Statistical change point inference for ergodic diffusion processes based on high frequency data

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Abstract

In this thesis, we consider statistical change point inference for ergodic diffusion processes using high frequency data. Our aim is to provide the adaptive inference for changes in the diffusion and drift parameters in ergodic diffusion processes, and the procedure of our inference is as follows. We first consider the change detection of the diffusion parameter regardless of the presence or absence of change in the drift parameter. If a change in the diffusion parameter is detected, we estimate the time of the change. If no change is detected, we end the inference of the change in the diffusion parameter. We then infer the change in the drift parameter considering the presence or absence of change in the diffusion parameter. Furthermore, we reveal the asymptotic properties of the test statistics for change detection and the change point estimators. We also give some examples and simulation results of our test statistics and estimators to corroborate our results.

Contents

Chapter 1

Introduction

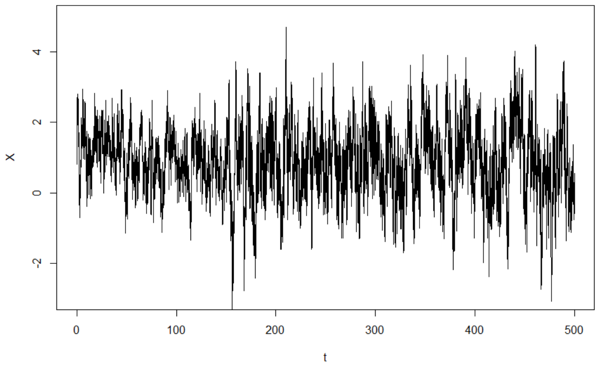
We consider the change point problem for a d-dimensional diffusion process $\{X_t\}_{t\geq 0}$ satisfying the stochastic differential equation (SDE)

$$\mathrm{d}X_t = b(X_t,\beta)\mathrm{d}t + a(X_t,\alpha)\mathrm{d}W_t, \quad X_0 = x_0 \tag{1.1}$$

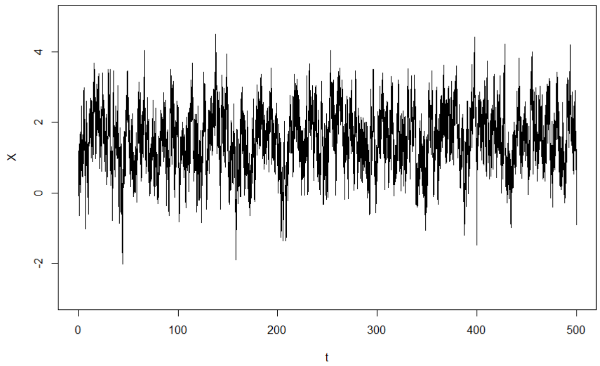
where the parameter space $\Theta = \Theta_A \times \Theta_B$ is a compact convex subset of $\mathbb{R}^p \times \mathbb{R}^q$, $\theta = (\alpha,\beta) \in \Theta$ is an unknown parameter and $\{W_t\}_{t\geq 0}$ is an r-dimensional standard Wiener process. The diffusion coefficient $a : \mathbb{R}^d \times \Theta_A \to \mathbb{R}^d \otimes \mathbb{R}^r$ and the drift coefficient $b : \mathbb{R}^d \times \Theta_B \to \mathbb{R}^d$ are known except for θ. We assume that the solution of SDE (1.1) exists, and $\mathbf{P}_\theta$ and $\mathbb{E}_\theta$ denote the law of the solution and the expectation with respect to $\mathbf{P}_\theta$, respectively. We use high frequency and long term discrete observations to infer the change point of the diffusion and drift parameters. Let $\{X_{t_i}\}_{i=0}^n$ be high frequency data, where $t_i = t_i^n = ih_n$ and $\{h_n\}$ is a positive sequence with $h_n \to 0$, $T = t_n = nh_n \to \infty$ and $nh_n^2 \to 0$ as $n \to \infty$.

The change point problem was originally addressed in the field of quality control and has been recently developed in various fields where changes are of interest, such as economics, finance, genetics, and medicine. For example, in finance, a diffusion process model is used as a stock price fluctuation model to make forecasts of stock prices. The stock market records show that all stock prices fluctuate on a daily basis. Most of them are normal fluctuations as expected in statistical models, but some of them are abnormal fluctuations and caused by political or economic influences. This abnormal fluctuations may have an effect on our assumed model. If one ignores this change, the stock price forecast may be worthless. For this reason, we need to investigate the presence of changes that affect the model, and if there are changes, when these changes occur. The change point problem plays a role in identifying this abnormal change.

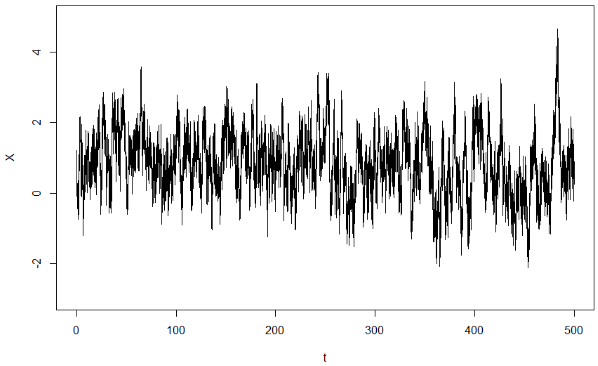
The change point problem for diffusion processes based on discrete observations has been developed by many researchers. For non-ergodic diffusion processes, see De Gregorio and Iacus (2008) and Iacus and Yoshida (2012). Since it is impossible to estimate the drift parameter β for the non-ergodic diffusion process model, one only deals with the change point inference for the diffusion parameter α. De Gregorio and Iacus (2008) studied the change point estimation for the diffusion parameter based on the least squares approach, and Iacus and Yoshida (2012) considered the quasi-maximum likelihood estimator of the change point of the diffusion parameter. As for ergodic diffusion processes, see Song and Lee (2009), Lee (2011), Negri and Nishiyama (2017) and Song (2020). Because it is possible to estimate the drift parameter in the ergodic diffusion process model, one can treat the change point inference for the diffusion and drift parameters. Song and Lee (2009) proposed the CUSUM type test statistic for changes in the diffusion parameter based on the estimator proposed by Kessler (1997) under the assumptions $nh_n^p \to 0$ and $nh_n^q \to \infty$ $(p > q > 4)$. Lee (2011) and Song (2020) considered the CUSUM type test statistic for changes in the diffusion parameter based on the residuals and trimmed-residuals under $nh_n^2 \to 0$, respectively. Negri and Nishiyama (2017) treated the joint test for changes in the diffusion and drift parameters based on the Z-process method.



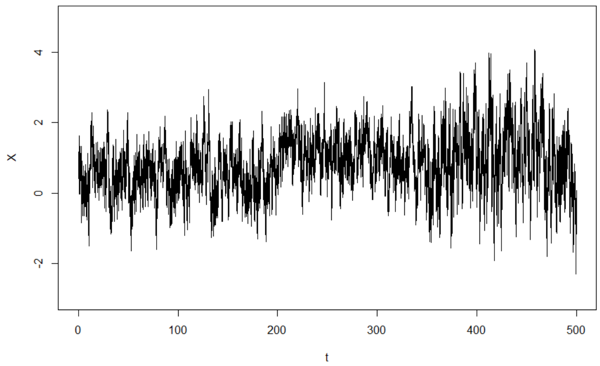
(a) α changes from 1 to 1.5 at $t = 150$, and $(\beta, \gamma) = (1, 1)$ does not change.



(b) $(\alpha, \beta, \gamma) = (1.2, 1, 1.5)$ does not change.



(c) $\alpha = 1$ does not change, and (β, γ) changes from $(1, 1)$ to $(0.5, 0.5)$ at $t = 250$.



(d) α changes from 1 to 1.5 at $t = 350$, and (β, γ) changes from $(1.3, 0.5)$ to $(1.3, 1)$ at $t = 200$.

Figure 1: Sample paths of the Ornstein-Uhlenbeck process $\mathrm{d}X_t = -\beta(X_t - \gamma)\mathrm{d}t + \alpha \mathrm{d}W_t$.

Our aim is to infer changes in the diffusion and drift parameters from given data. Therefore, this thesis provides a statistical method for parameter changes in ergodic diffusion processes. Specifically, we consider the detection of changes in the diffusion and drift parameters, and the estimation of the time of the change. By using our method, we can infer the parameter changes of the paths as shown in Figure 1. That is, we can infer that the diffusion parameter changes at $t = 150$ and $t = 350$ for (a) and (d), respectively, and that there is no change in the diffusion parameter for (b) and (c). Moreover, for the drift parameter, it can be inferred that there is a change point at $t = 250$ and $t = 200$ in (c) and (d), respectively, while there is no change in (a) and (b).

For simplicity, we assume that there is at most one change point for each diffusion and drift parameters throughout this thesis. That is, we consider the following four situations.

I. Neither parameter changes.

$$X_t = X_0 + \int_0^t b(X_s, \beta^*)\mathrm{d}s + \int_0^t a(X_s, \alpha^*)\mathrm{d}W_s, \quad t \in [0, T].$$

II. Only drift parameter changes.

$$X_t = \begin{cases} X_0 + \int_0^t b(X_s, \beta_1^*)\mathrm{d}s + \int_0^t a(X_s, \alpha^*)\mathrm{d}W_s, & t \in [0, \tau_*^\beta T), \\ X_{\tau_*^\beta T} + \int_{\tau_*^\beta T}^t b(X_s, \beta_2^*)\mathrm{d}s + \int_{\tau_*^\beta T}^t a(X_s, \alpha^*)\mathrm{d}W_s, & t \in [\tau_*^\beta T, T]. \end{cases}$$

III. Only diffusion parameter changes.

$$X_t = \begin{cases} X_0 + \int_0^t b(X_s, \beta^*)\mathrm{d}s + \int_0^t a(X_s, \alpha_1^*)\mathrm{d}W_s, & t \in [0, \tau_*^\alpha T), \\ X_{\tau_*^\alpha T} + \int_{\tau_*^\alpha T}^t b(X_s, \beta^*)\mathrm{d}s + \int_{\tau_*^\alpha T}^t a(X_s, \alpha_2^*)\mathrm{d}W_s, & t \in [\tau_*^\alpha T, T]. \end{cases}$$

IV. Both parameters change.

(i) Diffusion parameter changes after drift parameter does ($\tau_*^\beta < \tau_*^\alpha$).

$$X_t = \begin{cases} X_0 + \int_0^t b(X_s, \beta_1^*)\mathrm{d}s + \int_0^t a(X_s, \alpha_1^*)\mathrm{d}W_s, & t \in [0, \tau_*^\beta T), \\ X_{\tau_*^\beta T} + \int_{\tau_*^\beta T}^t b(X_s, \beta_2^*)\mathrm{d}s + \int_{\tau_*^\beta T}^t a(X_s, \alpha_1^*)\mathrm{d}W_s, & t \in [\tau_*^\beta T, \tau_*^\alpha T), \\ X_{\tau_*^\alpha T} + \int_{\tau_*^\alpha T}^t b(X_s, \beta_2^*)\mathrm{d}s + \int_{\tau_*^\alpha T}^t a(X_s, \alpha_2^*)\mathrm{d}W_s, & t \in [\tau_*^\alpha T, T]. \end{cases}$$

(ii) Drift parameter changes after diffusion parameter does ($\tau_*^\alpha < \tau_*^\beta$).

$$X_t = \begin{cases} X_0 + \int_0^t b(X_s, \beta_1^*)\mathrm{d}s + \int_0^t a(X_s, \alpha_1^*)\mathrm{d}W_s, & t \in [0, \tau_*^\alpha T), \\ X_{\tau_*^\alpha T} + \int_{\tau_*^\alpha T}^t b(X_s, \beta_1^*)\mathrm{d}s + \int_{\tau_*^\alpha T}^t a(X_s, \alpha_2^*)\mathrm{d}W_s, & t \in [\tau_*^\alpha T, \tau_*^\beta T), \\ X_{\tau_*^\beta T} + \int_{\tau_*^\beta T}^t b(X_s, \beta_2^*)\mathrm{d}s + \int_{\tau_*^\beta T}^t a(X_s, \alpha_2^*)\mathrm{d}W_s, & t \in [\tau_*^\beta T, T]. \end{cases}$$

(iii) Both parameters change at the same time ($\tau_*^\alpha = \tau_*^\beta$).

$$X_t = \begin{cases} X_0 + \int_0^t b(X_s, \beta_1^*)\mathrm{d}s + \int_0^t a(X_s, \alpha_1^*)\mathrm{d}W_s, & t \in [0, \tau_*^\alpha T), \\ X_{\tau_*^\alpha T} + \int_{\tau_*^\alpha T}^t b(X_s, \beta_2^*)\mathrm{d}s + \int_{\tau_*^\alpha T}^t a(X_s, \alpha_2^*)\mathrm{d}W_s, & t \in [\tau_*^\alpha T, T]. \end{cases}$$

Here, we set $\tau_*^\alpha, \tau_*^\beta \in (0,1)$, $\alpha^*, \alpha_1^*, \alpha_2^* \in \operatorname{Int}\Theta_A$, $\alpha_1^* \neq \alpha_2^*$, $\beta^*, \beta_1^*, \beta_2^* \in \operatorname{Int}\Theta_B$ and $\beta_1^* \neq \beta_2^*$. Hence, we will study the presence of parameter changes in each situation and consider the estimation of τ_*^α or τ_*^β.

This thesis is organized as follows.

In Chapter 2, we treat the change point inference for the diffusion parameter. In Section 2.1, we consider the hypothesis testing problem for detecting a change in the diffusion parameter and present asymptotic properties of the proposed test statistic. Section 2.2 gives the estimation method of the change point of the diffusion parameter in Situation III or IV, and shows the asymptotic properties of the estimator.

In Chapter 3, we study the change point inference for the drift parameter. In Section 3.1, we deal with the change point inference of the drift parameter in Situation I or II, that is, when there is no change in the diffusion parameter. We discuss the detection of parameter change in Subsection 3.1.1 and the estimation of the time of change in Subsection 3.1.2, respectively. We also treat the change point inference of the drift parameter when there is a change in the diffusion parameter in Section 3.2. Subsections 3.2.1 and 3.2.2 provide the change detection method and the change point estimation method, respectively. Moreover, we consider the case where the diffusion and drift parameters change at the same time in Subsection 3.2.3.

In Chapter 4, we consider two diffusion process models and conduct numerical simulations in order to verify the asymptotic behavior of the proposed test statistics and estimators in the above four situations.

In Appendix A, we provide some remarks on change point inference, such as the estimation method of the nuisance parameters in change point estimation and models that satisfy the assumptions.

Appendix B is devoted to the proofs of our results.

Chapter 2

Change point inference for diffusion parameter

In this chapter, we consider the change point detection and estimation for the diffusion parameter. We first provide the test statistic for detecting a change in the diffusion parameter. We then show that the null distribution of the test statistic is the supremum of the absolute value of a Brownian bridge and the test is consistent. Next, we estimate the time of the change in the diffusion parameter when a change in the diffusion parameter is detected. We treat two cases according to the level of change in the diffusion parameter and give the asymptotic properties of the estimator. In particular, we show that the asymptotic distribution of the estimator is the distribution given in Lemma 1.6.3 of Csörgö and Horváth (1997) in the case where $|\alpha_1^* - \alpha_2^*| \to 0$, see Case A_α in Section 2.2 below.

We set the following notations.

1. For a matrix M, $M^\top$ denotes the transpose of M and let $M^{\otimes 2} = MM^\top$. Let I_d be the d-dimensional identity matrix.

2. Let $A(x, \alpha) = a(x, \alpha)^{\otimes 2}$ and $\Delta_i X = X_{t_i} - X_{t_{i-1}}$.

3. For $k \in \mathbb{N}$, $\mathbb{B}_k$ denotes a k-dimensional standard Brownian motion.

4. For $k \in \mathbb{N}$, $\mathbb{B}_k^0$ denotes a k-dimensional Brownian bridge on $[0, 1]$, which is defined by $\mathbb{B}_k^0(s) = \mathbb{B}_k(s) - s\mathbb{B}_k(1)$. For $\epsilon \in (0, 1)$, let $w_k(\epsilon)$ be the upper-ϵ point of $\sup_{0\le s\le 1} |\mathbb{B}_k^0(s)|$, that is,
$$\mathbf{P}\Bigl(\sup_{0\le s\le 1} |\mathbb{B}_k^0(s)| > w_k(\epsilon)\Bigr) = \epsilon.$$

5. Let $\mathbb{W}$ be a two-sided standard Wiener process.

6. For $x = (x^1, \dots, x^d) \in \mathbb{R}^d$ and $f : \mathbb{R}^d \to \mathbb{R}$, we write
$$\partial_{x^j} f(x) = \frac{\partial}{\partial x^j} f(x), \quad \partial_x f(x) = (\partial_{x^1} f(x), \dots, \partial_{x^d} f(x)), \quad \partial_x^2 f(x) = (\partial_{x^i}\partial_{x^j} f(x))_{i,j=1}^d.$$

7. Let $C_{\uparrow}^{k,l}(\mathbb{R}^d \times \Theta)$ be the space of all functions f satisfying the following conditions.

(1) f is continuously differentiable with respect to $x \in \mathbb{R}^d$ up to order k for all $\theta \in \Theta$,

(2) f and all its x-derivatives up to order k are l times continuously differentiable with respect to $\theta \in \Theta$,

(3) f and all derivatives are of polynomial growth in $x \in \mathbb{R}^d$ uniformly in $\theta \in \Theta$, where g is of polynomial growth in $x \in \mathbb{R}^d$ uniformly in $\theta \in \Theta$ if for some $C > 0$,

$$\sup_{\theta \in \Theta} |g(x,\theta)| \le C(1+|x|)^C.$$

8. Let $\xrightarrow{\mathbf{P}}$ and $\xrightarrow{\mathrm{d}}$ be the convergence in probability and the convergence in distribution, respectively.

We make the following assumptions throughout this thesis.

[A1] There exists a constant $C > 0$ such that for any $x, y \in \mathbb{R}^d$,

$$\sup_{\alpha \in \Theta_A} |a(x,\alpha) - a(y,\alpha)| + \sup_{\beta \in \Theta_B} |b(x,\beta) - b(y,\beta)| \le C|x-y|.$$

[A2] $\sup_t \mathbb{E}_\theta[|X_t|^k] < \infty$ for all $k \ge 0$ and $\theta \in \Theta$.

[A3] $\inf_{x,\alpha} \det A(x,\alpha) > 0$.

[A4] $a \in C_{\uparrow}^{4,4}(\mathbb{R}^d \times \Theta_A)$ and $b \in C_{\uparrow}^{4,4}(\mathbb{R}^d \times \Theta_B)$.

[A5] There exists a unique invariant measure μ_θ such that for any μ_θ-integrable f,

$$\frac{1}{T}\int_0^T f(X_t)\mathrm{d}t \xrightarrow{\mathbf{P}} \int_{\mathbb{R}^d} f(x)\mathrm{d}\mu_\theta(x) \quad \text{as } T \to \infty.$$

Moreover, for any polynomial growth function f and $\theta_n \to \theta_0$,

$$\int_{\mathbb{R}^d} f(x)\mathrm{d}\mu_{\theta_n}(x) \to \int_{\mathbb{R}^d} f(x)\mathrm{d}\mu_{\theta_0}(x).$$

Remark 1. *[A1]-[A4] and the first part in [A5] are general assumptions in statistical inference for ergodic diffusion processes. The second part in [A5] is satisfied if the probability density* $\mathrm{d}\mu_\theta(x)/\mathrm{d}x$ *is continuous in* θ.

2.1 Change point detection

We first investigate the presence of a change in the diffusion parameter α. To this end, we consider the following hypothesis testing problem.

$$H_0^\alpha : \text{the diffusion parameter does not change over } [0,T]$$
$$\text{vs.}$$
$$H_1^\alpha : \text{there exists } \tau_*^\alpha \in (0,1) \text{ such that } \alpha^* = \begin{cases} \alpha_1^*, & t \in [0, \tau_*^\alpha T), \\ \alpha_2^*, & t \in [\tau_*^\alpha T, T], \end{cases}$$

where $\alpha_1^*, \alpha_2^* \in \operatorname{Int}\Theta_A$ and $\alpha_1^* \neq \alpha_2^*$.

We make the following assumptions.

[B1] Under H_0^α, there exists an estimator $\hat{\alpha}$ such that $\sqrt{n}(\hat{\alpha} - \alpha^*) = O_{\mathbf{P}}(1)$.

[C1] Under H_1^α, there exist $\alpha' \in \operatorname{Int}\Theta_A$ and an estimator $\hat{\alpha}$ such that $\hat{\alpha} - \alpha' = o_{\mathbf{P}}(1)$.

Let $\mathcal{F}(\alpha, \alpha') = \displaystyle\int_{\mathbb{R}^d} \operatorname{Tr}[A^{-1}(x, \alpha')A(x, \alpha)] \mathrm{d}\mu_\alpha(x)$.

[C2] $\mathcal{F}(\alpha_1^*, \alpha') \neq \mathcal{F}(\alpha_2^*, \alpha')$ under H_1^α.

[D1] Under H_1^α, $\vartheta_\alpha = |\alpha_1^* - \alpha_2^*|$ depends on n, and $\vartheta_\alpha \to 0$, $n\vartheta_\alpha^2 \to \infty$ as $n \to \infty$.

[D2] Under H_1^α, there exists an estimator $\hat{\alpha}$ such that $\vartheta_\alpha^{-1}(\hat{\alpha} - \alpha_0) = O_{\mathbf{P}}(1)$.

[D3] Under H_1^α, $\left(\int_{\mathbb{R}^d} (\operatorname{Tr}(A^{-1}\partial_{\alpha^l} A(x, \alpha_0)))_{l=1}^p \mathrm{d}\mu_{\alpha_0}(x)\right)^\mathsf{T} (c_1 - c_2) \neq 0$, where $c_k = \lim_{n\to\infty} \vartheta_\alpha^{-1}(\alpha_k^* - \alpha_0)$.

Remark 2. *For the construction of the estimators that appear in this thesis, see Kessler (1997), Uchida and Yoshida (2011, 2012, 2014), Yoshida (2011), Kamatani and Uchida (2015), Kaino and Uchida (2018) or Appendix A.*

By setting

$$F_i(\alpha) = \operatorname{Tr}\left(A^{-1}(X_{t_{i-1}}, \alpha)\frac{(\Delta_i X)^{\otimes 2}}{h_n}\right) + \log\det A(X_{t_{i-1}}, \alpha), \quad U_n(\alpha) = \sum_{i=1}^n F_i(\alpha)$$

as the contrast function of the diffusion parameter, $n^{-1/2}\partial_\alpha U_n(\alpha^*)$ has asymptotic normality. Notice that

$$\partial_{\alpha^l} U_n(\alpha) = \sum_{i=1}^n \operatorname{Tr}\left[A^{-1}(X_{t_{i-1}}, \alpha)\partial_{\alpha^l} A(X_{t_{i-1}}, \alpha)\left(A^{-1}(X_{t_{i-1}}, \alpha)\frac{(\Delta_i X)^{\otimes 2}}{h_n} - I_d\right)\right]$$

and

$$\frac{1}{\sqrt{2dn}} \sum_{i=1}^{[ns]} \operatorname{Tr}\left(A^{-1}(X_{t_{i-1}}, \alpha^*)\frac{(\Delta_i X)^{\otimes 2}}{h_n} - I_d\right) \xrightarrow{\mathrm{w}} \mathbb{B}_1(s) \quad \text{in } \mathbb{D}[0,1],$$

where $Y_n(\cdot) \xrightarrow{\mathrm{w}} Y(\cdot)$ in $\mathbb{D}[0,1]$ denotes that $Y_n(\cdot)$ weakly converges to $Y(\cdot)$ in the Skorohod space on $[0,1]$. Therefore, we define the test statistic to detect a change in the diffusion parameter by

$$\mathcal{T}_n^\alpha = \frac{1}{\sqrt{2dn}} \max_{1\le k\le n} \left|\sum_{i=1}^k \hat{\eta}_i - \frac{k}{n}\sum_{i=1}^n \hat{\eta}_i\right|, \quad \hat{\eta}_i = \operatorname{Tr}\left(A^{-1}(X_{t_{i-1}}, \hat{\alpha})\frac{(\Delta_i X)^{\otimes 2}}{h_n}\right).$$

The following theorem gives the asymptotic null distribution and the consistency of the test statistic $\mathcal{T}_n^\alpha$.

Theorem 1. *Suppose that [A1]-[A5] hold.*

(1) If [B1] is satisfied, then $\mathcal{T}_n^\alpha \xrightarrow{\mathrm{d}} \sup_{0\le s\le 1} |\mathbb{B}_1^0(s)|$ *under* H_0^α.

(2) If either (a) [C1] and [C2] or (b) [D1]-[D3] is satisfied, then for $\epsilon \in (0,1)$, $\lim_{n\to\infty} \mathbf{P}(\mathcal{T}_n^\alpha > w_1(\epsilon)) = 1$ *under* H_1^α.

2.2 Change point estimation

If H_0^α is rejected, in other words, if a change of the diffusion parameter α is detected, then we estimate the time of the change. In this section, we study the estimation problem of τ_*^α for the SDE in Situation III or IV.

We consider the following two cases.

Case A$_\alpha$: $\vartheta_\alpha = |\alpha_1^* - \alpha_2^*|$ depends on n, and as $n \to \infty$,

$$\vartheta_\alpha \to 0, \quad n\vartheta_\alpha^2 \to \infty, \quad \alpha_1^* \to \alpha_0 \in \operatorname{Int}\Theta_A.$$

Case B$_\alpha$: $|\alpha_1^* - \alpha_2^*|$ is fixed.

We make the following assumptions.

[E1] There exist estimators $\hat\alpha_k$ $(k = 1, 2)$ and a constant $\tau_*^\alpha \in (0,1)$ such that

$$\sqrt{n}(\hat\alpha_k - \alpha_k^*) = O_{\mathbf{P}}(1), \quad \alpha = \begin{cases} \alpha_1^*, & t \in [0, \tau_*^\alpha T), \\ \alpha_2^*, & t \in [\tau_*^\alpha T, T]. \end{cases}$$

[F1] $h_n/\vartheta_\alpha^2 \to \infty$ and $T\vartheta_\alpha \to 0$ as $n \to \infty$, and $\vartheta_\alpha^{-1}(\alpha_k^* - \alpha_0) = O(1)$.

Let

$$\begin{aligned} \Xi^\alpha(x,\alpha) &= \left[\operatorname{Tr}(A^{-1}\partial_{\alpha^{l_1}} A A^{-1}\partial_{\alpha^{l_2}} A(x,\alpha))\right]_{l_1,l_2=1}^p, \\ \Gamma^\alpha(x,\alpha_1,\alpha_2) &= \operatorname{Tr}(A^{-1}(x,\alpha_1)A(x,\alpha_2) - I_d) - \log\det A^{-1}(x,\alpha_1)A(x,\alpha_2), \end{aligned}$$

and $Q(x,\theta) = L_\theta^2\varphi(y|x)|_{y=x}$, $\varphi(y|x) = (y-x)^{\otimes 2}$, where the operator L_θ is defined as follows. For $\mathbb{R}^r \times \mathbb{R}^r$-valued C^2 functions $f = (f_{i,j})_{i,j=1}^r$ on $\mathbb{R}^d$,

$$L_\theta f(x) = \left(\partial_x f_{i,j}(x) b(x,\beta) + \frac{1}{2}\operatorname{Tr}[\partial_x^2 f_{i,j}(x) A(x,\alpha)]\right)_{i,j=1}^r.$$

[F2] Let $f(x)$ be the following three functions, (a) $\Xi^\alpha(x,\alpha_0)$, (b) $\partial_\alpha \Xi^\alpha(x,\alpha_0)$, (c) $\partial_{\alpha_1}^3 \Gamma^\alpha(x,\alpha_0,\alpha_0)$. For any $\delta \in (1,2)$ such that $nh_n^\delta \to \infty$,

$$\max_{[n^{1/\delta}]\le k\le n-[n\tau_*^\alpha]} \left| \frac{1}{k}\sum_{i=[n\tau_*^\alpha]+1}^{[n\tau_*^\alpha]+k} f(X_{t_{i-1}}) - \int_{\mathbb{R}^d} f(x)\mathrm{d}\mu_{(\alpha_0,\beta)}(x) \right| \xrightarrow{\mathbf{P}} 0.$$

[G1] $\inf_x \Gamma^\alpha(x, \alpha_1^*, \alpha_2^*) > 0$.

[G2] There exists a constant $C > 0$ such that

(a) $\sup_{x,\alpha_k}\Big(|\partial_{\alpha_1}\Gamma^\alpha(x, \alpha_1, \alpha_2)| \vee |\partial_{\alpha_2}\Gamma^\alpha(x, \alpha_1, \alpha_2)|\Big) < C$,

(b) $\sup_{x,\alpha_k}\Big|[\mathrm{Tr}\,(\{A^{-1}(x, \alpha_1) - A^{-1}(x, \alpha_2)\}\partial_{\alpha^l} A(x, \alpha_3))]_{l=1}^p\Big| < C$,

(c) $\sup_{x,\theta} |Q(x, \theta)| < C$.

Remark 3.

(1) If the diffusion coefficient is $a(x, \alpha) = \alpha$, then $h_n/\vartheta_\alpha^2 \to \infty$ is not required in [F1].

(2) [F2] is the assumption that the convergence corresponding to Lemma 4.3 in Song and Lee (2009) is valid for the three functions (a)-(c). This is the key convergence for the change point problems for ergodic diffusion processes.

(3) When $d = 1$, Q in [G2] can be expressed as

$$Q(x, \theta) = \Big(2b(x, \beta) + \partial_x A(x, \alpha)\Big)b(x, \beta) + \Big(2\partial_x b(x, \beta) + \partial_x^2 A(x, \alpha)\Big)A(x, \alpha),$$

and thus if $\partial_x^k A(x, \alpha)$ ($k = 0, 1, 2$) and $\partial_x^l b(x, \beta)$ ($l = 0, 1$) are bounded with respect to x and $\theta = (\alpha, \beta)$, [G2](c) is fulfilled.

Let

$$\Phi_n(\tau : \alpha_1, \alpha_2) = \sum_{i=1}^{[n\tau]} F_i(\alpha_1) + \sum_{i=[n\tau]+1}^{n} F_i(\alpha_2).$$

We define the change point estimator for the diffusion parameter by

$$\hat{\tau}_n^\alpha = \operatorname*{argmin}_{\tau\in[0,1]} \Phi_n(\tau : \hat{\alpha}_1, \hat{\alpha}_2).$$

Remark 4. *The change point estimator $\hat{\tau}_n^\alpha$ requires the estimators $\hat{\alpha}_1$ and $\hat{\alpha}_2$ of the parameters α_1 and α_2 before and after the change. In order to construct these estimators, we need to find intervals with α_1^* and α_2^*, respectively. We will discuss the method for finding these intervals in Appendix A.*

In Case A_α, we define

$$e_\alpha = \lim_{n\to\infty} \vartheta_\alpha^{-1}(\alpha_1^* - \alpha_2^*), \quad \mathcal{J}_\alpha = \frac{1}{2} e_\alpha^\top \int_{\mathbb{R}^d} \Xi^\alpha(x, \alpha_0)\mathrm{d}\mu_{\alpha_0}(x) e_\alpha,$$
$$\mathbb{F}(v) = -2\mathcal{J}_\alpha^{1/2}\mathbb{W}(v) + \mathcal{J}_\alpha|v| \quad \text{for } v \in \mathbb{R}.$$

We get the following result on the asymptotic behavior of the estimator $\hat{\tau}_n^\alpha$.

Theorem 2. *Suppose that [A1]-[A5] and [E1] hold.*

(1) Under [F1] and [F2], $n\vartheta_\alpha^2(\hat{\tau}_n^\alpha - \tau_*^\alpha) \xrightarrow{\mathrm{d}} \operatorname*{argmin}_{v\in\mathbb{R}} \mathbb{F}(v)$ *in Case* A_α*.*

(2) Under [G1] and [G2], $n(\hat{\tau}_n^\alpha - \tau_*^\alpha) = O_{\mathbf{P}}(1)$ *in Case* B_α*.*

(3) Under [G1], [G2](a) and (b), $n^{\epsilon_1}(\hat{\tau}_n^\alpha - \tau_*^\alpha) = o_{\mathbf{P}}(1)$ *for* $\epsilon_1 \in [0, 1/2)$ *in Case* B_α*.*

Remark 5. *For* $v \in \mathbb{R}$*, let* $\widehat{\mathbb{W}}(v) = \mathbb{W}(v) - |v|/2$ *and* $\hat{\eta} = \inf\{\eta \in \mathbb{R} | \widehat{\mathbb{W}}(\eta) = \sup_{v\in\mathbb{R}} \widehat{\mathbb{W}}(v)\}$*. Since* $\mathbb{F}(v) \stackrel{\mathrm{d}}{=} -2\widehat{\mathbb{W}}(\mathcal{J}_\alpha v)$*, the asymptotic distribution of (1) in Theorem 2 can be expressed as* $\hat{\eta}/\mathcal{J}_\alpha$*. For the probability density function of the distribution of* $\hat{\eta}$*, see Lemma 1.6.3 of Csörgö and Horváth (1997).*

Chapter 3

Change point inference for drift parameter

The aim of this chapter is to detect and estimate the change point of the drift parameter. First, we propose the change detection method of the drift parameter when no change in the diffusion parameter is detected and the change point estimation method when a change in the drift parameter is detected, and present the asymptotic properties of the statistics. We also consider change point inference for the drift parameter when a change in the diffusion parameter is detected. Moreover, we discuss the case where the diffusion and drift parameters change at the same time.

3.1 Change point inference under no significant change in diffusion parameter

In this section, we investigate the presence of a change in the drift parameter β when no change in the diffusion parameter α is detected, and if there is the change, estimate the time of the change. For this setting, we make the following assumption throughout this section.

[E2] α does not change over $[0, T]$, and there exists an estimator $\hat{\alpha}$ such that $\sqrt{n}(\hat{\alpha} - \alpha^*) = O_{\mathbf{P}}(1)$.

3.1.1 Change point detection

In order to investigate a change in the drift parameter, we first consider the following hypothesis testing problem.

$$H_0^\beta : \text{the drift parameter does not change over } [0, T]$$
$$\text{vs.}$$
$$H_1^\beta : \text{there exists } \tau_*^\beta \in (0, 1) \text{ such that } \beta^* = \begin{cases} \beta_1^*, & t \in [0, \tau_*^\beta T), \\ \beta_2^*, & t \in [\tau_*^\beta T, T], \end{cases}$$

where $\beta_1^*, \beta_2^* \in \operatorname{Int}\Theta_B$ and $\beta_1^* \neq \beta_2^*$.

We assume the following conditions.

[B2] Under H_0^β, there exists an estimator $\hat{\beta}$ such that $\sqrt{T}(\hat{\beta} - \beta^*) = O_{\mathbf{P}}(1)$.

[H1] Under H_1^β, there exist $\beta' \in \Theta_B$ and an estimator $\hat{\beta}$ such that $\hat{\beta} - \beta' = o_{\mathbf{P}}(1)$.

For $\alpha \in \Theta_A, \beta_1, \beta_2 \in \Theta_B$, let

$$\mathcal{G}(\alpha, \beta_1, \beta_2) = \int_{\mathbb{R}^d} 1_d^\top a^{-1}(x, \alpha)(b(x, \beta_1) - b(x, \beta_2)) \mathrm{d}\mu_{(\alpha, \beta_1)}(x).$$

[H2] $\mathcal{G}(\alpha^*, \beta_1^*, \beta') \neq \mathcal{G}(\alpha^*, \beta_2^*, \beta')$ under H_1^β.

[I1] Under H_1^β, $\vartheta_\beta = |\beta_1^* - \beta_2^*|$ depends on n, and $\vartheta_\beta \to 0$, $T\vartheta_\beta^2 \to \infty$ as $n \to \infty$.

[I2] Under H_1^β, there exists $\beta^{(0)} \in \operatorname{Int}\Theta_B$ such that for $k = 1, 2$, $\vartheta_\beta^{-1}(\beta_k^* - \beta^{(0)}) \to d_k$ as $n \to \infty$.

[I3] Under H_1^β, there exist β' with $\beta' - \beta^{(0)} = o(1)$ and an estimator $\hat{\beta}$ such that $\sqrt{T}(\hat{\beta} - \beta') = O_{\mathbf{P}}(1)$.

[I4] $\displaystyle\int_{\mathbb{R}^d} 1_d^\top a^{-1}(x, \alpha^*) \partial_\beta b(x, \beta^{(0)}) \mathrm{d}\mu_{(\alpha^*, \beta^{(0)})}(x)(d_1 - d_2) \neq 0$ under H_1^β.

Define

$$G_i(\beta|\alpha) = \operatorname{Tr}\left(A^{-1}(X_{t_{i-1}}, \alpha) \frac{(\Delta_i X - h_n b(X_{t_{i-1}}, \beta))^{\otimes 2}}{h_n}\right), \quad V_n(\beta|\alpha) = \sum_{i=1}^n G_i(\beta|\alpha).$$

Since $T^{-1/2}\partial_\beta V_n(\beta^*|\alpha^*)$ is asymptotically normal and

$$\partial_{\beta^l} V_n(\beta|\alpha) = \sum_{i=1}^n \partial_{\beta^l} b(X_{t_{i-1}}, \beta)^\top A^{-1}(X_{t_{i-1}}, \alpha)(\Delta_i X - h_n b(X_{t_{i-1}}, \beta)),$$

we find that for $r = d = 1$,

$$\frac{1}{\sqrt{T}} \sum_{i=1}^{[ns]} \frac{\Delta_i X - h_n b(X_{t_{i-1}}, \beta^*)}{a(X_{t_{i-1}}, \alpha^*)} \xrightarrow{\mathrm{w}} \mathbb{B}(s) \quad \text{in } \mathbb{D}[0, 1]. \tag{3.1}$$

Hence we consider the case $r = d$ and define

$$\hat{\xi}_i = 1_d^\top a^{-1}(X_{t_{i-1}}, \hat{\alpha})(\Delta_i X - h_n b(X_{t_{i-1}}, \hat{\beta})),$$

which is a simple extension to multiple dimensions. The test statistic for detecting a change in the drift parameter is as follows.

$$\mathcal{T}_{1,n}^\beta = \frac{1}{\sqrt{dT}} \max_{1 \le k \le n} \left| \sum_{i=1}^k \hat{\xi}_i - \frac{k}{n} \sum_{i=1}^n \hat{\xi}_i \right|.$$

We then obtain the following theorem.

Theorem 3. *Suppose that [A1]-[A5] and [E2] hold.*

(1) If [B2] is satisfied, then $\mathcal{T}^{\beta}_{1,n} \xrightarrow{\mathrm{d}} \sup_{0\le s\le 1} |\mathbb{B}^0_1(s)|$ *under* H^{β}_0.

(2) If either (a) [H1] and [H2] or (b) [I1]-[I4] is satisfied, then for $\epsilon \in (0,1)$, $\lim_{n\to\infty} \mathbf{P}(\mathcal{T}^{\beta}_{1,n} > w_1(\epsilon)) = 1$ *under* H^{β}_1.

$\mathcal{T}^{\beta}_{1,n}$ is a simple test statistic, but for the 1-dimensional Ornstein-Uhlenbeck process defined by $\mathrm{d}X_t = -\beta(X_t - \gamma)\mathrm{d}t + \alpha \mathrm{d}W_t$ ($\alpha,\beta > 0$, $\gamma \in \mathbb{R}$), if β changes and γ does not change, this test statistic does not satisfy the identifiability condition [H2] (see Appendix A). For this reason, we introduce another test statistic. Let

$$\hat{\zeta}_i = \partial_\beta b(X_{t_{i-1}},\hat{\beta})^\top A^{-1}(X_{t_{i-1}},\hat{\alpha})(\Delta_i X - h_n b(X_{t_{i-1}},\hat{\beta})),$$

$$\mathcal{I}_n = \frac{1}{n}\sum_{i=1}^n \partial_\beta b(X_{t_{i-1}},\hat{\beta})^\top A^{-1}(X_{t_{i-1}},\hat{\alpha})\partial_\beta b(X_{t_{i-1}},\hat{\beta}),$$

$$\mathcal{T}^{\beta}_{2,n} = \frac{1}{\sqrt{T}} \max_{1\le k\le n} \left| \mathcal{I}_n^{-1/2}\left(\sum_{i=1}^k \hat{\zeta}_i - \frac{k}{n}\sum_{i=1}^n \hat{\zeta}_i\right)\right|.$$

We additionally make the following assumptions.

[B3] There exists an integer $\mathrm{m}_1 \ge 3$ such that $nh_n^{\mathrm{m}_1/(\mathrm{m}_1-1)} \to \infty$ and $b \in C^{4,\mathrm{m}_1+1}_{\uparrow}(\mathbb{R}^d \times \Theta_B)$.

[H3] Under H^{β}_1, there exist $\beta' \in \Theta_B$ and an estimator $\hat{\beta}$ such that $\sqrt{T}(\hat{\beta} - \beta') = O_{\mathbf{P}}(1)$.

For $\alpha \in \Theta_A, \beta_1, \beta_2 \in \Theta_B$, let

$$\mathcal{H}(\alpha,\beta_1,\beta_2) = \int_{\mathbb{R}^d} \partial_\beta b(x,\beta_2)^\top A^{-1}(x,\alpha)(b(x,\beta_1) - b(x,\beta_2))\mathrm{d}\mu_{(\alpha,\beta_1)}(x).$$

[H4] $\mathcal{H}(\alpha^*,\beta^*_1,\beta') \ne \mathcal{H}(\alpha^*,\beta^*_2,\beta')$ under H^{β}_1.

[I5] There exists an integer $\mathrm{m}_2 \ge 2$ such that $h_n^{-1/2}\vartheta_\beta^{\mathrm{m}_2} \to 0$ and $b \in C^{4,\mathrm{m}_2+1}_{\uparrow}(\mathbb{R}^d \times \Theta_B)$.

As in Theorem 3, we obtain the following result.

Theorem 4. *Suppose that [A1]-[A5] and [E2] hold.*

(1) If [B2] and [B3] are satisfied, then $\mathcal{T}^{\beta}_{2,n} \xrightarrow{\mathrm{d}} \sup_{0\le s\le 1} |\mathbb{B}^0_q(s)|$ *under* H^{β}_0.

(2) If either (a) [B3], [H3] and [H4] or (b) [I1]-[I3] and [I5] is satisfied, then for $\epsilon \in (0,1)$, $\lim_{n\to\infty} \mathbf{P}(\mathcal{T}^{\beta}_{2,n} > w_q(\epsilon)) = 1$ *under* H^{β}_1.

Remark 6. *Since* $h_n \to 0$, $nh_n \to \infty$ *and* $nh_n^2 \to 0$, *if* $h_n = O(n^{-\delta})$ *for some* $\delta \in (1/2, 1)$, *then there exists an integer* $\mathrm{m}_1 > 1/(1-\delta) > 2$ *such that* $nh_n^{\mathrm{m}_1/(\mathrm{m}_1-1)} \to \infty$. *Therefore, in [B3], we make the assumption of the smoothness of the drift coefficient b with respect to* β *up to order* $\mathrm{m}_1 + 1$ (≥ 4) *when the drift coefficient b is general. In particular, if* $b \in C^{4,\infty}_{\uparrow}(\mathbb{R}^d, \Theta_B)$, *then [B3] is satisfied. [I5] is also the assumption on the smoothness of b.*

3.1.2 Change point estimation

This subsection provides an estimator of the time of change of the drift parameter for the SDE such as Situation II.

We consider the following two cases.

Case A$_\beta$: $\vartheta_\beta = |\beta_1^* - \beta_2^*|$ depends on n, and as $n \to \infty$,

$$\vartheta_\beta \to 0, \quad T\vartheta_\beta^2 \to \infty, \quad \beta_1^* \to \beta_0 \in \operatorname{Int}\Theta_B.$$

Case B$_\beta$: $|\beta_1^* - \beta_2^*|$ is fixed.

We assume the following conditions.

[E3] There exist estimators $\hat{\beta}_k$ $(k = 1, 2)$ and a constant $\tau_*^\beta \in (0, 1)$ such that

$$\sqrt{T}(\hat{\beta}_k - \beta_k^*) = O_{\mathbf{P}}(1), \quad \beta = \begin{cases} \beta_1^*, & t \in [0, \tau_*^\beta T), \\ \beta_2^*, & t \in [\tau_*^\beta T, T]. \end{cases}$$

[J1] $T\vartheta_\beta^4 \to 0$ as $n \to \infty$ and $\vartheta_\beta^{-1}(\beta_k^* - \beta_0) = O(1)$ for $k = 1, 2$.

[J2] Let $f(x)$ be the following three functions: (a) $\Xi^\beta(x, \alpha^*, \beta_0)$, (b) $\partial_\beta \Xi^\beta(x, \alpha^*, \beta_0)$, (c) $\partial_{\beta_1}^3 \Gamma^\beta(x, \alpha^*, \beta_0, \beta_0)$. For any $\delta \in (1, 2)$ such that $nh_n^\delta \to \infty$,

$$\max_{[n^{1/\delta}] \le k \le n - [n\tau_*^\beta]} \left| \frac{1}{k} \sum_{i=[n\tau_*^\beta]+1}^{[n\tau_*^\beta]+k} f(X_{t_{i-1}}) - \int_{\mathbb{R}^d} f(x) \mathrm{d}\mu_{(\alpha^*, \beta_0)}(x) \right| \xrightarrow{\mathbf{P}} 0.$$

[J3] There exists $\mathrm{m}_3 \ge 3$ such that $nh_n^{\mathrm{m}_3/(\mathrm{m}_3-1)} \to \infty$, $h_n^{-1/2}\vartheta_\beta^{\mathrm{m}_3-1} \to 0$ and $b \in C_\uparrow^{4, \mathrm{m}_3+1}(\mathbb{R}^d \times \Theta_B)$.

