x_n) = (x_0 \cdot \exp(2q_0 \pi \sqrt{-1}/q), \cdots, x_n \cdot \exp(2q_n \pi \sqrt{-1}/q)).
\]

Then \( S^{2n+1}(q; q_0, \cdots, q_n) \) denotes the space \( S^{2n+1} \) with this action,

\[
L_q^{2n+1}(q_0, \cdots, q_n) = S^{2n+1}(q; q_0, \cdots, q_n)/(\mathbb{Z}/q)
\]

and \( L_q^{2n}(q_0, \cdots, q_n) \) is the subspace of \( L_q^{2n+1}(q_0, \cdots, q_n) \) defined by

\[
L_q^{2n}(q_0, \cdots, q_n) = \{ [x_0, \cdots, x_n] \in L_q^{2n+1}(q_0, \cdots, q_n) \mid x_n \text{ is real } \geq 0 \}.
\]

For \( 0 \leq n < m \leq 2l+1 \), we set

\[
L_q^n/L_q^0(q_0, \cdots, q_l) = L_q^0(q_0, \cdots, q_l)/L_q^0(q_0, \cdots, q_l),
\]

which is called a stunted lens space \( \pmod{q} \). Then we have

\[
(2.2) \quad L_q^n/L_q^0(1, \cdots, 1) = L_q^n/L_q^n.
\]

Considering the \(( \mathbb{Z}/q)\)-action on \( S^{2l+1}(q; q_0, \cdots, q_l) \times \mathbb{C}^k \) given by

\[
\exp(2\pi \sqrt{-1}/q)(x, w_0, \cdots, w_k) = (\exp(2\pi \sqrt{-1}/q) \cdot x, w_0 \cdot \exp(2a_0 \pi \sqrt{-1}/q), \cdots, w_k \cdot \exp(2a_k \pi \sqrt{-1}/q))
\]

for \( (x, w_0, \cdots, w_k) \in S^{2l+1}(q; q_0, \cdots, q_l) \times \mathbb{C}^k \), we have a complex \( k \)-dimensional vector bundle

\[
\eta(a_1, \cdots, a_k): (S^{2l+1}(q; q_0, \cdots, q_l) \times \mathbb{C}^k)/(\mathbb{Z}/q) \to L_q^{2l+1}(q_0, \cdots, q_l).
\]

We use same symbol for the restriction of \( \eta(a_1, \cdots, a_k) \) to \( L_q^n(q_0, \cdots, q_l) \) \( (n \leq 2l+1) \) and denote the complex line bundle \( \eta(1) \) by \( \eta \). Then we have

\[
(2.3) \quad \eta(a_1, \cdots, a_k) \simeq \eta^s \oplus \cdots \oplus \eta^{s_k}.
\]

Let \( E(\xi) \to X \) be a real vector bundle over a finite \( CW \)-complex \( X \) with disk bundle \( B(\xi) \) and sphere bundle \( S(\xi) \). Then the Thom complex \( X^\xi \) of \( \xi \) is defined as the quotient space \( B(\xi)/S(\xi) \). Define \( f: S^{2n+1} \times D^k \to S^{2k+2n+1} \) by

\[
f(x_0, \cdots, x_n, w) = (w, (1 - ||w||^2)^{1/2} x_0, \cdots, (1 - ||w||^2)^{1/2} x_n)
\]

for \( (x_0, \cdots, x_n, w) \in S^{2n+1} \times D^k \), where \( D^k \subset \mathbb{C}^k \) is the unit disk. Then we have

\[
(2.4) \quad \text{Let } a_1, \cdots, a_k \text{ be integers relatively prime to } q. \text{ Then } f \text{ induces the following homeomorphisms.}
\]
We define the function \( h(q, k) \) by setting
\[
(2.5) \quad h(q, k) = \text{ord} \langle J(r(\eta) - 2) \rangle,
\]
where \( J(r(\eta) - 2) \) is the image of \( r(\eta) - 2 \in K\tilde{O}(L^q) \) by the \( J \)-homomorphism \( J: K\tilde{O}(L^q) \to J(L^q) \).

**REMARK.** The function \( h(q, k) \) have been determined completely (cf. [8]).

Spaces \( X \) and \( Y \) are said to have the same stable homotopy type \( (X \cong Y) \) if there exist non-negative integers \( a \) and \( b \) such that \( S^a X \) and \( S^b Y \) have the same homotopy type. For stunted generalized lens spaces, we have

(2.6)  
(1) If \( mn \equiv 0 \pmod{2} \), then \( L^m | L^q(q_0, \ldots, q_{m/2}) \) and \( L^n | L^q \) have the same stable homotopy type. In particular, \( L^m | L^q(q_0, \ldots, q_{m/2}) \) and \( L^n \) have the same stable homotopy type.

(2) Let \( a_1, \ldots, a_k, b_1, \ldots, b_k, q_0, \ldots, q_n, r_0, \ldots, r_n \) and \( a \) be integers relatively prime to \( q \).

i) Assume \( q_0 \cdots q_n \equiv \pm a^{q+1} r_0 \cdots r_n \pmod{q} \). Then
\[
L^q | L^q(q_0, \ldots, q_{m/2}) \quad \text{and} \quad L^q | L^q(b_1, \ldots, b_k, r_0, \ldots, r_n)
\]
and \( L^q | L^q | L^q(b_1, \ldots, b_k, r_0, \ldots, r_n) \) have the same stable homotopy type.

ii) If \( k \equiv 0 \pmod{h(q, 2m+1)} \), then the spaces
\[
L^q | L^q(q_0, \ldots, q_{m/2}) \quad \text{and} \quad L^q | L^q(b_1, \ldots, b_k, r_0, \ldots, r_n)
\]
and \( L^q | L^q | L^q \) have the same stable homotopy type.

