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ON THE ISOMORPHISM THEOREM OF THE MEROMORPHIC FUNCTION FIELDS

Dedicated to Professor Kentaro Murata on his 60th birthday

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1. Introduction

It is one of beautiful theorems in classical function theory that compact Riemann surfaces are determined by the fields of functions meromorphic on them. More precisely, let R and S be compact Riemann surfaces and let $M(R)$ and $M(S)$ be the fields of functions meromorphic on them, respectively. Then, R and S are conformally equivalent if and only if $M(R)$ and $M(S)$ are $\boldsymbol{C}$-isomorphic, i.e. there is an isomorphism of $M(R)$ onto $M(S)$ whose restriction to the field of complex numbers is the identity or the conjugate map.

For open Riemann surfaces, an isomorphism of these meromorphic function fields as abstract fields is necessarily reduced to a $\boldsymbol{C}$-isomorphism of them (Iss'sa [2]). In spite of this fact, there is a pair of compact Riemann surfaces R and S of genus g ($\geqq 1$) which are not conformally equivalent and $M(R)$ and $M(S)$ are isomorphic as abstract fields. It was noted by Heins [1] for $g=1$ and the author [3] for $g\geqq 2$. Nakai and Sario [5], however, showed that if $M(R)$ and $M(S)$ are isomorphic, then R and S are topologically equivalent.

Let I_g be the set of compact Riemann surfaces of genus g satisfying the following condition: R is an element of I_g if for any Riemann surface S such that $M(R)$ is isomorphic to $M(R)$, S and R are conformally equivalent. In this paper we shall study how many elements of I_g do there exist in H_g the space of hyperelliptic Riemann surfaces of genus g.

2. Statement of Theorem

For the sake of simplicity, henceforce, we restrict the meaning of the terms *isomorphism* and *conformal* to the following: *isomorphisms* of fields we mean are *direct* isomorphisms, i.e. $\sqrt{-1}$ is mapped to $\sqrt{-1}$, and *conformal* does not mean *indirect* (or *anti-*) conformal.

Let $a_1, \cdots, a_{2g-1}$ ($g\geqq 2$) be mutually distinct complex numbers. Then, there is a hyperelliptic Riemann surface of genus g defined by the equation

$$y^2 = x(x-1)(x-a_1)\cdots(x-a_{2g-1})\,. \tag{1}$$

Conversely, for a hyperelliptic Riemann surface of genus g, there is a set of mutually distinct complex numbers $b_1, \cdots, b_{2g-1}$ such that the surface is defined by

$$y^2 = x(x-1)(x-b_1)\cdots(x-b_{2g-1})\,. \tag{2}$$

There are at most finitely many such $(2g-1)$-tuples that define the surface by the equation of type (2) and the number of these tuples is uniformly bounded on H_g. That is, there is a mapping π of $(\boldsymbol{C}-\{0, 1\})^{2g-1}$-the diagonal onto H_g and there is a positive number N such that $\#\pi^{-1}(R)\leqq N$ for every R in H_g. Then, we shall prove:

Theorem. *The set $\pi^{-1}(I_g\cap H_g)$ is of Lebesgue measure zero in $\boldsymbol{C}^{2g-1}$.*

3. Proof of Theorem.

To prove Theorem we shall need several lemmas.

Lemma 1 (cf. Kato [3]). *Let R and S be hyperelliptic Riemann surfaces of genus $g(\geqq 2)$ defined by*

$$y^2 = \prod_{j=1}^{2g+2}(x-a_j) \qquad \text{and} \qquad Y^2 = \prod_{j=1}^{2g+2}(X-b_j)\,,$$

respectively. Then, R is conformally equivalent to S if and only if there is a linear transformation T such that $T(\{a_1, \cdots, a_{2g+2}\}) = \{b_1, \cdots, b_{2g+2}\}$.