Let

$$\Xi^\beta(x, \alpha, \beta) = \left[\partial_{\beta^{l_1}} b(x, \beta)^\mathsf{T} A^{-1}(x, \alpha) \partial_{\beta^{l_2}} b(x, \beta)\right]_{l_1, l_2 = 1}^q,$$
$$\Gamma^\beta(x, \alpha, \beta_1, \beta_2) = \operatorname{Tr}\left[A^{-1}(x, \alpha)(b(x, \beta_1) - b(x, \beta_2))^{\otimes 2}\right].$$

[K1] $\inf_x \Gamma^\beta(x, \alpha^*, \beta_1^*, \beta_2^*) > 0$.

[K2] There exists a constant $C > 0$ such that

(a) $\displaystyle\sup_{x, \alpha, \beta_k} \left(|\partial_\alpha \Gamma^\beta(x, \alpha, \beta_1, \beta_2)| \vee |\partial_{\beta_1} \Gamma^\beta(x, \alpha, \beta_1, \beta_2)| \vee |\partial_{\beta_2} \Gamma^\beta(x, \alpha, \beta_1, \beta_2)|\right) < C$,

(b) $\displaystyle\sup_{x, \alpha, \beta_k} \left| \left[\partial_{\beta^l} b(x, \beta_1)^\mathsf{T} A^{-1}(x, \alpha)(b(x, \beta_2) - b(x, \beta_3))\right]_{l=1}^q \right| < C$.

We define

$$\Psi_n(\tau:\beta_1,\beta_2|\alpha,k_n,l_n)=\sum_{i=k_n+1}^{[n\tau]}G_i(\beta_1|\alpha)+\sum_{i=[n\tau]+1}^{l_n}G_i(\beta_2|\alpha)$$

where $0\le k_n<[n\tau]$ and $[n\tau]<l_n\le n$. Set

$$\hat{\tau}_n^\beta=\underset{\tau\in[0,1]}{\mathrm{argmin}}\,\Psi_n(\tau:\hat{\beta}_1,\hat{\beta}_2|\hat{\alpha},0,n)$$

as an estimator of τ_*^β.

In Case A_β, we set

$$e_\beta=\lim_{n\to\infty}\vartheta_\beta^{-1}(\beta_1^*-\beta_2^*),\quad \mathcal{J}_\beta(\alpha)=e_\beta^\top\int_{\mathbb{R}^d}\Xi^\beta(x,\alpha,\beta_0)\mathrm{d}\mu_{(\alpha,\beta_0)}(x)e_\beta,$$
$$\mathbb{G}(v:\alpha)=-2\mathcal{J}_\beta(\alpha)^{1/2}\mathbb{W}(v)+\mathcal{J}_\beta(\alpha)|v|\quad\text{for } v\in\mathbb{R}.$$

Then, we have the following asymptotic properties of the estimator $\hat{\tau}_n^\beta$.

Theorem 5. *Suppose that [A1]-[A5], [E2] and [E3] hold.*

(1) Under [J1]-[J3], $T\vartheta_\beta^2(\hat{\tau}_n^\beta-\tau_*^\beta)\xrightarrow{\mathrm{d}}\underset{v\in\mathbb{R}}{\mathrm{argmin}}\,\mathbb{G}(v:\alpha^*)$ *in Case* A_β.

(2) Under [K1] and [K2], $T(\hat{\tau}_n^\beta-\tau_*^\beta)=O_{\mathbf{P}}(1)$ *in Case* B_β.

3.2 Change point inference with a change in diffusion parameter

In this section, we infer the change point in the drift parameter β when there is a change point in the diffusion parameter α. The estimator of the time of change of the diffusion parameter is already given in Section 2.2, and thus we assume [E1] and the existence of the estimator throughout this section.

[E4] There exists $\epsilon_1\in(0,1)$ such that $n^{\epsilon_1}(\hat{\tau}_n^\alpha-\tau_*^\alpha)=o_{\mathbf{P}}(1)$.

For $k=1,2$, let $\alpha_k^*\to\alpha_k^{(0)}\in\mathrm{Int}\,\Theta_A$ as $n\to\infty$, where we note that α_k^* may depend on n.

3.2.1 Change point detection

In this subsection, we treat the change detection for the drift parameter of the SDE in Situation III or IV. Since we can estimate the change point $\tau_*^\alpha T$ according to Section 2.2, we divide the intervals into two parts based on the estimated time of the change in diffusion parameters, and investigate the change in the drift parameter in each interval. Therefore, we consider the following two hypothesis testing problems.

$$H_0^{(1)}: \text{the drift parameter does not change over } [0,\tau_*^\alpha T]$$
vs.
$$H_1^{(1)}: \text{there exists } \tau_*^\beta\in(0,\tau_*^\alpha) \text{ such that } \beta^*=\begin{cases}\beta_{1,1}^*, & t\in[0,\tau_*^\beta T),\\ \beta_{1,2}^*, & t\in[\tau_*^\beta T,\tau_*^\alpha T],\end{cases}$$

$$H_0^{(2)}: \text{the drift parameter does not change over } [\tau_*^\alpha T,T]$$
vs.
$$H_1^{(2)}: \text{there exists } \tau_*^\beta\in(\tau_*^\alpha,1) \text{ such that } \beta^*=\begin{cases}\beta_{2,1}^*, & t\in[\tau_*^\alpha T,\tau_*^\beta T),\\ \beta_{2,2}^*, & t\in[\tau_*^\beta T,T],\end{cases}$$

where $\beta_{k,1}^*,\beta_{k,2}^*\in\operatorname{Int}\Theta_B$, $\beta_{k,1}^*\neq\beta_{k,2}^*$ for $k=1,2$.

Let $k=1,2$. We make the following assumptions.

[B2′]$_k$ There exists an estimator $\hat\beta_k$ such that $\sqrt{T}(\hat\beta_k-\beta^*)=O_{\mathbf{P}}(1)$ under $H_0^{(k)}$.

[B4] Let $f(x)$ be the following two functions: (a) $1_d^\top a^{-1}(x,\alpha_2^*)\partial_\beta b(x,\beta_0)$, (b) $\partial_\alpha(1_d^\top a^{-1}(x,\alpha_2^*))b(x,\beta_0)$. For any $\delta\in(1,2)$ such that $nh_n^\delta\to\infty$ and $M_n=[n(\tau_*^\alpha+2n^{-\epsilon_1})]$,

$$\max_{[n^{1/\delta}]\le k\le n-M_n}\left|\frac{1}{k}\sum_{i=M_n+1}^{M_n+k}f(X_{t_{i-1}})-\int_{\mathbb{R}^d}f(x)\mathrm{d}\mu_{(\alpha_2^*,\beta_0)}(x)\right|\xrightarrow{\mathbf{P}}0.$$

[H1′]$_k$ There exist $\beta_k'\in\Theta_B$ and an estimator $\hat\beta_k$ such that $\hat\beta_k-\beta_k'=o_{\mathbf{P}}(1)$ under $H_1^{(k)}$.

[H2′]$_k$ $\mathcal{G}(\alpha_k^{(0)},\beta_{k,1}^*,\beta_k')\neq\mathcal{G}(\alpha_k^{(0)},\beta_{k,2}^*,\beta_k')$ under $H_1^{(k)}$.

[I1′]$_k$ $\vartheta_{\beta_k}=|\beta_{k,1}^*-\beta_{k,2}^*|$ depends on n, and $\vartheta_{\beta_k}\to0$, $T\vartheta_{\beta_k}^2\to\infty$ as $n\to\infty$ under $H_1^{(k)}$.

[I2′]$_k$ There exists $\beta_k^{(0)}\in\operatorname{Int}\Theta_B$ such that $\vartheta_{\beta_k}^{-1}(\beta_{k,l}^*-\beta_k^{(0)})\to d_{k,l}\in\mathbb{R}^q$ as $n\to\infty$ for $l=1,2$.

[I3′]$_k$ There exist β_k' with $\beta_k'-\beta_k^{(0)}=o(1)$ and an estimator $\hat\beta_k$ such that $\sqrt{T}(\hat\beta_k-\beta_k')=O_{\mathbf{P}}(1)$ under $H_1^{(k)}$.

[I4′]$_k$ $\displaystyle\int_{\mathbb{R}^d}1_d^\top a^{-1}(x,\alpha_k^{(0)})\partial_\beta b(x,\beta_k^{(0)})\mathrm{d}\mu_{(\alpha_k^{(0)},\beta_k^{(0)})}(x)(d_{k,1}-d_{k,2})\neq0$ under $H_1^{(k)}$.

[I5′]$_k$ There exists $\mathrm{m}_2'\ge3$ such that $n^{-\mathrm{m}_2'}h_n^{-(\mathrm{m}_2'+1)}=O(1)$, $h_n^{-1/2}\vartheta_{\beta_k}^{\mathrm{m}_2'}\to0$ and $b\in C_\uparrow^{4,\mathrm{m}_2'+1}(\mathbb{R}^d\times\Theta_B)$.

Remark 7. *Since $|\alpha_1^*-\alpha_2^*|$ depends on n and satisfies $n|\alpha_1^*-\alpha_2^*|^2\to\infty$ in Case A_α and $|\alpha_1^*-\alpha_2^*|$ is fixed in Case B_α, it is obvious from Theorem 2 that there exists $\epsilon_1\in(0,1)$ such that [E4] holds. In practice, since $|\alpha_1^*-\alpha_2^*|$ is unknown, we obtain an estimator $\hat\epsilon_1$ of ϵ_1 satisfying [E4], for example, as follows. Let $\hat\alpha_1$, $\hat\alpha_2$ be estimators of α_1^*, α_2^*. If $n|\alpha_1^*-\alpha_2^*|^2\to\infty$, then $n|\hat\alpha_1-\hat\alpha_2|^2\to\infty$ in probability, in other words, the probability of $n|\hat\alpha_1-\hat\alpha_2|^2<1$ converges to zero. According to Theorem 2, for a sufficiently large n, we get $\hat\epsilon_1$ such that $\hat\epsilon_1=0.45\wedge(0.9\log(n|\hat\alpha_1-\hat\alpha_2|^2)/\log n)$.*

Let $\underline{\tau}_n = \hat{\tau}_n^\alpha - n^{-\epsilon_1}$ and $\overline{\tau}_n = \hat{\tau}_n^\alpha + n^{-\epsilon_1}$. For $r = d$, we define the test statistics for detecting a change in the drift parameter as follows.

$$\mathcal{T}_{1,n}^{(1)} = \frac{1}{\sqrt{d\underline{\tau}_n T}} \max_{1\le k\le [n\underline{\tau}_n]} \left| \sum_{i=1}^{k} \hat{\xi}_{1,i} - \frac{k}{[n\underline{\tau}_n]} \sum_{i=1}^{[n\underline{\tau}_n]} \hat{\xi}_{1,i} \right|,$$

$$\mathcal{T}_{1,n}^{(2)} = \frac{1}{\sqrt{d(1-\overline{\tau}_n)T}} \max_{1\le k\le n-[n\overline{\tau}_n]} \left| \sum_{i=[n\overline{\tau}_n]+1}^{[n\overline{\tau}_n]+k} \hat{\xi}_{2,i} - \frac{k}{n-[n\overline{\tau}_n]} \sum_{i=[n\overline{\tau}_n]+1}^{n} \hat{\xi}_{2,i} \right|,$$

where $\hat{\xi}_{k,i} = 1_d^\top a^{-1}(X_{t_{i-1}}, \hat{\alpha}_k)(\Delta_i X - h_n b(X_{t_{i-1}}, \hat{\beta}_k))$.

The following theorem provides the results on the asymptotic properties of $\mathcal{T}_{1,n}^{(1)}$ and $\mathcal{T}_{1,n}^{(2)}$.

Theorem 6. *Let $k = 1, 2$. Suppose that [A1]-[A5], [E1] and [E4] hold.*

(1) If $[B2']_1$ is satisfied, then $\mathcal{T}_{1,n}^{(1)} \xrightarrow{d} \sup_{0\le s\le 1} |\mathbb{B}_1^0(s)|$ under $H_0^{(1)}$.

(2) If $[B2']_2$ and [B4] hold, then $\mathcal{T}_{1,n}^{(2)} \xrightarrow{d} \sup_{0\le s\le 1} |\mathbb{B}_1^0(s)|$ under $H_0^{(2)}$.

(3) If either (a) $[H1']_k$ and $[H2']_k$ or (b) $[I1']_k$-$[I5']_k$ is satisfied, then for $\epsilon \in (0,1)$, $\lim_{n\to\infty} \mathbf{P}(\mathcal{T}_{1,n}^{(k)} > w_1(\epsilon)) = 1$ under $H_1^{(k)}$.

Moreover, we consider other test statistics as follows.

$$\mathcal{T}_{2,n}^{(1)} = \frac{1}{\sqrt{\underline{\tau}_n T}} \max_{1\le k\le [n\underline{\tau}_n]} \left| \mathcal{I}_{1,n}^{-1/2} \left(\sum_{i=1}^{k} \hat{\zeta}_{1,i} - \frac{k}{[n\underline{\tau}_n]} \sum_{i=1}^{[n\underline{\tau}_n]} \hat{\zeta}_{1,i} \right) \right|,$$

$$\mathcal{T}_{2,n}^{(2)} = \frac{1}{\sqrt{(1-\overline{\tau}_n)T}} \max_{1\le k\le n-[n\overline{\tau}_n]} \left| \mathcal{I}_{2,n}^{-1/2} \left(\sum_{i=[n\overline{\tau}_n]+1}^{[n\overline{\tau}_n]+k} \hat{\zeta}_{2,i} - \frac{k}{n-[n\overline{\tau}_n]} \sum_{i=[n\overline{\tau}_n]+1}^{n} \hat{\zeta}_{2,i} \right) \right|,$$

where

$$\hat{\zeta}_{k,i} = \partial_\beta b(X_{t_{i-1}}, \hat{\beta}_k)^\top A^{-1}(X_{t_{i-1}}, \hat{\alpha}_k)\Big(\Delta_i X - h_n b(X_{t_{i-1}}, \hat{\beta}_k)\Big),$$

$$\mathcal{I}_{1,n} = \frac{1}{[n\underline{\tau}_n]} \sum_{i=1}^{[n\underline{\tau}_n]} \partial_\beta b(X_{t_{i-1}}, \hat{\beta}_1)^\top A^{-1}(X_{t_{i-1}}, \hat{\alpha}_1)\partial_\beta b(X_{t_{i-1}}, \hat{\beta}_1),$$

$$\mathcal{I}_{2,n} = \frac{1}{n-[n\overline{\tau}_n]} \sum_{i=[n\overline{\tau}_n]+1}^{n} \partial_\beta b(X_{t_{i-1}}, \hat{\beta}_2)^\top A^{-1}(X_{t_{i-1}}, \hat{\alpha}_2)\partial_\beta b(X_{t_{i-1}}, \hat{\beta}_2).$$

We additionally make the following assumptions.

[B5] Let $f(x)$ be the following three functions:

(a) $\partial_\beta^2 b(x, \beta_0) A^{-1}(x, \alpha_2^*) b(x, \beta_0)$, (b) $\partial_\beta b(x, \beta_0)^\top A^{-1}(x, \alpha_2^*) \partial_\beta b(x, \beta_0)$,

(c) $\partial_\beta b(x,\beta_0)\partial_\alpha A^{-1}(x,\alpha_2^*)b(x,\beta_0)$.

For any $\delta \in (1,2)$ such that $nh_n^\delta \to \infty$ and $M_n = [n(\tau_*^\alpha + 2n^{-\epsilon_1})]$,

$$\max_{[n^{1/\delta}]\le k\le n-M_n}\left|\frac{1}{k}\sum_{i=M_n+1}^{M_n+k} f(X_{t_{i-1}}) - \int_{\mathbb{R}^d} f(x)\mathrm{d}\mu_{(\alpha_2^*,\beta_0)}(x)\right| \xrightarrow{\mathbf{P}} 0.$$

For $k = 1,2$, we make the following assumptions.

[H3′]$_k$ There exist $\beta_k' \in \Theta_B$ and an estimator $\hat\beta_k$ such that $\sqrt{T}(\hat\beta_k - \beta_k') = O_{\mathbf{P}}(1)$ under $H_1^{(k)}$.

[H4′]$_k$ $\mathcal{H}(\alpha_k^{(0)},\beta_{k,1}^*,\beta_k') \neq \mathcal{H}(\alpha_k^{(0)},\beta_{k,2}^*,\beta_k')$ under $H_1^{(k)}$.

[I6]$_k$ $n^{\epsilon_1}\vartheta_{\beta_k} \to \infty$.

We obtain the following theorem.

Theorem 7. *Let $k = 1,2$. Suppose that [A1]-[A5], [E1] and [E4] hold.*

(1) If [B2′]$_1$ and [B3] are satisfied, then $\mathcal{T}_{2,n}^{(1)} \xrightarrow{\mathrm{d}} \sup_{0\le s\le 1}|\mathbb{B}_q^0(s)|$ under $H_0^{(1)}$.

(2) If [B2′]$_2$, [B3] and [B5] are satisfied, then $\mathcal{T}_{2,n}^{(2)} \xrightarrow{\mathrm{d}} \sup_{0\le s\le 1}|\mathbb{B}_q^0(s)|$ under $H_0^{(2)}$.

(3) If either (a) [B3], [H3′]$_k$ and [H4′]$_k$ or (b) [I1′]$_k$-[I3′]$_k$, [I5′]$_k$ and [I6′]$_k$ is satisfied, then for $\epsilon \in (0,1)$, $\mathbf{P}(\mathcal{T}_{2,n}^{(k)} > w_q(\epsilon)) \to 1$ under $H_1^{(k)}$.

Remark 8. *When the drift parameter changes at the same time point as the diffusion parameter does, these tests are unable to detect a change in the drift parameter. In other words, even if the null hypotheses $H_0^{(1)}$ and $H_0^{(2)}$ are not rejected, it is possible that the drift parameter changes at the same time point as the diffusion parameter does. We will discuss the change in the drift parameters when neither $H_0^{(1)}$ nor $H_0^{(2)}$ is rejected in Subsection 3.2.3.*

3.2.2 Change point estimation

In this subsection, we estimate the time of the change of the drift parameter for the SDE such as Situation IV-(i) or (ii). As in Subsection 3.1.2, we address two cases with different levels of change in the parameter.

Let $k = 1,2$. We make the following assumptions.

[J2′]$_k$ Let $f(x)$ be the following three functions: (a) $\Xi^\beta(x,\alpha_k^*,\beta_0)$, (b) $\partial_\beta\Xi^\beta(x,\alpha_k^*,\beta_0)$, (c) $\partial_{\beta_1}^3\Gamma^\beta(x,\alpha_k^*,\beta_0,\beta_0)$.
For any $\delta \in (1,2)$ such that $nh_n^\delta \to \infty$,

$$\max_{[n^{1/\delta}]\le m\le n-[n\tau_*^\beta]}\left|\frac{1}{m}\sum_{i=[n\tau_*^\beta]+1}^{[n\tau_*^\beta]+m} f(X_{t_{i-1}}) - \int_{\mathbb{R}^d} f(x)\mathrm{d}\mu_{(\alpha_k^*,\beta_0)}(x)\right| \xrightarrow{\mathbf{P}} 0.$$

[K1′]$_k$ $\inf_x \Gamma^\beta(x, \alpha_k^{(0)}, \beta_1^*, \beta_2^*) > 0$.

Consider $\tau_*^\beta < \tau_*^\alpha$. Define

$$\hat{\tau}_{1,n}^\beta = \underset{\tau\in[0,\underline{\tau}_n]}{\operatorname{argmin}}\, \Psi_n(\tau : \hat{\beta}_1, \hat{\beta}_2 | \hat{\alpha}_1, 0, [n\underline{\tau}_n])$$

as an estimator of τ_*^β.

Then, we obtain the following result of the asymptotic properties of the estimator $\hat{\tau}_{1,n}^\beta$.

Theorem 8. *Let $\tau_*^\beta < \tau_*^\alpha$. Suppose that [A1]-[A5], [E1], [E3] and [E4] hold.*

(1) Under [J1], [J2′]$_1$ and [J3], $T\vartheta_\beta^2(\hat{\tau}_{1,n}^\beta - \tau_*^\beta) \xrightarrow{\mathrm{d}} \underset{v\in\mathbb{R}}{\operatorname{argmin}}\, \mathbb{G}(v : \alpha_1^{(0)})$ in Case A_β.

(2) Under [K1′]$_1$ and [K2], $T(\hat{\tau}_{1,n}^\beta - \tau_*^\beta) = O_{\mathbf{P}}(1)$ in Case B_β.

Consider $\tau_*^\alpha < \tau_*^\beta$. Set

$$\hat{\tau}_{2,n}^\beta = \underset{\tau\in[\overline{\tau}_n,1]}{\operatorname{argmin}}\, \Psi_n(\tau : \hat{\beta}_1, \hat{\beta}_2 | \hat{\alpha}_2, [n\overline{\tau}_n], n)$$

as an estimator of τ_*^β.

As in Theorem 8, we have the following result.

Theorem 9. *Let $\tau_*^\alpha < \tau_*^\beta$. Suppose that [A1]-[A5], [E1], [E3] and [E4] hold.*

(1) Under [J1], [J2′]$_2$ and [J3], $T\vartheta_\beta^2(\hat{\tau}_{2,n}^\beta - \tau_*^\beta) \xrightarrow{\mathrm{d}} \underset{v\in\mathbb{R}}{\operatorname{argmin}}\, \mathbb{G}(v : \alpha_2^{(0)})$ in Case A_β.

(2) Under [K1′]$_2$ and [K2], $T(\hat{\tau}_{2,n}^\beta - \tau_*^\beta) = O_{\mathbf{P}}(1)$ in Case B_β.

Remark 9. *We proposed the methods to detect changes in the drift parameter and to estimate the change point of the drift parameter, which assume the existence of a change point estimator of the diffusion parameter. This method is based on the fact that the tests utilizing normalization of errors are distribution free under no change in the parameters. For this thesis, we employed the method using the change point estimator of the diffusion parameter so that the asymptotic null distribution of the test statistics for changes in the drift parameter is the distribution of the supremum of the norm of a Brownian bridge by using the convergence in* (3.1)*. For example, instead of $\hat{\xi}_{k,i}$ or $\hat{\zeta}_{k,i}$, one could consider a test statistic without a change point estimator of the diffusion parameter and the diffusion term. In this case, however, the test generally dose not converge to the supremum of a Brownian bridge under the null hypothesis that the drift parameter does not change. The discussion of methods to detect changes in the drift parameter independent of the diffusion parameter and to estimate its change point is a subject for future work.*

3.2.3 Change in diffusion and drift parameters at the same time

Since $\mathcal{T}_{1,n}^{(1)}$ and $\mathcal{T}_{1,n}^{(2)}$ (or, $\mathcal{T}_{2,n}^{(1)}$ and $\mathcal{T}_{2,n}^{(2)}$) are tests for the change of the drift parameter in $[0, \underline{\tau}_n T]$ and $[\overline{\tau}_n T, T]$, respectively, neither test can detect the change when the drift parameter changes in $[\underline{\tau}_n T, \overline{\tau}_n T]$, i.e., $\tau_*^\beta = \tau_*^\alpha$. Therefore, in this subsection, we consider how to investigate whether the drift parameter changes at the same time as the diffusion parameter. In other words, we consider a method for detecting a change in the drift parameter for the SDE in Situation IV-(iii).

If neither $H_0^{(1)}$ nor $H_0^{(2)}$ is rejected, we construct the estimators $\check{\beta}_1$ and $\check{\beta}_2$ for β_1^* and β_2^* with data from the intervals $[0, \underline{\tau}_n T]$ and $[\overline{\tau}_n T, T]$, respectively. Notice that the estimators $\check{\beta}_1$ and $\check{\beta}_2$ can be constructed to satisfy

$$\sqrt{T}(\check{\beta}_1 - \beta_1^*) = O_{\mathbf{P}}(1), \qquad \sqrt{T}(\check{\beta}_2 - \beta_2^*) = O_{\mathbf{P}}(1).$$

Since one then has

$$\sqrt{T}|\beta_1^* - \beta_2^*| \le \sqrt{T}|\check{\beta}_1 - \beta_1^*| + \sqrt{T}|\check{\beta}_2 - \beta_2^*| + \sqrt{T}|\check{\beta}_1 - \check{\beta}_2| = O_{\mathbf{P}}(1) + \sqrt{T}|\check{\beta}_1 - \check{\beta}_2|$$

and

$$\sqrt{T}|\check{\beta}_1 - \check{\beta}_2| \le \sqrt{T}|\check{\beta}_1 - \beta_1^*| + \sqrt{T}|\check{\beta}_2 - \beta_2^*| + \sqrt{T}|\beta_1^* - \beta_2^*| = O_{\mathbf{P}}(1) + \sqrt{T}|\beta_1^* - \beta_2^*|,$$

$\sqrt{T}|\check{\beta}_1 - \check{\beta}_2| = O_{\mathbf{P}}(1)$ is equivalent to $\sqrt{T}|\beta_1^* - \beta_2^*| = O(1)$. Note that if $\sqrt{T}|\check{\beta}_1 - \check{\beta}_2| \to \infty$, then $\sqrt{T}|\check{\beta}_1 - \check{\beta}_2| \ne O_{\mathbf{P}}(1)$, and if $\sqrt{T}|\beta_1^* - \beta_2^*|$ is monotone, then $\sqrt{T}|\beta_1^* - \beta_2^*| \ne O(1)$ is equivalent to $\sqrt{T}|\beta_1^* - \beta_2^*| \to \infty$. Hence, we have the following assertions.

$$\text{If } \sqrt{T}|\check{\beta}_1 - \check{\beta}_2| = O_{\mathbf{P}}(1), \text{ then } \sqrt{T}|\beta_1^* - \beta_2^*| = O(1). \tag{3.2}$$

$$\text{If } \sqrt{T}|\check{\beta}_1 - \check{\beta}_2| \to \infty, \text{ then } \sqrt{T}|\beta_1^* - \beta_2^*| \to \infty. \tag{3.3}$$

The facts imply that if $\sqrt{T}|\check{\beta}_1 - \check{\beta}_2|$ is sufficiently large, then we infer that the drift parameter changes at $\tau_*^\alpha T$. Here we note that $\tau_*^\alpha T$ is the same time at which the diffusion parameter changes.

Remark 10. *The change in the drift parameter that satisfies the assumption [I1′]$_k$ can be detected by the test $\mathcal{T}_{1,n}^{(1)}$ or $\mathcal{T}_{1,n}^{(2)}$ if the change does not occur at the same time as the diffusion parameter, and can also be detected by the above method based on* (3.3) *if the change occurs at the same time. As we saw above, we can theoretically determine whether the drift parameter changes at the same time as the diffusion parameter by investigating $\sqrt{T}|\check{\beta}_1 - \check{\beta}_2|$, but it would be difficult to determine whether the drift parameter changes simultaneously with the diffusion parameter in practice. See the numerical simulations in Chapter 4.*

Chapter 4

Numerical simulations

In this chapter, we consider the SDEs of the four situations in the introduction and verify our main results by numerical simulations. The number of iterations is 1000 for all situations. In the hypothesis testing problem, the significance level ϵ is 0.05 or 0.10, and the corresponding critical values are obtained from the following: the Brownian bridge is generated by taking 10^4 points on the interval $[0, 1]$, and the maximum value of its norm is recorded. This is repeated 10^4 times. As a result, we have $w_1(0.05) = 1.3617$, $w_2(0.05) = 1.5736$, $w_1(0.10) = 1.2232$ and $w_2(0.10) = 1.4437$, where $w_k(\epsilon)$ is the upper-ϵ point of $\sup_{0\le s\le 1} |\mathbb{B}_k^0(s)|$.

4.1 Model 1 : Ornstein-Uhlenbeck process

We consider the one-dimensional Ornstein-Uhlenbeck process defined by

$$\mathrm{d}X_t = -\beta(X_t - \gamma)\mathrm{d}t + \alpha \mathrm{d}W_t, \quad X_0 = x_0,$$

where $\alpha, \beta > 0$ and $\gamma \in \mathbb{R}$.

4.1.1 Situation I : neither parameter changes

In order to verify Theorems 1, 3 and 4, we treat the following situation.

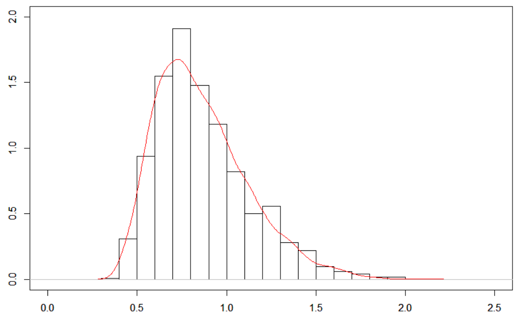
$$X_t = X_0 - \int_0^t \beta^*(X_s - \gamma^*)\mathrm{d}s + \alpha^* W_t, \quad t \in [0, T],$$

where $X_0 = 1$, $\alpha^* = 1$, $\beta^* = 1$, $\gamma^* = 1$. In this simulation, we set that the sample size of the data $\{X_{t_i}\}_{i=0}^n$ is $n = 10^5$ or 10^6, $h_n = n^{-3/5}$, $T = nh_n = n^{2/5}$, $nh_n^2 = n^{-1/5}$ and the significant level is $\epsilon = 0.10$.

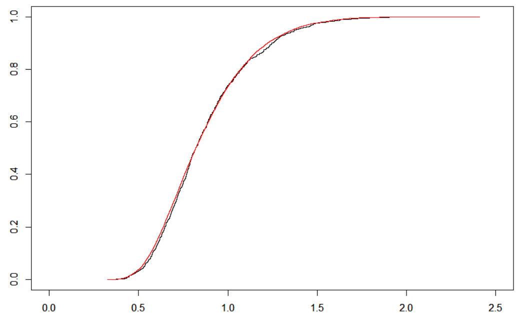
We verified the performance of the test statistics $\mathcal{T}_n^\alpha$, $\mathcal{T}_{1,n}^\beta$ and $\mathcal{T}_{2,n}^\beta$. Table 4.1 and Figure 1 show the empirical sizes, the histograms and the empirical distribution functions (EDFs) of $\mathcal{T}_n^\alpha$, $\mathcal{T}_{1,n}^\beta$ and $\mathcal{T}_{2,n}^\beta$. We find from Table 4.1 and Figure 1 that the proportions of the test statistics that exceed the critical values are close to $\epsilon = 0.10$ and the distribution of the test statistics almost corresponds with the null distribution, which implies that the test statistics have good performance.

Table 4.1: Proportions over the corresponding critical value in Situation I of Model 1.

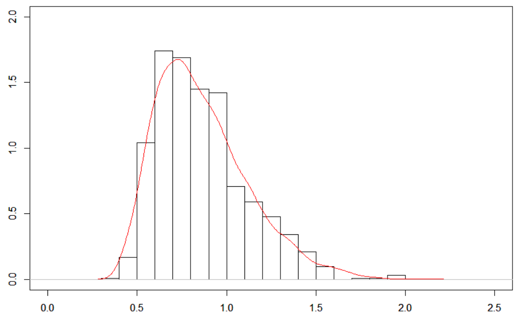
n	T	h_n	$\mathcal{T}_n^{\alpha}$	$\mathcal{T}_{1,n}^{\beta}$	$\mathcal{T}_{2,n}^{\beta}$
10^5	100	10^{-3}	0.098	0.095	0.094
10^6	251.19	2.51×10^{-4}	0.118	0.104	0.091



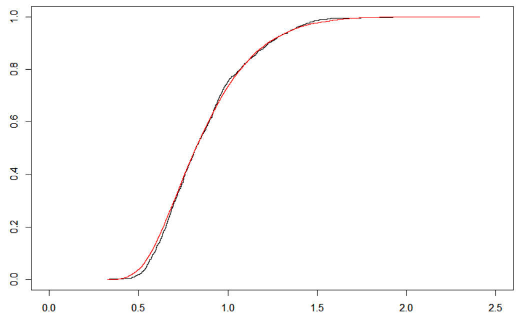
(a) Histogram of $\mathcal{T}_n^{\alpha}$ with $n = 10^6$.



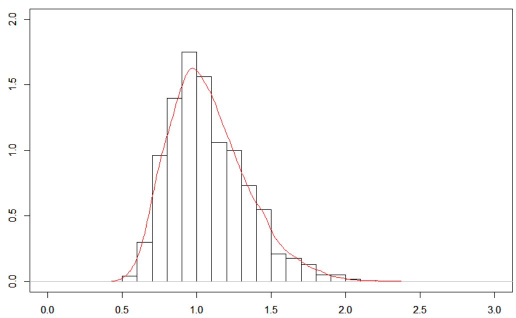
(b) EDF of $\mathcal{T}_n^{\alpha}$ with $n = 10^6$.



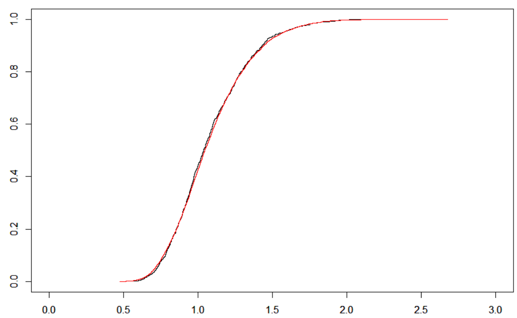
(c) Histogram of $\mathcal{T}_{1,n}^{\beta}$ with $n = 10^6$.



(d) EDF of $\mathcal{T}_{1,n}^{\beta}$ with $n = 10^6$.



(e) Histogram of $\mathcal{T}_{2,n}^{\beta}$ with $n = 10^6$.



(f) EDF of $\mathcal{T}_{2,n}^{\beta}$ with $n = 10^6$.

Figure 1: Histogram (black line) versus theoretical density function (red line) and empirical distribution function (black line) versus theoretical distribution function (red line) in Situation I of Model 1.

4.1.2 Situation II : only drift parameter changes

We consider the following situation to support Theorems 3-5.

$$X_t = \begin{cases} X_0 - \displaystyle\int_0^t \beta^*(X_s - \gamma_1^*)\mathrm{d}s + \alpha^* W_t, & t \in [0, \tau_*^\beta T), \\ X_{\tau_*^\beta T} + \displaystyle\int_{\tau_*^\beta T}^t \beta^*(X_s - \gamma_2^*)\mathrm{d}s + \alpha^*(W_t - W_{\tau_*^\beta T}), & t \in [\tau_*^\beta T, T], \end{cases}$$

where $X_0 = 5$, $\alpha^* = 0.5, \beta^* = 2.5$, $\gamma_1^* = 5 + \vartheta_\beta$, $\gamma_2^* = 5$, $\tau_*^\beta = 0.5$. In this simulation, we set the sample size of the data $\{X_{t_i}\}_{i=0}^n$ being $n = 10^6$, $h_n = n^{-4/7}$, $T = n^{3/7}$, $nh_n^2 = n^{-1/7}$, $\vartheta_\beta = n^{-1/8}$ and the significant level $\epsilon = 0.10$.

We first verified the performance of the test statistics $\mathcal{T}_n^\alpha$, $\mathcal{T}_{1,n}^\beta$ and $\mathcal{T}_{2,n}^\beta$. The simulation results of the test statistics can be found in Table 4.2 and Figure 2. Table 4.2 shows that the proportion of $\mathcal{T}_n^\alpha$ that exceed the critical value is close to $\epsilon = 0.10$ and the change in the drift parameter is detected in all iterations.

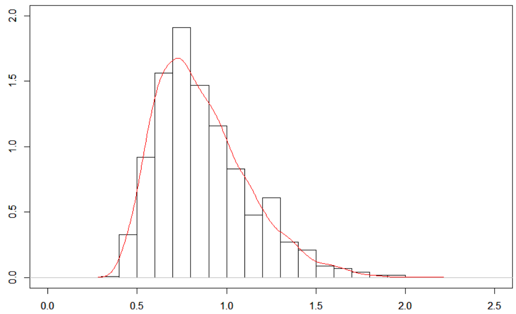
Table 4.2: Proportions over the corresponding critical value in Situation II of Model 1.

n	T	h_n	$\mathcal{T}_n^\alpha$	$\mathcal{T}_{1,n}^\beta$	$\mathcal{T}_{2,n}^\beta$
10^6	372.76	3.73×10^{-4}	0.117	1.000	1.000

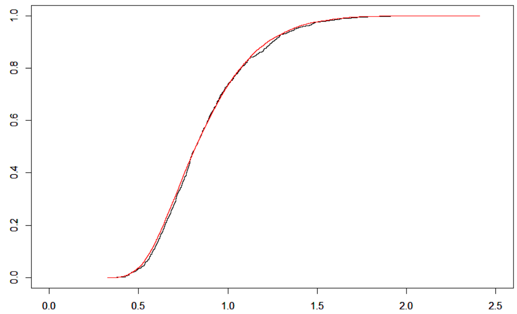
Next, we estimated the change point of the drift parameter. In all iterations, the change point was detected in the intervals $[T/4, T]$ and $[0, 3T/4]$. Therefore, we estimated β_1^* and β_2^* from $[0, T/4]$ and $[3T/4, T]$, respectively. The estimates of α^*, β_1^*, β_2^* and τ_*^β are reported in Table 4.3, and the histogram and the EDF of the estimator $\hat{\tau}_n^\beta$ are illustrated in Figure 3. From Figure 3, we can see that the distribution of the estimator almost corresponds with the asymptotic distribution in (1) of Theorem 5 and the estimators have good performance.

Table 4.3: Mean and standard deviation of the estimators in Situation II of Model 1. True values: $\alpha^* = 0.5, \beta^* = 2.5$, $\gamma_1^* = 5.1778$, $\gamma_2^* = 5$, $\tau_*^\beta = 0.5$.

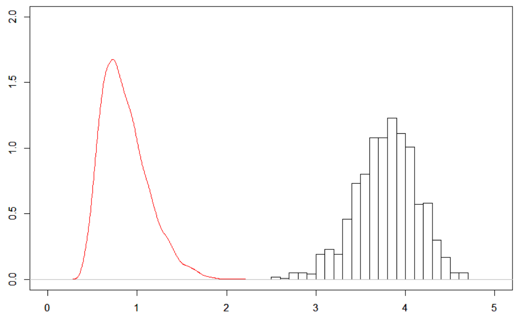
n	T	h_n	$\hat{\alpha}$	$\hat{\beta}_1$	$\hat{\gamma}_1$	$\hat{\beta}_2$	$\hat{\gamma}_2$	$\hat{\tau}_n^\beta$
10^6	372.76	3.73×10^{-4}	0.5001	2.5498	5.1773	2.5431	4.9998	0.4980
			(0.0004)	(0.2257)	(0.0205)	(0.2468)	(0.0203)	(0.0134)



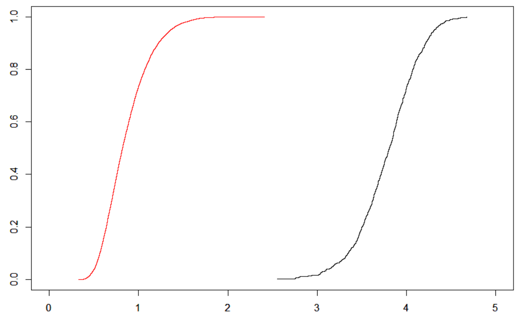
(a) Histogram of $\mathcal{T}_n^{\alpha}$ with $n = 10^6$.



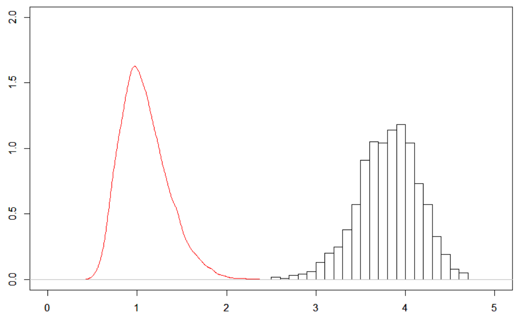
(b) EDF of $\mathcal{T}_n^{\alpha}$ with $n = 10^6$.



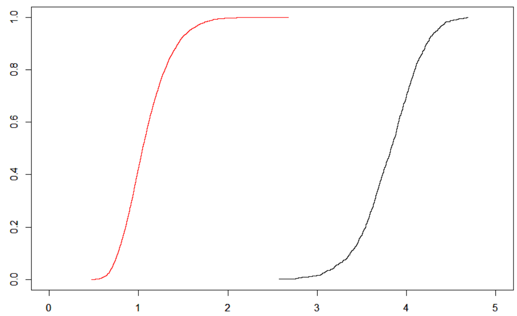
(c) Histogram of $\mathcal{T}_{1,n}^{\beta}$ with $n = 10^6$.



(d) EDF of $\mathcal{T}_{1,n}^{\beta}$ with $n = 10^6$.

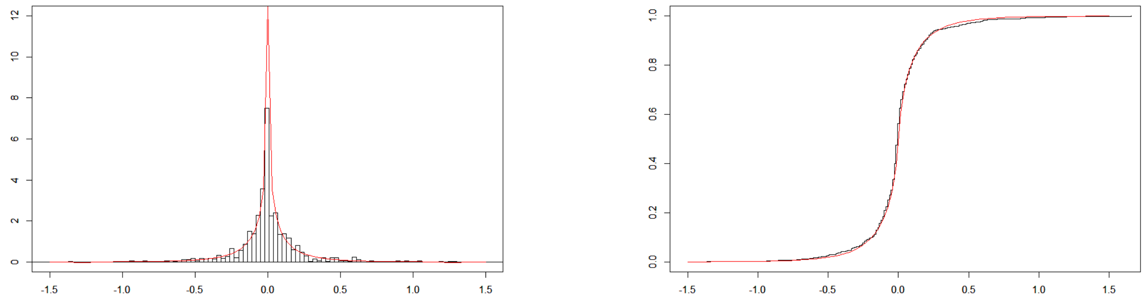


(e) Histogram of $\mathcal{T}_{2,n}^{\beta}$ with $n = 10^6$.



(f) EDF of $\mathcal{T}_{2,n}^{\beta}$ with $n = 10^6$.

Figure 2: Histogram (black line) versus theoretical density function (red line) and empirical distribution function (black line) versus theoretical distribution function (red line) in Situation II of Model 1.



(a) Histogram of $T\vartheta_\beta^2(\hat{\tau}_n^\beta - \tau_*^\beta)$.

(b) EDF of $T\vartheta_\beta^2(\hat{\tau}_n^\beta - \tau_*^\beta)$.

Figure 3: Histogram (black line) versus theoretical density function (red line) and empirical distribution function (black line) versus theoretical distribution function (red line) with $n = 10^7$ in Situation II of Model 1.

4.1.3 Situation III : only diffusion parameter changes

We deal with the following situation in support of Theorems 2, 6, 7 and (3.2).

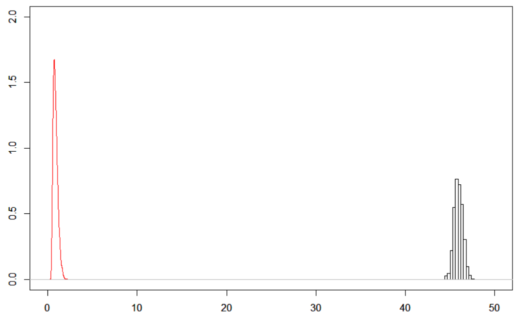
$$X_t = \begin{cases} X_0 - \displaystyle\int_0^t \beta^*(X_s - \gamma^*)\mathrm{d}s + \alpha_1^* W_t, & t \in [0, \tau_*^\alpha T), \\ X_{\tau_*^\alpha T} - \displaystyle\int_{\tau_*^\alpha T}^t \beta^*(X_s - \gamma^*)\mathrm{d}s + \alpha_2^*(W_t - W_{\tau_*^\alpha T}), & t \in [\tau_*^\alpha T, T], \end{cases}$$

where $X_0 = 2$, $\tau_*^\alpha = 0.8$, $\alpha_1^* = 1$, $\alpha_2^* = 1.2$. We set the sample size of the data $\{X_{t_i}\}_{i=0}^n$ being $n = 10^6$ or 10^7, $h_n = n^{-13/25}$, $T = nh_n = n^{12/25}$, $nh_n^2 = n^{-1/25}$ and the significant level $\epsilon = 0.05$.

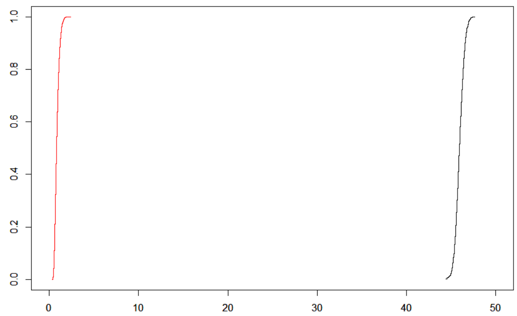
We first tested for a change in the diffusion parameter in the interval $[0, T]$. As a result, the change was detected in all 1000 iterations and Figure 4 shows the histogram and the EDF of the test statistic $\mathcal{T}_n^\alpha$. In order to estimate the parameters before and after the change, we tested for the change in the diffusion parameter in the interval $[0.125T, 0.875T]$. Since the results indicated that the change is detected in all 1000 iterations, we estimated α_1^* and α_2^* using the data obtained from the intervals $[0, 0.125T]$ and $[0.875T, T]$, respectively, and estimated τ_*^α using the estimators $\hat{\alpha}_1$ and $\hat{\alpha}_2$. Table 4.4 and Figure 5 show the simulation results of the estimates of α_1^*, α_2^* and τ_*^α. In this case, we chose $\epsilon_1 = 0.45$ for all iterations. It seems from Figure 5 that $n^{\epsilon_1}(\hat{\tau}_n^\alpha - \tau_*^\alpha) = o_{\mathbf{P}}(1)$ in this example.