**Proof.** Set \( X = L^q(q_0, \ldots, q_n) \). According to [17], we have
\[
K(X) \simeq Z[\eta]/(\eta^q - 1, (\eta^{-1})^{q+1})
\]
and \( \text{ord} \tilde{K}(X) = q^n \). If \( a \) is an integer relatively prime to \( q \), then
\[
\bigcap_{i} (\sum j^{(i)}(\psi^i - 1) \tilde{K}(X)) = \sum_{i} (\bigcap_{i} j^{(i)}(\psi^i - 1) \tilde{K}(X))
\]
\[
\supseteq \bigcap_{i} a^i(\psi^a - 1) \tilde{K}(X) = (\psi^a - 1) \tilde{K}(X)
\]
\[
\subseteq (\psi^a - 1) (\eta - 1) = \eta^a - \eta.
\]
Since the Adams operations are compatible with the real restriction \( r: K \to KO \) [4],
\[
    r(\eta^a - \eta) = r(\eta^a) - r(\eta) \in \bigcap_j (\sum_j j^{(i)} (\psi^j - 1) \tilde{KO}(X)).
\]
By [2], [3] and [19], this implies that
\[
    J(r(\eta^a - 2)) = J(r(\eta) - 2)
\]
in \( J(X) \). If \( a_1, \ldots, a_k \) are integers relatively prime to \( q \),
\[
    J(r(\eta; a_1, \ldots, a_k)) - 2k = J(k(r(\eta)) - 2k)
\]
in \( J(X) \). Suppose that \( a_1, \ldots, a_k, b_1, \ldots, b_h, q_0, \ldots, q_n, r_0, \ldots, r_n \) and \( a \) be integers relatively prime to \( q \). According to [5, Proposition (2.6)], we have
\[
    L_q^{2n+2k+1}/L_q^{2k-1}(a_1, \ldots, a_k, q_0, \ldots, q_n) \cong L_q^{2n+2k+1}/L_q^{2k-1}(a, \ldots, a, q_0, \ldots, q_n)
\]
and
\[
    L_q^{2n+2k+1}/L_q^{2k-1}(b_1, \ldots, b_h, r_0, \ldots, r_n) \cong L_q^{2n+2k+1}/L_q^{2k-1}(1, \ldots, 1, r_0, \ldots, r_n).
\]
Suppose that \( q_0 \cdots q_n \equiv \pm a^{a+1} r_0 \cdots r_n \pmod{q} \). Identify \( S^{2n+2k+1} \) with the iterated join
\[
    S^1 \ast \cdots \ast S^1 = \{ \lambda_0 x_0 + \cdots + \lambda_{n+k} x_{n+k} \mid \sum_j \lambda_j = 1, \lambda_j \geq 0, x_j \in S^1 \}.
\]
Choose integers \( q_i (0 \leq i \leq n) \) with \( q_i q_i \equiv 1 \pmod{q} \). Denote the generator \( \exp(2\pi i/|q|) \) of \( \mathbb{Z}/q \) by \( g \). Then, the map
\[
    f: S^{2n+2k+1}(q; a, \ldots, a, q_0, \ldots, q_n) \to S^{2n+2k+1}(q; 1, \ldots, 1, r_0, \ldots, r_n)
\]
defined by
\[
    f = 1 \ast \cdots \ast 1 \ast (ar_0 q_0) \ast \cdots \ast (ar_n q_n),
\]
is a map of degree \( a^{a+1} r_0 \cdots r_n q_0 \cdots q_n = \pm 1 \pmod{q} \) with \( f g = g^a f \). Modify \( f \) to get a map \( h \) of degree \( \pm 1 \) with \( h g = g^a h \) and
\[
    h|S^{2k-1}(q; a, \ldots, a) = f|S^{2k-1}(q; a, \ldots, a)
\]
(see the proof of [6, (29.4)] for the detail). Then \( h \) induces a homotopy equivalence
\[
    \tilde{h}: L_q^{2n+2k+1}/L_q^{2k-1}(a, \ldots, a, q_0, \ldots, q_n) \to L_q^{2n+2k+1}/L_q^{2k-1}(1, \ldots, 1, r_0, \ldots, r_n).
\]
This completes the proof of (2) i).

Now we turn to the proof of (1). Since \( J(X) \) has a finite order, we may assume that \( k > 1 \). Let \( a \) be an integer with \( a q_0 \cdots q_n \equiv 1 \pmod{q} \). Then
By the proof above, there exists an equivariant map

\[ h: S^{2n+2k+1}(q; 1, \ldots, 1) \to S^{2n+2k+1}(q; 1, \ldots, 1, a, q_0, \ldots, q_n) \]

of degree 1, which induces homotopy equivalences

\[ h: L^{2n+2k+1}/L^{2k}_q(a_1, \ldots, a_k, q_0, \ldots, q_n) \to L^{2n+2k+1}/L^{2k}_q(1, \ldots, 1, a, q_0, \ldots, q_n) \]

and

\[ h|_{L^{2n+2k}/L^{2k}_q}: L^{2n+2k}/L^{2k}_q(1, \ldots, 1, a, q_0, \ldots, q_n) \to L^{2n+2k}/L^{2k}_q(a_1, \ldots, a_k, q_0, \ldots, q_n) \]

Thus \( L^{2n+2k+1}/L^{2k}_q \) is stably homotopically equivalent to \( L^{2n+2k+1}/L^{2k}_q \) and \( L^{2n+2k}/L^{2k}_q \) is stably homotopically equivalent to \( L^{2n+2k}/L^{2k}_q \).

The equivariant map

\[ f = 1 * \cdots * 1 * q_0 * \cdots * q_n: S^{2n+2k+1}(q; 1, \ldots, 1) \to S^{2n+2k+1}(q; 1, \ldots, 1, a, q_0, \ldots, q_n) \]

induces a homotopy equivalence

\[ f|_{L^{2n+2k}/L^{2k-1}_q}: L^{2n+2k}/L^{2k-1}_q(1, \ldots, 1, a, q_0, \ldots, q_n) \to L^{2n+2k}/L^{2k-1}_q(1, \ldots, 1, a, q_0, \ldots, q_n) \]

Thus \( L^{2n+2k}/L^{2k-1}_q \) is stably homotopically equivalent to \( L^{2n+2k}/L^{2k-1}_q \). This completes the proof of (1).

Finally, we prove the part ii) of (2). Suppose that \( k \equiv 0 \pmod{h(q, 2n+1)} \). Since the order of \( f(\eta) - 2 \in \tilde{f}(X) \) coincides with \( h(q, 2n+1) \), we have

\[ L^{2n+2k+1}/L^{2k-1}_q(a_1, \ldots, a_k, q_0, \ldots, q_n) \]

\[ \cong L^{2n+2k+1}/L^{2k-1}_q(1, \ldots, 1, q_0, \ldots, q_n) \]

\[ \cong (L^{2n+1}(q_0, \ldots, q_n))^{2k} \]

\[ \cong S^{2k} L^{2n+1}(q_0, \ldots, q_n) \cup S^{2k} \]

\[ \cong (L^{2n+1} S^{2k} \cong (L^{2n+1})^{(2k)} \cong L^{2n+2k+1}/L^{2k-1}_q, \]

by [5, Proposition (2.6)], (2.4) and (2.6) (1). This completes the proof of the part ii) of (2).

q.e.d.