Lemma 2. *Let σ be an automorphism of $\boldsymbol{C}$. Suppose that R and S are compact Riemann surfaces defined by the equations*

$$\sum_{j=0}^{n}\sum_{i=0}^{m(j)} a_{ij}x^i y^j = 0 \tag{3}$$

and

$$\sum_{j=0}^{n}\sum_{i=0}^{m(j)} A_{ij}X^i Y^j = 0\,, \tag{4}$$

respectively, where a_{ij} and A_{ij} are complex numbers and $A_{ij}=\sigma(a_{ij})$, $(i=0, \cdots, m(j), j=0, \cdots, n)$. Then σ can be extended to an isomorphism of $M(R)$ onto $M(S)$.

Proof. Define $\hat{\sigma}$ by $\hat{\sigma}(x)=X$, $\hat{\sigma}(y)=Y$ and $\hat{\sigma}|_{\boldsymbol{C}}=\sigma$. Since σ is an automorphism of $\boldsymbol{C}$, (3) is irreducible if and only if (4) is irreducible. Hence, $\hat{\sigma}$ is an isomorphism of $M(R)$ onto $M(S)$ and is an extension of σ.

As a corollary to Lemma 2 we have the following fact. This, however, will not be used later.

Corollary. *Suppose that R and S are Riemann surfaces such that $M(R)$ and $M(S)$ are isomorphic. If R is hyperelliptic, then so is S, if R is trigonal, then so is S and so on.*

Lemma 3. *Let $n\geqq 4$. Suppose that A_n is the set of points $(a_1, \cdots, a_n)$ in $\boldsymbol{C}^n$ such that $a_1, \cdots, a_n$ are mutually distinct and there is a non-trivial linear transformation T which maps $\{a_1, \cdots, a_n, \infty\}$ onto itself. Then, A_n is of Lebesgue measure zero in $\boldsymbol{C}^n$.*

Proof. Choose $a_1, \cdots, a_{n-1}$ arbitrarily and fix. Let K be the extension field over $\boldsymbol{Q}$ obtained by adjoining $a_1, \cdots, a_{n-1}$, where $\boldsymbol{Q}$ is the field of real rational numbers. Suppose that $(a_1, \cdots, a_n)$ is in A_n. Then, there is a non-trivial linear transformation $T(x)=(\alpha x+\beta)/(\gamma x+\delta)$ which maps $\{a_1, \cdots, a_n, \infty\}$ onto itself. Since there are at least three points among $\{a_1, \cdots, a_{n-1}, \infty\}$, say a_1, a_2 and a_3, such that $T(a_1)$, $T(a_2)$ and $T(a_3)$ are contained in $\{a_1, \cdots, a_{n-1}, \infty\}$, we may choose α, β, γ, δ from K. Hence, $T(K)=K$. Therefore, a_n is either an element of K or a fixed point of T. There are finite number of linear transformations which maps $\{a_1, \cdots, a_{n-1}, \infty\}$ onto itself and K is a countable set. Hence, for fixed $a_1, \cdots, a_{n-1}$, the set $\{a_n | (a_1, \cdots, a_n)\in A_n\}$ is of Lebesgue measure zero in $\boldsymbol{C}$. Thus, we have the desired result.

Lemma 4. *Let K be a subfield of $\boldsymbol{C}$ which contains $\boldsymbol{Q}(\sqrt{-1})$, the extension field over $\boldsymbol{Q}$ obtained by adjoining $\sqrt{-1}$. For every α not in K, there is a non-trivial automorphism σ of $\boldsymbol{C}$ so that the restriction of σ to K is the identity and α is not a fixed point of σ. In other words, the set of points fixed by every automorphism of $\boldsymbol{C}$ whose restriction to K is the identity is K itself.*

Proof. This proof mainly appeals to the construction of a non-trivial automorphism of $\boldsymbol{C}$. Any non-trivial automorhpism of a subfield of $\boldsymbol{C}$ can be extended to an automorphism of $\boldsymbol{C}$ (Kestelman [4, p. 6]). By the definition of $K(\alpha)$, there is a non-trivial automorphism σ of $K(\alpha)$ such that $\sigma(\alpha)\neq\alpha$. Hence, we have the desired result.