Next, we tested for a change in the drift parameter in the intervals $[0, \underline{\tau}_n T]$ and $[\overline{\tau}_n T, T]$. Table 4.5 and Figure 6 show the results of the tests for the change in the drift parameter. From Table 4.5 and Figures 6, we can see that the proportions of the test statistics that exceed the critical values are close to the significance level $\epsilon = 0.05$, and the distribution of the test statistics almost corresponds with the null distribution, which implies that the test statistics have good performance.

Finally, we constructed estimators $\check{\boldsymbol{\beta}}_1 = (\check{\beta}_1, \check{\gamma}_1)$ and $\check{\boldsymbol{\beta}}_2 = (\check{\beta}_2, \check{\gamma}_2)$ using the data obtained from the intervals $[0, \underline{\tau}_n T]$ and $[\overline{\tau}_n T, T]$, respectively when the test statistics $\mathcal{T}_{1,n}^{(1)}$ and $\mathcal{T}_{1,n}^{(2)}$ did not exceed the critical value, and investigate $\sqrt{T}|\check{\boldsymbol{\beta}}_1 - \check{\boldsymbol{\beta}}_2|$. Table 4.6 and Figure 7 show the results of the estimates of $\boldsymbol{\beta}_1^*$ and $\boldsymbol{\beta}_2^*$. It appears from Figure 7 that $\sqrt{T}|\check{\boldsymbol{\beta}}_1 - \check{\boldsymbol{\beta}}_2|$ is bounded in probability.



(a) Histogram of $\mathcal{T}_n^\alpha$ with $n = 10^6$.

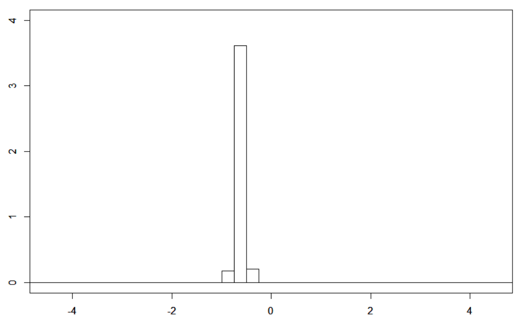


(b) EDF of $\mathcal{T}_n^\alpha$ with $n = 10^6$.

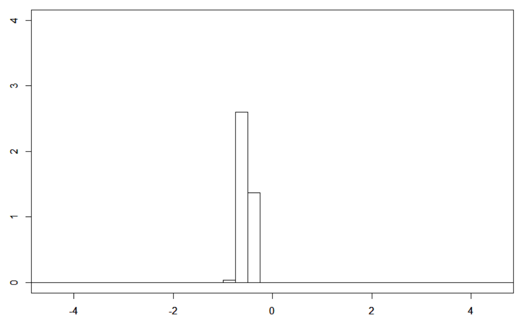
Figure 4: Histogram (black line) versus theoretical density function (red line) and empirical distribution function (black line) versus theoretical distribution function (red line) in Situation III of Model 1.

Table 4.4: Mean and standard deviation of the estimators in Situation III of Model 1. True values: $\alpha_1^* = 1$, $\alpha_2^* = 1.2$, $\tau_*^\alpha = 0.8$.

n	T	h_n	$\hat{\alpha}_1$	$\hat{\alpha}_2$	$\hat{\tau}_n^\alpha$
10^6	758.58	7.59×10^{-4}	1.0002	1.2002	0.7988
			(0.0021)	(0.0024)	(0.0002)
10^7	2290.87	2.29×10^{-4}	1.0001	1.2001	0.7996
			(0.0006)	(0.0008)	(0.0001)



(a) $n = 10^6$.

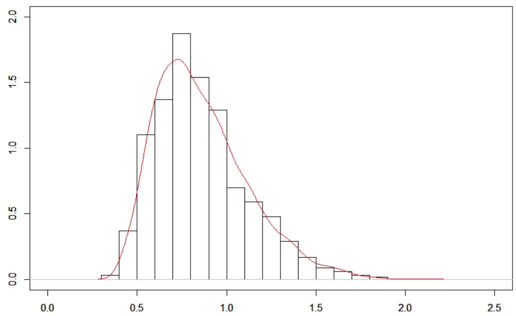


(b) $n = 10^7$.

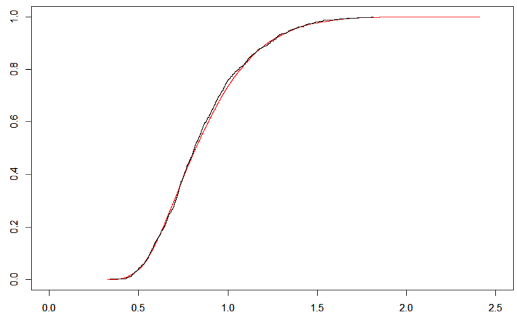
Figure 5: Histogram of $n^{\epsilon_1}(\hat{\tau}_n^\alpha - \tau_*^\alpha)$ in Situation III of Model 1.

Table 4.5: Proportions over the corresponding critical value in Situation III of Model 1.

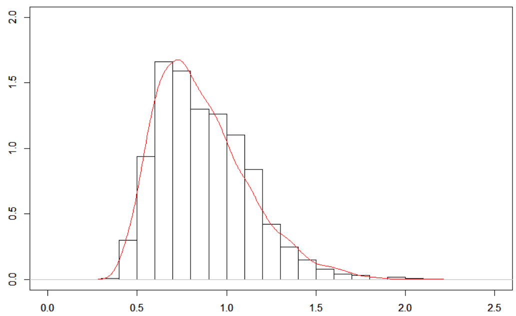
n	T	h_n	$\mathcal{T}^{(1)}_{1,n}$	$\mathcal{T}^{(1)}_{2,n}$	$\mathcal{T}^{(2)}_{1,n}$	$\mathcal{T}^{(2)}_{2,n}$
10^6	758.58	7.59×10^{-4}	0.040	0.034	0.045	0.048
10^7	2290.87	2.29×10^{-4}	0.048	0.053	0.040	0.046



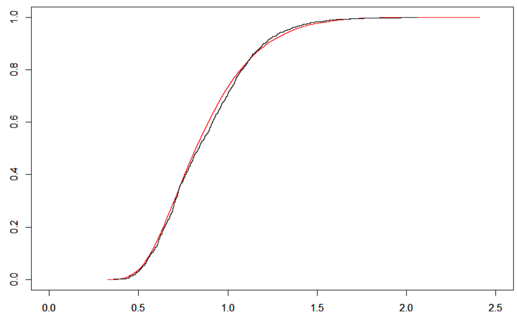
(a) Histogram of $\mathcal{T}^{(1)}_{1,n}$ with $n = 10^7$.



(b) EDF of $\mathcal{T}^{(1)}_{1,n}$ with $n = 10^7$.



(c) Histogram of $\mathcal{T}^{(2)}_{1,n}$ with $n = 10^7$.

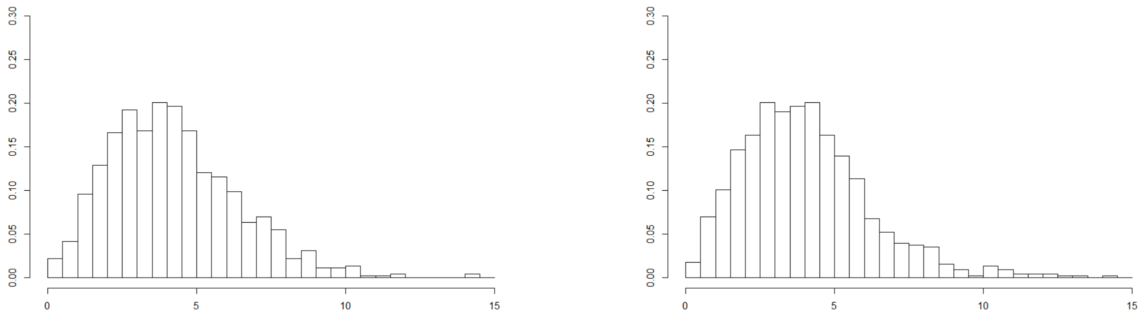


(d) EDF of $\mathcal{T}^{(2)}_{1,n}$ with $n = 10^7$.

Figure 6: Histogram (black line) versus theoretical density function (red line) and empirical distribution function (black line) versus theoretical distribution function (red line) in Situation III of Model 1.

Table 4.6: Mean and standard deviation of the estimators in Situation III of Model 1. True values: $\beta^* = 1, \gamma^* = 2$.

n	T	h_n	$\check{\beta}_1$	$\check{\gamma}_1$	$\check{\beta}_2$	$\check{\gamma}_2$
10^6	758.58	7.59×10^{-4}	1.0070	1.9985	1.0270	1.9999
			(0.0587)	(0.0408)	(0.1217)	(0.0959)
10^7	2290.87	2.29×10^{-4}	1.0021	2.0002	1.0084	1.9980
			(0.0332)	(0.0242)	(0.0654)	(0.0544)



(a) $n = 10^6$. (b) $n = 10^7$.

Figure 7: Histogram of $\sqrt{T}|\check{\boldsymbol{\beta}}_1 - \check{\boldsymbol{\beta}}_2|$ in Situation III of Model 1.

4.1.4 Situation IV-(i) : diffusion parameter changes after drift parameter does

In order to corroborate Theorems 6-8, we consider the following situation.

$$X_t = \begin{cases} X_0 - \displaystyle\int_0^t \beta^*(X_s - \gamma_1^*)\mathrm{d}s + \alpha_1^* W_t, & t \in [0, \tau_*^\beta T), \\ X_{\tau_*^\beta T} - \displaystyle\int_{\tau_*^\beta T}^t \beta^*(X_s - \gamma_2^*)\mathrm{d}s + \alpha_1^*(W_t - W_{\tau_*^\beta T}), & t \in [\tau_*^\beta T, \tau_*^\alpha T), \\ X_{\tau_*^\alpha T} - \displaystyle\int_{\tau_*^\alpha T}^t \beta^*(X_s - \gamma_2^*)\mathrm{d}s + \alpha_2^*(W_t - W_{\tau_*^\alpha T}), & t \in [\tau_*^\alpha T, T], \end{cases}$$

where $X_0 = 2$, $\tau_*^\alpha = 0.8$, $\tau_*^\beta = 0.4$, $\alpha_1^* = 1$, $\alpha_2^* = 1.2$, $\beta^* = 1$, $\gamma_1^* = 2 - \vartheta_\beta$, $\gamma_2^* = 2$. We set the sample size of the data $\{X_{t_i}\}_{i=0}^n$ being $n = 10^6$ or 10^7, $h_n = n^{-13/25}$, $T = nh_n = n^{12/25}$, $nh_n^2 = n^{-1/25}$, $\vartheta_\beta = n^{-1/10}$ and the significant level $\epsilon = 0.05$.

Table 4.7: Mean and standard deviation of the estimators in Situation IV-(i) of Model 1. True values: $\alpha_1^* = 1$, $\alpha_2^* = 1.2$, $\tau_*^\alpha = 0.8$.

n	T	h_n	$\hat{\alpha}_1$	$\hat{\alpha}_2$	$\hat{\tau}_n^\alpha$
10^6	758.58	7.59×10^{-4}	1.0002	1.2002	0.7988
			(0.0021)	(0.0024)	(0.0002)
10^7	2290.87	2.29×10^{-4}	1.0001	1.2001	0.7996
			(0.0006)	(0.0008)	(0.0001)

We first tested for a change in the diffusion parameter in the interval $[0, T]$. As a result, the change was detected in all 1000 iterations. We estimated α_1^*, α_2^* and τ_*^α in the same way as in Subsection 4.1.3, and investigated the change in the drift parameter. The results can be found in Tables 4.7 and 4.8 and Figure 8. Here we chose $\epsilon_1 = 0.45$ for all iterations. We find from Table 4.8 and (a)-(d) of Figure 8 that the proportions of the test statistics $\mathcal{T}_{1,n}^{(1)}$ and $\mathcal{T}_{2,n}^{(1)}$ that exceed the critical value approach 1.000 as n increases, and the distribution of the test statistics diverges. We can also see from (e)-(f) of Figure 8 that the distribution of the test statistic $\mathcal{T}_{1,n}^{(2)}$ almost corresponds with the null distribution.

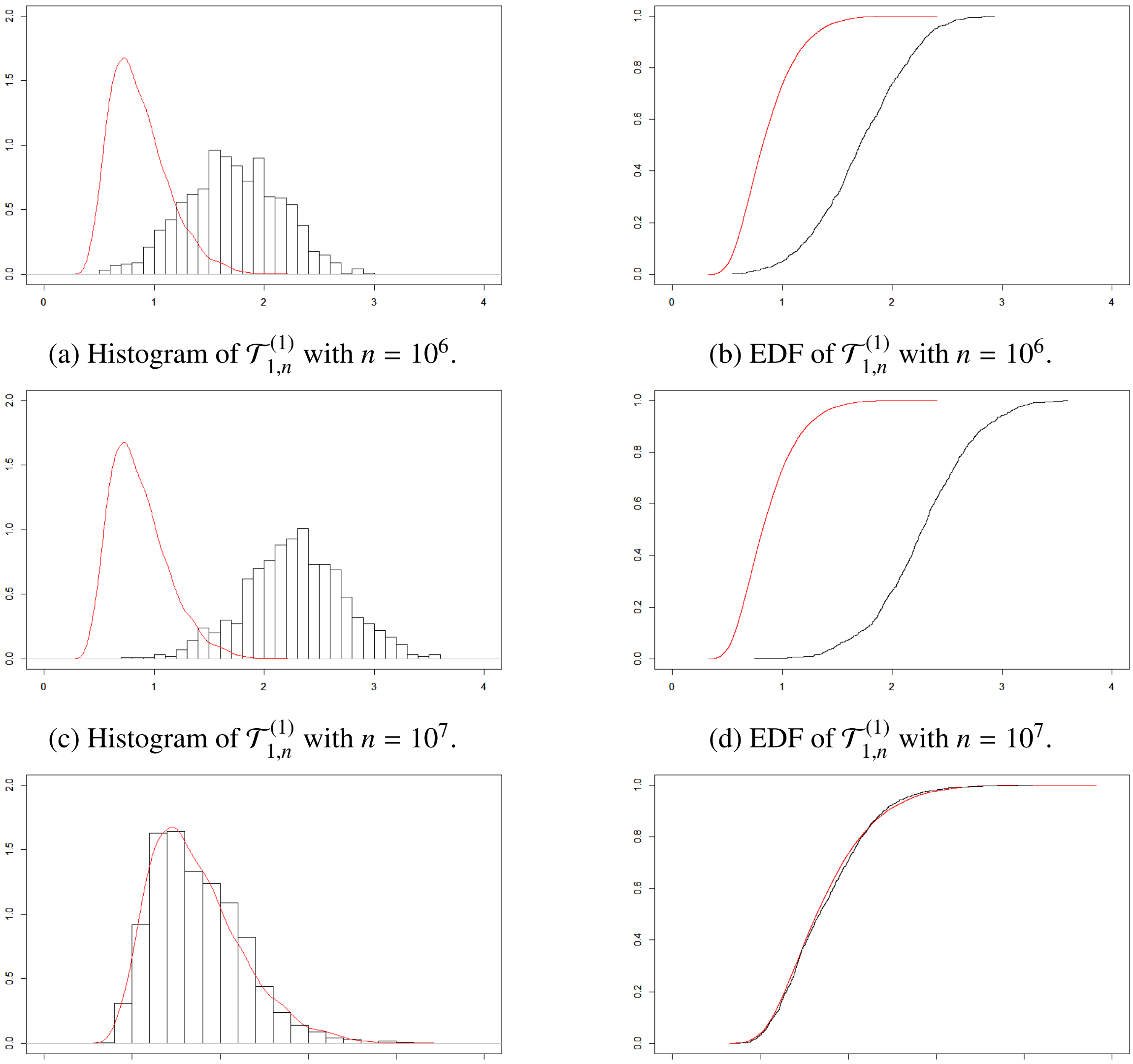
Table 4.8: Proportions over the corresponding critical value in Situation IV-(i) of Model 1.

n	T	h_n	$\mathcal{T}_{1,n}^{(1)}$	$\mathcal{T}_{2,n}^{(1)}$	$\mathcal{T}_{1,n}^{(2)}$	$\mathcal{T}_{2,n}^{(2)}$
10^6	758.58	7.59×10^{-4}	0.784	0.704	0.045	0.046
10^7	2290.87	2.29×10^{-4}	0.981	0.944	0.040	0.045

We finally estimated the drift parameters before and after the change of the diffusion parameter, and also estimated the change point of the drift parameter when the test statistic $\mathcal{T}_{1,n}^{(1)}$ exceeded the critical value. Here, we constructed the estimators by looking for the intervals with no change point. Specifically, we first tested for changes in the drift parameter in $[0.25\underline{\tau}_n T, 0.75\underline{\tau}_n T]$. When the change was detected, we constructed $\hat{\boldsymbol{\beta}}_1$ from $[0, 0.25\underline{\tau}_n T]$ and $\hat{\boldsymbol{\beta}}_2$ from $[0.75\underline{\tau}_n T, \underline{\tau}_n T]$. If no change was detected, we next expanded the test interval to $[0.125\underline{\tau}_n T, 0.875\underline{\tau}_n T]$, $[0.0625\underline{\tau}_n T, 0.9375\underline{\tau}_n T]$, and $[0.01\underline{\tau}_n T, 0.99\underline{\tau}_n T]$ and when the change was detected in the expanded interval, we estimated $\boldsymbol{\beta}_1^*$ and $\boldsymbol{\beta}_2^*$ using the data in the intervals that were not used in the test. The results of these estimates are shown in Table 4.9 and Figure 9. We can see that the distribution of the estimator almost corresponds with the asymptotic distribution and the estimator has good performance.

Table 4.9: Mean and standard deviation of the estimators in Situation IV-(i) of Model 1. True values: $\beta^* = 1$, $\gamma_2^* = 2$, $\tau_*^\beta = 0.4$, $\gamma_1^* \approx 1.7488$ and 1.8005 for $n = 10^6$ and 10^7, respectively.

n	T	h_n	$\hat{\beta}_1$	$\hat{\gamma}_1$	$\hat{\beta}_2$	$\hat{\gamma}_2$	$\hat{\tau}_{1,n}^{\beta}$
10^6	758.58	7.59×10^{-4}	1.0932	1.7353	1.1195	2.0080	0.4079
			(0.3319)	(0.1493)	(0.4638)	(0.1718)	(0.1237)
10^7	2290.87	2.29×10^{-4}	1.0110	1.7984	1.0213	2.0002	0.4023
			(0.0957)	(0.0562)	(0.0920)	(0.0637)	(0.0703)



(a) Histogram of $\mathcal{T}_{1,n}^{(1)}$ with $n = 10^6$.

(b) EDF of $\mathcal{T}_{1,n}^{(1)}$ with $n = 10^6$.

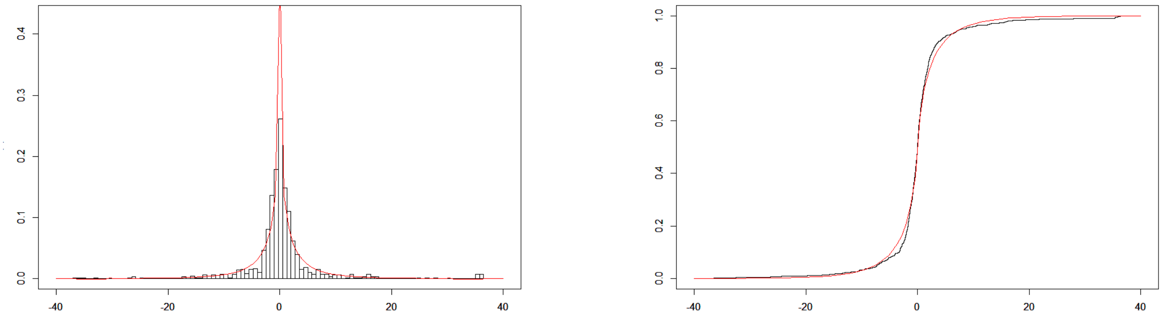
(c) Histogram of $\mathcal{T}_{1,n}^{(1)}$ with $n = 10^7$.

(d) EDF of $\mathcal{T}_{1,n}^{(1)}$ with $n = 10^7$.

(e) Histogram of $\mathcal{T}_{1,n}^{(2)}$ with $n = 10^7$.

(f) EDF of $\mathcal{T}_{1,n}^{(2)}$ with $n = 10^7$.

Figure 8: Histogram (black line) versus theoretical density function (red line) and empirical distribution function (black line) versus theoretical distribution function (red line) in Situation IV-(i) of Model 1.



(a) Histogram of $T\vartheta_\beta^2(\hat{\tau}_{1,n}^\beta - \tau_*^\beta)$. (b) EDF of $T\vartheta_\beta^2(\hat{\tau}_{1,n}^\beta - \tau_*^\beta)$.

Figure 9: Histogram (black line) versus theoretical density function (red line) and empirical distribution function (black line) versus theoretical distribution function (red line) with $n = 10^7$ in Situation IV-(i) of Model 1.

4.1.5 Situation IV-(iii) : both parameters change at the same time

We treat the following situation in order to verify Theorems 6, 7 and (3.3).

$$X_t = \begin{cases} X_0 - \displaystyle\int_0^t \beta^*(X_s - \gamma_1^*)\mathrm{d}s + \alpha_1^* W_t, & t \in [0, \tau_*^\alpha T), \\ X_{\tau_*^\alpha T} - \displaystyle\int_{\tau_*^\alpha T}^t \beta^*(X_s - \gamma_2^*)\mathrm{d}s + \alpha_2^*(W_t - W_{\tau_*^\alpha T}), & t \in [\tau_*^\alpha T, T], \end{cases}$$

where $X_0 = 2$, $\tau_*^\alpha = 0.8$, $\alpha_1^* = 1$, $\alpha_2^* = 1.2$, $\beta^* = 1$, $\gamma_1^* = 2 - \vartheta_\beta$, $\gamma_2^* = 2$. We set the sample size of the data $\{X_{t_i}\}_{i=0}^n$ being $n = 10^6$ or 10^7, $h_n = n^{-13/25}$, $T = nh_n = n^{12/25}$, $nh_n^2 = n^{-1/25}$, $\vartheta_\beta = n^{-1/10}$ and the significant level $\epsilon = 0.05$.

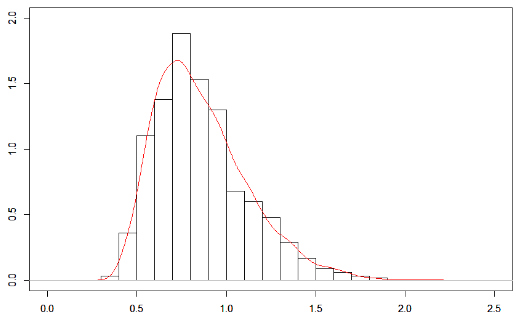
We first tested for a change in the diffusion parameter in the interval $[0, T]$. As a result, the change was detected in all 1000 iterations. In the same way as in Subsection 4.1.3, we estimated α_1^*, α_2^* and τ_*^α, and investigated the change in the drift parameter. The results are shown in Tables 4.10, 4.11 and Figure 10. Here we chose $\epsilon_1 = 0.45$ for all iterations. It can be seen that the results are similar to those of Subsection 4.1.3.

Table 4.10: Mean and standard deviation of the estimators in Situation IV-(iii) of Model 1. True values: $\alpha_1^* = 1$, $\alpha_2^* = 1.2$, $\tau_*^\alpha = 0.8$

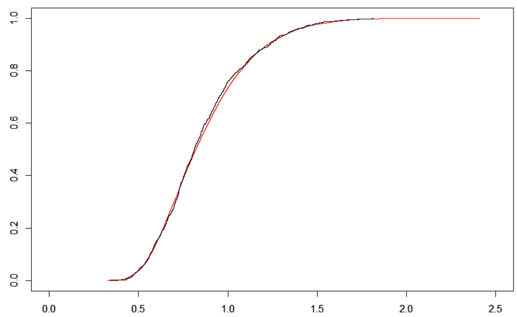
n	T	h_n	$\hat{\alpha}_1$	$\hat{\alpha}_2$	$\hat{\tau}_n^\alpha$
10^6	758.58	7.59×10^{-4}	1.0002	1.2002	0.7988
			(0.0021)	(0.0024)	(0.0002)
10^7	2290.87	2.29×10^{-4}	1.0001	1.2001	0.7996
			(0.0006)	(0.0008)	(0.0001)

Table 4.11: Proportions over the corresponding critical value in Situation IV-(iii) of Model 1.

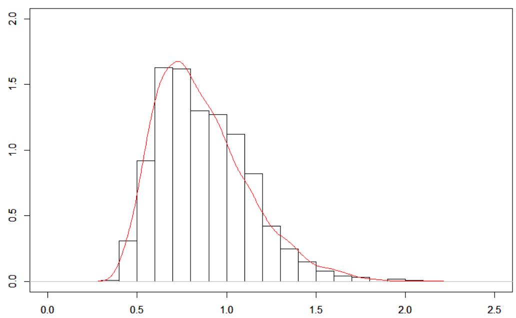
n	T	h_n	$\mathcal{T}^{(1)}_{1,n}$	$\mathcal{T}^{(1)}_{2,n}$	$\mathcal{T}^{(2)}_{1,n}$	$\mathcal{T}^{(2)}_{2,n}$
10^6	758.58	7.59×10^{-4}	0.040	0.034	0.046	0.048
10^7	2290.87	2.29×10^{-4}	0.048	0.053	0.040	0.046



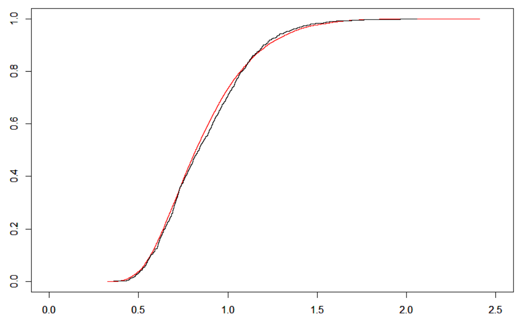
(a) Histogram of $\mathcal{T}^{(1)}_{1,n}$ with $n = 10^7$.



(b) EDF of $\mathcal{T}^{(1)}_{1,n}$ with $n = 10^7$.



(c) Histogram of $\mathcal{T}^{(2)}_{1,n}$ with $n = 10^7$.



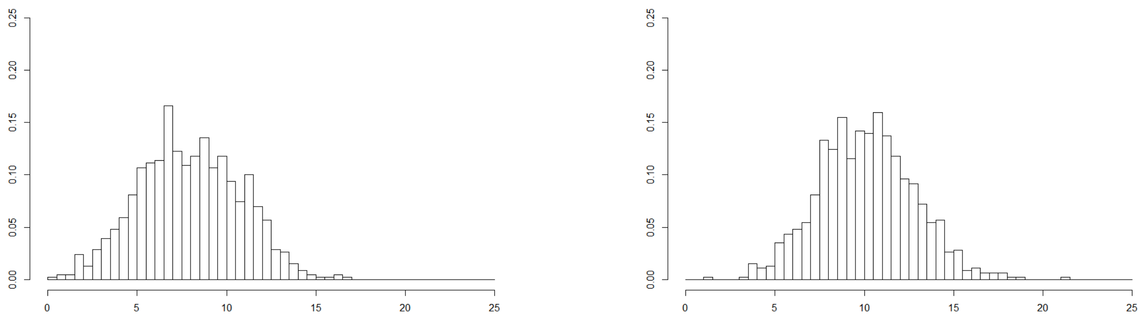
(d) EDF of $\mathcal{T}^{(2)}_{1,n}$ with $n = 10^7$.

Figure 10: Histogram (black line) versus theoretical density function (red line) and empirical distribution function (black line) versus theoretical distribution function (red line) in Situation IV-(iii) of Model 1.

Finally, we constructed the estimators $\check{\boldsymbol{\beta}}_1 = (\check{\beta}_1, \check{\gamma}_1)$ and $\check{\boldsymbol{\beta}}_2 = (\check{\beta}_2, \check{\gamma}_2)$ using the data obtained from the intervals $[0, \underline{\tau}_n T]$ and $[\overline{\tau}_n T, T]$, respectively when the test statistics $\mathcal{T}^{(1)}_{1,n}$ and $\mathcal{T}^{(2)}_{1,n}$ did not exceed the critical value, and investigated $\sqrt{T}|\check{\boldsymbol{\beta}}_1 - \check{\boldsymbol{\beta}}_2|$. Table 4.12 and Figure 11 show the results of the estimates of $\boldsymbol{\beta}^*_1$ and $\boldsymbol{\beta}^*_2$. It can be seen that $\sqrt{T}|\check{\boldsymbol{\beta}}_1 - \check{\boldsymbol{\beta}}_2|$ tends to increase as n does.

Table 4.12: Mean and standard deviation of the estimators in Situation IV-(iii) of Model 1. True values: $\beta^* = 1$, $\gamma^*_2 = 2$, $\gamma^*_1 \approx 1.7488$ and 1.8005 for $n = 10^6$ and 10^7, respectively.

n	T	h_n	$\check{\beta}_1$	$\check{\gamma}_1$	$\check{\beta}_2$	$\check{\gamma}_2$
10^6	758.58	7.59×10^{-4}	1.0071	1.7472	1.0273	1.9999
			(0.0587)	(0.0408)	(0.1214)	(0.0961)
10^7	2290.87	2.29×10^{-4}	1.0022	1.8007	1.0084	1.9980
			(0.0333)	(0.0242)	(0.0654)	(0.0543)



(a) $n = 10^6$. (b) $n = 10^7$.

Figure 11: Histogram of $\sqrt{T}|\check{\boldsymbol{\beta}}_1 - \check{\boldsymbol{\beta}}_2|$ in Situation IV-(iii) of Model 1.

4.2 Model 2 : hyperbolic diffusion model

We treat the hyperbolic diffusion model defined by

$$\mathrm{d}X_t = \left(\beta - \frac{\gamma X_t}{\sqrt{1+X_t^2}}\right)\mathrm{d}t + \alpha \mathrm{d}W_t, \quad X_0 = x_0,$$

where $\alpha > 0, \beta \in \mathbb{R}$ and $|\beta| < \gamma$.

4.2.1 Situation I : neither parameter changes

In order to corroborate Theorems 1, 3 and 4, we consider the following situation.

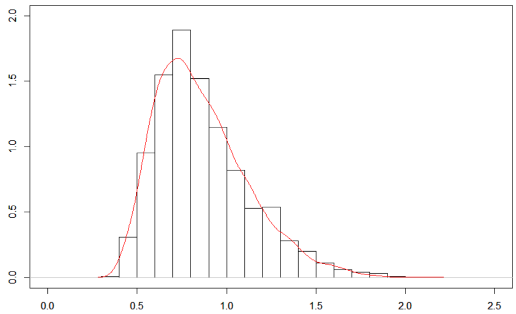
$$X_t = X_0 + \int_0^t \left(\beta^* - \frac{\gamma^* X_s}{\sqrt{1+X_s^2}}\right)\mathrm{d}s + \alpha^* W_t, \quad t \in [0, T],$$

where $X_0 = 0.5$, $\alpha^* = 0.5$, $\beta^* = 0.5$, $\gamma^* = 1$. We set the sample size of the data $\{X_{t_i}\}_{i=0}^n$ being $n = 10^5$ or 10^6, $h_n = n^{-7/12}$, $T = n^{5/12}$, $nh_n^2 = n^{-1/6}$ and the significant level $\epsilon = 0.10$.

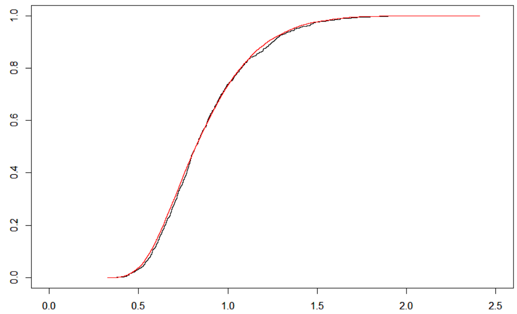
We investigated the presence of the change in the diffusion or drift parameter. Table 4.13 and Figure 12 show the empirical sizes, the histograms and the EDFs of $\mathcal{T}_n^\alpha$, $\mathcal{T}_{1,n}^\beta$ and $\mathcal{T}_{2,n}^\beta$. We see that the proportions of the test statistics that exceed the critical values are close to $\epsilon = 0.10$ and the distribution of the test statistics almost corresponds with the null distribution, which implies that the test statistics have good performance.

Table 4.13: Proportions over the corresponding critical value in Situation of Model 2.

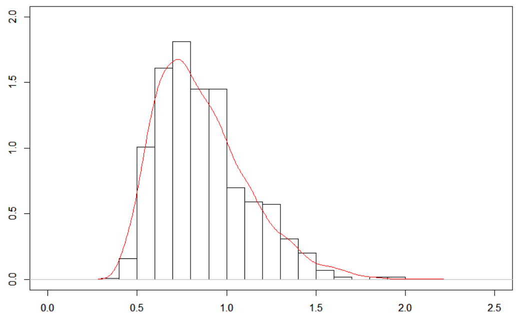
n	T	h_n	$\mathcal{T}_n^\alpha$	$\mathcal{T}_{1,n}^\beta$	$\mathcal{T}_{2,n}^\beta$
10^5	121.15	1.21×10^{-3}	0.098	0.094	0.090
10^6	316.23	3.16×10^{-4}	0.118	0.106	0.092



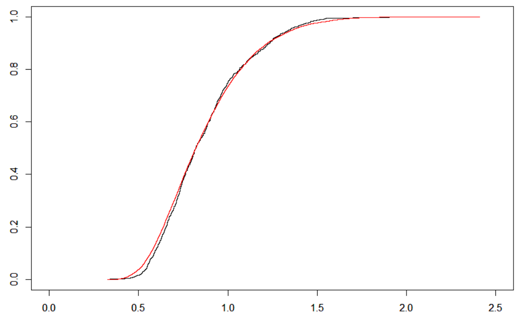
(a) Histogram of $\mathcal{T}_n^{\alpha}$ with $n = 10^6$.



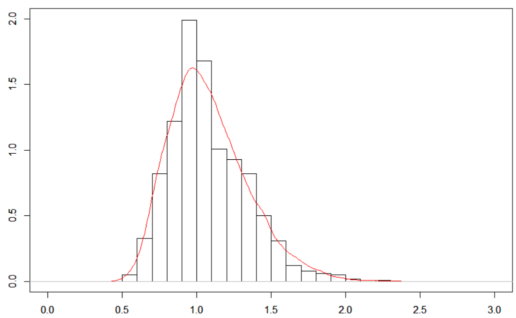
(b) EDF of $\mathcal{T}_n^{\alpha}$ with $n = 10^6$.



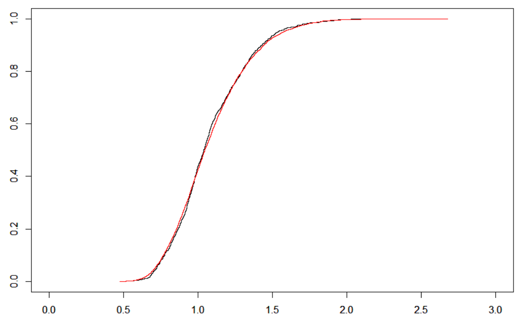
(c) Histogram of $\mathcal{T}_{1,n}^{\beta}$ with $n = 10^6$.



(d) EDF of $\mathcal{T}_{1,n}^{\beta}$ with $n = 10^6$.



(e) Histogram of $\mathcal{T}_{2,n}^{\beta}$ with $n = 10^6$.



(f) EDF of $\mathcal{T}_{2,n}^{\beta}$ with $n = 10^6$.

Figure 12: Histogram (black line) versus theoretical density function (red line) and empirical distribution function (black line) versus theoretical distribution function (red line) in Situation I of Model 2.

4.2.2 Situation II : only drift parameter changes

We consider the following situation and verify Theorems 3-5.

$$X_t = \begin{cases} X_0 + \int_0^t \left(\beta_1^* - \frac{\gamma^* X_s}{\sqrt{1+X_s^2}}\right) ds + \alpha^* W_t, & t \in [0, \tau_*^\beta T), \\ X_{\tau_*^\beta T} + \int_{\tau_*^\beta T}^t \left(\beta_2^* - \frac{\gamma^* X_s}{\sqrt{1+X_s^2}}\right) ds + \alpha^*(W_t - W_{\tau_*^\beta T}), & t \in [\tau_*^\beta T, T], \end{cases}$$

where $X_0 = 0.25$, $\alpha^* = 0.2$, $\beta_1^* = 0.25$, $\beta_2^* = -0.25$, $\gamma^* = 1.2$, $\tau_*^\beta = 0.5$. We set the sample size of the data $\{X_{t_i}\}_{i=0}^n$ being $n = 10^6$ or 10^7, $h_n = n^{-4/7}$, $T = n^{3/7}$, $nh_n^2 = n^{-1/7}$ and the significant level $\epsilon = 0.10$.

First, we verified the performance of test statistics $\mathcal{T}_n^\alpha$, $\mathcal{T}_{1,n}^\beta$ and $\mathcal{T}_{2,n}^\beta$. The simulation results of the test statistics can be found in Table 4.14 and Figure 13. Table 4.14 shows that the proportion of $\mathcal{T}_n^\alpha$ that exceed the critical value is close to $\epsilon = 0.10$ and the change in the drift parameter is detected in all iterations.

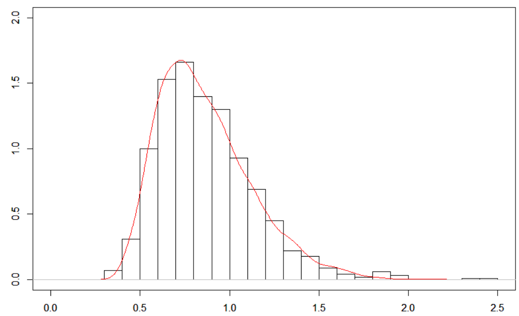
Table 4.14: Proportions over the corresponding critical value in Situation II of Model 2.

n	T	h_n	$\mathcal{T}_n^\alpha$	$\mathcal{T}_{1,n}^\beta$	$\mathcal{T}_{2,n}^\beta$
10^6	372.76	3.73×10^{-4}	0.119	1.000	1.000
10^7	10^3	10^{-4}	0.095	1.000	1.000

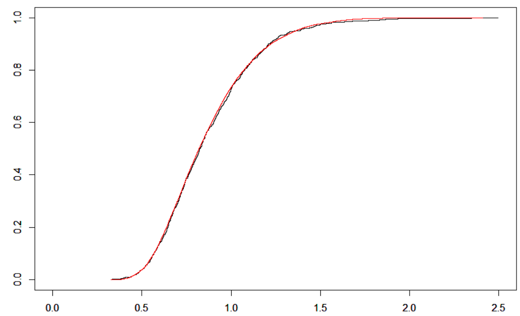
Next, we estimated the change point of the drift parameter. Since the change point was detected in the intervals $[T/4, T]$ and $[0, 3T/4]$, we estimated β_1^* and β_2^* using the data obtained from the intervals $[0, T/4]$ and $[3T/4, T]$ in all iterations, respectively. The estimates of α^*, β_1^*, β_2^* and τ_*^β are reported in Table 4.15. It seems from Figure 14 that $T(\hat{\tau}_n^\beta - \tau_*^\beta) = O_{\mathbf{P}}(1)$.

Table 4.15: Mean and standard deviation of the estimators in Situation II of Model 2. True values: $\alpha^* = 0.2$, $\beta_1^* = 0.25$, $\beta_2^* = -0.25$, $\gamma^* = 1.2$, $\tau_*^\beta = 0.5$.

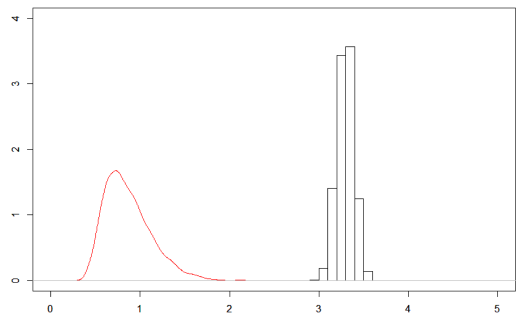
n	T	h_n	$\hat{\alpha}$	$\hat{\beta}_1$	$\hat{\gamma}_1$	$\hat{\beta}_2$	$\hat{\gamma}_2$	$\hat{\tau}_n^\beta$
10^6	372.76	3.73×10^{-4}	0.2000	0.2596	1.2485	−0.2600	1.2468	0.4988
			(1.42×10^{-4})	(0.0414)	(0.1674)	(0.0445)	(0.1844)	(0.0019)
10^7	10^3	10^{-4}	0.2000	0.2522	1.2121	−0.2549	1.2216	0.5000
			(4.44×10^{-5})	(0.0257)	(0.1028)	(0.0245)	(0.0997)	(0.0006)



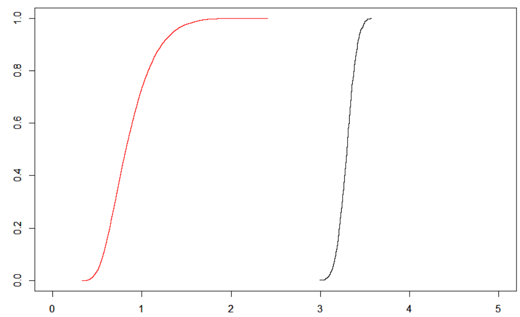
(a) Histogram of $\mathcal{T}_n^{\alpha}$ with $n = 10^7$.



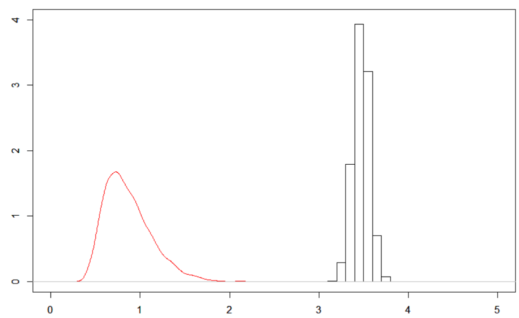
(b) EDF of $\mathcal{T}_n^{\alpha}$ with $n = 10^7$.



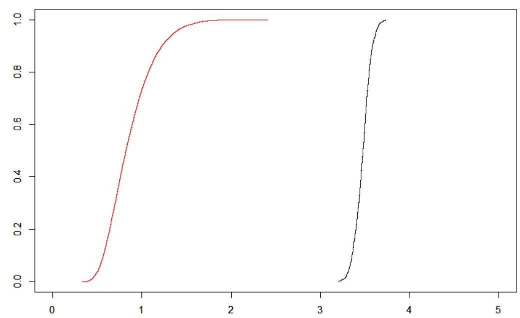
(c) Histogram of $\mathcal{T}_{1,n}^{\beta}$ with $n = 10^6$.



(d) EDF of $\mathcal{T}_{1,n}^{\beta}$ with $n = 10^6$.

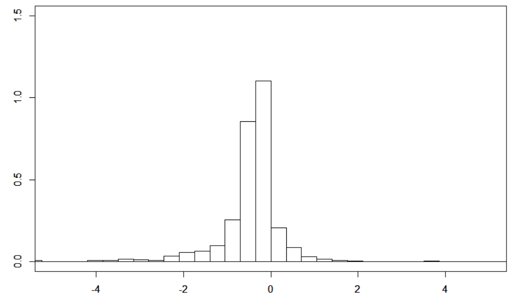


(e) Histogram of $\mathcal{T}_{1,n}^{\beta}$ with $n = 10^7$.

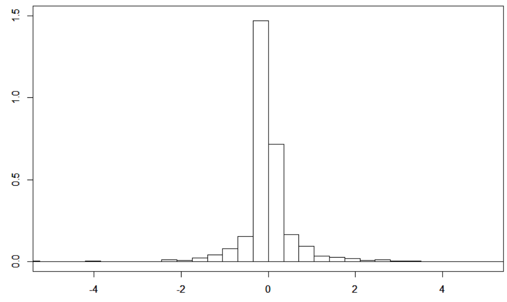


(f) EDF of $\mathcal{T}_{1,n}^{\beta}$ with $n = 10^7$.

Figure 13: Histogram (black line) versus theoretical density function (red line) and empirical distribution function (black line) versus theoretical distribution function (red line) in Situation II of Model 2.



(a) Histogram of $T(\hat{\tau}_n^\beta - \tau_*^\beta)$.



(b) Histogram of $T(\hat{\tau}_n^\beta - \tau_*^\beta)$.

Figure 14: Histogram (black line) versus theoretical density function (red line) and empirical distribution function (black line) versus theoretical distribution function (red line) with $n = 10^7$ in Situation II of Model 2.

4.2.3 Situation III : only diffusion parameter changes

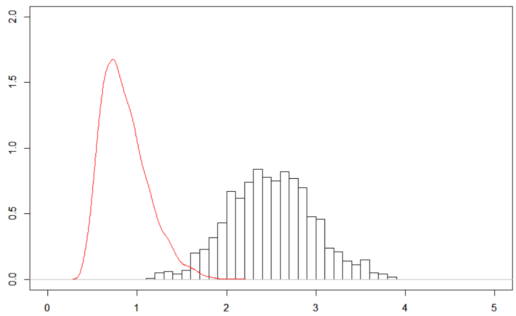
In order to verify Theorems 1, 2, 6, 7 and (3.2), we deal with the following situation.

$$X_t = \begin{cases} X_0 + \displaystyle\int_0^t \Big(\beta^* - \frac{\gamma^* X_s}{\sqrt{1+X_s^2}}\Big)\mathrm{d}s + \alpha_1^* W_t, & t \in [0, \tau_*^\alpha T), \\ X_{\tau_*^\alpha T} + \displaystyle\int_{\tau_*^\alpha T}^t \Big(\beta^* - \frac{\gamma^* X_s}{\sqrt{1+X_s^2}}\Big)\mathrm{d}s + \alpha_2^*(W_t - W_{\tau_*^\alpha T}), & t \in [\tau_*^\alpha T, T], \end{cases}$$

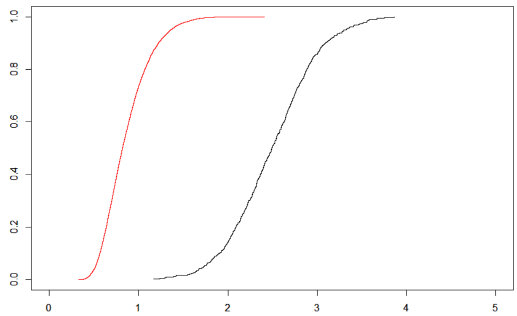
where $X_0 = 1$, $\tau_*^\alpha = 0.4$, $\alpha_1^* = 1 + n^{-9/25}$, $\alpha_2^* = 1$, $\beta^* = 1$, $\gamma^* = 2$. We set the sample size of the data $\{X_{t_i}\}_{i=0}^n$ being $n = 10^6$ or 10^7, $h_n = n^{-5/8}$, $T = n^{3/8}$, $nh_n^2 = n^{-1/4}$ and the significant level $\epsilon = 0.05$.