Let \( X \) and \( Y \) be pointed CW-complexes. The stable homotopy group from \( X \) to \( Y \) is defined by

\[ \{X, Y\} = \lim_{\longrightarrow} [S^n X, S^n Y], \]

The following assertion is proved by making use of Puppe exact sequences.

(2.8) (1) If \( k > m \), then \( \{L^m/L^n(q_0, \ldots, q_{(m/2)}), S^k\} \cong 0. \)
(2) If \( k \leq n \), then \( \{S^k, L^n_q/L^n_q(q_0, \cdots, q_{(m/2)}) \} \cong 0 \).

(3) Suppose that \( [m/2] > [(n+1)/2] \). Then we have

i) \( \{S^{n+1}, L^n_q/L^n_q(q_0, \cdots, q_{(m/2)}) \} = \langle \{i_{n+1} \} \rangle \cong \{ \mathbb{Z} \ (n: \text{odd}) \}
\mathbb{Z}/q \ (n: \text{even}) \),

where \( i_{n+1} \) is the composition

\[ S^{n+1} \cong L^{n+1}_q/L^n_q(q_0, \cdots, q_{(m/2)}) \subseteq L^n_q/L^n_q(q_0, \cdots, q_{(m/2)}). \]

ii) \( \{L^n_q/L^n_q(q_0, \cdots, q_{(m/2)}), S^m \} = \langle \{p_m \} \rangle \cong \{ \mathbb{Z} \ (m: \text{odd}) \}
\mathbb{Z}/q \ (m: \text{even}) \),

where \( p_m \) is the composition

\[ L^n_q/L^n_q(q_0, \cdots, q_{(m/2)}) \rightarrow L^n_q/L^{n-1}_q(q_0, \cdots, q_{(m/2)}) \cong S^m. \]

The following lemma implies that the family of stunted lens spaces is closed under S-duality.

**Lemma 2.9.** Suppose that \( k = 2[m/2] + 1 - 2[(n+1)/2] \geq 3 \), \( N \equiv 0 \ (\text{mod} \ 2h(q,k)) \) and \( N > m + 1 \). Then the S-dual of

\[ L^n_q/L^n_q(a_1, \cdots, a_{(n+1)/2}, q_0, \cdots, q_{(m/2)}) \]

is \( L^{N-n-2}/L^{N-m-2}(b_1, \cdots, b_{(N-m-1)/2}, q_0, \cdots, q_{(m/2)}) \).

**Proof.** If \( nm \equiv 1 \ (\text{mod} \ 2) \), then the S-dual of

\[ X = L^n_q/L^n_q(a_1, \cdots, a_{(n+1)/2}, q_0, \cdots, q_{(m/2)}) \]

\[ \cong (L^n_q(q_0, \cdots, q_{(n+1)/2}))^{((n+1)/2)r(q)} \]

is \((L^n_q(q_0, \cdots, q_{(n+1)/2}))^{-\tau - ((n+1)/2)r(q)}\), where \( \tau \) denotes the tangent bundle of \( L^n_q(q_0, \cdots, q_{(n+1)/2}) \) (cf. [5, Theorem (3.5)]). Since

\[ \tau + 1 \cong r(q_0, \cdots, q_{(n+1)/2}) \]

it follows from [5, Proposition (2.6)] and (2.6) that

\[ (L^n_q(q_0, \cdots, q_{(n+1)/2}))^{-(n+1)/2)r(q)} \]

\[ \cong (L^n_q(q_0, \cdots, q_{(n+1)/2}))^{(N/2)r(q)} - ((n+1)/2)r(q) - ((n+1)/2)r(q)} \]

\[ = (L^n_q(q_0, \cdots, q_{(n+1)/2}))^{((N-m-1)/2)r(q)} \]

\[ \cong L^{N-n-2}/L^{N-m-2}(b_1, \cdots, b_{(N-m-1)/2}, q_0, \cdots, q_{(n+1)/2}) = Y. \]

This implies that there are positive integers \( a, b \) and \( M \) such that the \( M \)-dual of \( S^a X \) is \( S^a Y \). Let

\[ u: S^a X \wedge S^b Y \rightarrow S^M \]
be a $M$-duality pairing. Then the homomorphism
\[ \Gamma^i: \{S^i, S^sX\} \to \{S^{i+s}Y, S^m\} \]
defined by $\Gamma^i(\{f\}) = \{S^i w(f \wedge 1)\}$ for $f: S^{i+s} \to S^{i+s} X$, is an isomorphism. It follows from (2.8) that $M=a+b+N-1$ and
\[ \{S^{s+1}, X\} = \langle \{i_{s+1}\} \rangle \cong \{Y, S^{N-s-2}\} = \langle \{p_{N-s-2}\} \rangle \cong \mathbb{Z}. \]
Hence, by the isomorphism
\[ \{S^{s+1}, X\} \cong \{S^{s+s+1}, S^sX\} \xrightarrow{\Gamma^s+i} \{S^{s+b+N-1} Y, S^{s+s+N-1}\} \cong \{Y, S^{N-s-2}\}, \]
\[ \{i_{s+1}\} \] corresponds to either $\{p_{N-s-2}\}$ or $-\{p_{N-s-2}\}$. This implies that there exists a homotopy commutative diagram
\[
\begin{array}{ccc}
S^{t+s+N+1} \wedge S^s Y & \xrightarrow{1 \wedge S^t(p_{N-s-2})} & S^{t+s+N+1} \wedge S^{t+N+s-2} \\
\downarrow & & \downarrow \\
S^{t+s} (i_{s+1}) \wedge 1 & \approx & S^t u \\
\downarrow & & \downarrow \\
S^{t+s} X \wedge S^s Y & \xrightarrow{1} & S^{t+s+N-1},
\end{array}
\]
where $v$ and $S^t u$ are $(l+a+b+N-1)$-duality pairings. It follows that $C_{S^{t+s}(i_{s+1})}$ is the $(l+a+b+N)$-dual of $C_{S^t(p_{N-s-2})}$. Since
\[ C_{S^{t+s}(i_{s+1})} \cong S^{t+s}(L^m_q/L^{n+1}_q(a_1, \ldots, a_{(s+1)/2}, q_0, \ldots, q_{[s/2]})) \]
and
\[ C_{S^t(p_{N-s-2})} \cong S^{t+1}(L^{N-s-3}_q/L^{N-s-2}_q(b_1, \ldots, b_{(N-s-1)/2}, q_0, \ldots, q_{[s/2]})), \]
this implies that $L^m_q/L^{n+1}_q$ is the $S$-dual of $L^{N-s-3}_q/L^{N-s-2}_q$. In the similar way it is shown that $L^{N-s-2}_q/L^{m-1}_q$ is the $S$-dual of $L^{m-1}_q/L^{n}_q$. Using this fact, in the similar way it is shown that $L^{m-1}_q/L^{n+1}_q$ is the $S$-dual of $L^{N-s-3}_q/L^{N-s-2}_q$. q.e.d.