Proof of Theorem. Suppose that $a_1, \cdots, a_{2g-2}$ are complex numbers such that $a_i\neq a_j$ $(i\neq j)$, $a_i\neq 0$, 1 $(i=1, \cdots, 2g-2)$ and that $(0, 1, a_1, \cdots, a_{2g-2})$ is not contained in A_{2g}. By Lemma 4, for every number α not in $\boldsymbol{Q}(\sqrt{-1}, a_1, \cdots, a_{2g-2})$, there is an automorphism σ of $\boldsymbol{C}$ such that $\sigma(\alpha)\neq\alpha$ and the restriction of σ to $\boldsymbol{Q}(\sqrt{-1}, a_1, \cdots, a_{2q-2})$ is the identity. Let R and S be Riemann surfaces defined by

$$y^2 = x(x-1)(x-a_1)\cdots(x-a_{2g-2})(x-\alpha)$$

and

$$Y^2 = X(X-1)(X-a_1)\cdots(X-a_{2g-2})\,(X-\sigma(\alpha))\,,$$

respectively. Since $\sigma(a_i)=a_i$ $(i=1, \cdots, 2g-2)$, by Lemma 2, σ can be extended to an isomorphism of $M(R)$ onto $M(S)$. Suppose that R and S are conformally equivalent. Then, there is a linear transformation T which maps $\{0, 1, \infty, a_1, \cdots, a_{2g-2}, \alpha\}$ onto $\{0, 1, \infty, a_1, \cdots, a_{2g-2}, \sigma(\alpha)\}$. Since

$$^{\sharp}(\{0, 1, \infty, a_1, \cdots, a_{2g-2}\} - \{T(\alpha), T^{-1}(\sigma(\alpha))\}) \geqq 2g-1 \geqq 3 ,$$

T maps $\boldsymbol{Q}(\sqrt{-1}, a_1, \cdots, a_{2g-2}) \cup \{\infty\}$ onto itself. Hence, $T(\alpha)=\sigma(\alpha)\neq\alpha$. Since $(0, 1, a_1, \cdots, a_{2g-2})$ is not contained in A_{2g}, this is a contradiction. Thus, $(a_1, \cdots, a_{2g-2}, \alpha)$ is not contained in $\pi^{-1}(I_g \cap H_g)$. Since $\boldsymbol{Q}(\sqrt{-1}, a_1, \cdots, a_{2g-2})$ is countable, the set $\{\alpha \mid (a_1, \cdots, a_{2g-2}, \alpha) \in \pi^{-1}(I_g \cap H_g)\}$ is of measure zero in $\boldsymbol{C}$ for every fixed $(a_1, \cdots, a_{2g-2})$ such that $(0, 1, a_1, \cdots, a_{2g-2})$ is not in A_{2g}. Hence, $\pi^{-1}(I_g \cap H_g)$ is of measure zero in $\boldsymbol{C}^{2g-1}$.

4. Closing remark

We close the present paper by remarking on our Theorem and the problem posed by Nakai and Sario [5]. In the first place, our previous paper [3] and our Theorem asserted the affirmative answer to Problem 1° of them [5].

Although the measure of $\pi^{-1}(I_g \cap H_g)$ is zero, the set is dense in $\boldsymbol{C}^{2g-1}$.

Indeed, if $a_1, \cdots, a_{2g-1}$ are complex rational numbers, then the Riemann surface defined by (1) is in $I_g \cap H_g$.

By virtue of Lemma 4, for any transcendental complex numbers α and β, there is an isomorphism σ of $\boldsymbol{C}$ which satisfies $\sigma(\alpha)=\beta$. Hence, related to Problem 2° [5], we can assert that one parameter family of distinct fields of meromorphic functions on the surface of genus one is, if exists, countable. Then, we have the following problem: Is $(3g-3)$ parameter family of field for genus $g(\geqq 2)$, if exists, countable?

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