We tested for changes in the diffusion parameter in the interval $[0, T]$ in 1000 iterations. The change was detected 990 times when $n = 10^6$ and 1000 times when $n = 10^7$. Figure 15 shows the histograms and the EDFs of $\mathcal{T}_n^\alpha$. When the change in the diffusion parameter was detected, we estimated the parameters α_1^* and α_2^* in the same way to estimate $\boldsymbol{\beta}_1^*$ and $\boldsymbol{\beta}_2^*$ in Subsection 4.1.3, and estimated τ_*^α using the estimators $\hat{\alpha}_1$ and $\hat{\alpha}_2$. The estimates of α_1^*, α_2^* and τ_*^α are shown in Table 4.16. We find from Figure 16 that the distribution of $n\vartheta_\alpha^2(\hat{\tau}_n^\alpha - \tau_*^\alpha)$ almost corresponds with the theoretical distribution in Theorem 2-(1) and the estimators have good performance. In this case, we chose $\epsilon_1 = 0.9 + 1.8 \log|\hat{\alpha}_1 - \hat{\alpha}_2| / \log n$ for all iterations.

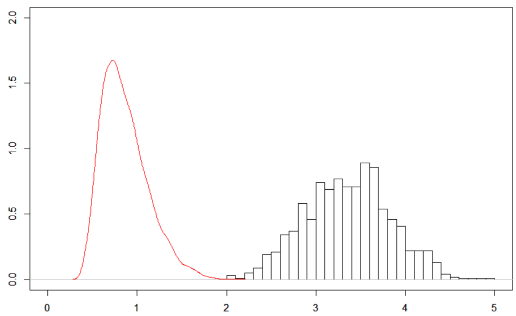
Next, we tested for changes in the drift parameter in the intervals $[0, \underline{\tau}_n T]$ and $[\overline{\tau}_n T, T]$. Table 4.17 and Figure 17 show the simulation results of the tests for changes in the drift parameter. It can be seen that the test statistics have good performance. Hence, we constructed $\check{\boldsymbol{\beta}}_1$ and $\check{\boldsymbol{\beta}}_2$ using the data obtained from the intervals $[0, \underline{\tau}_n T]$ and $[\overline{\tau}_n T, T]$, respectively when the test statistics $\mathcal{T}_{1,n}^{(1)}$ and $\mathcal{T}_{1,n}^{(2)}$ did not exceed the critical value, and investigated $\sqrt{T}|\check{\boldsymbol{\beta}}_1 - \check{\boldsymbol{\beta}}_2|$. The result of the estimates of $\boldsymbol{\beta}_1^*$ and $\boldsymbol{\beta}_2^*$ can be found in Table 4.18 and Figure 18. It can be seen that $\sqrt{T}|\check{\boldsymbol{\beta}}_1 - \check{\boldsymbol{\beta}}_2| = O_{\mathbf{P}}(1)$.



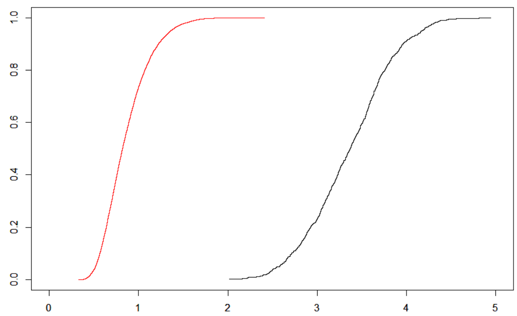
(a) Histogram of $\mathcal{T}_n^\alpha$ with $n = 10^6$.



(b) EDF of $\mathcal{T}_n^\alpha$ with $n = 10^6$.



(c) Histogram of $\mathcal{T}_n^\alpha$ with $n = 10^7$.



(d) EDF of $\mathcal{T}_n^\alpha$ with $n = 10^7$.

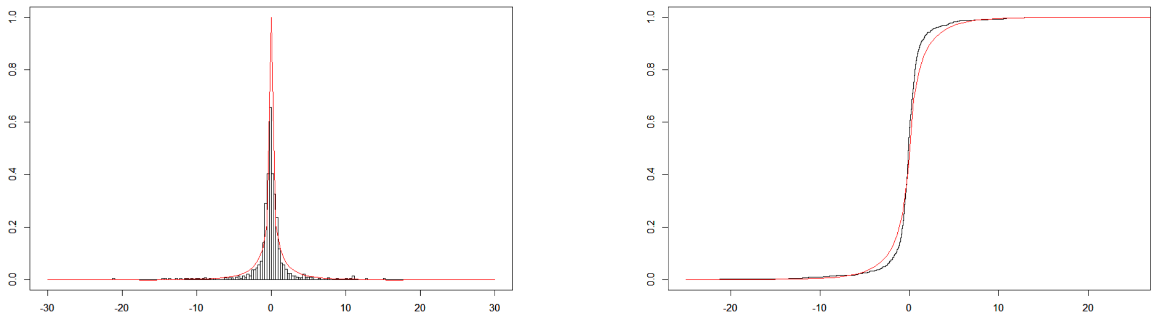
Figure 15: Histogram (black line) versus theoretical density function (red line) and empirical distribution function (black line) versus theoretical distribution function (red line) in Situation III of Model 2.

Table 4.16: Mean and standard deviation of the estimators in Situation III of Model 2. True values: $\alpha_2^* = 1$, $\tau_*^\alpha = 0.4$, $\alpha_1^* = 1.0069$ and 1.0030 for $n = 10^6$ and 10^7, respectively

n	T	h_n	$\hat{\alpha}_1$	$\hat{\alpha}_2$	$\hat{\tau}_n^\alpha$
10^6	177.83	1.78×10^{-4}	1.0070	1.0000	0.3986
			(0.0017)	(0.0017)	(0.0663)
10^7	421.70	4.22×10^{-4}	1.0030	1.0000	0.3986
			(0.0005)	(0.0005)	(0.0258)

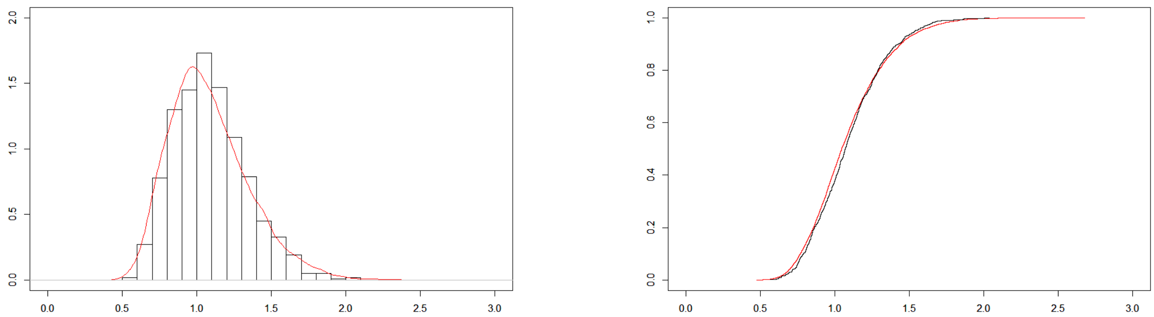
Table 4.17: Proportions over the corresponding critical value in Situation III of Model 2.

n	T	h_n	$\mathcal{T}_{1,n}^{(1)}$	$\mathcal{T}_{2,n}^{(1)}$	$\mathcal{T}_{1,n}^{(2)}$	$\mathcal{T}_{2,n}^{(2)}$
10^6	177.83	1.78×10^{-4}	0.035	0.043	0.060	0.051
10^7	421.70	4.22×10^{-4}	0.038	0.040	0.038	0.031

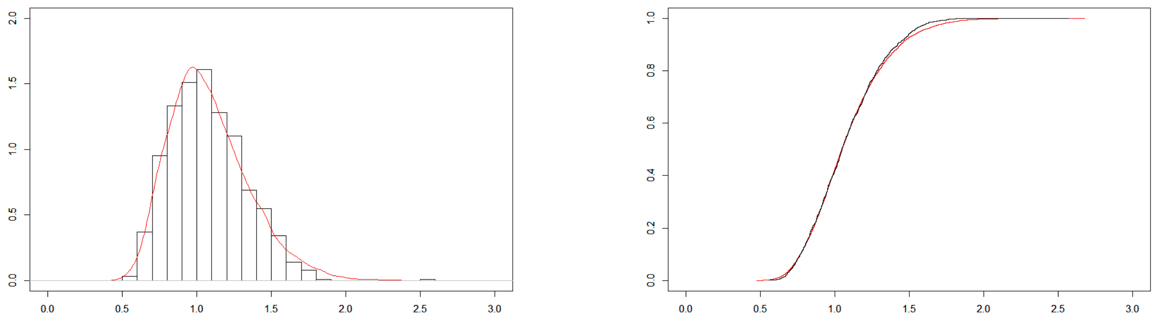


(a) Histogram of $n\vartheta_\alpha^2(\hat{\tau}_n^\alpha - \tau_*^\alpha)$. (b) EDF of $n\vartheta_\alpha^2(\hat{\tau}_n^\alpha - \tau_*^\alpha)$.

Figure 16: Histogram (black line) versus theoretical density function (red line) and empirical distribution function (black line) versus theoretical distribution function (red line) with $n = 10^7$ in Situation III of Model 2.



(a) Histogram of $\mathcal{T}_{2,n}^{(1)}$ with $n = 10^7$. (b) EDF of $\mathcal{T}_{2,n}^{(1)}$ with $n = 10^7$.

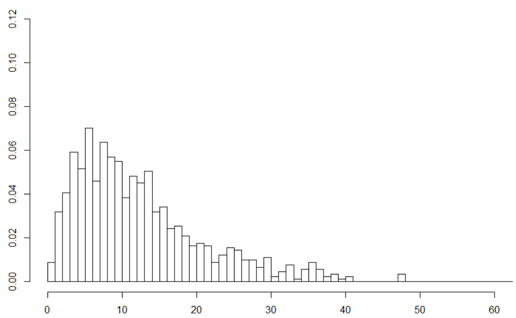


(c) Histogram of $\mathcal{T}_{2,n}^{(2)}$ with $n = 10^7$. (d) EDF of $\mathcal{T}_{2,n}^{(2)}$ with $n = 10^7$.

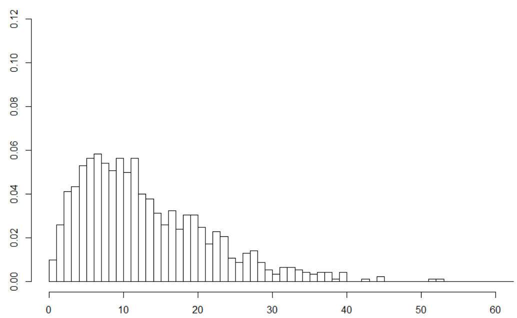
Figure 17: Histogram (black line) versus theoretical density function (red line) and empirical distribution function (black line) versus theoretical distribution function (red line) in Situation III in Model 2.

Table 4.18: Mean and standard deviation of the estimators in Situation III of Model 2. True values: $\beta^* = 1, \gamma^* = 2$.

n	T	h_n	$\check{\beta}_1$	$\check{\gamma}_1$	$\check{\beta}_2$	$\check{\gamma}_2$
10^6	177.83	1.78×10^{-4}	1.0993	2.1873	1.0509	2.0980
			(0.5332)	(0.7292)	(0.3476)	(0.4881)
10^7	421.70	4.22×10^{-4}	1.0179	2.0413	1.0165	2.0323
			(0.1370)	(0.2114)	(0.1110)	(0.1722)



(a) $n = 10^6$.



(b) $n = 10^7$.

Figure 18: Histogram of $\sqrt{T}|\check{\boldsymbol{\beta}}_1 - \check{\boldsymbol{\beta}}_2|$ in Situation III in Model 2.

4.2.4 Situation IV-(ii) : drift parameter changes after diffusion parameter does

In support of Theorems 6, 7 and 9, we consider the following situation.

$$X_t = \begin{cases} X_0 + \displaystyle\int_0^t \left(\beta_1^* - \frac{\gamma^* X_s}{\sqrt{1+X_s^2}}\right) \mathrm{d}s + \alpha_1^* W_t, & t \in [0, \tau_*^\alpha T), \\ X_{\tau_*^\alpha T} + \displaystyle\int_{\tau_*^\alpha T}^t \left(\beta_1^* - \frac{\gamma^* X_s}{\sqrt{1+X_s^2}}\right) \mathrm{d}s + \alpha_2^* (W_t - W_{\tau_*^\alpha T}), & t \in [\tau_*^\alpha T, \tau_*^\beta T), \\ X_{\tau_*^\beta T} + \displaystyle\int_{\tau_*^\beta T}^t \left(\beta_2^* - \frac{\gamma^* X_s}{\sqrt{1+X_s^2}}\right) \mathrm{d}s + \alpha_2^* (W_t - W_{\tau_*^\beta T}), & t \in [\tau_*^\beta T, T], \end{cases}$$

where $X_0 = 1$, $\tau_*^\alpha = 0.4$, $\tau_*^\beta = 0.7$, $\alpha_1^* = 1 + n^{-9/25}$, $\alpha_2^* = 1, \beta_1^* = 1, \beta_2^* = 0.5, \gamma^* = 2$. We set the sample size of the data $\{X_{t_i}\}_{i=0}^n$ being $n = 10^6$ or 10^7, $h_n = n^{-5/8}$, $T = n^{3/8}$, $nh_n^2 = n^{-1/4}$ and the significant level $\epsilon = 0.05$.

We tested for changes in the diffusion parameter in the interval $[0, T]$ in 1000 iterations. The change was detected 990 times when $n = 10^6$ and 1000 times when $n = 10^7$. As in Subsection 4.2.3, we estimated τ_*^α using the estimators $\hat{\alpha}_1$ and $\hat{\alpha}_2$. The estimates of α_1^*, α_2^* and τ_*^α are shown in Table 4.19. In this case, we chose $\epsilon_1 = 0.9 + 1.8 \log|\hat{\alpha}_1 - \hat{\alpha}_2| / \log n$ for all iterations.

Table 4.19: Mean and standard deviation of the estimators in Situation IV-(ii) of Model 2. True values: $\alpha_2^* = 1$, $\tau_*^\alpha = 0.4$, $\alpha_1^* = 1.0069$ and 1.0030 for $n = 10^6$ and 10^7, respectively

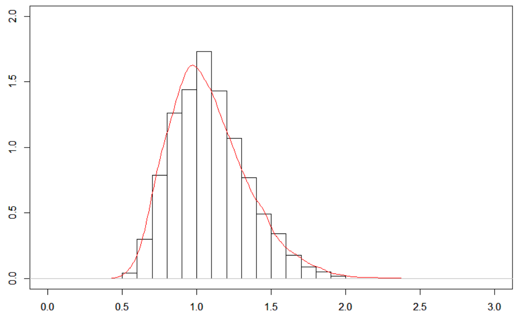
n	T	h_n	$\hat{\alpha}_1$	$\hat{\alpha}_2$	$\hat{\tau}_n^\alpha$
10^6	177.83	1.78×10^{-4}	1.0071	0.9999	0.3929
			(0.0018)	(0.0015)	(0.0600)
10^7	421.70	4.22×10^{-4}	1.0030	1.0000	0.3987
			(0.0005)	(0.0005)	(0.0273)

Table 4.20: Proportions over the corresponding critical value in Situation IV-(ii) of Model 2.

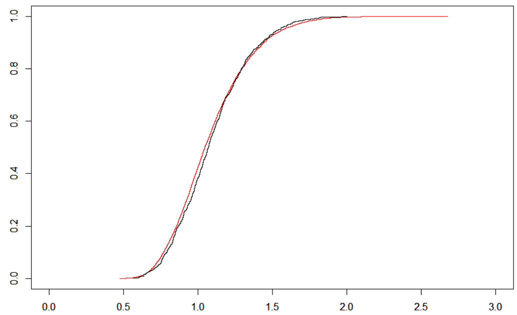
n	T	h_n	$\mathcal{T}_{1,n}^{(1)}$	$\mathcal{T}_{2,n}^{(1)}$	$\mathcal{T}_{1,n}^{(2)}$	$\mathcal{T}_{2,n}^{(2)}$
10^6	177.83	1.78×10^{-4}	0.034	0.060	0.510	0.414
10^7	421.70	4.22×10^{-4}	0.040	0.040	0.941	0.887

Table 4.21: Mean and standard deviation of the estimators in Situation IV-(ii) of Model 2. True values: $\beta_1^* = 1, \beta_2^* = 0.5$, $\gamma^* = 2$, $\tau_*^\beta = 0.7$.

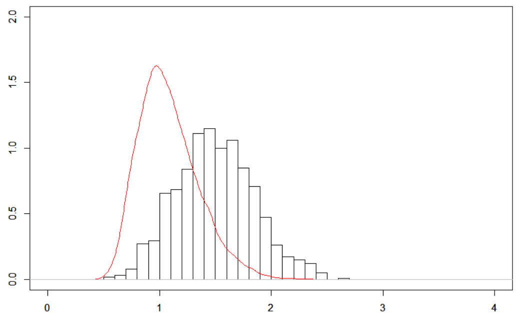
n	T	h_n	$\hat{\beta}_1$	$\hat{\gamma}_1$	$\hat{\beta}_2$	$\hat{\gamma}_2$	$\hat{\tau}_{2,n}^\beta$
10^6	177.83	1.78×10^{-4}	1.7297	3.0951	0.5300	2.9141	0.7063
			(1.8108)	(2.1513)	(0.8434)	(1.8332)	(0.1318)
10^7	421.70	4.22×10^{-4}	1.1436	2.2546	0.5143	2.1135	0.6987
			(0.5965)	(0.8216)	(0.2102)	(0.4559)	(0.0698)



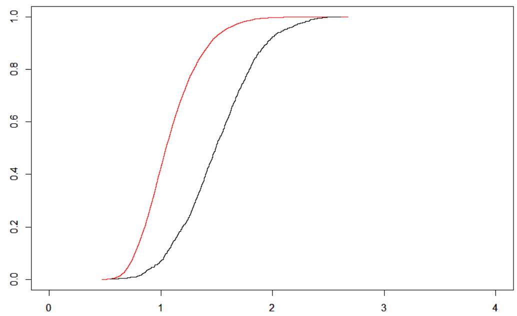
(a) Histogram of $\mathcal{T}_{2,n}^{(1)}$ with $n = 10^7$.



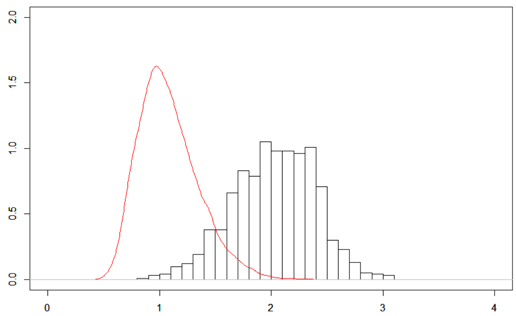
(b) EDF of $\mathcal{T}_{2,n}^{(1)}$ with $n = 10^7$.



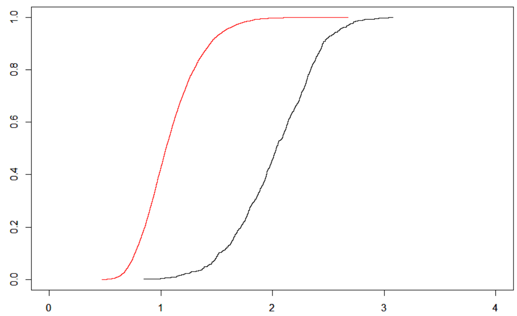
(c) Histogram of $\mathcal{T}_{2,n}^{(2)}$ with $n = 10^6$.



(d) EDF of $\mathcal{T}_{2,n}^{(2)}$ with $n = 10^6$.



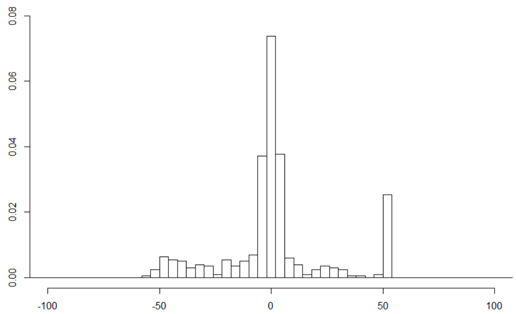
(e) Histogram of $\mathcal{T}_{2,n}^{(2)}$ with $n = 10^7$.



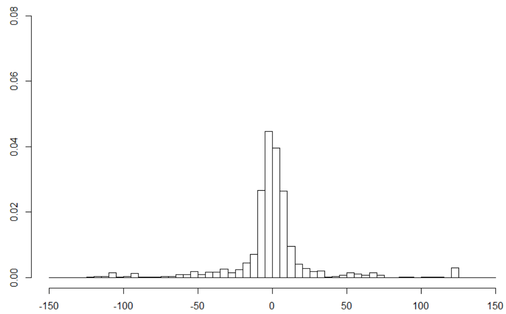
(f) EDF of $\mathcal{T}_{2,n}^{(2)}$ with $n = 10^7$.

Figure 19: Histogram (black line) versus theoretical density function (red line) and empirical distribution function (black line) versus theoretical distribution function (red line) in Situation IV-(ii) of Model 2.

Next, we tested for changes in the drift parameter in the intervals $[0, \underline{\tau}_n T]$ and $[\overline{\tau}_n T, T]$. From Table 4.20 and (a)-(b) of Figure 19, we see that the distribution of the test statistic $\mathcal{T}_{2,n}^{(1)}$ almost corresponds with the null distribution. Moreover, it can be seen from Table 4.20 and (c)-(f) of Figure 19 that the proportions of the test statistics $\mathcal{T}_{1,n}^{(2)}$ and $\mathcal{T}_{2,n}^{(2)}$ that exceed the critical value approach 1.000 as n increases, and the distribution of the test statistics $\mathcal{T}_{2,n}^{(2)}$ diverges. Therefore, we estimated the drift parameters before and after the change of the diffusion parameter, and also estimated the change point when the test statistic $\mathcal{T}_{1,n}^{(2)}$ exceeded the critical value. Here, we constructed the estimators $\hat{\boldsymbol{\beta}}_1$ and $\hat{\boldsymbol{\beta}}_2$ in the same way as in Subsection 4.1.4. The results of these estimates are shown in Table 4.21 and Figure 20. We can see that the distribution of the estimator does not diverge when n increases from 10^6 to 10^7, which implies that the estimator has good performance.



(a) $n = 10^6$.



(b) $n = 10^7$.

Figure 20: Histogram of $T(\hat{\tau}_{2,n}^{\beta} - \tau_*^{\beta})$ in Situation IV-(ii) of Model 2.

4.2.5 Situation IV-(iii) : both parameters change at the same time

We treat the following situation and corroborate Theorems 6, 7 and (3.3).

$$X_t = \begin{cases} X_0 - \displaystyle\int_0^t \left(\beta_1^* - \frac{\gamma^* X_s}{\sqrt{1+X_s^2}}\right) \mathrm{d}s + \alpha_1^* W_t, & t \in [0, \tau_*^\alpha T), \\ X_{\tau_*^\alpha T} - \displaystyle\int_{\tau_*^\alpha T}^t \left(\beta_2^* - \frac{\gamma^* X_s}{\sqrt{1+X_s^2}}\right) \mathrm{d}s + \alpha_2^* (W_t - W_{\tau_*^\alpha T}), & t \in [\tau_*^\alpha T, T], \end{cases}$$

where $X_0 = 1$, $\tau_*^\alpha = 0.4$, $\alpha_1^* = 1 + n^{-9/25}$, $\alpha_2^* = 1$, $\beta_1^* = 1$, $\beta_2^* = 0.5$, $\gamma^* = 2$. We set the sample size of the data $\{X_{t_i}\}_{i=0}^n$ being $n = 10^6$ or 10^7, $h_n = n^{-5/8}$, $T = n^{3/8}$, $nh_n^2 = n^{-1/4}$ and the significant level $\epsilon = 0.05$.

We tested for changes in the diffusion parameter in the interval $[0, T]$ in 1000 iterations. The change was detected 990 times when $n = 10^6$ and 1000 times when $n = 10^7$. When the change in the diffusion parameter was detected, we estimated the parameters α_1^* and α_2^* in the same way to estimate $\boldsymbol{\beta}_1^*$ and $\boldsymbol{\beta}_2^*$ in Subsection 4.1.3, and estimated τ_*^α using the estimators $\hat{\alpha}_1$ and $\hat{\alpha}_2$. The estimates of α_1^*, α_2^* and τ_*^α are shown in Table 4.16. We find from Figure 16 that the distribution of $n\vartheta_\alpha^2(\hat{\tau}_n^\alpha - \tau_*^\alpha)$ almost corresponds with the theoretical distribution in (1) of Theorem 2 and the estimators have good performance. In this case, we chose $\epsilon_1 = 0.9 + 1.8 \log |\hat{\alpha}_1 - \hat{\alpha}_2| / \log n$ for all iterations.

Table 4.22: Mean and standard deviation of the estimators in Situation IV-(iii) of Model 2. True values: $\alpha_2^* = 1$, $\tau_*^\alpha = 0.4$, $\alpha_1^* = 1.0069$ and 1.0030 for $n = 10^6$ and 10^7, respectively

n	T	h_n	$\hat{\alpha}_1$	$\hat{\alpha}_2$	$\hat{\tau}_n^\alpha$
10^6	177.83	1.78×10^{-4}	1.0070	1.0000	0.3992
			(0.0017)	(0.0017)	(0.0677)
10^7	421.70	4.22×10^{-4}	1.0030	1.0000	0.3986
			(0.0005)	(0.0005)	(0.0264)

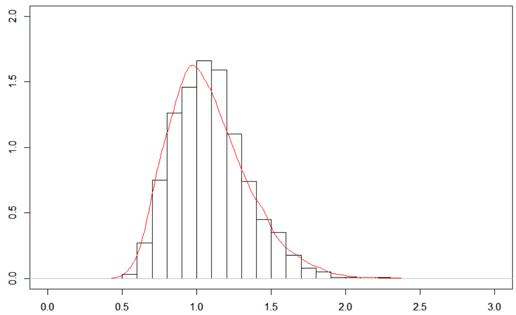
Next, we tested for changes in the drift parameter in the intervals $[0, \underline{\tau}_n T]$ and $[\overline{\tau}_n T, T]$. Table 4.23 and Figure 21 show the simulation results of the tests for changes in the drift parameter. It can be seen that the test statistics have good performance. Hence, we constructed $\check{\boldsymbol{\beta}}_1$ and $\check{\boldsymbol{\beta}}_2$ using the data obtained from the intervals $[0, \underline{\tau}_n T]$ and $[\overline{\tau}_n T, T]$, respectively when the test statistics $\mathcal{T}_{1,n}^{(1)}$ and $\mathcal{T}_{1,n}^{(2)}$ did not exceed the critical value, and investigate $\sqrt{T}|\check{\boldsymbol{\beta}}_1 - \check{\boldsymbol{\beta}}_2|$. The result of the estimates of $\boldsymbol{\beta}_1^*$ and $\boldsymbol{\beta}_2^*$ is summarized in Table 4.24 and Figure 22. It can be inferred from Figure 22 that $\sqrt{T}|\check{\boldsymbol{\beta}}_1 - \check{\boldsymbol{\beta}}_2|$ tends to increase as n does.

Table 4.23: Proportions over the corresponding critical value in Situation IV-(iii) of Model 2.

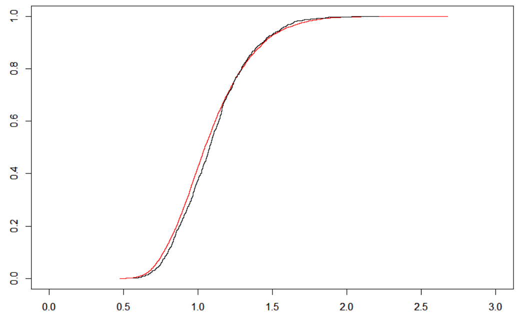
n	T	h_n	$\mathcal{T}_{1,n}^{(1)}$	$\mathcal{T}_{2,n}^{(1)}$	$\mathcal{T}_{1,n}^{(2)}$	$\mathcal{T}_{2,n}^{(2)}$
10^6	177.83	1.78×10^{-4}	0.043	0.071	0.052	0.065
10^7	421.70	4.22×10^{-4}	0.043	0.049	0.044	0.048

Table 4.24: Mean and standard deviation of the estimators in Situation IV-(iii) of Model 2. True values: $\beta_1^* = 1, \beta_1^* = 0.5, \gamma^* = 2$.

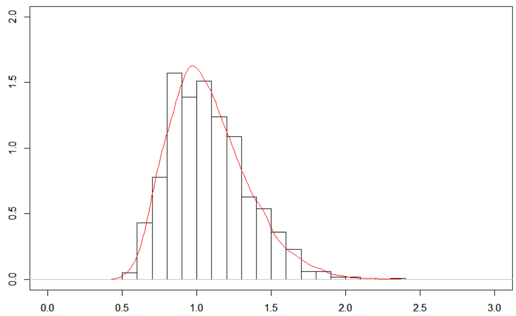
n	T	h_n	$\check{\beta}_1$	$\check{\gamma}_1$	$\check{\beta}_2$	$\check{\gamma}_2$
10^6	177.83	1.78×10^{-4}	1.0720	2.1524	0.5222	2.0737
			(0.4282)	(0.6173)	(0.2731)	(0.4723)
10^7	421.70	4.22×10^{-4}	1.0142	2.0395	0.5064	2.0227
			(0.1374)	(0.2129)	(0.0769)	(0.1502)



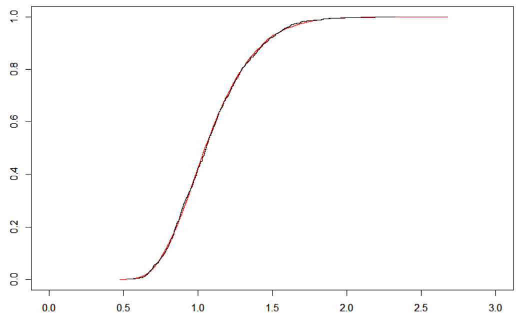
(a) Histogram of $\mathcal{T}_{2,n}^{(1)}$ with $n = 10^7$.



(b) EDF of $\mathcal{T}_{2,n}^{(1)}$ with $n = 10^7$.

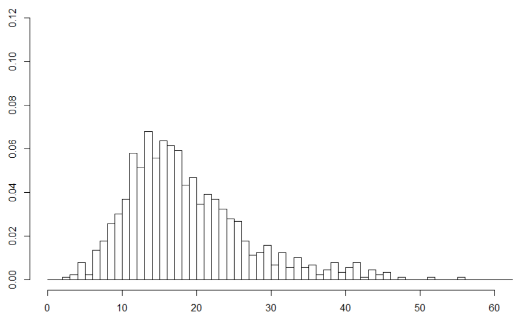


(c) Histogram of $\mathcal{T}_{2,n}^{(2)}$ with $n = 10^7$.

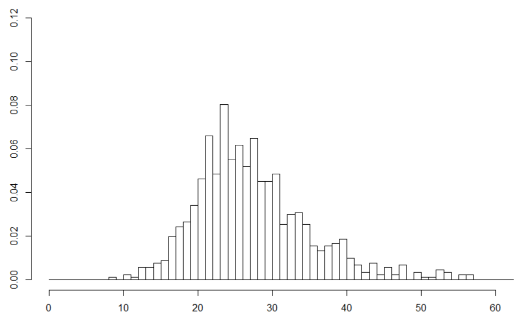


(d) EDF of $\mathcal{T}_{2,n}^{(2)}$ with $n = 10^7$.

Figure 21: Histogram (black line) versus theoretical density function (red line) and empirical distribution function (black line) versus theoretical distribution function (red line) in Situation IV-(iii) of Model 2.



(a) $n = 10^6$.



(b) $n = 10^7$.

Figure 22: Histogram of $\sqrt{T}|\check{\beta}_1 - \check{\beta}_2|$ in Situation IV-(iii) of Model 2.

Appendix A

Estimation of the nuisance parameters

When one considers change point estimation, it is necessary to estimate the parameters α_k^* and β_k^*. Here we discuss the estimation of nuisance parameters α_k^* and β_k^*.

First, we need the following information to estimate α_k^* or β_k^*:

[L1] There exist $\underline{\tau}^\alpha, \overline{\tau}^\alpha \in (0,1)$ such that $\tau_*^\alpha \in [\underline{\tau}^\alpha, \overline{\tau}^\alpha]$.

[L2] There exist $\underline{\tau}^\beta, \overline{\tau}^\beta \in (0,1)$ such that $\tau_*^\beta \in [\underline{\tau}^\beta, \overline{\tau}^\beta]$.

If this information is obtained, one can see that there is no change point in interval $[0, \underline{\tau}^\alpha T]$ nor $[\overline{\tau}^\alpha T, T]$, and estimate α_1^* from the data of $[0, \underline{\tau}^\alpha T]$ and α_2^* from the data of $[\overline{\tau}^\alpha T, T]$.

Next, we discuss how to find $\underline{\tau}$ and $\overline{\tau}$ that satisfy [L1] or [L2]. To find them, we use the test statistics to detect a change in the diffusion or drift parameters. Specifically, one can detect a change in the diffusion or drift parameters in the interval $[\tau_1 T, \tau_2 T]$ by the following test statistics.

$$\mathcal{T}_n^\alpha(\tau_1,\tau_2) = \frac{1}{\sqrt{2d([n\tau_2]-[n\tau_1])}} \max_{1\le k\le [n\tau_2]-[n\tau_1]} \left| \sum_{i=[n\tau_1]+1}^{[n\tau_1]+k} \hat{\eta}_i - \frac{k}{[n\tau_2]-[n\tau_1]} \sum_{i=[n\tau_1]+1}^{[n\tau_2]} \hat{\eta}_i \right|,$$

$$\mathcal{T}_{1,n}^\beta(\tau_1,\tau_2) = \frac{1}{\sqrt{dT(\tau_2-\tau_1)}} \max_{1\le k\le [n\tau_2]-[n\tau_1]} \left| \sum_{i=[n\tau_1]+1}^{[n\tau_1]+k} \hat{\xi}_i - \frac{k}{[n\tau_2]-[n\tau_1]} \sum_{i=[n\tau_1]+1}^{[n\tau_2]} \hat{\xi}_i \right|,$$

$$\mathcal{T}_{2,n}^\beta(\tau_1,\tau_2) = \frac{1}{\sqrt{T(\tau_2-\tau_1)}} \max_{1\le k\le [n\tau_2]-[n\tau_1]} \left| \mathcal{I}_n^{-1/2}(\tau_1,\tau_2) \left(\sum_{i=[n\tau_1]+1}^{[n\tau_1]+k} \hat{\zeta}_i - \frac{k}{[n\tau_2]-[n\tau_1]} \sum_{i=[n\tau_1]+1}^{[n\tau_2]} \hat{\zeta}_i \right) \right|,$$

where

$$\mathcal{I}_n(\tau_1,\tau_2) = \frac{1}{[n\tau_2]-[n\tau_1]} \sum_{i=[n\tau_1]+1}^{[n\tau_2]} \partial_\beta b(X_{t_{i-1}},\hat{\beta})^\top A^{-1}(X_{t_{i-1}},\hat{\alpha}) \partial_\beta b(X_{t_{i-1}},\hat{\beta}).$$

Finally, we describe how to find $\underline{\tau}$ and $\overline{\tau}$. Assume that a change is detected in the interval $[0, T]$.

U$_1$) Choose $\tau_1^U \in (0,1)$, and investigate a change point in the interval $[0, \tau_1^U T]$.

(1) If a change is detected, set $\overline{\tau} = \tau_1^U$ and go to step L$_1$.

(2) If not detected, go to step U_2.

$\vdots$

U_k) Choose $\tau_k^U \in (\tau_{k-1}^U, 1)$, and investigate a change point in the interval $[0, \tau_k^U T]$.

(1) If a change is detected, set $\overline{\tau} = \tau_k^U$ and go to step L_1.

(2) If not detected, go to step U_{k+1}.

Assume that τ_k^U is chosen as $\overline{\tau}$ in step U_k.

L_1) Choose $\tau_1^L \in (0, \tau_k^U)$, and investigate a change point in the interval $[\tau_1^L T, T]$.

(1) If a change is detected, set $\underline{\tau} = \tau_1^L$.

(2) If not detected, go to step L_2.

$\vdots$

L_m) Choose $\tau_m^L \in (0, \tau_{m-1}^L)$, and investigate a change point in the interval $[\tau_m^L T, T]$.

(1) If a change is detected, set $\underline{\tau} = \tau_m^L$.

(2) If not detected, go to step L_{m+1}.

We may also choose $\underline{\tau}$ and $\overline{\tau}$ at the same time, that is,

1) Choose $\tau_1 \in (0, 1/2)$, and investigate a change point in the interval $[\tau_1 T, (1-\tau_1)T]$.

(1) If a change is detected, set $\underline{\tau} = \tau_1$ and $\overline{\tau} = 1 - \tau_1$.

(2) If not detected, go to step 2.

***k*)** Choose $\tau_k \in (0, \tau_{k-1})$, and investigate a change point in the interval $[\tau_k T, (1-\tau_k)T]$.

(1) If a change is detected, set $\underline{\tau} = \tau_k$ and $\overline{\tau} = 1 - \tau_k$.

(2) If not detected, go to step $k+1$.

We can choose $\underline{\tau}$ and $\overline{\tau}$ in the above manner.

Sufficient condition of the assumptions

A process $\{X_t\}_{t\geq 0}$ with a change point can be expressed as follows. There exists a process $\{\tilde{X}_t\}_{t\geq 0}$ such that $X_t^{(1)} = \tilde{X}_t(\theta_1^*)$, $X_0^{(1)} = x_0^{(1)}$, $X_t^{(2)} = \tilde{X}_t(\theta_2^*)$, $X_0^{(2)} = x_0^{(2)}$, $X_{\tau T}^{(1)} = X_{\tau T}^{(2)}$ and

$$X_t = \begin{cases} X_t^{(1)}, & t \in [0, \tau T), \\ X_t^{(2)}, & t \in [\tau T, T]. \end{cases}$$

If the process $\{X_t^{(2)}\}_{t\ge 0}$ is stationary and $\theta_2^* \to \theta_0$, then one has the result that for $f \in C^1_{\uparrow}(\mathbb{R}^d)$,

$$
\begin{aligned}
&\max_{[n^{1/r}]\le k\le n-[n\tau]}\left|\frac{1}{k}\sum_{i=[n\tau]+1}^{[n\tau]+k} f(X_{t_{i-1}}) - \int_{\mathbb{R}^d} f(x)\mathrm{d}\mu_{\theta_0}(x)\right|\\
&= \max_{[n^{1/r}]\le k\le n-[n\tau]}\left|\frac{1}{k}\sum_{i=[n\tau]+1}^{[n\tau]+k} f(X^{(2)}_{t_{i-1}}) - \int_{\mathbb{R}^d} f(x)\mathrm{d}\mu_{\theta_0}(x)\right|\\
&\overset{\mathrm{d}}{=} \max_{[n^{1/r}]\le k\le n-[n\tau]}\left|\frac{1}{k}\sum_{i=1}^{k} f(X^{(2)}_{t_{i-1}}) - \int_{\mathbb{R}^d} f(x)\mathrm{d}\mu_{\theta_0}(x)\right| \xrightarrow{\mathbf{P}} 0,
\end{aligned}
$$

and thus [F2], [J2], [B4] and [B5] hold. The same is true for $[\mathrm{J2}']_k$. In this case, one can remove $T\vartheta_\alpha \to 0$ in [F1] and $T\vartheta_\beta^4 \to 0$ in [J1].

Model which satisfies [D2]

As an example of a model that satisfies [D2], we consider the d-dimensional diffusion process

$$
X_t = \begin{cases} X_0 + \displaystyle\int_0^t b(X_s,\beta)\mathrm{d}s + \int_0^t \sigma(X_s)\delta(\boldsymbol{\alpha}_1^*)\mathrm{d}W_s, & t \in [0, \tau_*^\alpha T),\\ X_{\tau_*^\alpha T} + \displaystyle\int_{\tau_*^\alpha T}^t b(X_s,\beta)\mathrm{d}s + \int_{\tau_*^\alpha T}^t \sigma(X_s)\delta(\boldsymbol{\alpha}_2^*)\mathrm{d}W_s, & t \in [\tau_*^\alpha T, T],\end{cases}
$$

where $\sigma : \mathbb{R}^d \to \mathbb{R}^d \otimes \mathbb{R}^d$, $\delta(\boldsymbol{\alpha}) = \mathrm{diag}\,(\alpha_1,\dots,\alpha_d)$, $\boldsymbol{\alpha} = (\alpha_1,\dots,\alpha_d)^\top$, $\alpha_1,\dots,\alpha_d > 0$. The true values of the parameters are $\boldsymbol{\alpha}_1^* = (\alpha_{1,1}^*,\dots,\alpha_{1,d}^*)^\top$, $\boldsymbol{\alpha}_2^* = (\alpha_{2,1}^*,\dots,\alpha_{2,d}^*)^\top$, which converge to $\boldsymbol{\alpha}_0 = (\alpha_{0,1},\dots,\alpha_{0,d})^\top \neq 0$. We define the estimator $\hat{\boldsymbol{\alpha}} = \mathrm{arginf}_{\boldsymbol{\alpha}}\, U_n(\boldsymbol{\alpha})$. Then, we have

$$
\vartheta_\alpha^{-1}(\hat{\boldsymbol{\alpha}} - \boldsymbol{\alpha}_0) = O_{\mathbf{P}}(1). \tag{A.1}
$$

Proof of (A.1). $\hat{\boldsymbol{\alpha}} = (\hat{\alpha}_j)_{j=1}^d$ can be expressed as follows.

$$
\hat{\alpha}_j = \sqrt{\frac{1}{n}\sum_{i=1}^n \mathrm{Tr}\left([\sigma(X_{t_{i-1}})^\top]^{-1}\delta(e_j)\sigma(X_{t_{i-1}})^{-1}\frac{(\Delta_i X)^{\otimes 2}}{h_n}\right)},
$$

where $e_1 = (1,0,\dots,0)^\top,\dots,e_d = (0,\dots,0,1)^\top$. Define

$$
\underline{U}_n(\boldsymbol{\alpha}) = \sum_{i=1}^{[n\tau_*^\alpha]} F_i(\boldsymbol{\alpha}), \quad \overline{U}_n(\boldsymbol{\alpha}) = \sum_{i=[n\tau_*^\alpha]+1}^{n} F_i(\boldsymbol{\alpha}),
$$

$\hat{\boldsymbol{\alpha}}_1 = \mathrm{arginf}_{\boldsymbol{\alpha}}\, \underline{U}_n(\boldsymbol{\alpha})$ and $\hat{\boldsymbol{\alpha}}_2 = \mathrm{arginf}_{\boldsymbol{\alpha}}\, \overline{U}_n(\boldsymbol{\alpha})$. We find that $\hat{\boldsymbol{\alpha}}_1 = (\hat{\alpha}_{1,j})_{j=1}^d$ and $\hat{\boldsymbol{\alpha}}_2 = (\hat{\alpha}_{2,j})_{j=1}^d$ satisfy $\sqrt{n}(\hat{\boldsymbol{\alpha}}_k - \boldsymbol{\alpha}_k^*) = O_{\mathbf{P}}(1)$ and

$$
\hat{\alpha}_{1,j} = \sqrt{\frac{1}{[n\tau_*^\alpha]}\sum_{i=1}^{[n\tau_*^\alpha]} \mathrm{Tr}\left([\sigma(X_{t_{i-1}})^\top]^{-1}\delta(e_j)\sigma(X_{t_{i-1}})^{-1}\frac{(\Delta_i X)^{\otimes 2}}{h_n}\right)},
$$

$$\hat{\alpha}_{2,j} = \sqrt{\frac{1}{n-[n\tau_*^\alpha]}\sum_{i=[n\tau_*^\alpha]+1}^{n}\mathrm{Tr}\left([\sigma(X_{t_{i-1}})^\top]^{-1}\delta(e_j)\sigma(X_{t_{i-1}})^{-1}\frac{(\Delta_i X)^{\otimes 2}}{h_n}\right)}.$$

Noting that $\vartheta_\alpha^{-1}(\hat{\boldsymbol{\alpha}}_k - \boldsymbol{\alpha}_0) = O_{\mathbf{P}}(1)$ and

$$\hat{\alpha}_j = \sqrt{\frac{[n\tau_*^\alpha]}{n}\hat{\alpha}_{1,j}^2 + \frac{n-[n\tau_*^\alpha]}{n}\hat{\alpha}_{2,j}^2},$$

we obtain

$$\vartheta_\alpha^{-1}|\hat{\boldsymbol{\alpha}} - \boldsymbol{\alpha}_0| \le \sum_{j=1}^{d}\vartheta_\alpha^{-1}|\hat{\alpha}_j - \alpha_{0,j}| \le \sum_{j=1}^{d}\left(\frac{|\hat{\alpha}_{1,j}+\alpha_{0,j}|}{|\hat{\alpha}_j+\alpha_{0,j}|}\vartheta_\alpha^{-1}|\hat{\alpha}_{1,j}-\alpha_{0,j}| + \frac{|\hat{\alpha}_{2,j}+\alpha_{0,j}|}{|\hat{\alpha}_j+\alpha_{0,j}|}\vartheta_\alpha^{-1}|\hat{\alpha}_{2,j}-\alpha_{0,j}|\right),$$

which together with $\hat{\boldsymbol{\alpha}}_k \xrightarrow{\mathbf{P}} \boldsymbol{\alpha}_0$ and $\hat{\boldsymbol{\alpha}} \xrightarrow{\mathbf{P}} \boldsymbol{\alpha}_0$ yields $\vartheta_\alpha^{-1}(\hat{\boldsymbol{\alpha}} - \boldsymbol{\alpha}_0) = O_{\mathbf{P}}(1)$. □

From the above, the one-dimensional Ornstein-Uhlenbeck process and the hyperbolic diffusion model satisfy [D2] because the diffusion coefficient is $a(x,\alpha) = \alpha$.