REMARK. The partial results for the case where $q$ is a prime of this lemma have been obtained in [18].

It follows from (2.6) and Lemma 2.9 that, in the following cases,
\[ L^m_q/L^{n+1}_q(a_1, \ldots, a_{(s+1)/2}, q_0, \ldots, q_{[s/2]}), \]
and $L^m_q/L^n_q$ have the same stable homotopy type:
(1) $q=2, 3, 4$ or 6,
(2) $mn \equiv 0 \pmod{2}$,
(3) $q^0 \cdots q^{[m/2]-(s+1)/2} \equiv \pm q^{[m/2]-(n-1)/2} \pmod{q}$. 

(4) \( n+1 \equiv 0 \pmod{2h(q, m-n-1)} \),
(5) \( m+1 \equiv 0 \pmod{2h(q, m-n-1)} \).

**Question.** Is it true that \( L^n_q/L^n_q \) and \( L^{n+1}_q/L^{n+1}_q \) have the same stable homotopy type for any case? If it is not, then how many stable homotopy types are there for fixed \( m, n \) and \( q \)?

From now on, we restrict ourselves to standard stunted lens spaces. According to [5, Propositions (2.6) and (2.9)], (2.4) and Lemma 2.9, we obtain the following.

(2.10) Set \( k = m-2 \lceil (n+1)/2 \rceil \) and \( l = 2 \lfloor m/2 \rfloor - n \).

(1) If \( t \equiv 0 \pmod{2h(q, k)} \), then \( L^n_q/L_q^t \) and \( L^{n+1}_q/L^{n+1}_q \) have the same stable homotopy type.

(2) If \( k \geq 2 \) and \( n+1 \equiv 0 \pmod{2h(q, k)} \), then \( t \equiv 0 \pmod{2h(q, k)} \) if and only if \( L^n_q/L_q^t \) and \( L^{n+1}_q/L^{n+1}_q \) have the same stable homotopy type.

(3) If \( t \equiv 0 \pmod{2h(q, l)} \), then \( L^n_q/L_q^t \) and \( L^{n+1}_q/L_q^{n+1} \) have the same stable homotopy type.

(4) If \( l \geq 2 \) and \( m+1 \equiv 0 \pmod{2h(q, l)} \), then \( t \equiv 0 \pmod{2h(q, l)} \) if and only if \( L^n_q/L_q^t \) and \( L^{n+1}_q/L_q^{n+1} \) have the same stable homotopy type.

(2.11) (1) ([12, I; Theorem 1.1]) Let \( p \) be a prime and \( r \) a positive integer with \( p^r > 2 \). Suppose that \( k = m-2 \lceil (n+1)/2 \rceil \geq 2 \). Then \( t \equiv 0 \pmod{2p^{(k-2)/2(p-1)}} \) if \( L^n_q/L_q^t \) and \( L^{n+1}_q/L_q^{n+1} \) have the same stable homotopy type.

(2) Let \( r \geq 2 \) be a positive integer and set \( k = m-2 \lceil (n+1)/2 \rceil \). Then \( \nu_q(t) \geq \lceil k/2 \rceil + \beta_q(k, n) \) if \( L^n_q/L_q^t \) and \( L^{n+1}_q/L_q^{n+1} \) have the same stable homotopy type, where \( \beta_q \) is the function defined by (1.5).

(3) Suppose that \( q \equiv 0 \pmod{2} \) and \( m \geq n+2 \). Then \( \nu_q(t) \geq \lceil \log_2 (m-n-1) \rceil \) if \( L^n_q/L_q^t \) and \( L^{n+1}_q/L_q^{n+1} \) have the same stable homotopy type.

**Proof.** Suppose that \( q \equiv 0 \pmod{2} \) and \( m \geq n+2 \). It is well known that

\[
H^*(L^n_q; \mathbb{Z}/2) \cong \begin{cases} \langle (\mathbb{Z}/2) [u]/(u^{m+1}) \rangle & (q \equiv 2 \pmod{4}) \\
\langle (\mathbb{Z}/2) [u, v]/(u^2, v^{m/2+1}, uv^{m-1/2}) \rangle & (q \equiv 0 \pmod{4}) \end{cases},
\]

where \( \deg u = 1 \) and \( \deg v = 2 \). The action of the Steenrod squares is given by

\[
\begin{align*}
S_q^0(u^i) &= \left( \begin{array}{c} i \\ 0 \end{array} \right) u^{i+t} & (q \equiv 2 \pmod{4}) \\
S_q^0(v^i) &= \left( \begin{array}{c} i \\ 0 \end{array} \right) v^{i+t} & (q \equiv 0 \pmod{4}) \\
S_q^0(uv^i) &= \left( \begin{array}{c} i \\ 0 \end{array} \right) uv^{i+t} & (q \equiv 0 \pmod{4}) \\
S_q^{2i+1} &= 0 & (q \equiv 0 \pmod{4}).
\end{align*}
\]
Assume that $L^n_q/L^n_r$ and $L^{n+t}_q/L^{n+t}_r$ have the same stable homotopy type. Then $t \equiv 0 \pmod{2}$. It follows from the naturality of the Steenrod squares that we have

$$
\begin{align*}
\left\{ \begin{array}{l}
\binom{n+1}{i} \equiv \binom{n+t+1}{i} \pmod{2} \\
\binom{[(n+1)/2]}{i} \equiv \binom{[(n+t+1)/2]}{i} \pmod{2}
\end{array} \right. \\
(1 \leq i \leq m-n-1, q \equiv 2 \pmod{4}) \\
(2 \leq 2i \leq m-n-1, q \equiv 0 \pmod{4}).
\end{align*}
$$

Let $s$ be the integer with $2^s \leq m-n-1 < 2^{s+1}$. By Lemma 2.1, (2.12) implies that $v_2(t) \geq s+1 = \log_2 (m-n-1)$. This completes the proof of (3).

(2) It follows from [11, Theorem 1.1] that

$$
v_2(t) \geq \frac{m}{2} - \frac{(n+1)}{2} + C(m, n)
$$

if $L^n_q/L^n_r$ and $L^{n+t}_q/L^{n+t}_r$ have the same stable homotopy type, where $C(m, n)$ is the function defined by

$$
C(m, n) = \begin{cases} 
1 & (n \equiv 1, 5 \text{ or } 6 \pmod{8} \text{ and } m \equiv 1, 4 \text{ or } 5 \pmod{8}) \\
0 & \text{(otherwise)}.
\end{cases}
$$

Then (2) is obtained by making use of the S-duality (Lemma 2.9).

Proposition 2.15 ([15, Theorem 3]). Let $j, m$ and $n$ be non-negative in-
tgers with \( m > n \) and \( j \equiv n+1 \equiv 0 \pmod{4} \). Then we have

\[
J(S^i(L^n_\mathbb{L}/L_\mathbb{L})) \simeq \begin{cases} 
\mathbb{Z}/m((n+j+1)/2) \cdot 2^i \oplus \mathbb{Z}[2^{i+1}] \oplus \mathbb{Z}[2^i] & (b(j, m, n) \geq 0) \\
\mathbb{Z}/m((n+j+1)/2) & (b(j, m, n) < 0),
\end{cases}
\]

where \( i, k, c \) and \( d \) are integers defined by

\[
i = \begin{cases} 
\min \{ \nu_2(n+1) - 1, a(j, m, n) \} & (n+j \equiv 7 \pmod{8}) \\
\min \{ \nu_2(n+1), a(j, m, n) \} & (n+j \equiv 3 \pmod{8})
\end{cases}
\]

\[
k = \min \{ \nu_2(n+1) - 1, b(j, m, n) \}
\]

\[
c = \max \{ a(j, m, n) - i, b(j, m, n) - k \}
\]

\[
d = \min \{ a(j, m, n) - i, b(j, m, n) - k \}
\]

In order to state the next proposition, we set

\[
a_a(m, n) = [(m-2)/8] - [(n+5)/8]
\]

\[
a_b(m, n) = [m/8] - [(n+7)/8]
\]

\[
a_c(m, n) = a_a(m, n) + [m/8] - [(m-4)/8]
\]

\[
a_d(m, n) = [(m+4)/8] + [(n-2)/8] - [(n+1)/4].
\]

\[
b_a(j, m, n) = \min \{ \nu - a_a(n, n+4), a_a(m, n) \}
\]

\[
b_b(j, m, n) = \min \{ \nu - a_a(n-5, n), a_a(m, n) \}
\]

\[
b_c(j, m, n) = \min \{ \nu+1, a_a(m, n) \}
\]

where \( \nu \) is the integer defined in (2.14).

**Proposition 2.19** ([16, Theorem 2]). Let \( j, m \) and \( n \) be non-negative integers with \( m > n \) and \( j \equiv 0 \pmod{8} \).

1. If \( n \equiv 3 \pmod{4} \) and \( m \geq 2 [(n+6)/8] + 2[n/8] + 4[(n+15)/8] \), then we have

\[
J(S^i(L^n_\mathbb{L}/L_\mathbb{L})) \simeq \begin{cases} 
\oplus_{i=1}^{3} \mathbb{Z}/2^{i(j,m,n)} \oplus \mathbb{Z}/2 & (n \equiv 2 \pmod{8}) \\
\oplus_{i=1}^{3} \mathbb{Z}/2^{i(j,m,n)} & (otherwise)
\end{cases}
\]

2. If \( n \equiv 3 \pmod{4} \) and \( 2 [(n+6)/8] + 2[n/8] + 4 [(n+15)/8] > m > n \), then we have

\[
J(S^i(L^n_\mathbb{L}/L_\mathbb{L})) \simeq \begin{cases} 
\mathbb{Z}[2^{2j+1}/2^{i+1}] \oplus \mathbb{Z}[2^{i+1}] \oplus \mathbb{Z}[2^{i}] \oplus \mathbb{Z}/2 & (n \equiv 2 \pmod{8} \text{ and } m \geq n+6) \\
\mathbb{Z}/8 & (n \equiv 1 \pmod{8} \text{ and } m \geq n+3)
\end{cases}
\]

3. If \( n \equiv 3 \pmod{4} \) and \( m \geq n+5 \), then we have

\[
J(S^i(L^n_\mathbb{L}/L_\mathbb{L})) \simeq \mathbb{Z}/m((n+j+1)/2) \cdot 2^i \oplus \mathbb{Z}/2^{i+1} \oplus \mathbb{Z}/2^i \oplus \mathbb{Z}/2^i(j,m,n),
\]

where \( i_1, i_2, c_1 \) and \( c_2 \) are integers defined by
If \( n \equiv 3 \pmod{4} \) and \( n+5 > m > n \), then we have

\[
\mathcal{F}(L_b^m | L_b^n) \cong \begin{cases} 
\mathbb{Z}/m((n+j+1)/2) \oplus \widetilde{KO}(S^j(L_b^n | L_b^{n+1})) & (m \geq n+2) \\
\mathbb{Z}/m((n+j+1)/2) & (m = n+1).
\end{cases}
\]

In order to state the next proposition, we set

\[
\begin{pmatrix}
\sigma_i' = \eta^i - 1 \\
\sigma_i = \sigma_i' \sigma_2 \\
\sigma_{2i+1} = \sigma_{2i} \sigma_1
\end{pmatrix}
\]

(1)

(2) Let \( F(x) \) denote the free abelian group generated by \( x_1, x_2, x_3, x_4, x_5, x_6 \) and \( x_7 \). Then \( X_1(n) \) and \( X_2(n) \) (1 \( \leq i \leq 7 \), \( n \geq 0 \)) denote the elements of \( F(x) \) defined by

\[
\begin{align*}
X_1(n) &= 4x_1 + 2x_2 + 2x_3 + x_7, \\
X_2(n) &= 2x_2 + x_6, \quad X_3 = 2x_3 + x_7, \quad X_6 = x_6 + x_7, \quad X_i = x_i (i = 4, 5 \text{ or } 7),
\end{align*}
\]

(3) Let \( \varphi : F(x) \rightarrow K(L_b^n) \) be the homomorphism defined by setting \( \varphi(x_i) = \sigma_i \) (1 \( \leq i \leq 7 \)).

**Proposition 2.22** (Kobayashi and Sugawara [12]). The homomorphism \( \varphi \) is an epimorphism, and the kernel of \( \varphi \) coincides with the subgroup of \( F(x) \) generated by \( \{X_i(m) | 1 \leq i \leq 7 \} \).