Model which satisfies [D3]

First, as an example of a model that satisfies [D3], we consider the d-dimensional diffusion process with the diffusion coefficient

$$a(x,\boldsymbol{\alpha}) = \sigma(x)\mathrm{diag}\,(\boldsymbol{\alpha}),$$

where $\sigma : \mathbb{R}^d \to \mathbb{R}^d \otimes \mathbb{R}^d$, $\boldsymbol{\alpha} = (\alpha_1,\dots,\alpha_d)^\top$, $\alpha_1,\dots,\alpha_d > 0$. The true values of the parameters are $\boldsymbol{\alpha}_1^* = \boldsymbol{\alpha}_0 + \vartheta_\alpha \boldsymbol{c}_1$ and $\boldsymbol{\alpha}_2^* = \boldsymbol{\alpha}_0 + \vartheta_\alpha \boldsymbol{c}_2$, where $\boldsymbol{\alpha}_0 = (\alpha_{0,1},\dots,\alpha_{0,3})^\top \ne 0$, $\boldsymbol{c}_1 = (c_{1,1},\dots,c_{1,d})^\top$, $\boldsymbol{c}_2 = (c_{2,1},\dots,c_{2,d})^\top$. We have from $\mathrm{Tr}\,(A^{-1}\partial_{\alpha_j}A(x,\boldsymbol{\alpha})) = 2/\alpha_j$ the result that

$$\int_{\mathbb{R}^d}[\mathrm{Tr}\,(A^{-1}\partial_{\alpha^l}A(x,\boldsymbol{\alpha}_0))]_l \mathrm{d}\mu_{\alpha_0}(x)(\boldsymbol{c}_1 - \boldsymbol{c}_2) = \sum_{j=1}^{d}\frac{2(c_{1,j}-c_{2,j})}{\alpha_{0,j}}.$$

Therefore if

$$\sum_{j=1}^{d}\frac{c_{1,j}-c_{2,j}}{\alpha_j} \ne 0, \tag{A.2}$$

then [D3] holds. In particular, we have (A.2) if any of the following cases:

1. $c_{1,j} - c_{2,j} \ge 0$ for all $1 \le j \le d$, and $c_{1,j} - c_{2,j} > 0$ for some $1 \le j \le d$.

2. $c_{1,j} - c_{2,j} \le 0$ for all $1 \le j \le d$, and $c_{1,j} - c_{2,j} < 0$ for some $1 \le j \le d$.

This means that [D3] holds when only α_j $(1 \le j \le d)$ changes. Therefore, the one-dimensional Ornstein-Uhlenbeck process and the hyperbolic diffusion model satisfy [D3] because the diffusion coefficient is $a(x,\alpha) = \alpha$.

Ornstein-Uhlenbeck process

We consider the one-dimensional Ornstein-Uhlenbeck process

$$\mathrm{d}X_t = -\beta(X_t - \gamma)\mathrm{d}t + \alpha \mathrm{d}W_t, \quad X_0 = x_0, \tag{A.3}$$

where $\alpha, \beta > 0$, $\gamma \in \mathbb{R}$.

First, consider the consistency of the test $\mathcal{T}_n^{\alpha}$. Since the invariant measure of the solution in (A.3) is $\mu_\theta \sim N(\gamma, \frac{\alpha^2}{2\beta})$, $\theta = (\alpha, \beta, \gamma)$, we have $\mathcal{F}(\alpha, \alpha') = (\alpha/\alpha')^2$. Therefore, it follows from $\alpha > 0$ that $\mathcal{F}(\alpha_1^*, \alpha') \neq \mathcal{F}(\alpha_2^*, \alpha')$ for $\alpha_1^* \neq \alpha_2^*$, and the test $\mathcal{T}_n^{\alpha}$ has consistency according to Theorem 1 when $|\alpha_1^* - \alpha_2^*|$ is fixed. For the case where $|\alpha_1^* - \alpha_2^*|$ shrinks, we have already discussed above.

Next, we investigate the consistency of the tests $\mathcal{T}_{1,n}^{\beta}$ and $\mathcal{T}_{2,n}^{\beta}$. For the drift parameter $\boldsymbol{\beta}_1^* = (\beta_1^*, \gamma_1^*)$, we have

$$\mathcal{G}(\alpha, \boldsymbol{\beta}_1^*, \boldsymbol{\beta}') = \int_{\mathbb{R}} \frac{1}{\alpha}\Big(-(\beta_1^* - \beta')x + (\beta_1^*\gamma_1^* - \beta'\gamma')\Big)\mathrm{d}\mu_{\theta_1}(x) = \frac{\beta'}{\alpha}(\gamma_1^* - \gamma'),$$

where $\theta_1 = (\alpha, \boldsymbol{\beta}_1^*)$. If $\beta_1^* \neq \beta_2^*$ and $\gamma_1^* = \gamma_2^*$, then

$$\mathcal{G}(\alpha^*, \boldsymbol{\beta}_1^*, \boldsymbol{\beta}') - \mathcal{G}(\alpha^*, \boldsymbol{\beta}_2^*, \boldsymbol{\beta}') = \frac{\beta'}{\alpha^*}(\gamma_1^* - \gamma_2^*) = 0,$$

and [H2] does not hold. If $\gamma_1^* \neq \gamma_2^*$, then

$$\mathcal{G}(\alpha^*, \boldsymbol{\beta}_1^*, \boldsymbol{\beta}') - \mathcal{G}(\alpha^*, \boldsymbol{\beta}_2^*, \boldsymbol{\beta}') = \frac{\beta'}{\alpha_0}(\gamma_1^* - \gamma_2^*) \neq 0,$$

and the test $\mathcal{T}_{1,n}^{\beta}$ is consistent according to Theorem 3 when $|\beta_1^* - \beta_2^*|$ is fixed. Consider the case where $|\beta_1^* - \beta_2^*|$ shrinks. Thus, we consider the SDE

$$X_t = \begin{cases} X_0 - \displaystyle\int_0^t \beta_1^*(X_s - \gamma_1^*)\mathrm{d}s + \alpha^* W_t, & t \in [0, \tau_*^{\beta}T), \\ X_{\tau_*^{\beta}T} - \displaystyle\int_{\tau_*^{\beta}T}^t \beta_2^*(X_s - \gamma_2^*)\mathrm{d}s + \alpha^*(W_t - W_{\tau_*^{\beta}T}), & t \in [\tau_*^{\beta}T, T], \end{cases}$$

where $\boldsymbol{\beta}_0 = (\beta_0, \gamma_0)^\mathsf{T}$, $\boldsymbol{d}_k = (d_{k,1}, d_{k,2})^\mathsf{T}$, $\boldsymbol{\beta}_k^* = (\beta_k^*, \gamma_k^*)^\mathsf{T} = \boldsymbol{\beta}_0 + \vartheta_\beta \boldsymbol{d}_k$, which implies that [I2] holds. Furthermore,

$$\int_{\mathbb{R}} \frac{1}{\alpha^*}(-x + \gamma_0, \beta_0)\mathrm{d}\mu_{(\alpha^*, \boldsymbol{\beta}_0)}(x)(\boldsymbol{d}_1 - \boldsymbol{d}_2) = \left(0, \frac{\beta_0}{\alpha^*}\right)\begin{pmatrix} d_{1,1} - d_{2,1} \\ d_{1,2} - d_{2,2} \end{pmatrix} = \frac{\beta_0}{\alpha^*}(d_{1,2} - d_{2,2}).$$

Therefore, if γ changes and β does not change, then [I4] holds, and the test $\mathcal{T}_{1,n}^{\beta}$ is consistent. However, when β changes and γ does not change, [I4] does not hold.

Since

$$\mathcal{H}(\alpha, \boldsymbol{\beta}_1^*, \boldsymbol{\beta}') = \frac{1}{\alpha^2}\int_{\mathbb{R}} \begin{pmatrix} -(x - \gamma') \\ \beta' \end{pmatrix} (-(\beta_1^* - \beta')x + (\beta_1^*\gamma_1^* - \beta'\gamma'))\mathrm{d}\mu_{\theta_1^*}(x)$$

$$= \frac{1}{\alpha^2}\begin{pmatrix} \alpha^2(1-\beta'/\beta_1^*)/2 - \beta'(\gamma_1^* - \gamma')^2 \\ (\beta')^2(\gamma_1^* - \gamma') \end{pmatrix}$$

and

$$\mathcal{H}(\alpha^*, \boldsymbol{\beta}_1^*, \boldsymbol{\beta}') - \mathcal{H}(\alpha^*, \boldsymbol{\beta}_2^*, \boldsymbol{\beta}') = \frac{\beta'}{(\alpha^*)^2}\begin{pmatrix} -(\alpha^*)^2((\beta_1^*)^{-1} - (\beta_2^*)^{-1})/2 - (\gamma_1^* - \gamma_2^*)(\gamma_1^* + \gamma_2^* - 2\gamma') \\ \beta'(\gamma_1^* - \gamma_2^*) \end{pmatrix},$$

we find from $\beta' > 0$ that $\mathcal{H}(\alpha^*, \boldsymbol{\beta}_1^*, \boldsymbol{\beta}') - \mathcal{H}(\alpha^*, \boldsymbol{\beta}_2^*, \boldsymbol{\beta}') \neq 0$ under $\gamma_1^* \neq \gamma_2^*$. It also holds that under $\beta_1^* \neq \beta_2^*$ and $\gamma_1^* = \gamma_2^*$,

$$\mathcal{H}(\alpha^*, \boldsymbol{\beta}_1^*, \boldsymbol{\beta}') - \mathcal{H}(\alpha^*, \boldsymbol{\beta}_2^*, \boldsymbol{\beta}') = -\frac{\beta'}{2}\left(\frac{1}{\beta_1^*} - \frac{1}{\beta_2^*}\right)\begin{pmatrix}1\\0\end{pmatrix} \neq 0.$$

Hence, the test $\mathcal{T}_{2,n}^{\beta}$ is consistent.

If $a(x, \alpha) = \sigma(x)c(\alpha)$ for $\sigma : \mathbb{R}^d \to \mathbb{R}^d \otimes \mathbb{R}^d$, $c : \mathbb{R}^p \to \mathbb{R}^d \otimes \mathbb{R}^d$, then [F2], [G1], [G2](a) and (b) are satisfied because the functions which appear in them do not depend on x. Therefore, we find that the Ornstein-Uhlenbeck process is an example of a model that satisfies the conditions in Case A_α.

Hyperbolic diffusion model

We consider the hyperbolic diffusion model

$$\mathrm{d}X_t = \left(\beta - \frac{\gamma X_t}{\sqrt{1+X_t^2}}\right)\mathrm{d}t + \alpha \mathrm{d}W_t, \quad X_0 = x_0, \tag{A.4}$$

where $\alpha > 0, \beta \in \mathbb{R}, \gamma > |\beta|$.

We study the consistency of the tests $\mathcal{T}_n^{\alpha}$ and $\mathcal{T}_{1,n}^{\beta}$ when the change in the parameter is small. Let $b(x, \boldsymbol{\beta}) = \beta - \gamma x/\sqrt{1+x^2}$ and $a(x, \alpha) = \alpha$. From (A.2), the test $\mathcal{T}_n^{\alpha}$ is consistent when $|\alpha_1^* - \alpha_2^*| \to 0$. Next, we investigate the consistency of the test $\mathcal{T}_{1,n}^{\beta}$ in Case A_β. That is, we consider the SDE

$$X_t = \begin{cases} X_0 + \int_0^t \left(\beta_1^* - \frac{\gamma_1^* X_s}{\sqrt{1+X_s^2}}\right)\mathrm{d}s + \alpha^* W_t, & t \in [0, \tau_*^{\beta} T), \\ X_{\tau_*^{\beta} T} + \int_{\tau_*^{\beta} T}^t \left(\beta_2^* - \frac{\gamma_2^* X_s}{\sqrt{1+X_s^2}}\right)\mathrm{d}s + \alpha^*(W_t - W_{\tau_*^{\beta} T}), & t \in [\tau_*^{\beta} T, T], \end{cases}$$

where $\boldsymbol{\beta}_0 = (\beta_0, \gamma_0)^\top$, $\boldsymbol{d}_k = (d_{k,1}, d_{k,2})^\top$, $\boldsymbol{\beta}_k^* = (\beta_k^*, \gamma_k^*)^\top = \boldsymbol{\beta}_0 + \vartheta_\beta \boldsymbol{d}_k$, which implies that [I2] holds. The invariant density of the solution in (A.4) is $\pi(x) = m(x)/M$, where

$$m(x) = \exp\left(\frac{2}{\alpha^2}\left(\beta x - \gamma\sqrt{1+x^2}\right)\right), \quad M = \int_{\mathbb{R}} m(x)\mathrm{d}x.$$

We then have

$$\int_{\mathbb{R}} \partial_\beta b(x, \boldsymbol{\beta})\pi(x)\mathrm{d}x = 1, \quad \int_{\mathbb{R}} \partial_\gamma b(x, \boldsymbol{\beta})\pi(x)\mathrm{d}x = -\frac{\beta}{\gamma}.$$

Therefore

$$\int_{\mathbb{R}} \frac{1}{\alpha^*} \partial_{\boldsymbol{\beta}} b(x, \boldsymbol{\beta}_0) \mathrm{d}\mu_{(\alpha^*, \boldsymbol{\beta}_0)}(x)(\boldsymbol{d}_1 - \boldsymbol{d}_2) = \frac{1}{\alpha^*}\left((d_{1,1} - d_{2,1}) - \frac{\beta_0}{\gamma_0}(d_{1,2} - d_{2,2})\right). \tag{A.5}$$

In the following cases, [I4] holds because equation (A.5) does not equal 0:

(1) β changes and γ does not change,

(2) $\beta_0 \neq 0$, γ changes and β does not change,

(3) $\beta_0 > 0$, $d_{1,1} - d_{2,1} < 0$ (resp. > 0) and $d_{1,2} - d_{2,2} > 0$ (resp. < 0),

(4) $\beta_0 < 0$, $d_{1,1} - d_{2,1} < 0$ (resp. > 0) and $d_{1,2} - d_{2,2} < 0$ (resp. > 0).

Finally, we confirm that the hyperbolic diffusion model is an example of a model that satisfies the conditions in Cases B_α and B_β. It was noted above that [G1], [G2](a) and (b) hold. Since $b(x, \boldsymbol{\beta})$ and $\partial_x b(x, \boldsymbol{\beta})$ are bounded, we see from Remark 3 that [G2](c) holds. Because of

$$\Gamma^{\beta}(x, \alpha, \boldsymbol{\beta}_1, \boldsymbol{\beta}_2) = \frac{1}{\alpha^2}\left((\beta_1 - \beta_2) - (\gamma_1 - \gamma_2)\frac{x}{\sqrt{1 + x^2}}\right)^2,$$

where $\boldsymbol{\beta}_k = (\beta_k, \gamma_k)^\top$ and $-1 < x/\sqrt{1 + x^2} < 1$ for $x \in \mathbb{R}$, we have $\sup_x |\Gamma^{\beta}(x, \alpha^*, \boldsymbol{\beta}_1^*, \boldsymbol{\beta}_2^*)| > 0$ in the following cases:

(1) $\gamma_1^* = \gamma_2^*$,

(2) $\gamma_1^* \neq \gamma_2^*$ and $\beta_1^* - \beta_2^* < -(\gamma_1^* - \gamma_2^*)$,

(3) $\gamma_1^* \neq \gamma_2^*$ and $\beta_1^* - \beta_2^* > \gamma_1^* - \gamma_2^*$,

and then [K1] holds. Furthermore, we see from boundedness of $x/\sqrt{1 + x^2}$ that

$$\sup_{x, \alpha, \boldsymbol{\beta}_k} \left(\left|\partial_\alpha \Gamma^{\beta}(x, \alpha, \boldsymbol{\beta}_1, \boldsymbol{\beta}_2)\right| \vee \left|\partial_{\boldsymbol{\beta}_1} \Gamma^{\beta}(x, \alpha, \boldsymbol{\beta}_1, \boldsymbol{\beta}_2)\right| \vee \left|\partial_{\boldsymbol{\beta}_2} \Gamma^{\beta}(x, \alpha, \boldsymbol{\beta}_1, \boldsymbol{\beta}_2)\right|\right) < C,$$

$$\sup_{x, \alpha, \boldsymbol{\beta}_k} \left|\frac{1}{\alpha^2} \partial_{\boldsymbol{\beta}} b(x, \boldsymbol{\beta})(b(x, \boldsymbol{\beta}_1) - b(x, \boldsymbol{\beta}_2))^2\right| < C$$

and thus [K2] holds. Therefore, we find that the hyperbolic diffusion model is an example of a model that satisfies the conditions in Cases B_α and B_β.

Appendix B

This chapter provides the proofs of the results of Chapters 2 and 3.

We set the following notations.

1. $\mathscr{G}_{i-1}^n = \sigma[\{W_s\}_{s\le t_{i-1}^n}]$.
2. For a measurable set A and an integrable random variable X, we define
$$\mathbb{E}[X : A] = \int_A X(\omega)\mathrm{d}\mathbf{P}(\omega).$$
3. For a function f on $\mathbb{R}^d \times \Theta$, we define $f_{i-1}(\theta) = f(X_{t_{i-1}}, \theta)$.
4. We define
$$A \otimes x^{\otimes k} = \sum_{l_1,\dots,l_k=1}^{d_1} A^{l_1,\dots,l_k} x^{l_1} \cdots x^{l_k}, \qquad \text{for } A \in \underbrace{\mathbb{R}^{d_1} \otimes \cdots \otimes \mathbb{R}^{d_1}}_{k},\ x \in \mathbb{R}^{d_1}.$$

Proof of Theorem 1

We first prepare some auxiliary results. Afterwards, we show Theorem 1.

Lemma 1 (Kessler,1997)**.** *Suppose that [A1]-[A4] hold. Then for $l_1, \dots, l_4 \in \{1, \dots, d\}$,*

(1) $\mathbb{E}_\theta[(\Delta_i X)^{l_1} | \mathscr{G}_{i-1}^n] = h_n b_{i-1}^{l_1}(\beta) + R_{i-1}(h_n^2)$,

(2) $\mathbb{E}_\theta[(\Delta_i X)^{l_1} (\Delta_i X)^{l_2} | \mathscr{G}_{i-1}^n] = h_n A_{i-1}^{l_1,l_2}(\alpha) + R_{i-1}(h_n^2)$,

(3) $\mathbb{E}_\theta[\prod_{j=1}^4 (\Delta_i X)^{l_j} | \mathscr{G}_{i-1}^n] = h_n^2 (A_{i-1}^{l_1,l_2} A_{i-1}^{l_3,l_4}(\alpha) + A_{i-1}^{l_1,l_3} A_{i-1}^{l_2,l_4}(\alpha) + A_{i-1}^{l_1,l_4} A_{i-1}^{l_2,l_3}(\alpha)) + R_{i-1}(h_n^3)$.

Let
$$\eta_i = \mathrm{Tr}\left(A_{i-1}^{-1}(\alpha^*) \frac{(\Delta X_i)^{\otimes 2}}{h_n}\right), \quad \kappa(x, \alpha) = 1_d^\top a^{-1}(x, \alpha), \quad \xi_i = \kappa_{i-1}(\alpha^*)(\Delta_i X - h_n b_{i-1}(\beta^*)),$$
$$\zeta_i = \partial_\beta b_{i-1}(\beta^*)^\top A_{i-1}^{-1}(\alpha^*)(\Delta_i X - h_n b_{i-1}(\beta^*)).$$

Lemma 2. *Suppose that [A1]-[A4] hold. Then,* *(1)* $\mathbb{E}_{\theta^*}[\eta_i|\mathscr{G}^n_{i-1}] = d + R_{i-1}(h_n)$,

(2) $\mathbb{E}_{\theta^*}[\eta_i^2|\mathscr{G}^n_{i-1}] = d^2 + 2d + R_{i-1}(h_n)$,

(3) $\mathbb{E}_{\theta^*}[\xi_i|\mathscr{G}^n_{i-1}] = R_{i-1}(h_n^2)$,

(4) $\mathbb{E}_{\theta^*}[\xi_i^2|\mathscr{G}^n_{i-1}] = dh_n + R_{i-1}(h_n^2)$,

(5) $\mathbb{E}_{\theta^*}[\zeta_i|\mathscr{G}^n_{i-1}] = R_{i-1}(h_n^2)$,

(6) $\mathbb{E}_{\theta^*}[\zeta_i\zeta_i^\top|\mathscr{G}^n_{i-1}] = h_n\partial_\beta b_{i-1}(\beta^*)^\top A_{i-1}^{-1}(\alpha^*)\partial_\beta b_{i-1}(\beta^*) + R_{i-1}(h_n^2)$.

Proof. (3) and (5) are obvious. We have from Lemma 1 the result that

$$\begin{aligned}
\mathbb{E}_{\theta^*}[\eta_i|\mathscr{G}^n_{i-1}] &= h_n^{-1}\mathrm{Tr}\,(A_{i-1}^{-1}(\alpha^*)\mathbb{E}_{\theta^*}[(\Delta_i X)^{\otimes 2}|\mathscr{G}^n_{i-1}]) = d + R_{i-1}(h_n),\\
\mathbb{E}_{\theta^*}[\eta_i^2|\mathscr{G}^n_{i-1}] &= h_n^{-2}\sum_{l_1,\dots,l_4=1}^{d}(A_{i-1}^{-1})^{l_1,l_2}(A_{i-1}^{-1})^{l_3,l_4}(\alpha^*)\mathbb{E}_{\theta^*}\left[\prod_{j=1}^{4}(\Delta_i X)^{l_j}\middle|\mathscr{G}^n_{i-1}\right]\\
&= \sum_{l_1,\dots,l_4=1}^{d}(A_{i-1}^{-1})^{l_1,l_2}(A_{i-1}^{-1})^{l_3,l_4}(A_{i-1}^{l_1,l_2}A_{i-1}^{l_3,l_4} + A_{i-1}^{l_1,l_3}A_{i-1}^{l_2,l_4} + A_{i-1}^{l_1,l_4}A_{i-1}^{l_2,l_3})(\alpha^*) + R_{i-1}(h_n)\\
&= d^2 + 2d + R_{i-1}(h_n),\\
\mathbb{E}_{\theta^*}[\xi_i^2|\mathscr{G}^n_{i-1}] &= \kappa_{i-1}(\alpha^*)\mathbb{E}_{\theta^*}[(\Delta_i X - h_n b_{i-1}(\beta^*))(\Delta_i X - h_n b_{i-1}(\beta^*))^\top|\mathscr{G}^n_{i-1}]\kappa_{i-1}^\top(\alpha^*)\\
&= \kappa_{i-1}(\alpha^*)(h_n A_{i-1}(\alpha^*) + R_{i-1}(h_n^2))\kappa_{i-1}^\top(\alpha^*)\\
&= dh_n + R_{i-1}(h_n^2),\\
\mathbb{E}_{\theta^*}[\zeta_i\zeta_i^\top|\mathscr{G}^n_{i-1}] &= \partial_\beta b_{i-1}(\beta^*)^\top A_{i-1}^{-1}(\alpha^*)\mathbb{E}_{\theta^*}[(\Delta_i X - h_n b_{i-1}(\beta^*))^{\otimes 2}|\mathscr{G}^n_{i-1}]A_{i-1}^{-1}(\alpha^*)\partial_\beta b_{i-1}(\beta^*)\\
&= \partial_\beta b_{i-1}(\beta^*)^\top A_{i-1}^{-1}(\alpha^*)(h_n A_{i-1}^{-1}(\alpha^*) + R_{i-1}(h_n^2))A_{i-1}^{-1}(\alpha^*)\partial_\beta b_{i-1}(\beta^*)\\
&= h_n\partial_\beta b_{i-1}(\beta^*)^\top A_{i-1}^{-1}(\alpha^*)\partial_\beta b_{i-1}(\beta^*) + R_{i-1}(h_n^2).
\end{aligned}$$

□

Lemma 3 (Song and Lee, 2009)**.** *Suppose that [A1], [A2], [A5] hold and a function f on $\mathbb{R}^d\times\Theta$ satisfies*

(i) *f is continuous in $\theta\in\Theta$ for all $x\in\mathbb{R}^d$,*

(ii) *$\partial_x f$ exists and $f, \partial_x f$ are of polynomial growth in $x\in\mathbb{R}^d$ uniformly $\theta\in\Theta$.*

Moreover, if $nh_n^r\to\infty$ for some $1<r<2$, then under H_0^α (or H_0^α and H_0^β) as $nh_n^2\to 0$,

$$\max_{[n^{1/r}]\le k\le n}\sup_{\theta\in\Theta}\left|\frac{1}{k}\sum_{i=1}^{k} f(X_{t_{i-1}},\theta) - \int_{\mathbb{R}^d} f(x,\theta)\mathrm{d}\mu_{\theta^*}(x)\right| \xrightarrow{\text{a.s.}} 0.$$

Lemma 4. *Suppose that [A1], [A2], [A5] hold and f satisfies the conditions (i), (ii) in Lemma 3. Then, under H_0^α (or H_0^α and H_0^β) as $nh_n^2\to 0$,*

$$\frac{1}{n}\max_{1\le k\le n}\sup_{\theta\in\Theta}\left|\sum_{i=1}^{k} f(X_{t_{i-1}},\theta) - \frac{k}{n}\sum_{i=1}^{n} f(X_{t_{i-1}},\theta)\right| = o_{\mathbf{P}}(1).$$

Proof. Under $nh_n \to \infty$ and $nh_n^2 \to 0$, there exists $1 < r < 2$ such that $nh_n^r \to \infty$. Since it follows from Lemma 3 that

$$
\begin{aligned}
&\max_{1\le k\le [n^{1/r}]} \sup_{\theta\in\Theta} \frac{1}{n} \left| \sum_{i=1}^{k} f_{i-1}(\theta) - \frac{k}{n}\sum_{i=1}^{n} f_{i-1}(\theta) \right| \\
&\le \frac{1}{n}\left(\sum_{i=1}^{[n^{1/r}]} \sup_{\theta\in\Theta} |f_{i-1}(\theta)| + \frac{[n^{1/r}]}{n} \sum_{i=1}^{n} \sup_{\theta\in\Theta} |f_{i-1}(\theta)| \right) \\
&= \frac{[n^{1/r}]}{n}\left(\frac{1}{[n^{1/r}]} \sum_{i=1}^{[n^{1/r}]} \sup_{\theta\in\Theta} |f_{i-1}(\theta)| + \frac{1}{n} \sum_{i=1}^{n} \sup_{\theta\in\Theta} |f_{i-1}(\theta)| \right) = o_{\mathbf{P}}(1)
\end{aligned}
$$

and

$$
\begin{aligned}
&\max_{[n^{1/r}]\le k\le n} \sup_{\theta\in\Theta} \frac{1}{n} \left| \sum_{i=1}^{k} f_{i-1}(\theta) - \frac{k}{n}\sum_{i=1}^{n} f_{i-1}(\theta) \right| \\
&= \max_{[n^{1/r}]\le k\le n} \sup_{\theta\in\Theta} \frac{k}{n} \left| \frac{1}{k}\sum_{i=1}^{k} f_{i-1}(\theta) - \frac{1}{n}\sum_{i=1}^{n} f_{i-1}(\theta) \right| \\
&\le \max_{[n^{1/r}]\le k\le n} \sup_{\theta\in\Theta} \left| \frac{1}{k}\sum_{i=1}^{k} f_{i-1}(\theta) - \int_{\mathbb{R}^d} f(x,\theta)\mathrm{d}\mu_{\theta^*}(x) - \frac{1}{n}\sum_{i=1}^{n} f_{i-1}(\theta) + \int_{\mathbb{R}^d} f(x,\theta)\mathrm{d}\mu_{\theta^*}(x) \right| \\
&\le 2 \max_{[n^{1/r}]\le k\le n} \sup_{\theta\in\Theta} \left| \frac{1}{k}\sum_{i=1}^{k} f_{i-1}(\theta) - \int_{\mathbb{R}^d} f(x,\theta)\mathrm{d}\mu_{\theta^*}(x) \right| \stackrel{\text{a.s.}}{\to} 0,
\end{aligned}
$$

we obtain

$$
\begin{aligned}
&\frac{1}{n} \max_{1\le k\le n} \sup_{\theta\in\Theta} \left| \sum_{i=1}^{k} f_{i-1}(\theta) - \frac{k}{n}\sum_{i=1}^{n} f_{i-1}(\theta) \right| \\
&\le \frac{1}{n} \max_{1\le k\le [n^{1/r}]} \sup_{\theta\in\Theta} \left| \sum_{i=1}^{k} f_{i-1}(\theta) - \frac{k}{n}\sum_{i=1}^{n} f_{i-1}(\theta) \right| + \frac{1}{n} \max_{[n^{1/r}]\le k\le n} \sup_{\theta\in\Theta} \left| \sum_{i=1}^{k} f_{i-1}(\theta) - \frac{k}{n}\sum_{i=1}^{n} f_{i-1}(\theta) \right| \\
&= o_{\mathbf{P}}(1).
\end{aligned}
$$

□

Lemma 5. *Suppose that [A1]-[A5] hold and f satisfies the conditions (i), (ii) in Lemma 3. Then, under H_0^α (or H_0^α and H_0^β) as $nh_n^2 \to 0$,*

$$
\frac{1}{T} \max_{1\le k\le n} \left| \sum_{i=1}^{k} f(X_{t_{i-1}},\theta^*)(\Delta_i X)^l - \frac{k}{n}\sum_{i=1}^{n} f(X_{t_{i-1}},\theta^*)(\Delta_i X)^l \right| = o_{\mathbf{P}}(1), \tag{B.6}
$$

$$
\frac{1}{T} \max_{1\le k\le n} \left| \sum_{i=1}^{k} f(X_{t_{i-1}},\theta^*)(\Delta_i X)^{l_1}(\Delta_i X)^{l_2} - \frac{k}{n}\sum_{i=1}^{n} f(X_{t_{i-1}},\theta^*)(\Delta_i X)^{l_1}(\Delta_i X)^{l_2} \right| = o_{\mathbf{P}}(1). \tag{B.7}
$$

Proof. We first show that

$$\mathcal{S}_n^{(1)} = \frac{1}{n} \max_{1\le k\le n} \left| \sum_{i=1}^k f_{i-1}(\theta^*)\frac{(\Delta_i X)^l}{h_n} - \sum_{i=1}^k \mathbb{E}_{\theta^*}\left[f_{i-1}(\theta^*)\frac{(\Delta_i X)^l}{h_n} \middle| \mathscr{G}_{i-1}^n \right] \right| = o_{\mathbf{P}}(1). \tag{B.8}$$

Let

$$M_i = f_{i-1}(\theta^*)\left(\frac{(\Delta_i X)^l}{h_n} - \mathbb{E}_{\theta^*}\left[\frac{(\Delta_i X)^l}{h_n} \middle| \mathscr{G}_{i-1}^n \right] \right), \quad \mathcal{M}_k = \sum_{i=1}^k M_i.$$

Because it follows from Lemma 1 that $\mathbb{E}_{\theta^*}[|M_i|^2|\mathscr{G}_{i-1}^n] = R_{i-1}(h_n^{-1}, \theta)$, we obtain from Theorem 2.11 of Hall and Heyde (1980) the result that

$$\frac{1}{n^2}\mathbb{E}_{\theta^*}\left[\max_{1\le k\le n} |\mathcal{M}_k|^2 \right] \lesssim \frac{1}{n^2}\left(\mathbb{E}_{\theta^*}\left[\sum_{i=1}^n \mathbb{E}_{\theta^*}[|M_i|^2|\mathscr{G}_{i-1}^n] \right] + \mathbb{E}_{\theta^*}\left[\max_{1\le k\le n} |M_k|^2 \right] \right) \lesssim \frac{1}{T} \to 0,$$

which implies that $\mathcal{S}_n^{(1)} = n^{-1}\max_{1\le k\le n}|\mathcal{M}_k| = o_{\mathbf{P}}(1)$.

Next, we show that

$$\mathcal{S}_n^{(2)} = \frac{1}{n} \max_{1\le k\le n} \left| \sum_{i=1}^k f_{i-1}(\theta^*)\mathbb{E}_{\theta^*}\left[\frac{(\Delta_i X)^l}{h_n} \middle| \mathscr{G}_{i-1}^n \right] - k\int_{\mathbb{R}^d} f(x,\theta^*)b^l(x,\beta^*)\mathrm{d}\mu_{\theta^*}(x) \right| = o_{\mathbf{P}}(1). \tag{B.9}$$

We have from Lemmas 1-4 the result that

$$\begin{aligned}
\mathcal{S}_n^{(2)} &= \frac{1}{n} \max_{1\le k\le n} \left| \sum_{i=1}^k f_{i-1}(\theta^*)(b_{i-1}^l(\beta^*) + R_{i-1}(h_n)) - k\int_{\mathbb{R}^d} f(x,\theta^*)b^l(x,\beta^*)\mathrm{d}\mu_{\theta^*}(x) \right| \\
&\le \frac{1}{n} \max_{1\le k\le n} \left| \sum_{i=1}^k f_{i-1}(\theta^*)b_{i-1}^l(\beta^*) - k\int_{\mathbb{R}^d} f(x,\theta^*)b^l(x,\beta^*)\mathrm{d}\mu_{\theta^*}(x) \right| + \frac{1}{n} \max_{1\le k\le n} \left| \sum_{i=1}^k R_{i-1}(h_n) \right| \\
&\le \frac{1}{n} \max_{1\le k\le n} \left| \sum_{i=1}^k f_{i-1}(\theta^*)b_{i-1}^l(\beta^*) - \frac{k}{n}\sum_{i=1}^n f_{i-1}(\theta^*)b_{i-1}^l(\beta^*) \right| \\
&\qquad + \frac{1}{n} \max_{1\le k\le n} \left| \frac{k}{n}\sum_{i=1}^n f_{i-1}(\theta^*)b_{i-1}^l(\beta^*) - k\int_{\mathbb{R}^d} f(x,\theta^*)b^l(x,\beta^*)\mathrm{d}\mu_{\theta^*}(x) \right| + o_{\mathbf{P}}(1) \\
&\le \frac{1}{n} \max_{1\le k\le n} \left| \sum_{i=1}^k f_{i-1}(\theta^*)b_{i-1}^l(\beta^*) - \frac{k}{n}\sum_{i=1}^n f_{i-1}(\theta^*)b_{i-1}^l(\beta^*) \right| \\
&\qquad + \left| \frac{1}{n}\sum_{i=1}^n f_{i-1}(\theta^*)b_{i-1}^l(\beta^*) - \int_{\mathbb{R}^d} f(x,\theta^*)b^l(x,\beta^*)\mathrm{d}\mu_{\theta^*}(x) \right| + o_{\mathbf{P}}(1) \\
&= o_{\mathbf{P}}(1).
\end{aligned}$$

Hence, we obtain from (B.8) and (B.9) the result that

$$\frac{1}{T} \max_{1\le k\le n} \left| \sum_{i=1}^k f_{i-1}(\theta^*)(\Delta_i X)^l - \frac{k}{n}\sum_{i=1}^n f_{i-1}(\theta^*)(\Delta_i X)^l \right|$$

$$
\begin{aligned}
&\le \frac{1}{n}\max_{1\le k\le n}\left|\sum_{i=1}^{k} f_{i-1}(\theta^*)\frac{(\Delta_i X)^l}{h_n} - k\int_{\mathbb{R}^d} f(x,\theta^*)b^l(x,\beta^*)\mathrm{d}\mu_{\theta^*}(x)\right| \\
&\qquad + \frac{1}{n}\max_{1\le k\le n}\left|k\int_{\mathbb{R}^d} f(x,\theta^*)b^l(x,\beta^*)\mathrm{d}\mu_{\theta^*}(x) - \frac{k}{n}\sum_{i=1}^{n} f_{i-1}(\theta^*)\frac{(\Delta_i X)^l}{h_n}\right| \\
&\le \frac{1}{n}\max_{1\le k\le n}\left|\sum_{i=1}^{k} f_{i-1}(\theta^*)\frac{(\Delta_i X)^l}{h_n} - k\int_{\mathbb{R}^d} f(x,\theta^*)b^l(x,\beta^*)\mathrm{d}\mu_{\theta^*}(x)\right| \\
&\qquad + \frac{1}{n}\left|n\int_{\mathbb{R}^d} f(x,\theta^*)b^l(x,\beta^*)\mathrm{d}\mu_{\theta^*}(x) - \sum_{i=1}^{n} f_{i-1}(\theta^*)\frac{(\Delta_i X)^l}{h_n}\right| \\
&\lesssim \frac{1}{n}\max_{1\le k\le n}\left|\sum_{i=1}^{k} f_{i-1}(\theta^*)\frac{(\Delta_i X)^l}{h_n} - k\int_{\mathbb{R}^d} f(x,\theta^*)b^l(x,\beta^*)\mathrm{d}\mu_{\theta^*}(x)\right| \\
&\le \mathcal{S}_n^{(1)} + \mathcal{S}_n^{(2)} = o_{\mathbf{P}}(1).
\end{aligned}
$$

This completes the proof of (B.6). In the same way, we have (B.7). □

Proof of Theorem 1. (1) Let

$$
\eta_i = \mathrm{Tr}\left(A_{i-1}^{-1}(\alpha^*)\frac{(\Delta X_i)^{\otimes 2}}{h_n}\right) = \sum_{l_1,l_2=1}^{d}(A_{i-1}^{-1}(\alpha^*))^{l_1,l_2}\frac{(\Delta_i X)^{l_1}(\Delta_i X)^{l_2}}{h_n}.
$$

Since it follows from the Taylor expansion that

$$
(A_{i-1}^{-1}(\hat\alpha))^{l_1,l_2} = (A_{i-1}^{-1}(\alpha^*))^{l_1,l_2} + \partial_\alpha(A_{i-1}^{-1}(\alpha^*))^{l_1,l_2}(\hat\alpha-\alpha^*) + (\hat\alpha-\alpha^*)^\mathsf{T}\mathcal{A}_{i-1}^{l_1,l_2}(\hat\alpha-\alpha^*),
$$

where

$$
\mathcal{A}_{i-1}^{l_1,l_2} = \int_0^1 (1-u)\partial_\alpha^2(A_{i-1}^{-1}(\alpha^*+u(\hat\alpha-\alpha^*))^{l_1,l_2}\mathrm{d}u,
$$

we have

$$
\begin{aligned}
\hat\eta_i &= \sum_{l_1,l_2=1}^{d}(A_{i-1}^{-1}(\hat\alpha))^{l_1,l_2}\frac{(\Delta_i X)^{l_1}(\Delta_i X)^{l_2}}{h_n} \\
&= \eta_i + \left(\frac{1}{\sqrt{n}}\sum_{l_1,l_2=1}^{d}\partial_\alpha(A_{i-1}^{-1}(\alpha^*))^{l_1,l_2}\frac{(\Delta_i X)^{l_1}(\Delta_i X)^{l_2}}{h_n}\right)\sqrt{n}(\hat\alpha-\alpha^*) \\
&\qquad + \sqrt{n}(\hat\alpha-\alpha^*)^\mathsf{T}\left(\frac{1}{n}\sum_{l_1,l_2=1}^{d}\mathcal{A}_{i-1}^{l_1,l_2}\frac{(\Delta_i X)^{l_1}(\Delta_i X)^{l_2}}{h_n}\right)\sqrt{n}(\hat\alpha-\alpha^*) \\
&=: \eta_i + \frac{1}{\sqrt{n}}\mathcal{H}_{1,i}\sqrt{n}(\hat\alpha-\alpha^*) + \frac{1}{n}\sqrt{n}(\hat\alpha-\alpha^*)^\mathsf{T}\mathcal{H}_{2,i}\sqrt{n}(\hat\alpha-\alpha^*).
\end{aligned}
$$

Therefore, it is enough to verify

$$\frac{1}{\sqrt{2dn}} \max_{1\le k\le n}\left|\sum_{i=1}^{k}\eta_i - \frac{k}{n}\sum_{i=1}^{n}\eta_i\right| \xrightarrow{\mathrm{d}} \sup_{0\le s\le 1}|\mathbb{B}_1^0(s)|, \tag{B.10}$$

$$\frac{1}{n} \max_{1\le k\le n}\left|\sum_{i=1}^{k}\mathcal{H}_{1,i} - \frac{k}{n}\sum_{i=1}^{n}\mathcal{H}_{1,i}\right| = o_{\mathbf{P}}(1), \tag{B.11}$$

$$\frac{1}{n^{3/2}} \max_{1\le k\le n}\left|\sum_{i=1}^{k}\mathcal{H}_{2,i} - \frac{k}{n}\sum_{i=1}^{n}\mathcal{H}_{2,i}\right| = o_{\mathbf{P}}(1) \tag{B.12}$$

for the proof of the first statement of Theorem 1. Thereafter, we show (B.10)-(B.12).

Proof of (B.10). We just show

$$\mathcal{U}_n(s) = \frac{1}{\sqrt{2dn}}\sum_{i=1}^{[ns]}(\eta_i - d) \xrightarrow{\mathrm{w}} \mathbb{B}_1(s) \quad \text{in } \mathbb{D}[0,1] \tag{B.13}$$

because it follows from the continuous mapping theorem that

$$\begin{aligned}
\frac{1}{\sqrt{2dn}} \max_{1\le k\le n}\left|\sum_{i=1}^{k}\eta_i - \frac{k}{n}\sum_{i=1}^{n}\eta_i\right| &= \max_{1\le k\le n}\left|\frac{1}{\sqrt{2dn}}\sum_{i=1}^{k}(\eta_i - d) - \frac{k}{n}\frac{1}{\sqrt{2dn}}\sum_{i=1}^{n}(\eta_i - d)\right| \\
&= \sup_{0\le s\le 1}\left|\frac{1}{\sqrt{2dn}}\sum_{i=1}^{[ns]}(\eta_i - d) - \frac{[ns]}{n}\frac{1}{\sqrt{2dn}}\sum_{i=1}^{n}(\eta_i - d)\right| \\
&= \sup_{0\le s\le 1}\left|\mathcal{U}_n(s) - \frac{[ns]}{n}\mathcal{U}_n(1)\right| \\
&\xrightarrow{\mathrm{d}} \sup_{0\le s\le 1}|\mathbb{B}_1^0(s)|
\end{aligned}$$

when the convergence (B.13) holds true.

Let us prove (B.13). First, we obtain from Lemma 1 the result that

$$\frac{1}{\sqrt{n}}\sum_{i=1}^{n}\mathbb{E}_{\theta^*}[\eta_i - d|\mathscr{G}_{i-1}^n] = \frac{1}{\sqrt{n}}\sum_{i=1}^{n}R_{i-1}(h_n) = \sqrt{nh_n^2}\times\frac{1}{n}\sum_{i=1}^{n}R_{i-1}(1) = o_{\mathbf{P}}(1). \tag{B.14}$$

We next show

$$\frac{1}{\sqrt{2dn}}\sum_{i=1}^{[ns]}(\eta_i - d - \mathbb{E}_{\theta^*}[\eta_i - d|\mathscr{G}_{i-1}^n]) \xrightarrow{\mathrm{w}} \mathbb{B}_1(s) \quad \text{in } \mathbb{D}[0,1] \tag{B.15}$$

in order to complete the proof of (B.13). Since it follows from Lemma 1 that

$$\mathbb{E}_{\theta^*}\Big[((\eta_i - d) - \mathbb{E}_{\theta^*}[\eta_i - d|\mathscr{G}_{i-1}^n])^2\Big|\mathscr{G}_{i-1}^n\Big] = 2d + R_{i-1}(h_n), \quad \mathbb{E}_{\theta^*}[\eta_i^4|\mathscr{G}_{i-1}^n] = R_{i-1}(1),$$

$$\frac{1}{2dn}\sum_{i=1}^{[ns]}\mathbb{E}_{\theta^*}\Big[((\eta_i - d) - \mathbb{E}_{\theta^*}[\eta_i - d|\mathscr{G}_{i-1}^n])^2\Big|\mathscr{G}_{i-1}^n\Big] = \frac{[ns]}{n}\frac{1}{2d[ns]}\sum_{i=1}^{[ns]}(2d + R_{i-1}(h_n)) \xrightarrow{\mathbf{P}} s$$

for all $s \in [0,1]$, and

$$\frac{1}{n^2}\sum_{i=1}^{[ns]}\mathbb{E}_{\theta^*}\Big[(\eta_i - d - \mathbb{E}_{\theta^*}[\eta_i - d|\mathscr{G}_{i-1}^n])^4\Big|\mathscr{G}_{i-1}^n\Big] \lesssim \frac{1}{n^2}\sum_{i=1}^{[ns]}\mathbb{E}_{\theta^*}[\eta_i^4 + d^4 + R_{i-1}(h_n^4)|\mathscr{G}_{i-1}^n]$$
$$= \frac{1}{n^2}\sum_{i=1}^{[ns]}R_{i-1}(1) = o_{\mathbf{P}}(1),$$

we obtain (B.15) from Corollary 3.8 of McLeish (1974). This concludes the proof of (B.10).