For each integer \( n \) with \( 0 \leq n < m \), we denote the inclusion map of \( L_b^n \) into \( L_b^m \).
\( L^n_8 \) by \( i^n_8 \), and denote the kernel of the homomorphism
\[
(i^n_8)^*: \tilde{K}(L^n_8) \rightarrow \tilde{K}(L^n_8)
\]
by \( V_n \). Set \( u = [(n + 1)/2] \) and \( S_i = \varphi(X_i(2u)) (1 \leq i \leq 7) \). Then \( V_{2u} \) is the subgroup of \( \tilde{K}(L^n_8) \) generated by \( S_i (1 \leq i \leq 7) \), and we have
\[
\tilde{K}(L^n_8)/\tilde{K}(L^n_8) \cong \begin{cases} V_{2u} & (n \equiv 0 \text{ (mod 2)}) \\ \mathbb{Z} \oplus V_{2u} & (n \equiv 1 \text{ (mod 2)}) \end{cases}.
\]
According to [1], we have the following lemma.

**Lemma 2.23.** The Adams operations are given by the following formulae, where \( s_i = \varphi(X_i) (1 \leq i \leq 7) \) and \( k \equiv 1 \text{ (mod 2)} \).

1. \( \psi^k(s_i) = s_i \) \( (i = 1, 2 \text{ or } 4) \).
2. \( \psi^k(s_3) = \begin{cases} s_3 & (k \equiv 1 \text{ (mod 4)}) \\ -s_3 - 2s_2 & (k \equiv 3 \text{ (mod 4)}) \end{cases} \).
3. \( \psi^k(s_5) = \begin{cases} s_5 & (k \equiv 1 \text{ (mod 4)}) \\ s_5 + s_6 & (k \equiv 3 \text{ (mod 4)}) \\ -s_5 - 2s_4 & (k \equiv 5 \text{ (mod 4)}) \\ -s_5 - 2s_4 - s_6 & (k \equiv 7 \text{ (mod 4)}) \end{cases} \).
4. \( \psi^k(s_6) = \begin{cases} s_6 & (k \equiv 1 \text{ (mod 8)}) \\ -s_6 & (k \equiv 3 \text{ (mod 8)}) \end{cases} \).
5. \( \psi^k(s_7) = \begin{cases} s_7 & (k \equiv 1 \text{ (mod 8)}) \\ -s_7 + 2s_4 & (k \equiv 3 \text{ (mod 8)}) \\ s_7 - 2s_6 & (k \equiv 5 \text{ (mod 8)}) \\ -s_7 + 2s_4 + 2s_6 & (k \equiv 7 \text{ (mod 8)}) \end{cases} \).

3. **Proofs of Theorems**

In this section we prove Theorems 1, 2, 3 and 4.

3.1. **Proof of Theorems 1 and 3.** According to [8], we have
\[
h(q, k) = \begin{cases} 2^{h(q,k)} & (q = 4) \\ 2^{h(q,k)} & (q = 8) \end{cases}.
\]
Then Theorems 1 and 3 follow from (2.10) for the case \( m \geq n + 3 \). Note that we have \( L^{n+1}_q/L^n_q \cong S^{*+1} \) and
\[
L^{n+2}_q/L^n_q \cong \begin{cases} S^{*+1} \setminus S^{*+2} & (n \equiv 1 \text{ (mod 2)}) \\ S^n L^2_q & (n \equiv 0 \text{ (mod 2)}) \end{cases}.
\]
This completes the proof of Theorems 1 and 3.

3.2. Proof of Theorem 2. The part (1) is obtained by (2.11) (2). Suppose that the spaces $L^n_s/L^s_t$ and $L^{n+s}_t/L^{s+t}_t$ have the same stable homotopy type, where $\nu(n+1) \geq i$ and $h(m-2 [(n+1)/2]) \geq i \geq 3$. Then $m \geq n+5$ and there exists a homotopy equivalence

$$f: S^i(L^n_s/L^s_t) \to S^{i-t}(L^{n+s}_t/L^{s+t}_t),$$

which induces an isomorphism

$$(3.1) \quad J(f^i): J(S^{i-t}(L^{n+s}_t/L^{s+t}_t)) \to J(S^i(L^n_s/L^s_t)).$$

We can assume that $\nu_d(j) \geq h_1(m-2 [(n+1)/2]) = h_1(m-n-1)$. By (2.11) (3), $t \equiv 0 \pmod{8}$. It follows from (2.11) (1) and Proposition 2.15 that we have $\nu_d(t) \equiv [(m-n-1)/2]$, $J(S^i(L^n_s/L^s_t)) \cong \mathbb{Z} \cdot m((n+j+1)/2) \cdot 2^i \oplus \mathbb{Z}/2^{d_1+i} \oplus \mathbb{Z}/2^{d_1}$ and

$$J(S^{i-t}(L^{n+s}_t/L^{s+t}_t)) \cong \mathbb{Z} \cdot m((n+j+1)/2) \cdot 2^i \oplus \mathbb{Z}/2^{d_1} \oplus \mathbb{Z}/2^{d_2},$$

where $i_1 = \min \{\nu_2(n+1)-1, h_1(m-n-1)\}$,

$$i_2 = \min \{\nu_2(n+t+1)-1, h_1(m-n-1)\},$$

$$k_1 = \min \{\nu_2(n+1)-1, [(m-n-1)/8] + [(m-n+5)/8]\},$$

$$k_2 = \min \{\nu_2(n+t+1)-1, [(m-n-1)/8] + [(m-n+5)/8]\},$$

$$c_l = h_1(m-n-1) - i_l \quad (l = 1 \text{ or } 2)$$

and $d_l = [(m-n-1)/8] + [(m-n+5)/8] - k_l \quad (l = 1 \text{ or } 2)$. Since $c_l \geq d_l \geq 0$ and $\nu_d(m((n+j+1)/2)) \geq i \geq k_l \quad (l = 1 \text{ or } 2)$, the isomorphism (3.1) implies that $c_1 = c_2$, and hence

$$\min \{\nu_2(n+1)-1, h_1(m-n-1)\} = \min \{\nu_2(n+t+1)-1, h_1(m-n-1)\}.$$ 

Since $\nu_2(n+1) \geq i$, this implies that we have $\nu_2(n+t+1) > i$ if $\nu_2(n+1) > i$ and $\nu_2(n+t+1) = i$ if $\nu_2(n+1) = i$. Thus we have $\nu_2(t) \geq i+1$. The proof of the part (2) of Theorem 2 is completed by making use of the $S$-duality (Lemma 2.9).

The part (3) is obtained by (2.11) (3) and the parts (1) and (2) of Theorem 2. This completes the proof of Theorem 2.