Proof of (B.11). Noting that

$$\mathcal{H}_{1,i} = \sum_{l_1,l_2=1}^{d}\partial_\alpha(A_{i-1}^{-1}(\alpha^*))^{l_1,l_2}\frac{(\Delta_i X)^{l_1}(\Delta_i X)^{l_2}}{h_n},$$

we have from Lemma 5 the result that

$$\frac{1}{n}\max_{1\le k\le n}\left|\sum_{i=1}^{k}\mathcal{H}_{1,i} - \frac{k}{n}\sum_{i=1}^{n}\mathcal{H}_{1,i}\right|$$
$$\le \sum_{l_1,l_2=1}^{d}\frac{1}{T}\max_{1\le k\le n}\left|\sum_{i=1}^{k}\partial_\alpha(A_{i-1}^{-1}(\alpha^*))^{l_1,l_2}(\Delta_i X)^{l_1}(\Delta_i X)^{l_2} - \frac{k}{n}\sum_{i=1}^{n}\partial_\alpha(A_{i-1}^{-1}(\alpha^*))^{l_1,l_2}(\Delta_i X)^{l_1}(\Delta_i X)^{l_2}\right|$$
$$= o_{\mathbf{P}}(1).$$

Proof of (B.12). Because of $\alpha^* \in \operatorname{Int}\Theta_A$, there exists an open neighborhood O_{α^*} of α^* such that $O_{\alpha^*} \subset \Theta_A$. Since it follows that on $\Omega_n = \{\hat{\alpha} \in O_{\alpha^*}\}$,

$$|\mathcal{H}_{2,i}| = \left|\sum_{l_1,l_2=1}^{d}\mathcal{A}_{i-1}^{l_1,l_2}\frac{(\Delta_i X)^{l_1}(\Delta_i X)^{l_2}}{h_n}\right| \le \sum_{l_1,l_2=1}^{d}\sup_{\alpha\in\Theta_A}|\partial_\alpha^2(A_{i-1}^{-1}(\alpha))^{l_1,l_2}|\left|\frac{(\Delta_i X)^{l_1}(\Delta_i X)^{l_2}}{h_n}\right|$$

and

$$\mathbb{E}_{\theta^*}\left[\frac{1}{n^{3/2}}\sum_{i=1}^{n}\sum_{l_1,l_2=1}^{d}\sup_{\alpha\in\Theta_A}|\partial_\alpha^2(A_{i-1}^{-1}(\alpha))^{l_1,l_2}|\left|\frac{(\Delta_i X)^{l_1}(\Delta_i X)^{l_2}}{h_n}\right|\right]$$
$$\le \frac{1}{n^{3/2}}\sum_{i=1}^{n}\sum_{l_1,l_2=1}^{d}\mathbb{E}_{\theta^*}\left[\sup_{\alpha\in\Theta_A}\left|\partial_\alpha^2(A_{i-1}^{-1}(\alpha))^{l_1,l_2}\right|^2\right]^{1/2}\mathbb{E}_{\theta^*}\left[\left|\frac{(\Delta_i X)^{l_1}(\Delta_i X)^{l_2}}{h_n}\right|^2\right]^{1/2} \lesssim \frac{1}{n^{1/2}} \to 0,$$

we have from [B1] the result that for all $\epsilon > 0$,

$$\mathbf{P}_{\theta^*}\left(\frac{1}{n^{3/2}}\sum_{i=1}^{n}|\mathcal{H}_{2,i}| \ge \epsilon\right)$$
$$\le \mathbf{P}_{\theta^*}\left(\left\{\frac{1}{n^{3/2}}\sum_{i=1}^{n}|\mathcal{H}_{2,i}| \ge \epsilon\right\}\cap\Omega_n\right) + \mathbf{P}_{\theta^*}\left(\left\{\frac{1}{n^{3/2}}\sum_{i=1}^{n}|\mathcal{H}_{2,i}| \ge \epsilon\right\}\cap\Omega_n^{\mathrm{c}}\right)$$

$$\le \mathbf{P}_{\theta^*}\left(\left\{\frac{1}{n^{3/2}}\sum_{i=1}^{n}\sum_{l_1,l_2=1}^{d}\sup_{\alpha\in\Theta_A}|\partial_\alpha^2(A_{i-1}^{-1}(\alpha))^{l_1,l_2}|\left|\frac{(\Delta_i X)^{l_1}(\Delta_i X)^{l_2}}{h_n}\right|\ge\epsilon\right\}\cap\Omega_n\right)+\mathbf{P}_{\theta^*}(\Omega_n^{\mathrm{c}})$$
$$\le \mathbf{P}_{\theta^*}\left(\frac{1}{n^{3/2}}\sum_{i=1}^{n}\sum_{l_1,l_2=1}^{d}\sup_{\alpha\in\Theta_A}|\partial_\alpha^2(A_{i-1}^{-1}(\alpha))^{l_1,l_2}|\left|\frac{(\Delta_i X)^{l_1}(\Delta_i X)^{l_2}}{h_n}\right|\ge\epsilon\right)+\mathbf{P}_{\theta^*}(\Omega_n^{\mathrm{c}})$$
$$\lesssim \frac{1}{\epsilon n^{1/2}}+\mathbf{P}_{\theta^*}(\Omega_n^{\mathrm{c}})\to 0.$$

Hence we get $n^{-3/2}\sum_{i=1}^n|\mathcal{H}_{2,i}|=o_{\mathbf{P}}(1)$ and

$$\frac{1}{n^{3/2}}\max_{1\le k\le n}\left|\sum_{i=1}^{k}\mathcal{H}_{2,i}-\frac{k}{n}\sum_{i=1}^{n}\mathcal{H}_{2,i}\right|\le\frac{1}{n^{3/2}}\max_{1\le k\le n}\left(\sum_{i=1}^{k}|\mathcal{H}_{2,i}|+\frac{k}{n}\sum_{i=1}^{n}|\mathcal{H}_{2,i}|\right)$$
$$\lesssim\frac{1}{n^{3/2}}\sum_{i=1}^{n}|\mathcal{H}_{2,i}|=o_{\mathbf{P}}(1).$$

(2) (a) Notice that

$$\mathcal{T}_n^\alpha\ge\sqrt{\frac{n}{2d}}\frac{[n\tau_\alpha^*]}{n}\left|\frac{1}{[n\tau_\alpha^*]}\sum_{i=1}^{[n\tau_\alpha^*]}\hat{\eta}_i-\frac{1}{n}\sum_{i=1}^{n}\hat{\eta}_i\right|=:\sqrt{\frac{n}{2d}}|\tilde{\mathcal{H}}_n|$$

and

$$\frac{1}{n}\sum_{i=1}^{n}\hat{\eta}_i=\frac{[n\tau_\alpha^*]}{n}\frac{1}{[n\tau_\alpha^*]}\sum_{i=1}^{[n\tau_\alpha^*]}\hat{\eta}_i+\frac{n-[n\tau_\alpha^*]}{n}\frac{1}{n-[n\tau_\alpha^*]}\sum_{i=[n\tau_\alpha^*]+1}^{n}\hat{\eta}_i.$$

Hence it is enough to show

$$\frac{1}{[n\tau_\alpha^*]}\sum_{i=1}^{[n\tau_\alpha^*]}\hat{\eta}_i\xrightarrow{\mathbf{P}}\mathcal{F}(\alpha_1^*,\alpha'),\tag{B.16}$$
$$\frac{1}{n-[n\tau_\alpha^*]}\sum_{i=[n\tau_\alpha^*]+1}^{n}\hat{\eta}_i\xrightarrow{\mathbf{P}}\mathcal{F}(\alpha_2^*,\alpha')\tag{B.17}$$

because it holds from (B.16), (B.17) and [C2] that

$$\frac{1}{n}\sum_{i=1}^{n}\hat{\eta}_i\xrightarrow{\mathbf{P}}\tau_\alpha^*\mathcal{F}(\alpha_1^*,\alpha')+(1-\tau_\alpha^*)\mathcal{F}(\alpha_2^*,\alpha')$$

and

$$\tilde{\mathcal{H}}_n=\frac{[n\tau_\alpha^*]}{n}\left(\frac{1}{[n\tau_\alpha^*]}\sum_{i=1}^{[n\tau_\alpha^*]}\hat{\eta}_i-\frac{1}{n}\sum_{i=1}^{n}\hat{\eta}_i\right)\xrightarrow{\mathbf{P}}\tau_\alpha^*(1-\tau_\alpha^*)(\mathcal{F}(\alpha_1^*,\alpha')-\mathcal{F}(\alpha_2^*,\alpha'))\ne 0,$$

which implies that $\mathbf{P}(\mathcal{T}_n^\alpha>w_1(\epsilon))\to 1$ for $\epsilon\in(0,1)$.

Let us show (B.16). We find from the Taylor expansion that

$$\hat{\eta}_i = \mathrm{Tr}\left(A_{i-1}^{-1}(\alpha')\frac{(\Delta_i X)^{\otimes 2}}{h_n}\right) + \sum_{l_1,l_2=1}^{d}\int_0^1 \partial_\alpha (A_{i-1}^{-1}(\alpha' + u(\hat{\alpha}-\alpha')))^{l_1,l_2}\mathrm{d}u \frac{(\Delta_i X)^{l_1}(\Delta_i X)^{l_2}}{h_n}(\hat{\alpha}-\alpha')$$
$$=: \eta_{1,i} + \eta_{2,i}(\hat{\alpha}-\alpha').$$

Since we have from Lemma 1 the result that

$$\frac{1}{[n\tau_\alpha^*]}\sum_{i=1}^{[n\tau_\alpha^*]}\mathbb{E}_{\alpha_1^*}[\eta_{1,i}|\mathscr{G}_{i-1}^n] = \frac{1}{[n\tau_\alpha^*]}\sum_{i=1}^{[n\tau_\alpha^*]}\mathrm{Tr}\left(A_{i-1}^{-1}(\alpha')A_{i-1}(\alpha_1^*) + R_{i-1}(h_n)\right)$$
$$\xrightarrow{\mathbf{P}} \int_{\mathbb{R}^d}\mathrm{Tr}\,[A^{-1}(x,\alpha')A(x,\alpha_1^*)]\mathrm{d}\mu_{\alpha_1^*}(x) = \mathcal{F}(\alpha_1^*,\alpha')$$

and

$$\frac{1}{[n\tau_\alpha^*]^2}\sum_{i=1}^{[n\tau_\alpha^*]}\mathbb{E}_{\alpha_1^*}[\eta_{1,i}^2|\mathscr{G}_{i-1}^n]$$
$$= \frac{1}{[n\tau_\alpha^*]^2}\sum_{i=1}^{[n\tau_\alpha^*]}\sum_{l_1,l_2=1}^{d}\sum_{l_3,l_4=1}^{d}(A_{i-1}^{-1}(\alpha'))^{l_1,l_2}(A_{i-1}^{-1}(\alpha'))^{l_3,l_4}\mathbb{E}_{\alpha_1^*}\left[\frac{(\Delta_i X)^{l_1}(\Delta_i X)^{l_2}(\Delta_i X)^{l_3}(\Delta_i X)^{l_4}}{h_n^2}\middle|\mathscr{G}_{i-1}^n\right]$$
$$= \frac{1}{[n\tau_\alpha^*]^2}\sum_{i=1}^{[n\tau_\alpha^*]}R_{i-1}(1) = o_{\mathbf{P}}(1),$$

it holds from Lemma 9 of Genon-Catalot and Jacod (1993) that

$$\frac{1}{[n\tau_\alpha^*]}\sum_{i=1}^{[n\tau_\alpha^*]}\eta_{1,i} \xrightarrow{\mathbf{P}} \mathcal{F}(\alpha_1^*,\alpha'). \tag{B.18}$$

On the other hand, we obtain from [C1] and $\frac{1}{[n\tau_\alpha^*]}\sum_{i=1}^{[n\tau_\alpha^*]}\eta_{2,i} = O_{\mathbf{P}}(1)$ the result that $\eta_{2,i}(\hat{\alpha}-\alpha') = o_{\mathbf{P}}(1)$, which together with (B.18) yields (B.16). In the same way, we have from $\alpha = \alpha_2^*$ under $[n\tau_\alpha^*]+1 \le i \le n$ the result that (B.17), which completes the proof.

(b) Note that

$$\mathcal{T}_n^\alpha \ge \sqrt{\frac{n\vartheta_\alpha^2}{2d}}\left|\frac{1}{n\vartheta_\alpha}\sum_{i=1}^{[n\tau_*^\alpha]}\hat{\eta}_i - \frac{[n\tau_*^\alpha]}{n}\frac{1}{n\vartheta_\alpha}\sum_{i=1}^{n}\hat{\eta}_i\right| =: \sqrt{\frac{n\vartheta_\alpha^2}{2d}}|\hat{\mathcal{H}}_n|.$$

By the Taylor expansion,

$$\hat{\eta}_i = \eta_{3,i} + \eta_{4,i}(\hat{\alpha}-\alpha_0) + \int_0^1 (1-u)\partial_\alpha^2 \mathrm{Tr}\left(A_{i-1}^{-1}(\alpha_0 + u(\hat{\alpha}-\alpha_0))\frac{(\Delta_i X)^{\otimes 2}}{h_n}\right)\mathrm{d}u \otimes (\hat{\alpha}-\alpha_0)^{\otimes 2},$$

where

$$\eta_{3,i} = \mathrm{Tr}\left(A_{i-1}^{-1}(\alpha_0)\frac{(\Delta_i X)^{\otimes 2}}{h_n}\right), \quad \eta_{4,i} = \left(\mathrm{Tr}\left(A^{-1}\partial_{\alpha^l}AA_{i-1}^{-1}(\alpha_0)\frac{(\Delta_i X)^{\otimes 2}}{h_n}\right)\right)_{l=1}^{p}.$$

Therefore we have from [D1], [D2] and $\mathbb{E}[h_n^{-1}(\Delta_i X)^{\otimes 2}] = O(1)$ the result that $\hat{\mathcal{H}}_n = \mathcal{H}_{3,i} + \mathcal{H}_{4,i}\vartheta_\alpha^{-1}(\hat{\alpha} - \alpha_0) + o_{\mathbf{P}}(1)$, where

$$\mathcal{H}_{3,i} = \sum_{i=1}^{n} c_{3,i}\eta_{3,i} := \left(1 - \frac{[n\tau_*^\alpha]}{n}\right)\frac{1}{n\vartheta_\alpha}\sum_{i=1}^{[n\tau_*^\alpha]}\eta_{3,i} - \frac{[n\tau_*^\alpha]}{n}\frac{1}{n\vartheta_\alpha}\sum_{i=[n\tau_*^\alpha]+1}^{n}\eta_{3,i},$$

$$\mathcal{H}_{4,i} = \sum_{i=1}^{n} c_{4,i}\eta_{4,i} := \left(1 - \frac{[n\tau_*^\alpha]}{n}\right)\frac{1}{n}\sum_{i=1}^{[n\tau_*^\alpha]}\eta_{4,i} - \frac{[n\tau_*^\alpha]}{n}\frac{1}{n}\sum_{i=[n\tau_*^\alpha]+1}^{n}\eta_{4,i}.$$

Because of $\mathbb{E}_{\alpha_k^*}[\eta_{3,i}|\mathscr{G}_{i-1}^n] = d + \left[\mathrm{Tr}\,(A^{-1}\partial_{\alpha^l}A_{i-1}(\alpha_0))\right]_l(\alpha_k^* - \alpha_0) + R_{i-1}(\vartheta_\alpha^2 \vee h_n)$, $\mathbb{E}[\eta_{3,i}^2] \lesssim 1$ and [D1], we have

$$\begin{aligned}\sum_{i=1}^{n} c_{3,i}\mathbb{E}[\eta_{3,i}|\mathscr{G}_{i-1}^n] &= \left(1 - \frac{[n\tau_*^\alpha]}{n}\right)\frac{1}{n}\sum_{i=1}^{[n\tau_*^\alpha]}[\mathrm{Tr}\,(A^{-1}\partial_{\alpha^l}A_{i-1}(\alpha_0))]_l\vartheta_\alpha^{-1}(\alpha_1^* - \alpha_0) \\ &\quad - \frac{[n\tau_*^\alpha]}{n}\frac{1}{n}\sum_{i=[n\tau_*^\alpha]+1}^{n}[\mathrm{Tr}\,(A^{-1}\partial_{\alpha^l}A_{i-1}(\alpha_0))]_l\vartheta_\alpha^{-1}(\alpha_2^* - \alpha_0) + o_{\mathbf{P}}(1) \\ &\xrightarrow{\mathbf{P}} \tau_*^\alpha(1 - \tau_*^\alpha)\left(\int_{\mathbb{R}^d}(\mathrm{Tr}\,(A^{-1}\partial_{\alpha^l}A(x,\alpha_0)))_{l=1}^p \mathrm{d}\mu_{\alpha_0}(x)\right)^\top(c_1 - c_2)\end{aligned}$$

and $\sum_{i=1}^n c_{3,i}^2\mathbb{E}[\eta_{3,i}^2|\mathscr{G}_{i-1}^n] \xrightarrow{\mathbf{P}} 0$. Similarly, it follows from $\mathbb{E}_{\alpha_k^*}[\eta_{4,i}|\mathscr{G}_{i-1}^n] = d + [\mathrm{Tr}\,(A^{-1}\partial_{\alpha^l}A_{i-1}(\alpha_0))]_l(\alpha_k^* - \alpha_0) + R_{i-1}(\vartheta_\alpha^2 \vee h_n)$ and $\mathbb{E}[|\eta_{4,i}|^2] \lesssim 1$ that

$$\sum_{i=1}^{n} c_{4,i}\mathbb{E}[\eta_{4,i}|\mathscr{G}_{i-1}^n] \xrightarrow{\mathbf{P}} 0, \quad \sum_{i=1}^{n} c_{4,i}^2\mathbb{E}[\eta_{4,i}^2|\mathscr{G}_{i-1}^n] \xrightarrow{\mathbf{P}} 0.$$

From Lemma 9 of Genon-Catalot and Jacod (1993), we obtain

$$\mathcal{H}_{3,i} \xrightarrow{\mathbf{P}} \tau_*^\alpha(1 - \tau_*^\alpha)\left(\int_{\mathbb{R}^d}(\mathrm{Tr}\,(A^{-1}\partial_{\alpha^l}A(x,\alpha_0)))_{l=1}^p \mathrm{d}\mu_{\alpha_0}(x)\right)^\top(c_1 - c_2), \quad \mathcal{H}_{4,i} \xrightarrow{\mathbf{P}} 0.$$

Hence we find from [D3] that $\hat{\mathcal{H}}_n$ converges to a non-zero constant in probability, which implies $\mathbf{P}(\mathcal{T}_n^\alpha > w_1(\epsilon)) \to 1$. □

Proof of Theorem 2

The following lemma presents a sufficient condition to specify the asymptotic distribution of the proposed estimators, which can be identified if (a) and (b) of Lemma 6 are fulfilled.

Lemma 6. *Let $\Upsilon_n(\tau : \theta_1, \theta_2)$ be a contrast function, and let $\hat{\theta}_1$, $\hat{\theta}_2$ be estimators of θ_1, θ_2, respectively, and let $\hat{\tau}_n = \mathrm{argmin}_{\tau\in[0,1]}\Upsilon_n(\tau : \hat{\theta}_1, \hat{\theta}_2)$ be the estimator of τ^*, and let $\hat{\mathbb{H}}_n(v) = \Upsilon_n(\tau^* + v/r_n : \hat{\theta}_1, \hat{\theta}_2) - \Upsilon_n(\tau^* : \hat{\theta}_1, \hat{\theta}_2)$. If there exist a positive sequence $\{r_n\}$ with $r_n \to \infty$ and a random field $\mathbb{H}(v)$ that satisfy the following conditions* *(a)* $r_n(\hat{\tau}_n - \tau^*) = O_{\mathbf{P}}(1)$*,*

(b) For all $L > 0$*,* $\hat{\mathbb{H}}_n(v) \xrightarrow{\mathrm{w}} \mathbb{H}(v)$ *in* $\mathbb{D}[-L, L]$*,*

then $r_n(\hat{\tau}_n - \tau^*) \xrightarrow{\mathrm{d}} \mathrm{argmin}_{v\in\mathbb{R}}\mathbb{H}(v)$*.*

Proof. Let $v^\dagger = \mathrm{argmin}_{v\in\mathbb{R}}\mathbb{H}(v)$. For all $x \in \mathbb{R}$,

$$\begin{aligned}\mathbf{P}(r_n(\hat{\tau}_n - \tau^*) \le x) \le \mathbf{P}\Big(&r_n(\hat{\tau}_n - \tau^*) \le x,\ r_n(\hat{\tau}_n - \tau^*) \in [-L, L],\ \inf_{v\in[-L,x]} \hat{\mathbb{H}}_n(v) > \inf_{v\in[x,L]} \hat{\mathbb{H}}_n(v)\Big) \\ &+ \mathbf{P}(r_n(\hat{\tau}_n - \tau^*) \notin [-L, L]) + \mathbf{P}\Big(\inf_{v\in[-L,x]} \hat{\mathbb{H}}_n(v) \le \inf_{v\in[x,L]} \hat{\mathbb{H}}_n(v)\Big). \qquad \text{(B.19)}\end{aligned}$$

If $r_n(\hat{\tau}_n - \tau^*) \in [-L, L]$ and $\inf_{v\in[-L,x]} \hat{\mathbb{H}}_n(v) > \inf_{v\in[x,L]} \hat{\mathbb{H}}_n(v)$, then $x < r_n(\hat{\tau}_n - \tau^*) \le L$. Therefore, we find that

$$\mathbf{P}\Big(r_n(\hat{\tau}_n - \tau^*) \le x,\ r_n(\hat{\tau}_n - \tau^*) \in [-L, L],\ \inf_{v\in[-L,x]} \hat{\mathbb{H}}_n(v) > \inf_{v\in[x,L]} \hat{\mathbb{H}}_n(v)\Big) = 0,$$

which together with (b) and (B.19) yields

$$\overline{\lim_{n\to\infty}}\, \mathbf{P}(r_n(\hat{\tau}_n - \tau^*) \le x) \le \sup_{n\in\mathbb{N}} \mathbf{P}(r_n(\hat{\tau}_n - \tau^*) \notin [-L, L]) + \mathbf{P}\Big(\inf_{v\in[-L,x]} \mathbb{H}(v) \le \inf_{v\in[x,L]} \mathbb{H}(v)\Big). \qquad \text{(B.20)}$$

Since

$$\begin{aligned}\mathbf{P}\Big(\inf_{v\in[-L,x]} \mathbb{H}(v) \le \inf_{v\in[x,L]} \mathbb{H}(v)\Big) \le \mathbf{P}\Big(&\inf_{v\in[-L,x]} \mathbb{H}(v) \le \inf_{v\in[x,L]} \mathbb{H}(v),\ v^\dagger \in [-L, L],\ v^\dagger > x\Big) \\ &+ \mathbf{P}(v^\dagger \notin [-L, L]) + \mathbf{P}(v^\dagger \le x),\end{aligned}$$

$$\mathbf{P}\Big(\inf_{v\in[-L,x]} \mathbb{H}(v) \le \inf_{v\in[x,L]} \mathbb{H}(v),\ v^\dagger \in [-L, L],\ v^\dagger > x\Big) \le \mathbf{P}(-L \le v^\dagger \le x,\ v^\dagger > x) = 0,$$

we obtain from (a) and (B.20) the result that as $L \to \infty$,

$$\overline{\lim_{n\to\infty}}\, \mathbf{P}(r_n(\hat{\tau}_n - \tau^*) \le x) \le \mathbf{P}(v^\dagger \le x).$$

In the same way, we have $\underline{\lim}_{n\to\infty} \mathbf{P}(r_n(\hat{\tau}_n - \tau^*) \le x) \ge \mathbf{P}(v^\dagger \le x)$ and thus the proof is complete. □

In Case A_α, define $\mathcal{D}_n^\alpha(v) = \hat{\mathbb{F}}_n(v) - \mathbb{F}_n(v)$, where

$$\mathbb{F}_n(v) = \Phi_n\left(\tau_*^\alpha + \frac{v}{n\vartheta_\alpha^2} : \alpha_1^*, \alpha_2^*\right) - \Phi_n\left(\tau_*^\alpha : \alpha_1^*, \alpha_2^*\right),$$

$$\hat{\mathbb{F}}_n(v) = \Phi_n\left(\tau_*^\alpha + \frac{v}{n\vartheta_\alpha^2} : \hat{\alpha}_1, \hat{\alpha}_2\right) - \Phi_n\left(\tau_*^\alpha : \hat{\alpha}_1, \hat{\alpha}_2\right).$$

Lemma 7. *Suppose that [A1]-[A5], [E1], [F1] and [F2] hold. Then, for all* $L > 0$*,*

$$\sup_{v\in[-L,L]} |\mathcal{D}_n^\alpha(v)| \xrightarrow{\mathbf{P}} 0$$

as $n \to \infty$*.*

Proof. We assume that $v > 0$. Since

$$F_i(\hat{\alpha}_k) = F_i(\alpha_k^*) + \partial_\alpha F_i(\alpha_k^*)(\hat{\alpha}_k - \alpha_k^*) + \int_0^1 \partial_\alpha^2 F_i(\alpha_k^* + u(\hat{\alpha}_k - \alpha_k^*))\mathrm{d}u \otimes (\hat{\alpha}_k - \alpha_k^*)^{\otimes 2},$$

[E1] and $n\vartheta_\alpha^2 \to \infty$, we have

$$\begin{aligned}\mathcal{D}_n^\alpha(v) &= \sum_{i=[n\tau_*^\alpha]+1}^{[n\tau_*^\alpha + v/\vartheta_\alpha^2]} (F_i(\hat{\alpha}_1) - F_i(\hat{\alpha}_2)) - \sum_{i=[n\tau_*^\alpha]+1}^{[n\tau_*^\alpha + v/\vartheta_\alpha^2]} (F_i(\alpha_1^*) - F_i(\alpha_2^*)) \\ &= \sum_{i=[n\tau_*^\alpha]+1}^{[n\tau_*^\alpha + v/\vartheta_\alpha^2]} (\partial_\alpha F_i(\alpha_1^*)(\hat{\alpha}_1 - \alpha_1^*) + \partial_\alpha F_i(\alpha_2^*)(\hat{\alpha}_2 - \alpha_2^*)) + \bar{o}_{\mathbf{P}}(1), \end{aligned} \tag{B.21}$$

where $Y_n(v) = \bar{o}_{\mathbf{P}}(1)$ denotes $\sup_{v\in[0,L]} |Y_n(v)| = o_{\mathbf{P}}(1)$. By Theorem 2.11 of Hall and Heyde (1980) and $\mathbb{E}_{\alpha_2^*}[|\partial_\alpha F_i(\alpha_k^*)|^2] \lesssim 1$, we have

$$\begin{aligned}&\mathbb{E}_{\alpha_2^*}\left[\frac{1}{n}\sup_{v\in[0,L]}\left|\sum_{i=[n\tau_*^\alpha]+1}^{[n\tau_*^\alpha + v/\vartheta_\alpha^2]} \Big(\partial_\alpha F_i(\alpha_k^*) - \mathbb{E}_{\alpha_2^*}[\partial_\alpha F_i(\alpha_k^*)|\mathscr{G}_{i-1}^n]\Big)\right|^2\right] \\ &\lesssim \frac{1}{n}\sum_{i=[n\tau_*^\alpha]+1}^{[n\tau_*^\alpha + L/\vartheta_\alpha^2]} \mathbb{E}_{\alpha_2^*}\left[\left|\partial_\alpha F_i(\alpha_k^*) - \mathbb{E}_{\alpha_2^*}[\partial_\alpha F_i(\alpha_k^*)|\mathscr{G}_{i-1}^n]\right|^2\right] \lesssim \frac{1}{n\vartheta_\alpha^2} \to 0.\end{aligned}$$

Moreover, we see from $\mathbb{E}_{\alpha_2^*}[\partial_\alpha F_i(\alpha_k^*)|\mathscr{G}_{i-1}^n] = \Xi_{i-1}^\alpha(\alpha_0)(\alpha_2^* - \alpha_k^*) + R_{i-1}(\vartheta_\alpha^2 \vee h_n)$ and [F2](a) that

$$\begin{aligned}&\sup_{v\in[0,L]}\left|\sum_{i=[n\tau_*^\alpha]+1}^{[n\tau_*^\alpha + v/\vartheta_\alpha^2]} \mathbb{E}_{\alpha_2^*}[\partial_\alpha F_i(\alpha_k^*)|\mathscr{G}_{i-1}^n]\right| |\hat{\alpha}_k - \alpha_k^*| \\ &\le \sup_{v\in[0,L]}\left|\sum_{i=[n\tau_*^\alpha]+1}^{[n\tau_*^\alpha + v/\vartheta_\alpha^2]} \Xi_{i-1}^\alpha(\alpha_0)\right| |\alpha_2^* - \alpha_k^*||\hat{\alpha}_k - \alpha_k^*| + O_{\mathbf{P}}\left(\frac{1}{\sqrt{n}}\sum_{i=[n\tau_*^\alpha]+1}^{[n\tau_*^\alpha + L/\vartheta_\alpha^2]} \vartheta_\alpha^2 \vee h_n\right) \\ &= O_{\mathbf{P}}\left(\frac{1}{\sqrt{n}\vartheta_\alpha} \vee \frac{1}{\sqrt{n}} \vee \frac{h_n}{\sqrt{n}\vartheta_\alpha^2}\right) = o_{\mathbf{P}}(1).\end{aligned}$$

Therefore, we have

$$\sum_{i=[n\tau_*^\alpha]+1}^{[n\tau_*^\alpha + v/\vartheta_\alpha^2]} \partial_\alpha F_i(\alpha_k^*)(\hat{\alpha}_k - \alpha_k^*) = \bar{o}_{\mathbf{P}}(1),$$

and we obtain from (B.21) the result that $\sup_{v\in[0,L]} |\mathcal{D}_n^\alpha(v)| \xrightarrow{\mathbf{P}} 0$. By a similar proof, we see $\sup_{v\in[-L,0]} |\mathcal{D}_n^\alpha(v)| \xrightarrow{\mathbf{P}} 0$ and this proof is complete. □

Lemma 8. *Suppose that [A1]-[A5], [E1], [F1] and [F2] hold. Then, for all $L > 0$,*

$$\mathbb{F}_n(v) \xrightarrow{\mathrm{w}} \mathbb{F}(v) \text{ in } \mathbb{D}[-L, L]$$

as $n \to \infty$.

Proof. We consider $v > 0$. Let

$$\mathbb{F}_{1,n}(v) = \sum_{i=[n\tau_*^\alpha]+1}^{[n\tau_*^\alpha+v/\vartheta_\alpha^2]} \partial_\alpha F_i(\alpha_2^*)(\alpha_1^* - \alpha_2^*), \quad \mathbb{F}_{2,n}(v) = \frac{1}{2}\sum_{i=[n\tau_*^\alpha]+1}^{[n\tau_*^\alpha+v/\vartheta_\alpha^2]} \partial_\alpha^2 F_i(\alpha_2^*) \otimes (\alpha_1^* - \alpha_2^*)^{\otimes 2}.$$

By the Taylor expansion and $|\alpha_1^* - \alpha_2^*| = \vartheta_\alpha$, we have

$$\mathbb{F}_n(v) = \sum_{i=[n\tau_*^\alpha]+1}^{[n\tau_*^\alpha+v/\vartheta_\alpha^2]} \big(F_i(\alpha_1^*) - F_i(\alpha_2^*)\big) = \mathbb{F}_{1,n}(v) + \mathbb{F}_{2,n}(v) + \bar{o}_{\mathbf{P}}(1).$$

It follows from $\mathbb{E}_{\alpha_2^*}[\partial_\alpha F_i(\alpha_2^*)|\mathscr{G}_{i-1}^n] = R_{i-1}(h_n)$ and $h_n/\vartheta_\alpha \to 0$ that

$$\sum_{i=[n\tau_*^\alpha]+1}^{[n\tau_*^\alpha+v/\vartheta_\alpha^2]} \mathbb{E}_{\alpha_2^*}[\partial_\alpha F_i(\alpha_2^*)|\mathscr{G}_{i-1}^n](\alpha_1^* - \alpha_2^*) = \bar{o}_{\mathbf{P}}(1). \tag{B.22}$$

Let $M_i = \partial_\alpha F_i(\alpha_2^*) - \mathbb{E}_{\alpha_2^*}[\partial_\alpha F_i(\alpha_2^*)|\mathscr{G}_{i-1}^n]$. Because of

$$\mathbb{E}_{\alpha_2^*}\Big[\big(M_i(\alpha_1^* - \alpha_2^*)\big)^2\Big|\mathscr{G}_{i-1}^n\Big] = 2\Xi_{i-1}^\alpha(\alpha_2^*) \otimes (\alpha_1^* - \alpha_2^*)^{\otimes 2} + R_{i-1}(h_n\vartheta_\alpha^2),$$
$$\mathbb{E}_{\alpha_2^*}\Big[\big(M_i(\alpha_1^* - \alpha_2^*)\big)^4\Big|\mathscr{G}_{i-1}^n\Big] = R_{i-1}(\vartheta_\alpha^4)$$

and [F2](a), we have

$$\sum_{i=[n\tau_*^\alpha]+1}^{[n\tau_*^\alpha+v/\vartheta_\alpha^2]} \mathbb{E}_{\alpha_2^*}\Big[\big(M_i(\alpha_1^* - \alpha_2^*)\big)^2\Big|\mathscr{G}_{i-1}^n\Big] \xrightarrow{\mathbf{P}} 2e_\alpha^\top \int_{\mathbb{R}^d} \Xi^\alpha(x, \alpha_0)\mathrm{d}\mu_{\alpha_0}(x)e_\alpha v = 4\mathcal{J}_\alpha v \tag{B.23}$$

$$\sum_{i=[n\tau_*^\alpha]+1}^{[n\tau_*^\alpha+v/\vartheta_\alpha^2]} \mathbb{E}_{\alpha_2^*}\Big[\big(M_i(\alpha_1^* - \alpha_2^*)\big)^4\Big|\mathscr{G}_{i-1}^n\Big] \xrightarrow{\mathbf{P}} 0. \tag{B.24}$$

According to Corollary 3.8 of McLeish (1974), we obtain from (B.23) and (B.24) the result that

$$\sum_{i=[n\tau_*^\alpha]+1}^{[n\tau_*^\alpha+v/\vartheta_\alpha^2]} M_i(\alpha_1^* - \alpha_2^*) \xrightarrow{\mathrm{w}} -2\mathcal{J}_\alpha^{1/2}\mathbb{W}(v) \text{ in } \mathbb{D}[0, L], \tag{B.25}$$

which together with (B.22) yields $\mathbb{F}_{1,n}(v) \xrightarrow{\mathrm{w}} -2\mathcal{J}_\alpha^{1/2}\mathbb{W}(v)$ in $\mathbb{D}[0, L]$.

Since it follows from Theorem 2.11 of Hall and Heyde (1980),

$$\mathbb{E}_{\alpha_2^*}[|\partial_\alpha F_i(\alpha_2^*)|^2] \lesssim 1, \quad \mathbb{E}_{\alpha_2^*}[\partial_\alpha^2 F_i(\alpha_2^*)|\mathscr{G}_{i-1}^n] = \Xi_{i-1}^\alpha(\alpha_2^*) + R_{i-1}(\vartheta_\alpha^2)$$

and [F2](a) that

$$\sup_{v\in[0,L]}\left|\sum_{i=[n\tau_*^\alpha]+1}^{[n\tau_*^\alpha+v/\vartheta_\alpha^2]} \Big(\partial_\alpha^2 F_i(\alpha_2^*) - \mathbb{E}_{\alpha_2^*}[\partial_\alpha^2 F_i(\alpha_2^*)|\mathscr{G}_{i-1}^n]\Big) \otimes (\alpha_1^* - \alpha_2^*)^{\otimes 2}\right| \xrightarrow{\mathbf{P}} 0$$

and

$$\sup_{v\in[0,L]}\left|\sum_{i=[n\tau_*^\alpha]+1}^{[n\tau_*^\alpha+v/\vartheta_\alpha^2]}\mathbb{E}_{\alpha_2^*}[\partial_\alpha^2 F_i(\alpha_2^*)|\mathscr{G}_{i-1}^n]\otimes(\alpha_1^*-\alpha_2^*)^{\otimes 2}-2\mathcal{J}_\alpha v\right|\stackrel{\mathbf{P}}{\to}0,$$

we have $\sup_{v\in[0,L]}|\mathbb{F}_{2,n}(v)-\mathcal{J}_\alpha v|\stackrel{\mathbf{P}}{\to}0$ and from the continuous mapping theorem,

$$\mathbb{F}_n(v)\stackrel{\mathrm{w}}{\to}-2\mathcal{J}_\alpha^{1/2}\mathbb{W}(v)+\mathcal{J}_\alpha v\ \text{ in }\ \mathbb{D}[0,L].$$

In the same way, it can be shown that $\mathbb{F}_n(v)\stackrel{\mathrm{w}}{\to}\mathbb{F}(v)$ in $\mathbb{D}[-L,0]$. □

By Lemmas 7 and 8, we find that (b) of Lemma 6 is satisfied. It remains for us to confirm the validity of (a) of Lemma 6 in Case A_α.

Proof of Theorem 2. (1) Let $M\geq 1$. Since

$$\begin{aligned}F_i(\alpha_1)-F_i(\alpha_2)&=F_i(\alpha_1)-F_i(\alpha_2)-\mathbb{E}_{\alpha_2^*}[F_i(\alpha_1)-F_i(\alpha_2)|\mathscr{G}_{i-1}^n]\\&\quad+\mathrm{Tr}\left(A_{i-1}^{-1}(\alpha_1)A_{i-1}(\alpha_2)-I_d\right)-\log\det A_{i-1}^{-1}(\alpha_1)A_{i-1}(\alpha_2)\\&\quad-\mathrm{Tr}\left(\left(A_{i-1}^{-1}(\alpha_1)-A_{i-1}^{-1}(\alpha_2)\right)\left(A_{i-1}(\alpha_2)-h^{-1}\mathbb{E}_{\alpha_2^*}[(\Delta_iX)^{\otimes 2}|\mathscr{G}_{i-1}^n]\right)\right),\end{aligned}$$

we see that for $\tau>\tau_*^\alpha$,

$$\begin{aligned}\Phi_n(\tau:\alpha_1,\alpha_2)-\Phi_n(\tau_*^\alpha:\alpha_1,\alpha_2)&=\sum_{i=[n\tau_*^\alpha]+1}^{[n\tau]}(F_i(\alpha_1)-F_i(\alpha_2))\\&=\mathcal{M}_n^\alpha(\tau:\alpha_1,\alpha_2)+\mathcal{A}_n^\alpha(\tau:\alpha_1,\alpha_2)+\varrho_n^\alpha(\tau:\alpha_1,\alpha_2),\end{aligned}$$

where

$$\begin{aligned}\mathcal{M}_n^\alpha(\tau:\alpha_1,\alpha_2)&=\sum_{i=[n\tau_*^\alpha]+1}^{[n\tau]}\left(F_i(\alpha_1)-F_i(\alpha_2)-\mathbb{E}_{\alpha_2^*}[F_i(\alpha_1)-F_i(\alpha_2)|\mathscr{G}_{i-1}^n]\right),\\\mathcal{A}_n^\alpha(\tau:\alpha_1,\alpha_2)&=\sum_{i=[n\tau_*^\alpha]+1}^{[n\tau]}\left(\mathrm{Tr}\left(A_{i-1}^{-1}(\alpha_1)A_{i-1}(\alpha_2)-I_d\right)-\log\det A_{i-1}^{-1}(\alpha_1)A_{i-1}(\alpha_2)\right),\\\varrho_n^\alpha(\tau:\alpha_1,\alpha_2)&=\sum_{i=[n\tau_*^\alpha]+1}^{[n\tau]}\mathrm{Tr}\left(\left(A_{i-1}^{-1}(\alpha_1)-A_{i-1}^{-1}(\alpha_2)\right)\left(A_{i-1}(\alpha_2)-h^{-1}\mathbb{E}_{\alpha_2^*}[(\Delta_iX)^{\otimes 2}|\mathscr{G}_{i-1}^n]\right)\right).\end{aligned}$$

Let $D_{n,M}^\alpha=\{\tau\in[0,1]|n\vartheta_\alpha^2(\tau-\tau_*^\alpha)>M\}$. For any $\delta>0$, we have

$$\begin{aligned}\mathbf{P}\left(n\vartheta_\alpha^2(\hat\tau_n-\tau_*^\alpha)>M\right)&\leq\mathbf{P}\left(\inf_{\tau\in D_{n,M}^\alpha}\Phi_n(\tau:\hat\alpha_1,\hat\alpha_2)\leq\Phi_n(\tau_*^\alpha:\hat\alpha_1,\hat\alpha_2)\right)\\&\leq\mathbf{P}\left(\sup_{\tau\in D_{n,M}^\alpha}\frac{|\mathcal{M}_n^\alpha(\tau:\hat\alpha_1,\hat\alpha_2)|}{\vartheta_\alpha^2([n\tau]-[n\tau_*^\alpha])}\geq\delta\right)+\mathbf{P}\left(\inf_{\tau\in D_{n,M}^\alpha}\frac{\mathcal{A}_n^\alpha(\tau:\hat\alpha_1,\hat\alpha_2)}{\vartheta_\alpha^2([n\tau]-[n\tau_*^\alpha])}\leq 2\delta\right)\end{aligned}$$

$$+\mathbf{P}\left(\sup_{\tau\in D^{\alpha}_{n,M}}\frac{|\varrho^{\alpha}_{n}(\tau:\hat{\alpha}_1,\hat{\alpha}_2)|}{\vartheta^2_{\alpha}([n\tau]-[n\tau^{\alpha}_{*}])}\geq\delta\right)$$
$$=: P^{\alpha}_{1,n}+P^{\alpha}_{2,n}+P^{\alpha}_{3,n}.$$

(i) Estimation of $P^{\alpha}_{1,n}$. Let $\epsilon > 0$ be an arbitrary number. Let O_{α} be an open neighborhood of α. Because $\partial_{\alpha}F_i(\alpha)$ is continuous with respect to $\alpha\in\Theta_A$, we can choose $\bar{\alpha}\in O_{\hat{\alpha}_2}$ such that

$$\mathcal{M}^{\alpha}_{n}(\tau:\hat{\alpha}_1,\hat{\alpha}_2)=\sum_{i=[n\tau^{\alpha}_{*}]+1}^{[n\tau]}\left(\partial_{\alpha}F_i(\bar{\alpha})-\mathbb{E}_{\alpha^*_2}[\partial_{\alpha}F_i(\alpha)|\mathscr{G}^n_{i-1}]\Big|_{\alpha=\bar{\alpha}}\right)(\hat{\alpha}_1-\hat{\alpha}_2).$$

If $\hat{\alpha}_k\in O_{\alpha_0}$, then

$$|\mathcal{M}^{\alpha}_{n}(\tau:\hat{\alpha}_1,\hat{\alpha}_2)|\leq\sup_{\alpha\in\Theta_A}\left|\sum_{i=[n\tau^{\alpha}_{*}]+1}^{[n\tau]}\left(\partial_{\alpha}F_i(\alpha)-\mathbb{E}_{\alpha^*_2}[\partial_{\alpha}F_i(\alpha)|\mathscr{G}^n_{i-1}]\right)\right||\hat{\alpha}_1-\hat{\alpha}_2|$$
$$=:\sup_{\alpha\in\Theta_A}|\mathbb{M}^{\alpha}_{n}(\tau:\alpha)||\hat{\alpha}_1-\hat{\alpha}_2|.$$

Hence we have

$$P^{\alpha}_{1,n}\leq\mathbf{P}\left(\sup_{\tau\in D^{\alpha}_{n,M}}\frac{|\mathcal{M}^{\alpha}_{n}(\tau:\hat{\alpha}_1,\hat{\alpha}_2)|}{\vartheta^2_{\alpha}([n\tau]-[n\tau^{\alpha}_{*}])}\geq\delta,\ |\hat{\alpha}_1-\hat{\alpha}_2|\leq 2\vartheta_{\alpha},\ \hat{\alpha}_1,\hat{\alpha}_2\in O_{\alpha_0}\right)$$
$$+\mathbf{P}(|\hat{\alpha}_1-\hat{\alpha}_2|>2\vartheta_{\alpha})+\sum_{k=1}^{2}\mathbf{P}(\hat{\alpha}_k\notin O_{\alpha_0})$$
$$\leq\mathbf{P}\left(\sup_{\tau\in D^{\alpha}_{n,M}}\frac{\sup_{\alpha\in\Theta_A}|\mathbb{M}^{\alpha}_{n}(\tau:\alpha)|}{[n\tau]-[n\tau^{\alpha}_{*}]}\geq\frac{\delta\vartheta_{\alpha}}{2}\right)+\mathbf{P}(|\hat{\alpha}_1-\hat{\alpha}_2|>2\vartheta_{\alpha})+\sum_{k=1}^{2}\mathbf{P}(\hat{\alpha}_k\notin O_{\alpha_0}).\quad\text{(B.26)}$$

By the uniform version on the Hájek-Renyi inequality in Lemma 2 of Iacus and Yoshida (2012), we obtain

$$\mathbf{P}\left(\sup_{\tau\in D^{\alpha}_{n,M}}\frac{\sup_{\alpha\in\Theta_A}|\mathbb{M}^{\alpha}_{n}(\tau:\alpha)|}{[n\tau]-[n\tau^{\alpha}_{*}]}\geq\frac{\delta\vartheta_{\alpha}}{2}\right)\lesssim\frac{1}{\delta^2 M}=:\gamma_{\alpha}(M).\quad\text{(B.27)}$$

Note that $\{|\hat{\alpha}_1-\hat{\alpha}_2|>2\vartheta_{\alpha}\}\subset\bigcup_{k=1}^{2}\{|\hat{\alpha}_k-\alpha^*_k|>\vartheta_{\alpha}/2\}$. Since $\mathbf{P}(|\hat{\alpha}_k-\alpha^*_k|>\vartheta_{\alpha}/2)<\epsilon/2$ for sufficiently large n because of $\vartheta^{-1}_{\alpha}(\hat{\alpha}_k-\alpha^*_k)=o_{\mathbf{P}}(1)$, we find

$$\mathbf{P}(|\hat{\alpha}_1-\hat{\alpha}_2|>2\vartheta_{\alpha})<\epsilon.\quad\text{(B.28)}$$

Therefore, we have from [E1] and [F1] the result that $\mathbf{P}(\hat{\alpha}_k\notin O_{\alpha_0})<\epsilon/2$ for large n, which together with (B.26)-(B.28) yields $P^{\alpha}_{1,n}\leq\gamma_{\alpha}(M)+2\epsilon$ for large n.