3.3. Proof of Theorem 4. Suppose that the spaces $L^n_s/L^s_t$ and $L^{n+s}_t/L^{s+t}_t$ have the same stable homotopy type. Then there exists a homotopy equivalence

$$f: S^i(L^n_s/L^s_t) \to S^{i-t}(L^{n+s}_t/L^{s+t}_t),$$

which induces isomorphisms
(3.2) \( f^*: \tilde{K}(S^{j-t}(L_8^{m+t}/L_8^{n+t})) \to \tilde{K}(S^t(L_8^n/L_8)) \)
and
(3.3) \( J(f^*): J(S^{j-t}(L_8^{m+t}/L_8^{n+t})) \to J(S^t(L_8^n/L_8)) \).

We can assume that \( v_2(j) \geq a_0(m, n) + 1 \). Suppose \( m \equiv n + 5 \). Then, by (2.11) (3), \( t \equiv 0 \pmod{8} \). If \( n \equiv 3 \pmod{4} \), then Proposition 2.19 asserts that the exponent of the group \( J(S^t(L_8^n/L_8)) \) is equal to \( 2^{t(j-i, m-n)} \) and the exponent of the group \( J(S^{j-t}(L_8^{m+t}/L_8^{n+t})) \) is equal to \( 2^{t(j-i, m-n+t)} \), and the isomorphism (3.3) implies that \( b_t(j, m, n) = b_t(j-t, m+t, n+t) \). In the case \( n \equiv 3 \pmod{4} \), \( f \) induces a homotopy equivalence
\[
f: S^t(L_8^n/L_8^{n+1}) \to S^{j-t}(L_8^{m+t}/L_8^{n+t+1}),
\]
which induces an isomorphism
\[
J(f^*): J(S^{j-t}(L_8^{m+t}/L_8^{n+t+1})) \to J(S^t(L_8^n/L_8^{n+1})).
\]

Since \( b_t(j, m, n) = b_t(j, m, n+1) \) and \( b_t(j-t, m+t, n+t) = b_t(j-t, m+t, n+t+1) \), the isomorphism \( J(f^*) \) implies that \( b_t(j, m, n) = b_t(j-t, m+t, n+t) \). In either case, we have \( b_t(j, m, n) = b_t(j-t, m+t, n+t) \). Hence \( v_2(j-t) - a_0(n, n+4) \equiv a_0(m, n) \), and
\[
v_2(t) \equiv \begin{cases} a_0(m, n) - 2 & (n \equiv 0, 1 \text{ or } 7 \pmod{8}) \\ a_0(m, n) - 1 & \text{(otherwise)}. \end{cases}
\]

By Lemma 2.9, (2.11) (2) and (3.4), the part (1) of Theorem 4 is obtained except for the case \( n \equiv m-n \equiv 2 \pmod{8} \). So, assume that \( n \equiv m-2 \equiv 2 \pmod{8} \). Let \( Y_i \) be the element of \( \tilde{K}(S^t(L_8^n/L_8)) \), which corresponds to \( I^{t/2}(S_i) \) by the isomorphism
\[
(p_t^n)^*: \tilde{K}(S^t(L_8^n/L_8)) \xrightarrow{\cong} I^{t/2}(V_5),
\]
where \( I \) denotes the Bott periodicity isomorphism \((1 \leq i \leq 7)\). Set \( u_i = Y_i \),
\[
\begin{align*}
u_2 &= Y_4 + Y_2 + 2(m-n-10)/8(2Y_2 + Y_1) + 2(3m-3n-6)/8 Y_1, \\
u_3 &= Y_7 - Y_2 - Y_1 + 2(m-n-10)/8(2Y_2 + Y_1 - Y_2) + 2(3m-3n-14)/8 Y_1 \\
&\quad - 2(3m-3n-12)/8 Y_1, \\
u_4 &= 2Y_4 - Y_1 + 2(m-n-2)/8(2Y_2 + Y_1) + 2(3m-3n+2)/8 Y_1, \\
u_5 &= Y_5 + Y_1 - 2(3m-3n-6)/8 Y_1, \\
u_6 &= 2Y_6 - Y_2
\end{align*}
\]
and \( u_7 = 2Y_7 + Y_3 - 2Y_2 + 2(m-n-2)/8(2Y_2 + Y_1 - Y_2) + 2(3m-3n-6)/8 Y_1 \). Then, by Proposition 2.22, we have
\[ \mathcal{K}(S^i(L_\delta^n|L_\delta^8)) = \langle \{u_i|1 \leq i \leq 7\} \rangle = \langle \{2^{\alpha(i)} u_i|1 \leq i \leq 7\} \rangle, \]

where
\[
\begin{align*}
a(1) &= (m-n+4)/2, \\
a(2) &= a(3) = (m-n+2)/4, \\
a(5) &= a(6) = (m-n-2)/8
\end{align*}
\]

and \(a(4) = a(7) = (m-n-10)/8\). According to Lemma 2.23, the Adams operation \(\psi^5\) is given by the following formulae.

\[
\begin{align*}
(1) & \quad \psi^5(u_1) = 5^{1/2}(u_4-u_6), \\
(2) & \quad \psi^5(u_2) = 5^{1/2} u_2, \\
(3) & \quad \psi^5(u_3) = 5^{1/2}(u_3-u_6), \\
(4) & \quad \psi^5(u_4) = 5^{1/2} u_4, \\
(5) & \quad \psi^5(u_5) = 5^{1/2}(2^{2(m-n-2)/8}(2u_2-u_4)-u_4-u_6), \\
(6) & \quad \psi^5(u_6) = 5^{1/2}(-u_4), \\
(7) & \quad \psi^5(u_7) = 5^{1/2}(u_2-2u_6).
\end{align*}
\]

Choose \(v_i \in \mathcal{K}(S^i-(L_\delta^{n+1}|L_\delta^{n+1}))\) similarly as \(u_i \in \mathcal{K}(S^i(L_\delta^n|L_\delta^8))\) \((1 \leq i \leq 7)\), and set

\[ f^*(v_i) = \sum_{k=1}^{s} a_{ik} u_k \quad (1 \leq i \leq 7). \]

By the equality \(\psi^5 \circ f^* = f^* \circ \psi^5\), we have

\[
\begin{align*}
5^{1/2} a_{11} & \equiv 5^{(i-n)/2}(a_{11}-a_{01}) \pmod{2^{\alpha(i)}}, \\
5^{1/2} a_{1i} & \equiv 5^{(i-n)/2}(a_{1i}-a_{01}) \pmod{2^{\alpha(i)}}.
\end{align*}
\]

Since \(\nu_2(a_{01}) \geq a(1) = a(3) = (m-n+6)/4\), (3.7) implies that \(a_{01} \equiv 0 \pmod{2^{\alpha(i)}}\). It follows from (3.6) that we have \(5^{1/2} a_{11} \equiv 0 \pmod{2^{\alpha(i)}}\). Note that \(a_{11} \equiv 1 \pmod{2}\). According to [15, Lemma 3.1], we see that \(\nu_2(t)+1 \equiv a(1) = (m-n+4)/2\). Hence

\[ \nu_2(t) \equiv (m-n+2)/2 = [m/2] - [(n+1)/2] + 1. \]

This completes the proof of the part (1) of Theorem 4.

Suppose that \(\nu_2(n+1) \geq i\) and \(h_2(m-2([(n+1)/2]) \geq i \geq 3\). Then \(m \geq n+5\) and \(t \equiv 0 \pmod{8}\) by (2.11) (3). It follows from (2.11) (1) and Proposition 2.19 that we have \(\nu_2(t) \geq [(m-n-1)/2], \)

\[ J(S^i(L_\delta^n|L_\delta^8)) \cong \mathbb{Z}/m(\langle n+j+1/2 \rangle \cdot 2^{s_i} \oplus \mathbb{Z} / 2^{s_i+1} \oplus \mathbb{Z} / 2^{s_i} \oplus \mathbb{Z} / 2^{s_i}), \]

and

\[ J(S^i-(L_\delta^{n+1}|L_\delta^{n+1})) \cong \mathbb{Z}/m(\langle n+j+1/2 \rangle \cdot 2^{s_i} \oplus \mathbb{Z} / 2^{s_i+1} \oplus \mathbb{Z} / 2^{s_i} \oplus \mathbb{Z} / 2^{s_i}), \]

where \(i = \min \{\nu_2(n+1)-1, h_2(m-m-1)\}, \)

\[ k_i = \min \{\nu_2(n+t+1)-1, h_2(m-n-1)\}, \]
\[ i_2 = \min \{ \nu_2(n+1)-2, [(m-n+3)/8]+[(m-n-3)/8] \} , \]
\[ k_2 = \min \{ \nu_2(n+t+1)-2, [(m-n+3)/8]+[(m-n-3)/8] \} , \]
\[ i_2 = k_3 = [(m-n-1)/8] , \]
\[ c_1 = h_2(m-n-1)-i_1 , \]
\[ d_1 = h_3(m-n-1)-k_1 , \]
\[ c_2 = [(m-n+3)/8]+[(m-n-3)/8]-i_2 , \]
\[ d_2 = [(m-n+3)/8]+[(m-n-3)/8]-k_2 . \]

Since \( c_1 \geq c_2 \geq 0 \), \( c_2+i_1 \geq i_3 \), and \( d_2 = [(m-n+3)/8]+[(m-n-3)/8]-k_2 \).

\[ \nu_2(m((n+j+1)/2)) \geq i_2 \geq i_1 , \]
\[ \nu_2(m((n+j+1)/2))+c_1 \geq h_2(m-n-1)+1 \]
and \( \max \{ d_2+k_1, k_2, k_3 \} \leq h_2(m-n-1) \), the isomorphism (3.3) implies that \( c_1 = d_1 \), and hence
\[ \min \{ \nu_2(n+1)-1, h_2(m-n-1) \} = \min \{ \nu_2(n+t+1)-1, h_2(m-n-1) \} . \]

Since \( \nu_2(n+1) \geq i \), this implies that we have \( \nu_2(n+t+1) \geq i \) if \( \nu_2(n+1) \geq i \) and \( \nu_2(n+t+1) = i \) if \( \nu_2(n+1) = i \). Thus we have \( \nu_2(t) \geq i+1 \). The proof of the part (2) of Theorem 4 is completed by making use of the \( S \)-duality (Lemma 2.9).

By (2.11) (3) and the parts (1) and (2) of Theorem 4, the part (3) of Theorem 4 is obtained except for the case \( n \equiv 0 \pmod{8} \) and \( m = n+6 \). So, assume that \( n \equiv 0 \pmod{8} \) and \( m = n+6 \). Let \( Y \) be the element of \( \mathcal{K}(S^i(L^a_n/L^a_b)) \), which corresponds to \( I^{1/2}(S^3) \) by the isomorphism
\[ (p^a_n)^*: \mathcal{K}(S^i(L^a_n/L^a_b)) \rightarrow I^{1/2}(V_a) \]
\((1 \leq i \leq 7)\). Set \( u_1 = Y_5, u_2 = Y_6+2Y_5 \) and \( u_3 = Y_7+2Y_5 \). Then, by Proposition 2.22, we have
\[ \mathcal{K}(S^i(L^a_n/L^a_b)) \cong \langle \{ u_i | 1 \leq i \leq 3 \} \rangle \langle \{ 2^{a(i)} u_i | 1 \leq i \leq 3 \} \rangle , \]
where \( a(1) = 32 \) and \( a(2) = a(3) = 4 \). According to Lemma 2.23, we have the following formulae.

\[ \begin{align*}
(1) \quad & \psi^{-1}(u_1) = 5u_1+u_2 . \\
(2) \quad & \psi^3(u_1) = 5^{1/2}(17u_1+2u_2) . \\
(3) \quad & \psi^{-1}(u_2) = 8u_1-u_2 . \\
(4) \quad & \psi^3(u_2) = 5^{1/2}(16u_1+u_2) . \\
(5) \quad & \psi^{-1}(u_3) = -8u_1-u_3 . \\
(6) \quad & \psi^3(u_3) = 5^{1/2}(16u_1+u_3) .
\end{align*} \]

Choose \( v_i \in \mathcal{K}(S^i(L^a_n/L^a_b)) \) similarly as \( u_i \in \mathcal{K}(S^i(L^a_n/L^a_b)) \) \((1 \leq i \leq 3)\), and set
By the equality $\psi^{-1} \circ f^* = f^* \circ \psi^{-1}$, we have
\[ 8a_{12} - 8a_{13} \equiv a_{21} \pmod{32}. \]  
By the equality $\psi^5 \circ f^* = f^* \circ \psi^5$, we have
\[ 5^{t/2}(17a_{11} + 16a_{12} + 16a_{13}) \equiv 5^{t-n/2}(17a_{11} + 2a_{21}) \pmod{32}. \]
By (3.9), (3.10) and the fact $5^{t/2} \equiv 1 \pmod{2^{n(t+1)}}$, we have
\[ (5^{t/2} - 1) a_{11} \equiv 0 \pmod{32}. \]
Note that $a_{11} \equiv 1 \pmod{2}$. According to [15, Lemma 3.1], we see that $\nu_2(t) + 1 \geq 5$. Hence
\[ \nu_2(t) \geq 4 = \left[ \frac{m}{2} \right] - \left[ \frac{n+1}{2} \right] + 1. \]
This completes the proof of Theorem 4.

References


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