(ii) Estimation of $P^{\alpha}_{2,n}$. If $\hat{\alpha}_k\in O_{\alpha_0}$, then

$$\mathrm{Tr}\left(A^{-1}_{i-1}(\hat{\alpha}_1)A_{i-1}(\hat{\alpha}_2)-I_d\right)-\log\det A^{-1}_{i-1}(\hat{\alpha}_1)A_{i-1}(\hat{\alpha}_2)$$

$$= \Big(\frac{1}{2}\Xi_{i-1}^{\alpha}(\alpha_0) + \frac{1}{2}\partial_\alpha \Xi_{i-1}^{\alpha}(\alpha_0)\otimes(\hat{\alpha}_2-\alpha_0)$$
$$+ \frac{1}{3!}\partial_\alpha^3 \Gamma_{i-1}^{\alpha}(\alpha_0,\alpha_0)\otimes(\hat{\alpha}_1-\hat{\alpha}_2) + O(\vartheta_\alpha^2)\Big)\otimes(\hat{\alpha}_1-\hat{\alpha}_2)^{\otimes 2}$$
$$\geq \left(\frac{1}{2}\lambda_1[\Xi_{i-1}^{\alpha}(\alpha_0)] + r_{n,i-1}\right)|\hat{\alpha}_1-\hat{\alpha}_2|^2,$$

where $\lambda_1[M]$ denotes the minimum eigenvalue of a symmetric matrix M, and $r_{n,i-1}$ satisfies

$$\sup_{\tau\in D_{n,M}^{\alpha}}\left|\frac{1}{[n\tau]-[n\tau_*^{\alpha}]}\sum_{i=[n\tau_*^{\alpha}]+1}^{[n\tau]} r_{n,i-1}\right| = o_{\mathbf{P}}(1)$$

from [F1], [F2](b) and (c). We thus obtain

$$P_{2,n}^{\alpha} \leq \mathbf{P}\left(\inf_{\tau\in D_{n,M}^{\alpha}}\frac{\mathcal{A}_n^{\alpha}(\tau:\hat{\alpha}_1,\hat{\alpha}_2)}{\vartheta_\alpha^2([n\tau]-[n\tau_*^{\alpha}])} \leq 2\delta,\ |\hat{\alpha}_1-\hat{\alpha}_2| \geq \frac{\vartheta_\alpha}{2},\ \hat{\alpha}_k\in \mathcal{O}_{\alpha_0}\right)$$
$$+ P\left(|\hat{\alpha}_1-\hat{\alpha}_2| < \frac{\vartheta_\alpha}{2}\right) + \sum_{k=1}^{2}\mathbf{P}(\hat{\alpha}_k\notin\mathcal{O}_{\alpha_0})$$
$$\leq \mathbf{P}\left(\inf_{\tau\in D_{n,M}^{\alpha}}\frac{1}{[n\tau]-[n\tau_*^{\alpha}]}\sum_{i=[n\tau_*^{\alpha}]+1}^{[n\tau]}\left(\frac{1}{2}\lambda_1[\Xi_{i-1}^{\alpha}(\alpha_0)] + r_{n,i-1}\right) \leq 8\delta\right)$$
$$+ \mathbf{P}\left(|\hat{\alpha}_1-\hat{\alpha}_2| < \frac{\vartheta_\alpha}{2}\right) + \sum_{k=1}^{2}\mathbf{P}(\hat{\alpha}_k\notin\mathcal{O}_{\alpha_0}).$$

Choose $\delta = \frac{1}{19}\int_{\mathbb{R}^d}\lambda_1[\Xi^{\alpha}(x,\alpha_0)]\mathrm{d}\mu_{\alpha_0}(x) > 0$. It then follows from [F2](a) that for large n,

$$\mathbf{P}\left(\inf_{\tau\in D_{n,M}^{\alpha}}\frac{1}{[n\tau]-[n\tau_*^{\alpha}]}\sum_{i=[n\tau_*^{\alpha}]+1}^{[n\tau]}\Big(\frac{1}{2}\lambda_1[\Xi_{i-1}^{\alpha}(\alpha_0)] + r_{n,i-1}\Big) \leq 8\delta\right)$$
$$\leq \mathbf{P}\left(\inf_{\tau\in D_{n,M}^{\alpha}}\frac{1}{[n\tau]-[n\tau_*^{\alpha}]}\sum_{i=[n\tau_*^{\alpha}]+1}^{[n\tau]}\lambda_1[\Xi_{i-1}^{\alpha}(\alpha_0)] \leq 18\delta\right) + \mathbf{P}\left(\sup_{\tau\in D_{n,M}^{\alpha}}\left|\frac{1}{[n\tau]-[n\tau_*^{\alpha}]}\sum_{i=[n\tau_*^{\alpha}]+1}^{[n\tau]} r_{n,i-1}\right| \geq \delta\right)$$
$$\leq \mathbf{P}\left(\sup_{\tau\in D_{n,M}^{\alpha}}\left|\frac{1}{[n\tau]-[n\tau_*^{\alpha}]}\sum_{i=[n\tau_*^{\alpha}]+1}^{[n\tau]}\lambda_1[\Xi_{i-1}^{\alpha}(\alpha_0)] - 19\delta\right| \geq \delta\right) + \frac{\epsilon}{2} \leq \epsilon.$$

Noting that $\{|\hat{\alpha}_1-\hat{\alpha}_2| < \vartheta_\alpha/2\} \subset \bigcup_{k=1}^{2}\{|\hat{\alpha}_k-\alpha_k^*| > \vartheta_\alpha/4\}$ and $\mathbf{P}(|\hat{\alpha}_k-\alpha_k^*| > \vartheta_\alpha/4) < \epsilon/2$ for large n, we see $\mathbf{P}(|\hat{\alpha}_1-\hat{\alpha}_2| < \vartheta_\alpha/2) < \epsilon$. Hence, we obtain $P_{2,n}^{\alpha} \leq 3\epsilon$ for large n.

(iii) Estimation of $P_{3,n}^{\alpha}$. If $\hat{\alpha}_k\in\mathcal{O}_{\alpha_0}$, then it holds from [E1] and $\hat{\alpha}_1-\hat{\alpha}_2 = O_{\mathbf{P}}(\vartheta_\alpha)$ that

$$\mathrm{Tr}\left(\left(A_{i-1}^{-1}(\hat{\alpha}_1) - A_{i-1}^{-1}(\hat{\alpha}_2)\right)\left(A_{i-1}(\hat{\alpha}_2) - h_n^{-1}\mathbb{E}_{\alpha_2^*}[(\Delta_i X)^{\otimes 2}|\mathscr{G}_{i-1}^n]\right)\right)$$
$$\leq \Xi_{i-1}^{\alpha}(\alpha_0)\otimes(\hat{\alpha}_1-\hat{\alpha}_2)\otimes(\hat{\alpha}_2-\alpha_2^*) + R_{i-1}\Big(\frac{\vartheta_\alpha^2}{\sqrt{n}}\vee h_n\vartheta_\alpha\Big).$$

We have from [F1] and [F2](a) the result that

$$\begin{aligned}
&\sup_{\tau\in D^{\alpha}_{n,M}} \frac{|\varrho^{\alpha}_n(\tau:\hat{\alpha}_1,\hat{\alpha}_2)|}{\vartheta^2_{\alpha}([n\tau]-[n\tau^{\alpha}_*])} \\
&\leq \sup_{\tau\in D^{\alpha}_{n,M}} \left| \frac{1}{[n\tau]-[n\tau^{\alpha}_*]} \sum_{i=[n\tau^{\alpha}_*]+1}^{[n\tau]} \Xi^{\alpha}_{i-1}(\alpha_0) \right| |\hat{\alpha}_1-\hat{\alpha}_2||\hat{\alpha}_2-\alpha^*_2| + O_{\mathbf{P}}(\sqrt{n}\vartheta^2_{\alpha}\vee T\vartheta_{\alpha}) \\
&= O_{\mathbf{P}}\left(\frac{1}{\sqrt{n}\vartheta_{\alpha}}\vee \sqrt{n}\vartheta^2_{\alpha}\vee T\vartheta_{\alpha}\right) = o_{\mathbf{P}}(1),
\end{aligned}$$

which implies that $P^{\alpha}_{3,n}\leq 2\epsilon$ for large n.

(iv) From the estimations in Steps (i)-(iii), we have

$$\overline{\lim_{n\to\infty}} \mathbf{P}(n\vartheta^2_{\alpha}(\hat{\tau}^{\alpha}_n-\tau^{\alpha}_*)>M)\leq \gamma_{\alpha}(M)+7\epsilon$$

for any $M\geq 1$ and $\epsilon>0$. Hence

$$\overline{\lim_{M\to\infty}}\,\overline{\lim_{n\to\infty}} \mathbf{P}(n\vartheta^2_{\alpha}(\hat{\tau}^{\alpha}_n-\tau^{\alpha}_*)>M)\leq 7\epsilon.$$

In the same way as above, we see

$$\overline{\lim_{M\to\infty}}\,\overline{\lim_{n\to\infty}} \mathbf{P}(n\vartheta^2_{\alpha}(\tau^{\alpha}_*-\hat{\tau}^{\alpha}_n)>M)\leq 7\epsilon,$$

and thus, $\overline{\lim}_{M\to\infty}\overline{\lim}_{n\to\infty} \mathbf{P}(n\vartheta^2_{\alpha}|\hat{\tau}^{\alpha}_n-\tau^{\alpha}_*|>M)\leq 14\epsilon$, which shows

$$n\vartheta^2_{\alpha}(\hat{\tau}^{\alpha}_n-\tau^{\alpha}_*) = O_{\mathbf{P}}(1). \tag{B.29}$$

From Lemmas 6-8 and (B.29), we obtain

$$n\vartheta^2_{\alpha}(\hat{\tau}^{\alpha}_n-\tau^{\alpha}_*) \xrightarrow{\mathrm{d}} \underset{v\in\mathbb{R}}{\operatorname{argmin}}\,\mathbb{F}(v).$$

(2) Let $D^{\alpha}_{n,M}=\{\tau\in[0,1]|n(\tau-\tau^{\alpha}_*)>M\}$. Similarly, we have $\mathbf{P}(n(\hat{\tau}^{\alpha}_n-\tau^{\alpha}_*)>M)\leq P^{\alpha}_{1,n}+P^{\alpha}_{2,n}+P^{\alpha}_{3,n}$, where

$$\begin{aligned}
&P^{\alpha}_{1,n} = \mathbf{P}\left(\sup_{\tau\in D^{\alpha}_{n,M}} \frac{|\mathcal{M}^{\alpha}_n(\tau:\hat{\alpha}_1,\hat{\alpha}_2)|}{[n\tau]-[n\tau^{\alpha}_*]}\geq\delta\right), \quad P^{\alpha}_{2,n} = \mathbf{P}\left(\inf_{\tau\in D^{\alpha}_{n,M}} \frac{\mathcal{A}^{\alpha}_n(\tau:\hat{\alpha}_1,\hat{\alpha}_2)}{[n\tau]-[n\tau^{\alpha}_*]}\leq 2\delta\right), \\
&P^{\alpha}_{3,n} = \mathbf{P}\left(\sup_{\tau\in D^{\alpha}_{n,M}} \frac{|\varrho^{\alpha}_n(\tau:\hat{\alpha}_1,\hat{\alpha}_2)|}{[n\tau]-[n\tau^{\alpha}_*]}\geq\delta\right).
\end{aligned}$$

Let $\epsilon>0$. Since it follows from Lemma 2 of Iacus and Yoshida (2012) that

$$\mathbf{P}\left(\sup_{\tau\in D^{\alpha}_{n,M}} \frac{1}{[n\tau]-[n\tau^{\alpha}_*]} \sup_{\alpha_k\in O_{\alpha^*_k}} |\mathcal{M}^{\alpha}_n(\tau:\alpha_1,\alpha_2)|\geq\delta\right) \lesssim \frac{1}{\delta^2 M} =: \gamma_{\alpha}(M),$$

we find $P^\alpha_{1,n} \le \gamma_\alpha(M) + \epsilon$ for large n.

From [G2](a), if $\hat\alpha_k \in O_{\alpha_k^*}$, then there exists $c > 0$ independent of i such that

$$\Gamma^\alpha_{i-1}(\hat\alpha_1, \hat\alpha_2) \ge \Gamma^\alpha_{i-1}(\alpha_1^*, \alpha_2^*) - c(|\hat\alpha_1 - \alpha_1^*| + |\hat\alpha_2 - \alpha_2^*|). \tag{B.30}$$

Choose $\delta = \inf_x \Gamma^\alpha(x, \alpha_1^*, \alpha_2^*)/4 > 0$. It then holds from [G1] that for large n,

$$\mathbf{P}\left(\inf_{\tau\in D^\alpha_{n,M}} \frac{\mathcal{A}^\alpha_n(\tau : \hat\alpha_1, \hat\alpha_2)}{[n\tau] - [n\tau^\alpha_*]} \le 2\delta,\ \hat\alpha_1 \in O_{\alpha_1^*},\ \hat\alpha_2 \in O_{\alpha_2^*}\right) \le \mathbf{P}\left(\inf_x \Gamma^\alpha(x, \alpha_1^*, \alpha_2^*) \le 3\delta\right) + \epsilon = \epsilon,$$

and thus we have $P^\alpha_{2,n} \le 2\epsilon$.

Moreover, it follows that for $\hat\alpha_k \in O_{\alpha_k^*}$,

$$\begin{aligned}
&\mathrm{Tr}\left(\left(A^{-1}_{i-1}(\hat\alpha_1) - A^{-1}_{i-1}(\hat\alpha_2)\right)\left(A_{i-1}(\hat\alpha_2) - h_n^{-1}\mathbb{E}_{\alpha_2^*}[(\Delta_i X)^{\otimes 2}|\mathscr{G}^n_{i-1}]\right)\right)\\
&\le \sup_{x,\alpha_k}\left|[\mathrm{Tr}\left((A^{-1}(x,\alpha_1) - A^{-1}(x,\alpha_2))\partial_{\alpha^l}A(x,\alpha_3)\right)]_l\right||\hat\alpha_2 - \alpha_2^*|\\
&\qquad + h|A^{-1}_{i-1}(\hat\alpha_1) - A^{-1}_{i-1}(\hat\alpha_2)||Q_{i-1}(\theta^*)| + R_{i-1}(h_n^2),
\end{aligned}$$

and from [E1], [G2](b) and (c) that

$$\begin{aligned}
\sup_{\tau\in D^\alpha_{n,M}} \frac{|\varrho^\alpha_n(\tau : \hat\alpha_1, \hat\alpha_2)|}{[n\tau] - [n\tau^\alpha_*]} &\le \sup_{x,\alpha_k}\left|[\mathrm{Tr}\left((A^{-1}(x,\alpha_1) - A^{-1}(x,\alpha_2))\partial_{\alpha^l}A(x,\alpha_3)\right)]_l\right||\hat\alpha_2 - \alpha_2^*|\\
&\qquad + h_n \sup_{x,\alpha_k}\left|A^{-1}(x,\alpha_1) - A^{-1}(x,\alpha_2)\right| \sup_{x,\theta}|Q(x,\theta)| + O_{\mathbf{P}}(nh_n^2)\\
&= O_{\mathbf{P}}(n^{-1/2} \vee h_n \vee nh_n^2) = o_{\mathbf{P}}(1).
\end{aligned}$$

We hence see $P^\alpha_{3,n} \le 2\epsilon$ for large n.

Therefore we have

$$\overline{\lim_{M\to\infty}}\ \overline{\lim_{n\to\infty}}\ \mathbf{P}(n(\hat\tau^\alpha_n - \tau^\alpha_*) > M) \le 5\epsilon.$$

(3) It suffices to estimate the following probabilities for any $\epsilon_1 \in [0, 1/2)$ and $M > 0$.

$$P^\alpha_{1,n} = \mathbf{P}\left(\sup_{\tau\in D^\alpha_{n,M}} \frac{|\mathcal{M}^\alpha_n(\tau : \hat\alpha_1, \hat\alpha_2)|}{n^{-\delta_1}([n\tau] - [n\tau^\alpha_*])} \ge 1\right), \quad P^\alpha_{2,n} = \mathbf{P}\left(\inf_{\tau\in D^\alpha_{n,M}} \frac{\mathcal{A}^\alpha_n(\tau : \hat\alpha_1, \hat\alpha_2)}{n^{-\delta_1}([n\tau] - [n\tau^\alpha_*])} \le 2\right),$$

$$P^\alpha_{3,n} = \mathbf{P}\left(\sup_{\tau\in D^\alpha_{n,M}} \frac{|\varrho^\alpha_n(\tau : \hat\alpha_1, \hat\alpha_2)|}{n^{-\delta_1}([n\tau] - [n\tau^\alpha_*])} \ge 1\right),$$

where $D^\alpha_{n,M} = \{\tau \in [0,1] | n^{\epsilon_1}(\tau - \tau^\alpha_*) > M\}$.

Let $0 < \delta_1 < 1/2 - \epsilon_1$. For any $\epsilon > 0$, we have from Lemma 2 of Iacus and Yoshida (2012), [E1] and $\epsilon_1 + 2\delta_1 < 1$ the result that

$$P^\alpha_{1,n} \le \mathbf{P}\left(\sup_{\tau\in D^\alpha_{n,M}} \frac{\sup_{\alpha_k\in O_{\alpha_k^*}} |\mathcal{M}^\alpha_n(\tau : \alpha_1, \alpha_2)|}{[n\tau] - [n\tau^\alpha_*]} \ge n^{-\delta_1}\right) + \sum_{k=1}^{2} \mathbf{P}(\hat\alpha_k \notin O_{\alpha_k^*})$$

$$\lesssim \frac{n^{\epsilon_1+2\delta_1-1}}{M} + \sum_{k=1}^{2} \mathbf{P}(\hat{\alpha}_k \notin O_{\alpha_k^*}) \le 2\epsilon$$

for large n. By (B.30), [G1], [E1] and $\delta_1 < 1/2$

$$P_{2,n}^{\alpha} \le \mathbf{P}\left(\inf_x \Gamma^{\alpha}(x, \alpha_1^*, \alpha_2^*) \le 3n^{-\delta_1}\right) + \mathbf{P}\left(|\hat{\alpha}_1 - \alpha_1^*| + |\hat{\alpha}_2 - \alpha_2^*| \ge \frac{n^{-\delta_1}}{c}\right) + \sum_{k=1}^{2} \mathbf{P}(\hat{\alpha}_k \notin O_{\alpha_k^*}) \le 3\epsilon$$

for large n. Furthermore, if $\hat{\alpha}_k \in O_{\alpha_k^*}$, then

$$\begin{aligned}
&\mathrm{Tr}\left(\left(A_{i-1}^{-1}(\hat{\alpha}_1) - A_{i-1}^{-1}(\hat{\alpha}_2)\right)\left(A_{i-1}(\hat{\alpha}_2) - h_n^{-1}\mathbb{E}_{\alpha_2^*}[(\Delta_i X)^{\otimes 2}|\mathscr{G}_{i-1}^n]\right)\right) \\
&\le \sup_{x,\alpha_k}\left|[\mathrm{Tr}\left((A^{-1}(x,\alpha_1) - A^{-1}(x,\alpha_2))\partial_{\alpha^l} A(x,\alpha_3)\right)]_l\right| |\hat{\alpha}_2 - \alpha_2^*| + R_{i-1}(h_n),
\end{aligned}$$

that is, it holds from [E1], [G2](b) and $0 < \delta_1 < 1/2 - \epsilon_1$ that

$$\sup_{\tau \in D_{n,M}^{\alpha}} \frac{|\varrho_n^{\alpha}(\tau : \hat{\alpha}_1, \hat{\alpha}_2)|}{n^{-\delta_1}([n\tau] - [n\tau_*^{\alpha}])} = O_{\mathbf{P}}(n^{\delta_1 - 1/2} \vee n^{\epsilon_1+\delta_1} h_n) = o_{\mathbf{P}}(1),$$

which indicates that $P_{3,n}^{\alpha} \le 2\epsilon$ for large n.

Therefore, we have

$$\overline{\lim_{n\to\infty}} \mathbf{P}(n^{\epsilon_1}(\hat{\tau}_n^{\alpha} - \tau_*^{\alpha}) > M) \le \overline{\lim_{n\to\infty}} (P_{1,n}^{\alpha} + P_{2,n}^{\alpha} + P_{3,n}^{\alpha}) \le 7\epsilon$$

for any $M > 0$ and $\epsilon > 0$. □

Proofs of Theorems 3 and 4

Proof of Theorem 3. (1) By the Taylor expansion, we have

$$\kappa_{i-1}^l(\hat{\alpha}) = \kappa_{i-1}^l(\alpha^*) + \partial_{\alpha}\kappa_{i-1}^l(\alpha^*)(\hat{\alpha} - \alpha^*) + (\hat{\alpha} - \alpha^*)^{\mathsf{T}} \mathcal{K}_i^l (\hat{\alpha} - \alpha^*),$$

where $\mathcal{K}_i^l = \int_0^1 (1-u)\partial_{\alpha}^2 \kappa_{i-1}^l(\alpha^* + u(\hat{\alpha} - \alpha^*))\mathrm{d}u$. We then find that

$$\begin{aligned}
\hat{\xi}_i &= \sum_{l=1}^{d} \kappa_{i-1}^l(\hat{\alpha})(\Delta_i X - h_n b_{i-1}(\hat{\beta}))^l \\
&= \sum_{l=1}^{d} \kappa_{i-1}^l(\alpha^*)\Big(\Delta_i X - h_n b_{i-1}(\beta^*) - h_n(b_{i-1}(\hat{\beta}) - b_{i-1}(\beta^*))\Big)^l \\
&\quad + \left(\frac{1}{\sqrt{n}} \sum_{l=1}^{d} \partial_{\alpha}\kappa_{i-1}^l(\alpha^*)(\Delta_i X - h_n b_{i-1}(\hat{\beta}))^l\right) \sqrt{n}(\hat{\alpha} - \alpha^*) \\
&\quad + \sqrt{n}(\hat{\alpha} - \alpha^*)^{\mathsf{T}} \left(\frac{1}{n} \sum_{l=1}^{d} \mathcal{K}_i^l (\Delta_i X - h_n b_{i-1}(\hat{\beta}))^l\right) \sqrt{n}(\hat{\alpha} - \alpha^*)
\end{aligned}$$

$$
\begin{aligned}
&= \xi_i - \left(\sqrt{\frac{h_n}{n}} \sum_{l=1}^{d} \kappa_{i-1}^l(\alpha^*) \partial_\beta b_{i-1}^l(\beta^*) \right) \sqrt{T}(\hat{\beta} - \beta^*) \\
&\quad - \sqrt{T}(\hat{\beta} - \beta^*)^\top \left(\frac{1}{n} \sum_{l=1}^{d} \kappa_{i-1}^l(\alpha^*) \int_0^1 (1-u) \partial_\beta^2 b_{i-1}^l(\beta^* + u(\hat{\beta} - \beta^*)) \mathrm{d}u \right) \sqrt{T}(\hat{\beta} - \beta^*) \\
&\quad + \left(\frac{1}{\sqrt{n}} \sum_{l=1}^{d} \partial_\alpha \kappa_{i-1}^l(\alpha^*)(\Delta_i X)^l \right) \sqrt{n}(\hat{\alpha} - \alpha^*) - \left(\frac{h_n}{\sqrt{n}} \sum_{l=1}^{d} \partial_\alpha \kappa_{i-1}^l(\alpha^*) b_{i-1}^l(\hat{\beta}) \right) \sqrt{n}(\hat{\alpha} - \alpha^*) \\
&\quad + \sqrt{n}(\hat{\alpha} - \alpha^*)^\top \left(\frac{1}{n} \sum_{l=1}^{d} \mathcal{K}_i^l (\Delta_i X - h_n b_{i-1}(\hat{\beta}))^l \right) \sqrt{n}(\hat{\alpha} - \alpha^*) \\
&=: \xi_i + \sqrt{\frac{h_n}{n}} Q_{1,i} \sqrt{T}(\hat{\beta} - \beta^*) + \frac{1}{n} \sqrt{T}(\hat{\beta} - \beta^*)^\top Q_{2,i} \sqrt{T}(\hat{\beta} - \beta^*) \\
&\quad + \frac{1}{\sqrt{n}} Q_{3,i} \sqrt{n}(\hat{\alpha} - \alpha^*) + \frac{h_n}{\sqrt{n}} Q_{4,i} \sqrt{n}(\hat{\alpha} - \alpha^*) \\
&\quad + \frac{1}{n} \sqrt{n}(\hat{\alpha} - \alpha^*)^\top Q_{5,i} \sqrt{n}(\hat{\alpha} - \alpha^*).
\end{aligned}
$$

Therefore, it is enough to show

$$
\frac{1}{\sqrt{dT}} \max_{1 \le k \le n} \left| \sum_{i=1}^{k} \xi_i - \frac{k}{n} \sum_{i=1}^{n} \xi_i \right| \xrightarrow{\mathrm{d}} \sup_{0 \le s \le 1} |\mathbb{B}_1^0(s)|, \tag{B.31}
$$

$$
\frac{1}{n} \max_{1 \le k \le n} \left| \sum_{i=1}^{k} Q_{1,i} - \frac{k}{n} \sum_{i=1}^{n} Q_{1,i} \right| = o_{\mathbf{P}}(1), \tag{B.32}
$$

$$
\frac{1}{n\sqrt{T}} \max_{1 \le k \le n} \left| \sum_{i=1}^{k} Q_{2,i} - \frac{k}{n} \sum_{i=1}^{n} Q_{2,i} \right| = o_{\mathbf{P}}(1), \tag{B.33}
$$

$$
\frac{1}{n\sqrt{h_n}} \max_{1 \le k \le n} \left| \sum_{i=1}^{n} Q_{3,i} - \frac{k}{n} \sum_{i=1}^{n} Q_{3,i} \right| = o_{\mathbf{P}}(1), \tag{B.34}
$$

$$
\frac{\sqrt{h_n}}{n} \max_{1 \le k \le n} \left| \sum_{i=1}^{k} Q_{4,i} - \frac{k}{n} \sum_{i=1}^{n} Q_{4,i} \right| = o_{\mathbf{P}}(1), \tag{B.35}
$$

$$
\frac{1}{n\sqrt{T}} \max_{1 \le k \le n} \left| \sum_{i=1}^{k} Q_{5,i} - \frac{k}{n} \sum_{i=1}^{n} Q_{5,i} \right| = o_{\mathbf{P}}(1). \tag{B.36}
$$

(B.32) and (B.34) are easily shown by Lemmas 4 and 5.
Proof of (B.31). In order to show

$$
\mathcal{V}_n(s) = \frac{1}{\sqrt{dT}} \sum_{i=1}^{[ns]} \xi_i \xrightarrow{\mathrm{w}} \mathbb{B}_1(s) \quad \text{in } \mathbb{D}[0,1], \tag{B.37}
$$

we verify that

$$\frac{1}{\sqrt{dT}}\sum_{i=1}^{[ns]}(\xi_i - \mathbb{E}_{\theta^*}[\xi_i|\mathscr{G}_{i-1}^n]) \xrightarrow{\text{w}} \mathbb{B}_1(s) \quad \text{in } \mathbb{D}[0,1], \tag{B.38}$$

$$\frac{1}{\sqrt{T}}\sum_{i=1}^{n}\mathbb{E}_{\theta^*}[\xi_i|\mathscr{G}_{i-1}^n] = o_{\mathbf{P}}(1). \tag{B.39}$$

(B.39) is easily shown from Lemma 2. Moreover, we find from Lemma 2 that

$$\frac{1}{dT}\sum_{i=1}^{[ns]}\mathbb{E}_{\theta^*}[(\xi_i - \mathbb{E}_{\theta^*}[\xi_i|\mathscr{G}_{i-1}^n])^2|\mathscr{G}_{i-1}^n] = \frac{[ns]}{n}\frac{1}{d[ns]h_n}\sum_{i=1}^{[ns]}(dh_n + R_{i-1}(h_n^2)) \xrightarrow{\mathbf{P}} s \quad \text{for all } s \in [0,1],$$

$$\frac{1}{T^2}\sum_{i=1}^{[ns]}\mathbb{E}_{\theta^*}[(\xi_i - \mathbb{E}_{\theta^*}[\xi_i|\mathscr{G}_{i-1}^n])^4|\mathscr{G}_{i-1}^n] \lesssim \frac{1}{T^2}\sum_{i=1}^{[ns]}\mathbb{E}_{\theta^*}[\xi_i^4 + R_{i-1}(h_n^8)|\mathscr{G}_{i-1}^n] = \frac{1}{n^2}\sum_{i=1}^{[ns]}R_{i-1}(1) = o_{\mathbf{P}}(1).$$

and thus (B.38) follows from Corollary 3.8 of McLeish (1974).

Proofs of (B.33), (B.35) *and* (B.36). Since there exists an open neighborhood $\mathcal{O}_{\theta^*}$ of θ^* such that $\mathcal{O}_{\theta^*} \subset \Theta$, we see that on $\Omega_n = \{\hat{\theta} \in \mathcal{O}_{\theta^*}\}$,

$$|\mathcal{Q}_{2,i}| \le \sum_{l=1}^{d}\sup_{\alpha\in\Theta_A}|\kappa_{i-1}^l(\alpha)|\sup_{\beta\in\Theta_B}|\partial_\beta^2 b_{i-1}^l(\beta)|,$$

$$|\mathcal{Q}_{4,i}| \le \sum_{l=1}^{d}\sup_{\alpha\in\Theta_A}|\partial_\alpha\kappa_{i-1}^l(\alpha)|\sup_{\beta\in\Theta_B}|b_{i-1}^l(\beta)|,$$

$$|\mathcal{Q}_{5,i}| \le \sum_{l=1}^{d}\sup_{\alpha\in\Theta_A}|\partial_\alpha^2\kappa_{i-1}^l(\alpha)|\Big(|(\Delta_i X)^l| + h_n\sup_{\beta\in\Theta_B}|b_{i-1}^l(\beta)|\Big).$$

Noting that

$$\frac{1}{n\sqrt{T}}\sum_{i=1}^{n}\sum_{l=1}^{d}\mathbb{E}_{\theta^*}\Big[\sup_{\alpha\in\Theta_A}|\kappa_{i-1}^l(\alpha)|\sup_{\beta\in\Theta_B}|\partial_\beta^2 b_{i-1}^l(\beta)|\Big] \lesssim \frac{1}{\sqrt{T}} \to 0,$$

$$\frac{\sqrt{h_n}}{n}\sum_{i=1}^{n}\sum_{l=1}^{d}\mathbb{E}_{\theta^*}\Big[\sup_{\alpha\in\Theta_A}|\partial_\alpha\kappa_{i-1}^l(\alpha)|\sup_{\beta\in\Theta_B}|b_{i-1}^l(\beta)|\Big] \lesssim \sqrt{h_n} \to 0$$

and

$$\begin{aligned}
&\frac{1}{n\sqrt{T}}\sum_{i=1}^{n}\sum_{l=1}^{d}\mathbb{E}_{\theta^*}\Big[\sup_{\alpha\in\Theta_A}|\partial_\alpha^2\kappa_{i-1}^l(\alpha)|\Big(|(\Delta_i X)^l| + h_n\sup_{\beta\in\Theta_B}|b_{i-1}^l(\beta)|\Big)\Big] \\
&\lesssim \frac{1}{n\sqrt{T}}\sum_{i=1}^{n}\sum_{l=1}^{d}\mathbb{E}_{\theta^*}\Big[\sup_{\alpha\in\Theta_A}|\partial_\alpha^2\kappa_{i-1}^l(\alpha)|^2\Big]^{1/2}\Big(\mathbb{E}_{\theta^*}[((\Delta_i X)^l)^2] + h_n^2\mathbb{E}_{\theta^*}\Big[\sup_{\beta\in\Theta_B}|b_{i-1}^l(\beta)|^2\Big]\Big)^{1/2}
\end{aligned}$$

$$\lesssim \frac{1}{\sqrt{n}} \to 0,$$

we see from [E2] and [B2] that

$$\frac{1}{n\sqrt{T}}\sum_{i=1}^{n}|Q_{2,i}| = o_{\mathbf{P}}(1), \quad \frac{\sqrt{h_n}}{n}\sum_{i=1}^{n}|Q_{4,i}| = o_{\mathbf{P}}(1), \quad \frac{1}{n\sqrt{T}}\sum_{i=1}^{n}|Q_{5,i}| = o_{\mathbf{P}}(1).$$

(2) (a) Since

$$\mathcal{T}_{1,n}^{\beta} \geq \sqrt{\frac{T}{d}}\left|\frac{1}{T}\sum_{i=1}^{[n\tau_*^\beta]}\hat{\xi}_i - \frac{[n\tau_*^\beta]}{n}\frac{1}{T}\sum_{i=1}^{n}\hat{\xi}_i\right| =: \sqrt{\frac{T}{d}}|\tilde{X}_n|, \quad \tilde{X}_n = \frac{[n\tau_*^\beta]}{n}\left(\frac{1}{[n\tau_*^\beta]h_n}\sum_{i=1}^{[n\tau_*^\beta]}\hat{\xi}_i - \frac{1}{nh_n}\sum_{i=1}^{n}\hat{\xi}_i\right),$$

and [H2], we just show

$$\frac{1}{[n\tau_*^\beta]h_n}\sum_{i=1}^{[n\tau_*^\beta]}\hat{\xi}_i \xrightarrow{\mathbf{P}} \mathcal{G}(\alpha^*,\beta_1^*,\beta'), \quad \frac{1}{(n-[n\tau_*^\beta])h_n}\sum_{i=[n\tau_*^\beta]+1}^{n}\hat{\xi}_i \xrightarrow{\mathbf{P}} \mathcal{G}(\alpha^*,\beta_2^*,\beta'). \tag{B.40}$$

We show only the first part of (B.40). By the Taylor expansion,

$$\hat{\xi}_i = \xi_{1,i} + \sqrt{n}(\hat{\alpha}-\alpha^*)^\top\xi_{2,i} + \xi_{3,i}(\hat{\beta}-\beta'),$$

where $\xi_{1,i} = \kappa_{i-1}(\alpha^*)(\Delta_i X - h_n b_{i-1}(\beta'))$,

$$\xi_{2,i} = \frac{1}{\sqrt{n}}\int_0^1 \partial_\alpha\kappa_{i-1}(\alpha^* + u(\hat{\alpha}-\alpha^*))^\top \mathrm{d}u(\Delta_i X - h_n b_{i-1}(\hat{\beta}))$$

$$\xi_{3,i} = \kappa_{i-1}(\alpha^*)\int_0^1 \partial_\beta b_{i-1}(\beta' + u(\hat{\beta}-\beta'))\mathrm{d}u.$$

Let $\theta_1 = (\alpha^*,\beta_1^*)$. Since we have from Lemma 2 the result that

$$\frac{1}{[n\tau_*^\beta]h_n}\sum_{i=1}^{[n\tau_*^\beta]}\mathbb{E}_{\theta_1}[\xi_{1,i}|\mathscr{G}_{i-1}^n] \xrightarrow{\mathbf{P}} \mathcal{G}(\alpha^*,\beta_1^*,\beta'), \quad \frac{1}{[n\tau_*^\beta]h_n}\sum_{i=1}^{[n\tau_*^\beta]}\mathbb{E}_{\theta_1}[\xi_{1,i}^2|\mathscr{G}_{i-1}^n] \xrightarrow{\mathbf{P}} 0,$$

we find from Lemma 9 of Genon-Catalot and Jacod (1993) that

$$\frac{1}{[n\tau_*^\beta]h_n}\sum_{i=1}^{[n\tau_*^\beta]}\xi_{1,i} \xrightarrow{\mathbf{P}} \mathcal{G}(\alpha^*,\beta_1^*,\beta'). \tag{B.41}$$

Furthermore, it follows from Lemma 1, [E1] and [H1] that

$$\frac{1}{[n\tau_*^\beta]h_n}\sum_{i=1}^{[n\tau_*^\beta]}\xi_{2,i} = o_{\mathbf{P}}(1), \quad \frac{1}{[n\tau_*^\beta]h_n}\sum_{i=1}^{[n\tau_*^\beta]}\xi_{3,i} = O_{\mathbf{P}}(1).$$

Hence we obtain the desired result.

(b) Since

$$\mathcal{T}_{1,n}^{\beta} \geq \sqrt{\frac{T\vartheta_{\beta}^{2}}{d}}\left|\frac{1}{T\vartheta_{\beta}}\sum_{i=1}^{[n\tau_{*}^{\beta}]}\hat{\xi}_{i}-\frac{[n\tau_{*}^{\beta}]}{n}\frac{1}{T\vartheta_{\beta}}\sum_{i=1}^{n}\hat{\xi}_{i}\right| =: \sqrt{\frac{T\vartheta_{\beta}^{2}}{d}}|\hat{\mathcal{X}}_{n}|,$$

it suffices to show that $\hat{\mathcal{X}}_n$ converges to a non-zero constant in probability.

Let $\xi_{1,i} = \kappa_{i-1}(\alpha^*)(\Delta_i X - h_n b_{i-1}(\beta'))$ and

$$\mathcal{X}_n = \sum_{i=1}^{n} c_i \xi_{1,i} := \frac{1}{T\vartheta_{\beta}}\sum_{i=1}^{[n\tau_{*}^{\beta}]}\xi_{1,i}-\frac{[n\tau_{*}^{\beta}]}{n}\frac{1}{T\vartheta_{\beta}}\sum_{i=1}^{n}\xi_{1,i}.$$

We have from the Taylor expansion, [E2], [I3] and $\mathbb{E}[|\Delta_i X - h_n b_{i-1}(\beta)|] \lesssim h_n^{1/2}$ the result that $\hat{\mathcal{X}}_n = \mathcal{X}_n + o_{\mathbf{P}}(1)$. Because of $\mathbb{E}_{\beta_k^*}[\xi_{1,i}|\mathscr{G}_{i-1}^n] = h_n\kappa_{i-1}^{-1}(\alpha^*)\partial_\beta b_{i-1}(\beta^{(0)})(\beta_k^* - \beta^{(0)}) + R_{i-1}(h_n\vartheta_\beta^2)$, $\mathbb{E}[\xi_{1,i}^2] \lesssim h_n$, [I1] and [I2], there exists a non-zero constant c such that

$$\sum_{i=1}^{n} c_i \mathbb{E}[\xi_i|\mathscr{G}_{i-1}^n] \xrightarrow{\mathbf{P}} c, \quad \sum_{i=1}^{n} c_i^2 \mathbb{E}[\xi_i^2|\mathscr{G}_{i-1}^n] \xrightarrow{\mathbf{P}} 0.$$

Therefore, we obtain from Lemma 9 of Genon-Catalot and Jacod (1993) the result that $\mathcal{X}_n \xrightarrow{\mathbf{P}} c$ and $\hat{\mathcal{X}}_n \xrightarrow{\mathbf{P}} c$. □

Proof of Theorem 4. (1) Let $\zeta_i = \partial_\beta b_{i-1}(\beta^*)^\top A_{i-1}^{-1}(\alpha^*)(\Delta_i X - h_n b_{i-1}(\beta^*))$. By the Taylor expansion,

$$\begin{aligned}
\hat{\zeta}_i^l &= \sum_{l_1,l_2=1}^{d} \partial_{\beta^l} b_{i-1}^{l_1}(\hat{\beta})(A_{i-1}^{-1}(\hat{\alpha}))^{l_1,l_2}(\Delta_i X - h_n b_{i-1}(\hat{\beta}))^{l_2} \\
&= \sum_{l_1,l_2=1}^{d} \partial_{\beta^l} b_{i-1}^{l_1}(\hat{\beta})(A_{i-1}^{-1}(\alpha^*))^{l_1,l_2}(\Delta_i X - h_n b_{i-1}(\beta^*))^{l_2} \\
&\quad + h_n \sum_{l_1,l_2=1}^{d} \partial_{\beta^l} b_{i-1}^{l_1}(\hat{\beta})(A_{i-1}^{-1}(\alpha^*))^{l_1,l_2}(b_{i-1}^{l_2}(\beta^*) - b_{i-1}^{l_2}(\hat{\beta})) \\
&\quad + \frac{1}{\sqrt{n}} \sum_{l_1,l_2=1}^{d} \partial_{\beta^l} b_{i-1}^{l_1}(\hat{\beta})\partial_\alpha(A_{i-1}^{-1}(\alpha^*))^{l_1,l_2}(\Delta_i X - h_n b_{i-1}(\hat{\beta}))^{l_2}\sqrt{n}(\hat{\alpha} - \alpha^*) \\
&\quad + \sqrt{n}(\hat{\alpha} - \alpha^*)^\top \left(\frac{1}{n}\sum_{l_1,l_2=1}^{d} \partial_{\beta^l} b_{i-1}^{l_1}(\hat{\beta})\mathcal{A}_{i-1}^{l_1,l_2}(\Delta_i X - h_n b_{i-1}(\hat{\beta}))^{l_2}\right)\sqrt{n}(\hat{\alpha} - \alpha^*) \\
&=: J_1 + J_2 + J_3 + J_4.
\end{aligned}$$

Note that

$$\partial_{\beta^l} b_{i-1}^{l_1}(\hat{\beta}) = \partial_{\beta^l} b_{i-1}^{l_1}(\beta^*) + \sum_{j=1}^{\mathrm{m}_1-1} T^{-j/2}\partial_\beta^j \partial_{\beta^l} b_{i-1}^{l_1}(\beta^*) \otimes (\sqrt{T}(\hat{\beta}-\beta^*))^{\otimes j} + T^{-\mathrm{m}_1/2}\mathcal{B}_{\mathrm{m}_1,i-1}^{l,l_1} \otimes (\sqrt{T}(\hat{\beta}-\beta^*))^{\otimes \mathrm{m}_1},$$

where $\mathcal{B}^{l,k}_{\mathrm{m}_1,i-1} = \frac{1}{(\mathrm{m}_1-1)!}\int_0^1 (1-u)^{\mathrm{m}_1-1}\partial_\beta^{\mathrm{m}_1}\partial_{\beta^l} b^k_{i-1}(\beta^* + u(\hat{\beta}-\beta^*))\mathrm{d}u$. Since

$$\begin{aligned}
J_1 &= \sum_{l_1,l_2=1}^{d}\Bigg(\partial_{\beta^l} b^{l_1}_{i-1}(\beta^*) + \sum_{j=1}^{\mathrm{m}_1-1} T^{-j/2}\partial_\beta^j\partial_{\beta^l} b^{l_1}_{i-1}(\beta^*)\otimes(\sqrt{T}(\hat{\beta}-\beta^*))^{\otimes j} \\
&\qquad + T^{-\mathrm{m}_1/2}\mathcal{B}^{l,k}_{\mathrm{m}_1,i-1}\otimes(\sqrt{T}(\hat{\beta}-\beta^*))^{\otimes \mathrm{m}_1}\Bigg)(A^{-1}_{i-1}(\alpha^*))^{l_1,l_2}(\Delta_i X - h_n b_{i-1}(\beta^*))^{l_2} \\
&=: \zeta^l_i + \sum_{j=1}^{\mathrm{m}_1-1} T^{-j/2}\mathcal{Y}^l_{j,i}\otimes(\sqrt{T}(\hat{\beta}-\beta^*))^{\otimes j} + T^{-\mathrm{m}_1/2}\mathcal{Y}^l_{\mathrm{m}_1,i}\otimes(\sqrt{T}(\hat{\beta}-\beta^*))^{\otimes \mathrm{m}_1}, \\
J_2 &= h_n\sum_{l_1,l_2=1}^{d}\partial_{\beta^l} b^{l_1}_{i-1}(\beta^*)(A^{-1}_{i-1}(\alpha^*))^{l_1,l_2}\Bigg(-\frac{1}{\sqrt{T}}\partial_\beta b^{l_2}_{i-1}(\beta^*)\sqrt{T}(\hat{\beta}-\beta^*) \\
&\qquad - T^{-1}\int_0^1(1-u)\partial_\beta^2 b^{l_2}_{i-1}(\beta^*+u(\hat{\beta}-\beta^*))\mathrm{d}u\otimes(\sqrt{T}(\hat{\beta}-\beta^*))^{\otimes 2}\Bigg) \\
&\quad - \frac{1}{n}\left(\sum_{l_1,l_2=1}^{d}\partial_\beta\partial_{\beta^l} b^{l_1}_{i-1}(\beta^*)^\top(A^{-1}_{i-1}(\alpha^*))^{l_1,l_2}\int_0^1\partial_\beta b^{l_2}_{i-1}(\beta^*+u(\hat{\beta}-\beta^*))\mathrm{d}u\right)\otimes(\sqrt{T}(\hat{\beta}-\beta^*))^{\otimes 2} \\
&\quad + \frac{1}{n}\left(\sum_{l_1,l_2=1}^{d}\mathcal{B}^{l,k}_{2,i-1}(A^{-1}_{i-1}(\alpha^*))^{l_1,l_2}(b^{l_2}_{i-1}(\beta^*) - b^{l_2}_{i-1}(\hat{\beta}))\right)\otimes(\sqrt{T}(\hat{\beta}-\beta^*))^{\otimes 2} \\
&=: \sqrt{\frac{h_n}{n}}\mathcal{Z}^l_{1,i}\sqrt{T}(\hat{\beta}-\beta^*) + \frac{1}{n}\mathcal{Z}^l_{2,i}\otimes(\sqrt{T}(\hat{\beta}-\beta^*))^{\otimes 2}, \\
J_3 &= \frac{1}{\sqrt{n}}\sum_{l_1,l_2=1}^{d}\partial_{\beta^l} b^{l_1}_{i-1}(\beta^*)\partial_\alpha(A^{-1}_{i-1}(\alpha^*))^{l_1,l_2}(\Delta_i X)^{l_2}\sqrt{n}(\hat{\alpha}-\alpha^*) \\
&\quad - \frac{h_n}{\sqrt{n}}\sum_{l_1,l_2=1}^{d}\partial_{\beta^l} b^{l_1}_{i-1}(\beta^*)\partial_\alpha(A^{-1}_{i-1}(\alpha^*))^{l_1,l_2} b^{l_2}_{i-1}(\hat{\beta})\sqrt{n}(\hat{\alpha}-\alpha^*) \\
&\quad + \sqrt{T}(\hat{\beta}-\beta^*)^\top\frac{1}{n\sqrt{h_n}}\sum_{l_1,l_2=1}^{d}\int_0^1\partial_\beta\partial_{\beta^l} b^{l_1}_{i-1}(\beta^*+u(\hat{\beta}-\beta^*))^\top\mathrm{d}u \\
&\qquad \times\partial_\alpha(A^{-1}_{i-1}(\alpha^*))^{l_1,l_2}(\Delta_i X - h_n b_{i-1}(\hat{\beta}_n))^{l_2}\sqrt{n}(\hat{\alpha}-\alpha^*) \\
&=: \frac{1}{\sqrt{n}}\mathcal{Z}^l_{3,i}\sqrt{n}(\hat{\alpha}-\alpha^*) + \frac{h_n}{\sqrt{n}}\mathcal{Z}^l_{4,i}\sqrt{n}(\hat{\alpha}-\alpha^*) + \frac{1}{n\sqrt{h_n}}\sqrt{T}(\hat{\beta}-\beta^*)^\top\mathcal{Z}^l_{5,i}\sqrt{n}(\hat{\alpha}-\alpha^*), \\
J_4 &=: \frac{1}{n}\sqrt{n}(\hat{\alpha}-\alpha^*)^\top\mathcal{Z}^l_{6,i}\sqrt{n}(\hat{\alpha}-\alpha^*),
\end{aligned}$$

it is sufficient to show

$$\frac{1}{\sqrt{T}}\max_{1\le k\le n}\left|\mathcal{I}^{-1/2}\left(\sum_{i=1}^{k}\zeta_i - \frac{k}{n}\sum_{i=1}^{n}\zeta_i\right)\right| \xrightarrow{\mathrm{d}} \sup_{0\le s\le 1}|\mathbb{B}^0_q(s)|, \tag{B.42}$$

$$T^{-(j+1)/2}\max_{1\le k\le n}\left|\sum_{i=1}^{k}\mathcal{Y}^l_{j,i} - \frac{k}{n}\sum_{i=1}^{n}\mathcal{Y}^l_{j,i}\right| = o_{\mathbf{P}}(1), \qquad (1\le j\le \mathrm{m}_1-1) \tag{B.43}$$

$$T^{-(\mathrm{m}_1+1)/2}\max_{1\le k\le n}\left|\sum_{i=1}^{k}\mathcal{Y}^{l}_{m,i}-\frac{k}{n}\sum_{i=1}^{n}\mathcal{Y}^{l}_{m,i}\right|=o_{\mathbf{P}}(1), \tag{B.44}$$

$$\frac{1}{n}\max_{1\le k\le n}\left|\sum_{i=1}^{k}\mathcal{Z}^{l}_{1,i}-\frac{k}{n}\sum_{i=1}^{n}\mathcal{Z}^{l}_{1,i}\right|=o_{\mathbf{P}}(1), \tag{B.45}$$

$$\frac{1}{n\sqrt{T}}\max_{1\le k\le n}\left|\sum_{i=1}^{k}\mathcal{Z}^{l}_{2,i}-\frac{k}{n}\sum_{i=1}^{n}\mathcal{Z}^{l}_{2,i}\right|=o_{\mathbf{P}}(1), \tag{B.46}$$

$$\frac{1}{n\sqrt{h_n}}\max_{1\le k\le n}\left|\sum_{i=1}^{n}\mathcal{Z}^{l}_{3,i}-\frac{k}{n}\sum_{i=1}^{n}\mathcal{Z}^{l}_{3,i}\right|=o_{\mathbf{P}}(1), \tag{B.47}$$

$$\frac{\sqrt{h_n}}{n}\max_{1\le k\le n}\left|\sum_{i=1}^{k}\mathcal{Z}^{l}_{4,i}-\frac{k}{n}\sum_{i=1}^{n}\mathcal{Z}^{l}_{4,i}\right|=o_{\mathbf{P}}(1), \tag{B.48}$$

$$\frac{1}{n^{3/2}h_n}\max_{1\le k\le n}\left|\sum_{i=1}^{k}\mathcal{Z}^{l}_{5,i}-\frac{k}{n}\sum_{i=1}^{n}\mathcal{Z}^{l}_{5,i}\right|=o_{\mathbf{P}}(1), \tag{B.49}$$

$$\frac{1}{n\sqrt{T}}\max_{1\le k\le n}\left|\sum_{i=1}^{k}\mathcal{Z}^{l}_{6,i}-\frac{k}{n}\sum_{i=1}^{n}\mathcal{Z}^{l}_{6,i}\right|=o_{\mathbf{P}}(1). \tag{B.50}$$

(B.43), (B.45) and (B.47) are shown by Lemmas 4 and 5.

Proof of (B.42). In order to show

$$\mathcal{W}_n(s)=\frac{1}{\sqrt{T}}\sum_{i=1}^{[ns]}\mathcal{I}^{-1/2}\zeta_i\xrightarrow{\mathrm{w}}\mathbb{B}_q(s)\quad\text{in }\mathbb{D}[0,1], \tag{B.51}$$

we prove

$$\frac{1}{\sqrt{T}}\sum_{i=1}^{[ns]}\mathcal{I}^{-1/2}(\zeta_i-\mathbb{E}_{\theta^*}[\zeta_i|\mathscr{G}^n_{i-1}])\xrightarrow{\mathrm{w}}\mathbb{B}_q(s)\quad\text{in }\mathbb{D}[0,1], \tag{B.52}$$

$$\frac{1}{\sqrt{T}}\sum_{i=1}^{n}\mathbb{E}_{\theta^*}[\zeta_i|\mathscr{G}^n_{i-1}]=o_{\mathbf{P}}(1). \tag{B.53}$$

(B.53) is shown from Lemma 2. Furthermore, we have from Lemma 2 and $\mathbb{E}_{\theta^*}[|\zeta_i|^4|\mathscr{G}^n_{i-1}]=R_{i-1}(h_n^2)$ the result that for all $c\in\mathbb{R}^q$,

$$\begin{aligned}&\frac{1}{T}\sum_{i=1}^{[ns]}\mathbb{E}_{\theta^*}\Big[\big(c^{\mathsf{T}}\mathcal{I}^{-1/2}\,(\zeta_i-\mathbb{E}_{\theta^*}[\zeta_i|\mathscr{G}^n_{i-1}])\big)^2\Big|\mathscr{G}^n_{i-1}\Big]\\&=\frac{1}{T}\sum_{i=1}^{[ns]}c^{\mathsf{T}}\mathcal{I}^{-1/2}\Big(\mathbb{E}_{\theta^*}[\zeta_i\zeta_i^{\mathsf{T}}|\mathscr{G}^n_{i-1}]-\mathbb{E}_{\theta^*}[\zeta_i|\mathscr{G}^n_{i-1}]\mathbb{E}_{\theta^*}[\zeta_i|\mathscr{G}^n_{i-1}]^{\mathsf{T}}\Big)\mathcal{I}^{-1/2}c\end{aligned}$$

$$= \frac{[ns]}{n}\frac{1}{[ns]h_n}\sum_{i=1}^{[ns]} c^\top \mathcal{I}^{-1/2}\Big(h_n\partial_\beta b_{i-1}(\beta^*)^\top A_{i-1}^{-1}(\alpha^*)\partial_\beta b_{i-1}(\beta^*) + R_{i-1}(h_n^2)\Big)\mathcal{I}^{-1/2}c$$

$$\xrightarrow{\mathbf{P}} sc^\top \mathcal{I}^{-1/2}\mathcal{I}\mathcal{I}^{-1/2}c = |c|^2 s$$

for all $s \in [0, 1]$, and

$$\frac{1}{T^2}\sum_{i=1}^{[ns]} \mathbb{E}_{\theta^*}\Big[(c^\top \mathcal{I}^{-1/2}(\zeta_i - \mathbb{E}_{\theta^*}[\zeta_i|\mathscr{G}_{i-1}^n]))^4\Big|\mathscr{G}_{i-1}^n\Big]$$

$$\leq \frac{1}{T^2}\sum_{i=1}^{[ns]} \mathbb{E}_{\theta^*}\Big[|c^\top \mathcal{I}^{-1/2}|^4|\zeta_i - \mathbb{E}_{\theta^*}[\zeta_i|\mathscr{G}_{i-1}^n]|^4\Big|\mathscr{G}_{i-1}^n\Big]$$

$$\lesssim \frac{1}{T^2}\sum_{i=1}^{[ns]} \mathbb{E}_{\theta^*}[|\zeta_i|^4 + R_{i-1}(h_n^8)|\mathscr{G}_{i-1}^n] = o_{\mathbf{P}}(1).$$

Therefore we find from Corollary 3.8 of McLeish (1974) that

$$\frac{1}{\sqrt{T}}\sum_{i=1}^{[ns]} c^\top \mathcal{I}^{-1/2}(\zeta_i - \mathbb{E}_{\theta^*}[\zeta_i|\mathscr{G}_{i-1}^n]) \xrightarrow{\mathrm{w}} c^\top \mathbb{B}_q(s) \quad \text{in } \mathbb{D}[0,1],$$

which together with the Cramér-Wold theorem yields (B.52). This completes the proof of (B.42).

Proofs of (B.44), (B.46), (B.48), (B.49) *and* (B.50). Since there exists an open neighborhood O_{θ^*} of θ^* such that $O_{\theta^*} \subset \Theta$, it follows that on $\Omega_n = \{\hat{\theta} \in O_{\theta^*}\}$,

$$|\mathcal{Y}_{\mathrm{m}_1,i}^l| \leq \sum_{l_1,l_2=1}^d \sup_{\beta\in\Theta_B} |\partial_\beta^{\mathrm{m}_1}\partial_{\beta^l} b_{i-1}^{l_1}(\beta)| \sup_{\alpha\in\Theta_A} |(A_{i-1}^{-1}(\alpha))^{l_1,l_2}|\,|(\Delta_i X - h_n b_{i-1}(\beta^*))^{l_2}|,$$

$$|\mathcal{Z}_{2,i}^l| \leq \sum_{l_1,l_2=1}^d \Big(\sup_{\beta\in\Theta_B} |\partial_{\beta^l} b_{i-1}^{l_1}(\beta)| \sup_{\alpha\in\Theta_A} |(A_{i-1}^{-1}(\alpha))^{l_1,l_2}| \sup_{\beta\in\Theta_B} |\partial_\beta^2 b_{i-1}^{l_2}(\beta)|$$
$$+ \sup_{\beta\in\Theta_B} |\partial_\beta\partial_{\beta^l} b_{i-1}^{l_1}(\beta)| \sup_{\alpha\in\Theta_A} |(A_{i-1}^{-1}(\alpha))^{l_1,l_2}| \sup_{\beta\in\Theta_B} |\partial_\beta b_{i-1}^{l_2}(\beta)|$$
$$+ 2\sup_{\beta\in\Theta_B} |\partial_\beta^2\partial_{\beta^l} b_{i-1}^{l_1}(\beta)| \sup_{\beta\in\Theta_B} |b_{i-1}^{l_2}(\beta)|\Big),$$

$$|\mathcal{Z}_{4,i}^l| \leq \sum_{l_1,l_2=1}^d \sup_{\beta\in\Theta_B} |\partial_{\beta^l} b_{i-1}^{l_1}(\beta)| \sup_{\alpha\in\Theta_A} |\partial_\alpha(A_{i-1}^{-1}(\alpha))^{l_1,l_2}| \sup_{\beta\in\Theta_B} |b_{i-1}^{l_2}(\beta)|,$$

$$|\mathcal{Z}_{5,i}^l| \leq \sum_{l_1,l_2=1}^d \sup_{\beta\in\Theta_B} |\partial_\beta\partial_{\beta^l} b_{i-1}^{l_1}(\beta)| \sup_{\alpha\in\Theta_A} |\partial_\alpha(A_{i-1}^{-1}(\alpha))^{l_1,l_2}|\Big(|(\Delta_i X)^{l_2}| + h_n \sup_{\beta\in\Theta_B} |b_{i-1}^{l_2}(\beta)|\Big),$$

$$|\mathcal{Z}_{6,i}^l| \leq \sum_{l_1,l_2=1}^d \sup_{\beta\in\Theta_B} |\partial_{\beta^l} b_{i-1}^{l_1}(\beta)| \sup_{\alpha\in\Theta_A} |\partial_\alpha^2(A_{i-1}^{-1}(\alpha))^{l_1,l_2}|\Big(|(\Delta_i X)^{l_2}| + h_n \sup_{\beta\in\Theta_B} |b_{i-1}^{l_2}(\beta)|\Big).$$

Since

$$\frac{1}{T^{(\mathrm{m}_1+1)/2}}\sum_{i=1}^n \mathbb{E}_{\theta^*}[|\mathcal{Y}_{\mathrm{m}_1,i}^l| : \Omega_n] \leq \frac{1}{T^{(\mathrm{m}_1+1)/2}}\sum_{i=1}^n\sum_{l_1,l_2=1}^d \mathbb{E}_{\theta^*}\Big[\sup_{\beta\in\Theta_B} |\partial_\beta^{\mathrm{m}_1}\partial_{\beta^l} b_{i-1}^{l_1}(\beta)|^2 \sup_{\alpha\in\Theta_A} |(A_{i-1}^{-1}(\alpha))^{l_1,l_2}|^2\Big]^{1/2}$$

$$\times \mathbb{E}_{\theta^*}[|(\Delta_i X - h_n b_{i-1}(\beta^*))^{l_2}|^2]^{1/2}$$

$$\lesssim \frac{1}{(nh_n^{\mathrm{m}_1/(\mathrm{m}_1-1)})^{(\mathrm{m}_1-1)/2}} \to 0,$$

$$\frac{1}{n\sqrt{T}}\sum_{i=1}^n \mathbb{E}_{\theta^*}[|\mathcal{Z}_{2,i}^l| : \Omega_n] \lesssim \frac{1}{\sqrt{T}} \to 0, \quad \frac{\sqrt{h_n}}{n}\sum_{i=1}^n \mathbb{E}_{\theta^*}[|\mathcal{Z}_{4,i}^l| : \Omega_n] \lesssim \sqrt{h_n} \to 0,$$

$$\frac{1}{n^{3/2}h_n}\sum_{i=1}^n \mathbb{E}_{\theta^*}[|\mathcal{Z}_{5,i}^l| : \Omega_n] \lesssim \frac{1}{n^{3/2}h_n}\sum_{i=1}^n\sum_{l_1,l_2=1}^d \mathbb{E}_{\theta^*}\Big[\sup_{\beta\in\Theta_B}|\partial_\beta\partial_{\beta^l} b_{i-1}^{l_1}(\beta)|^2 \sup_{\alpha\in\Theta_A}|\partial_\alpha(A_{i-1}^{-1}(\alpha))^{l_1,l_2}|^2\Big]^{1/2}$$

$$\times \mathbb{E}_{\theta^*}\Big[|(\Delta_i X)^{l_2}|^2 + h_n^2 \sup_{\beta\in\Theta_B}|b_{i-1}^{l_2}(\beta)|^2\Big]^{1/2}$$

$$\lesssim \frac{1}{\sqrt{T}} \to 0,$$

$$\frac{1}{n\sqrt{T}}\sum_{i=1}^n \mathbb{E}_{\theta^*}[|\mathcal{Z}_{6,i}^l| : \Omega_n] \lesssim \frac{1}{n\sqrt{T}}\sum_{i=1}^n\sum_{l_1,l_2=1}^d \mathbb{E}_{\theta^*}\Big[\sup_{\beta\in\Theta_B}|\partial_{\beta^l} b_{i-1}^{l_1}(\beta)|^2 \sup_{\alpha\in\Theta_A}|\partial_\alpha^2(A_{i-1}^{-1}(\alpha))^{l_1,l_2}|^2\Big]^{1/2}$$

$$\times \mathbb{E}_{\theta^*}\Big[|(\Delta_i X)^{l_2}|^2 + h_n^2 \sup_{\beta\in\Theta_B}|b_{i-1}^{l_2}(\beta)|^2\Big]^{1/2}$$

$$\lesssim \frac{1}{\sqrt{n}} \to 0,$$

[E2] and [B2], we obtain the desired results.

(2) (a) We show

$$\frac{1}{[n\tau_*^\beta]h_n}\sum_{i=1}^{[n\tau_*^\beta]}\hat{\zeta}_i \xrightarrow{\mathbf{P}} \mathcal{H}(\alpha^*,\beta_1^*,\beta'). \tag{B.54}$$

Let $\zeta_{1,i} = \partial_\beta b_{i-1}(\beta')^\top A_{i-1}^{-1}(\alpha^*)(\Delta_i X - h_n b_{i-1}(\beta'))$. In the same way as in the proof of (2)-(a) of Theorem 3, one has from the Taylor expansion, [E2], [B3] and [H3] that

$$\frac{1}{[n\tau_*^\beta]h_n}\sum_{i=1}^{[n\tau_*^\beta]}\hat{\zeta}_i = \frac{1}{[n\tau_*^\beta]h_n}\sum_{i=1}^{[n\tau_*^\beta]}\zeta_{1,i} + o_{\mathbf{P}}(1), \quad \frac{1}{[n\tau_*^\beta]h_n}\sum_{i=1}^{[n\tau_*^\beta]}\zeta_{1,i} \xrightarrow{\mathbf{P}} \mathcal{H}(\alpha^*,\beta_1^*,\beta').$$

Hence we get the desired result.

(b) It is enough to show that

$$\hat{\mathcal{Z}}_n = \frac{1}{T\vartheta_\beta}\left(\sum_{i=1}^{[n\tau_*^\beta]}\hat{\zeta}_i - \frac{[n\tau_*^\beta]}{n}\sum_{i=1}^n\hat{\zeta}_i\right) \xrightarrow{\mathbf{P}} c$$

for some $c \neq 0$. Let $\zeta_{1,i} = \partial_\beta b_{i-1}(\beta')^\top A_{i-1}^{-1}(\alpha^*)(\Delta_i X - h_n b_{i-1}(\beta'))$. Since

$$\mathbb{E}[\Delta_i X - h_n b_{i-1}(\beta')] = O(h_n), \quad \mathbb{E}[|\Delta_i X - h_n b_{i-1}(\beta)|] \lesssim h_n^{1/2}, \quad \mathbb{E}[|\zeta_{1,i}|^2] \lesssim h_n,$$

$$\mathbb{E}[\zeta_{1,i}|\mathscr{G}_{i-1}^n] = h_n\partial_\beta b_{i-1}(\beta')^\top A_{i-1}^{-1}(\alpha^*)\partial_\beta b_{i-1}(\beta_0)(\beta_k^* - \beta') + R_{i-1}(h_n\vartheta_\beta^2),$$

[E2], [I3] and [I5], we find that

$$\frac{1}{T\vartheta_\beta}\sum_{i=1}^{[n\tau_*^\beta]}\hat{\zeta}_i = \frac{1}{T\vartheta_\beta}\sum_{i=1}^{[n\tau_*^\beta]}\zeta_{1,i} + o_{\mathbf{P}}(1).$$

Therefore there exists $c \neq 0$ such that

$$\frac{1}{T\vartheta_\beta}\left(\sum_{i=1}^{[n\tau_*^\beta]}\zeta_{1,i} - \frac{[n\tau_*^\beta]}{n}\sum_{i=1}^{n}\zeta_{1,i}\right) \xrightarrow{\mathbf{P}} c,$$

and we get $\hat{\mathcal{Z}}_n \xrightarrow{\mathbf{P}} c$. □

Proof of Theorem 5

In Case A_β, define $\mathcal{D}_n^\beta(v) = \hat{\mathbb{G}}_n(v) - \mathbb{G}_n(v)$, where

$$\mathbb{G}_n(v) = \Psi_n\left(\tau_*^\beta + \frac{v}{T\vartheta_\beta^2} : \beta_1^*, \beta_2^*\middle|\alpha^*\right) - \Psi_n(\tau_*^\beta : \beta_1^*, \beta_2^*|\alpha^*),$$

$$\hat{\mathbb{G}}_n(v) = \Psi_n\left(\tau_*^\beta + \frac{v}{T\vartheta_\beta^2} : \hat{\beta}_1, \hat{\beta}_2\middle|\hat{\alpha}\right) - \Psi_n(\tau_*^\beta : \hat{\beta}_1, \hat{\beta}_2|\hat{\alpha}).$$

Lemma 9. *Suppose that [A1]-[A5], [E2], [E3] and [J1]-[J3] hold. Then, for all $L > 0$,*

$$\sup_{v\in[-L,L]}|\mathcal{D}_n^\beta(v)| \xrightarrow{\mathbf{P}} 0$$

as $n \to \infty$.

Proof. It is enough to show $\sup_{v\in[0,L]}|\mathcal{D}_n^\beta(v)| \xrightarrow{\mathbf{P}} 0$. We have from the Taylor expansion, [E2], [E3], [J3] and $\mathbb{E}_{\beta_2^*}[|\partial_\beta^{\mathrm{m}_3}G_i(\beta|\alpha^*)|] \lesssim h_n^{1/2}$ the result that

$$G_i(\hat{\beta}_k|\alpha^*) = \sum_{j=0}^{\mathrm{m}_3-1}\frac{1}{j!}\partial_\beta^j G_i(\beta_k^*|\alpha^*) \otimes (\hat{\beta}_k - \beta_k^*)^{\otimes j} + R_{i-1}(h_n^{1/2}T^{-\mathrm{m}_3/2}).$$

Since $\partial_\alpha G_i(\hat{\beta}_1|\alpha) - \partial_\alpha G_i(\hat{\beta}_2|\alpha) = O_{\mathbf{P}}(\vartheta_\beta)$ and [J3], we see

$$\mathcal{D}_n^\beta(v) = \sum_{i=[n\tau_*^\beta]+1}^{[n\tau_*^\beta+v/h_n\vartheta_\beta^2]}\sum_{j=1}^{\mathrm{m}_3-1}\frac{1}{j!}\left(\partial_\beta^j G_i(\beta_1^*|\alpha^*) \otimes (\hat{\beta}_1 - \beta_1^*)^{\otimes j} - \partial_\beta^j G_i(\beta_2^*|\alpha^*) \otimes (\hat{\beta}_2 - \beta_2^*)^{\otimes j}\right) + \bar{o}_{\mathbf{P}}(1). \tag{B.55}$$

Because of $\mathbb{E}_{\beta_2^*}[|\partial_\beta^j G_i(\beta|\alpha^*)|^2] \lesssim h_n$,

$$\sum_{i=[n\tau_*^\beta]+1}^{[n\tau_*^\beta+v/h_n\vartheta_\beta^2]}\mathbb{E}_{\beta_2^*}[\partial_\beta G_i(\beta|\alpha^*)|\mathscr{G}_{i-1}^n] = -2h_n\sum_{i=[n\tau_*^\beta]+1}^{[n\tau_*^\beta+v/h_n\vartheta_\beta^2]}\Xi_{i-1}^\beta(\alpha^*, \beta_k^*) + \bar{o}_{\mathbf{P}}(1),$$

and $\mathbb{E}_{\beta_2^*}[\partial_\beta^j G_i(\beta|\alpha^*)] = O(h_n)$ for $j = 2, \dots, \mathrm{m}_3 - 1$, we have

$$\sum_{i=[n\tau_*^\beta]+1}^{[n\tau_*^\beta + v/h_n\vartheta_\beta^2]} \partial_\beta^j G_i(\beta_k^*|\alpha^*) \otimes (\hat{\beta}_k - \beta_k^*)^{\otimes j} = \bar{o}_{\mathbf{P}}(1). \tag{B.56}$$

Therefore, we obtain from (B.55) and (B.56) the desired convergence. □

Lemma 10. *Suppose that [A1]-[A5], [E2], [E3] and [J1]-[J3] hold. Then, for all $L > 0$,*

$$\mathbb{G}_n(v) \xrightarrow{\mathrm{w}} \mathbb{G}(v : \alpha^*) \text{ in } \mathbb{D}[-L, L]$$

as $n \to \infty$.

Proof. We show $\mathbb{G}_n(v) \xrightarrow{\mathrm{w}} \mathbb{G}(v)$ in $\mathbb{D}[0, L]$. Let

$$\mathbb{G}_{1,n}(v) = \sum_{i=[n\tau_*^\beta]+1}^{[n\tau_*^\beta + v/h_n\vartheta_\beta^2]} \partial_\beta G_i(\beta_2^*|\alpha^*)(\beta_1^* - \beta_2^*), \quad \mathbb{G}_{2,n}(v) = \frac{1}{2} \sum_{i=[n\tau_*^\beta]+1}^{[n\tau_*^\beta + v/h_n\vartheta_\beta^2]} \partial_\beta^2 G_i(\beta_2^*|\alpha^*) \otimes (\beta_1^* - \beta_2^*)^{\otimes 2}.$$

It follows from

$$\sum_{i=[n\tau_*^\beta]+1}^{[n\tau_*^\beta + v/h_n\vartheta_\beta^2]} \partial_\beta^j G_i(\beta_2^*|\alpha^*) \otimes (\beta_1^* - \beta_2^*)^{\otimes j} = \bar{o}_{\mathbf{P}}(1)$$

for $j \geq 3$ and [J3] that $\mathbb{G}_n(v) = \mathbb{G}_{1,n}(v) + \mathbb{G}_{2,n}(v) + \bar{o}_{\mathbf{P}}(1)$. Since $\mathbb{E}_{\beta_2^*}[\partial_\beta G_i(\beta_2^*|\alpha^*)|\mathscr{G}_{i-1}^n] = R_{i-1}(h_n^2)$,

$$\begin{aligned}
&\mathbb{E}_{\beta_2^*}\Big[\Big((\partial_\beta G_i(\beta_2^*|\alpha^*) - \mathbb{E}_{\beta_2^*}[\partial_\beta G_i(\beta_2^*|\alpha^*)|\mathscr{G}_{i-1}^n])(\beta_1^* - \beta_2^*)\Big)^2 \Big|\mathscr{G}_{i-1}^n\Big] \\
&\quad = 4h_n \Xi_{i-1}^\beta(\alpha^*, \beta_2^*) \otimes (\beta_1^* - \beta_2^*)^{\otimes 2} + R_{i-1}(h_n^2\vartheta_\beta^2), \\
&\mathbb{E}_{\beta_2^*}\Big[\Big((\partial_\beta G_i(\beta_2^*|\alpha^*) - \mathbb{E}_{\beta_2^*}[\partial_\beta G_i(\beta_2^*|\alpha^*)|\mathscr{G}_{i-1}^n])(\beta_1^* - \beta_2^*)\Big)^4 \Big|\mathscr{G}_{i-1}^n\Big] = R_{i-1}(h_n\vartheta_\beta^4),
\end{aligned}$$

we have

$$\sum_{i=[n\tau_*^\beta]+1}^{[n\tau_*^\beta + v/h_n\vartheta_\beta^2]} \mathbb{E}_{\beta_2^*}[\partial_\beta G_i(\beta_2^*|\alpha^*)|\mathscr{G}_{i-1}^n](\beta_1^* - \beta_2^*) = \bar{o}_{\mathbf{P}}(1),$$

$$\sum_{i=[n\tau_*^\beta]+1}^{[n\tau_*^\beta + v/h_n\vartheta_\beta^2]} \mathbb{E}_{\beta_2^*}\Big[\Big((\partial_\beta G_i(\beta_2^*|\alpha^*) - \mathbb{E}_{\beta_2^*}[\partial_\beta G_i(\beta_2^*|\alpha^*)|\mathscr{G}_{i-1}^n])(\beta_1^* - \beta_2^*)\Big)^2 \Big|\mathscr{G}_{i-1}^n\Big] \xrightarrow{\mathbf{P}} 4\mathcal{J}_\beta v$$

and

$$\sum_{i=[n\tau_*^\beta]+1}^{[n\tau_*^\beta + v/h_n\vartheta_\beta^2]} \mathbb{E}_{\beta_2^*}\Big[\Big((\partial_\beta G_i(\beta_2^*|\alpha^*) - \mathbb{E}_{\beta_2^*}[\partial_\beta G_i(\beta_2^*|\alpha^*)|\mathscr{G}_{i-1}^n])(\beta_1^* - \beta_2^*)\Big)^4 \Big|\mathscr{G}_{i-1}^n\Big] \xrightarrow{\mathbf{P}} 0.$$

Hence, it follows that

$$\mathbb{G}_{1,n}(v) \xrightarrow{\mathrm{w}} -2\mathcal{J}_\beta^{1/2}\mathcal{W}(v) \quad \text{in } \mathbb{D}[0,L]. \tag{B.57}$$

Besides, from Theorem 2.11 of Hall and Heyde (1980), $\mathbb{E}_{\beta_2^*}[|\partial_\beta^2 G_i(\beta_2^*|\alpha^*)|] \lesssim h_n$, $\mathbb{E}_{\beta_2^*}[\partial_\beta^2 G_i(\beta_2^*|\alpha^*)|\mathscr{G}_{i-1}^n] = 2h_n \Xi_{i-1}^\beta(\alpha^*,\beta_2^*) + o_{\mathbf{P}}(h_n\vartheta_\beta^2)$ and [J1](a), we see

$$\sup_{v\in[0,L]}\left|\sum_{i=[n\tau_*^\beta]+1}^{[n\tau_*^\beta+v/h_n\vartheta_\beta^2]}\left(\partial_\beta^2 G_i(\beta_2^*|\alpha^*) - \mathbb{E}_{\beta_2^*}[\partial_\beta^2 G_i(\beta_2^*|\alpha^*)|\mathscr{G}_{i-1}^n]\right)\otimes(\beta_1^*-\beta_2^*)^{\otimes 2}\right| \xrightarrow{\mathbf{P}} 0 \tag{B.58}$$

and

$$\sup_{v\in[0,L]}\left|\sum_{i=[n\tau_*^\beta]+1}^{[n\tau_*^\beta+v/h_n\vartheta_\beta^2]}\mathbb{E}_{\beta_2^*}[\partial_\beta^2 G_i(\beta_2^*|\alpha^*)|\mathscr{G}_{i-1}^n]\otimes(\beta_1^*-\beta_2^*)^{\otimes 2} - 2\mathcal{J}_\beta v\right| \xrightarrow{\mathbf{P}} 0. \tag{B.59}$$

Therefore we have from (B.58) and (B.59) the result that $\sup_{v\in[0,L]}|\mathbb{G}_{2,n}(v)-\mathcal{J}_\beta v| \xrightarrow{\mathbf{P}} 0$, which together with (B.57) yields the desired result. □

Proof of Theorem 5. (1) According to Lemmas 6, 9 and 10, it is sufficient to prove

$$\overline{\lim_{M\to\infty}}\ \overline{\lim_{n\to\infty}}\ \mathbf{P}(T\vartheta_\beta^2(\hat{\tau}_n^\beta - \tau_*^\beta) > M) = 0. \tag{B.60}$$

Noting that

$$\begin{aligned} G_i(\beta_1|\alpha) - G_i(\beta_2|\alpha) &= G_i(\beta_1|\alpha) - G_i(\beta_2|\alpha) - \mathbb{E}_{\beta_2^*}[G_i(\beta_1|\alpha) - G_i(\beta_2|\alpha)|\mathscr{G}_{i-1}^n] \\ &\quad + h_n \mathrm{Tr}\left(A_{i-1}^{-1}(\alpha)\,(b_{i-1}(\beta_1) - b_{i-1}(\beta_2))^{\otimes 2}\right) \\ &\quad + 2\mathrm{Tr}\left(A_{i-1}^{-1}(\alpha)\left(hb_{i-1}(\beta_2) - \mathbb{E}_{\beta_2^*}[\Delta_i X|\mathscr{G}_{i-1}^n]\right)(b_{i-1}(\beta_1) - b_{i-1}(\beta_2))^\top\right), \end{aligned}$$

we see that for $\tau > \tau_*^\beta$,

$$\begin{aligned} &\Psi_n(\tau:\beta_1,\beta_2|\alpha) - \Psi_n(\tau_*^\beta:\beta_1,\beta_2|\alpha) \\ &= \sum_{i=[n\tau_*^\beta]+1}^{[n\tau]}\left(G_i(\beta_1|\alpha) - G_i(\beta_2|\alpha) - \mathbb{E}_{\beta_2^*}[G_i(\beta_1|\alpha) - G_i(\beta_2|\alpha)|\mathscr{G}_{i-1}^n]\right) \\ &\quad + h_n \sum_{i=[n\tau_*^\beta]+1}^{[n\tau]}\mathrm{Tr}\left(A_{i-1}^{-1}(\alpha)\,(b_{i-1}(\beta_1) - b_{i-1}(\beta_2))^{\otimes 2}\right) \\ &\quad + 2\sum_{i=[n\tau_*^\beta]+1}^{[n\tau]}\mathrm{Tr}\left(A_{i-1}^{-1}(\alpha)\left(h_n b_{i-1}(\beta_2) - \mathbb{E}_{\beta_2^*}[\Delta_i X|\mathscr{G}_{i-1}^n]\right)(b_{i-1}(\beta_1) - b_{i-1}(\beta_2))^\top\right) \\ &=: \mathcal{M}_n^\beta(\tau:\beta_1,\beta_2|\alpha) + \mathcal{A}_n^\beta(\tau:\beta_1,\beta_2|\alpha) + \varrho_n^\beta(\tau:\beta_1,\beta_2|\alpha). \end{aligned}$$

Let $M \geq 1$, $D^{\beta}_{n,M} = \{\tau \in [0,1] | T\vartheta^2_{\beta}(\tau - \tau^{\beta}_*) > M\}$. For all $\delta > 0$, we have

$$\begin{aligned}\mathbf{P}\Big(T\vartheta^2_{\beta}(\hat{\tau}^{\beta}_n - \tau^{\beta}_*) > M\Big) &\leq \mathbf{P}\left(\sup_{\tau \in D^{\beta}_{n,M}} \frac{|\mathcal{M}^{\beta}_n(\tau : \hat{\beta}_1, \hat{\beta}_2 | \hat{\alpha})|}{h_n \vartheta^2_{\beta}([n\tau] - [n\tau^{\beta}_*])} \geq \delta\right) + \mathbf{P}\left(\inf_{\tau \in D^{\beta}_{n,M}} \frac{\mathcal{A}^{\beta}_n(\tau : \hat{\beta}_1, \hat{\beta}_2 | \hat{\alpha})}{h_n \vartheta^2_{\beta}([n\tau] - [n\tau^{\beta}_*])} \leq 2\delta\right) \\ &\quad + \mathbf{P}\left(\sup_{\tau \in D^{\beta}_{n,M}} \frac{|\varrho^{\beta}_n(\tau : \hat{\beta}_1, \hat{\beta}_2 | \hat{\alpha})|}{h_n \vartheta^2_{\beta}([n\tau] - [n\tau^{\beta}_*])} \geq \delta\right) \\ &=: P^{\beta}_{1,n} + P^{\beta}_{2,n} + P^{\beta}_{3,n}.\end{aligned}$$

Let $\epsilon > 0$ be an arbitrary number. In the same way as the proof of (1) of Theorem 2, we obtain $P^{\beta}_{1,n} \leq \gamma_{\beta}(M) + \epsilon$ for large n, where $\gamma_{\beta}(M) > 0$ satisfies $\gamma_{\beta}(M) \to 0$ as $M \to \infty$. Further, using [J1], [J2](b) and (c), we see that for $\hat{\alpha} \in O_{\alpha^*}$ and $\hat{\beta}_k \in O_{\beta_0}$,

$$\mathrm{Tr}\,(A^{-1}_{i-1}(\hat{\alpha})(b_{i-1}(\hat{\beta}_1) - b_{i-1}(\hat{\beta}_2))^{\otimes 2}) \geq (\lambda_1[\Xi^{\beta}_{i-1}(\alpha^*, \beta_0)] + r_{n,i-1})|\hat{\beta}_1 - \hat{\beta}_2|^2,$$

where $r_{n,i-1}$ satisfies

$$\sup_{\tau \in D^{\beta}_{n,M}} \left| \frac{1}{[n\tau] - [n\tau^{\beta}_*]} \sum_{i=[n\tau^{\beta}_*]+1}^{[n\tau]} r_{n,i-1} \right| = o_{\mathbf{P}}(1),$$

and therefore we estimate $P^{\beta}_{2,n} \leq \epsilon$ for some $\delta > 0$ and for large n. Since it follows from [E2], [E3] and $\hat{\beta}_1 - \hat{\beta}_2 = O_{\mathbf{P}}(\vartheta_{\beta})$ that for $\hat{\alpha} \in O_{\alpha^*}$ and $\hat{\beta}_k \in O_{\beta_0}$,

$$\begin{aligned}&\mathrm{Tr}\Big(A^{-1}_{i-1}(\hat{\alpha})(b_{i-1}(\hat{\beta}_2) - h_n\mathbb{E}[\Delta_i X | \mathscr{G}^n_{i-1}])(b_{i-1}(\hat{\beta}_1) - b_{i-1}(\hat{\beta}_2))^{\mathsf{T}}\Big) \\ &\leq h_n \Xi^{\beta}_{i-1}(\alpha^*, \beta_0) \otimes (\hat{\beta}_2 - \beta^*_2) \otimes (\hat{\beta}_1 - \hat{\beta}_2) + R_{i-1}\Big(\frac{h_n \vartheta^2_{\beta}}{\sqrt{T}} \vee h^2_n\Big),\end{aligned}$$

we have from [J1] and [J2](c) the result that

$$\begin{aligned}&\sup_{\tau \in D^{\beta}_{n,M}} \frac{|\varrho^{\beta}_n(\tau : \hat{\beta}_1, \hat{\beta}_2 | \hat{\alpha})|}{h_n \vartheta^2_{\beta}([n\tau] - [n\tau^{\beta}_*])} \\ &\leq \sup_{\tau \in D^{\beta}_{n,M}} \left| \frac{1}{[n\tau] - [n\tau^{\beta}_*]} \sum_{i=[n\tau^{\beta}_*]+1}^{[n\tau]} \Xi^{\beta}_{i-1}(\alpha^*, \beta_0) \right| |\hat{\beta}_2 - \beta^*_2| |\hat{\beta}_1 - \hat{\beta}_2| + O_{\mathbf{P}}(\sqrt{T}\vartheta^2_{\beta} \vee nh^2_n) \\ &= O_{\mathbf{P}}\Big(\frac{1}{\sqrt{T}\vartheta_{\beta}} \vee \sqrt{T}\vartheta^2_{\beta} \vee nh^2_n\Big) = o_{\mathbf{P}}(1),\end{aligned}$$

and we see $P^{\beta}_{3,n} \leq \epsilon$ for large n. From the estimations, we obtain (B.60).

(2) It is sufficient to control for the following probabilities for some $\delta > 0$.

$$P^{\beta}_{1,n} = \mathbf{P}\left(\sup_{\tau \in D^{\beta}_{n,M}} \frac{|\mathcal{M}^{\beta}_n(\tau : \hat{\beta}_1, \hat{\beta}_2 | \hat{\alpha})|}{h_n([n\tau] - [n\tau^{\beta}_*])} \geq \delta\right), \quad P^{\beta}_{2,n} = \mathbf{P}\left(\inf_{\tau \in D^{\beta}_{n,M}} \frac{\mathcal{A}^{\beta}_n(\tau : \hat{\beta}_1, \hat{\beta}_2 | \hat{\alpha})}{h_n([n\tau] - [n\tau^{\beta}_*])} \leq 2\delta\right),$$

$$P_{3,n}^{\beta}=\mathbf{P}\left(\sup_{\tau\in D_{n,M}^{\beta}}\frac{|\varrho_n^{\beta}(\tau:\hat\beta_1,\hat\beta_2|\hat\alpha)|}{h_n([n\tau]-[n\tau_*^{\beta}])}\geq\delta\right),$$

where $M\geq 1$, $D_{n,M}^{\beta}=\{\tau\in[0,1]|T(\tau-\tau_*^{\beta})>M\}$.

Let $\epsilon>0$. In the same way as the proof of (2) of Theorem 2, we have $P_{1,n}^{\beta}\leq\gamma_\beta(M)+\epsilon$ and $P_{2,n}^{\beta}\leq\epsilon$ for large n and $\gamma_\beta(M)$ such that $\gamma_\beta(M)\to 0$ as $M\to\infty$ by Lemma 2 of Iacus and Yoshida (2012), [E2], [E3], [K1] and [K2](a). Since

$$\begin{aligned}&\mathrm{Tr}\left(A_{i-1}^{-1}(\hat\alpha)(b_{i-1}(\hat\beta_2)-h_n\mathbb{E}_{\beta_2^*}[\Delta_iX|\mathscr{G}_{i-1}^n])(b_{i-1}(\hat\beta_1)-b_{i-1}(\hat\beta_2))^{\mathsf{T}}\right)\\&\leq h_n\sup_{x,\alpha,\beta_k}\left|[\partial_{\beta^l}b(x,\beta_1)^{\mathsf{T}}A^{-1}(x,\alpha)(b(x,\beta_2)-b(x,\beta_3))]_l\right||\hat\beta_2-\beta_2^*|+R_{i-1}(h_n^2)\end{aligned}$$

for $\hat\alpha\in O_{\alpha^*}$ and $\hat\beta_k\in O_{\beta_k^*}$, we find from [E2], [E3] and [K2](b) that

$$\sup_{\tau\in D_{n,M}^{\beta}}\frac{|\varrho_n^{\beta}(\tau:\hat\beta_1,\hat\beta_2)|}{h_n([n\tau]-[n\tau_*^{\beta}])}=o_{\mathbf{P}}(1),$$

which yields $P_{3,n}^{\beta}\leq\epsilon$ for large n. Thus, we obtain the desired result. □

Proofs of Theorems 6-9

Lemma 11. *Let $0\leq\tau_1<\tau_2\leq 1$, where τ_1,τ_2 may depend on n. Let $\{r_n\}_{n=1}^{\infty}$ be a sequence with $r_n^2([n\tau_2]-[n\tau_1])h_n\to 0$, and $\{\mathcal{M}_i\}_{i=1}^n$ be a martingale with $\mathbb{E}[|\mathcal{M}_i|^2]\lesssim h_n$. If $[n\tau_1]<k_n\leq[n\tau_2]$ and $[n\tau_1]\leq l_n<[n\tau_2]$ on Ω_n with $\mathbf{P}(\Omega_n)\to 1$, then*

$$r_n\left|\sum_{i=[n\tau_1]+1}^{k_n}\mathcal{M}_i\right|=o_{\mathbf{P}}(1),\quad r_n\left|\sum_{i=l_n+1}^{[n\tau_2]}\mathcal{M}_i\right|=o_{\mathbf{P}}(1).\tag{B.61}$$

Proof. Let $\mathcal{S}_n=r_n|\sum_{i=[n\tau_1]+1}^{k_n}\mathcal{M}_i|$. For all $\epsilon>0$,

$$\mathbf{P}(\mathcal{S}_n>\epsilon)\leq\mathbf{P}(\mathcal{S}_n>\epsilon,\Omega_n)+\mathbf{P}(\Omega_n^{\mathrm{c}})\leq\epsilon^{-2}\mathbb{E}[\mathcal{S}_n^2:\Omega_n]+\mathbf{P}(\Omega_n^{\mathrm{c}}).\tag{B.62}$$

From the Burkholder inequality, we have

$$\begin{aligned}\mathbb{E}[\mathcal{S}_n^2:\Omega_n]&\leq r_n^2\mathbb{E}\left[\max_{[n\tau_1]<k\leq[n\tau_2]}\left|\sum_{i=[n\tau_1]+1}^{k}\mathcal{M}_i\right|^2\right]\lesssim r_n^2\sum_{i=[n\tau_1]+1}^{[n\tau_2]}\mathbb{E}[|\mathcal{M}_i|^2]\\&=O(r_n^2([n\tau_2]-[n\tau_1])h_n)=o(1).\end{aligned}\tag{B.63}$$

Therefore, we see from (B.62), (B.63) and $\mathbf{P}(\Omega_n^{\mathrm{c}})\to 0$ that the first part of (B.61). According to

$$\left|\sum_{i=l_n+1}^{[n\tau_2]}\mathcal{M}_i\right|^2\leq 2\left(\left|\sum_{i=[n\tau_1]+1}^{[n\tau_2]}\mathcal{M}_i\right|^2+\left|\sum_{i=[n\tau_1]+1}^{l_n}\mathcal{M}_i\right|^2\right),$$

the second part of (B.61) is obtained in the same way. □

Let $\theta_k = (\alpha_k^*, \beta_k)$, $\tau_n^L = \tau_*^\alpha - 2n^{-\epsilon_1}$, $\tau_n^U = \tau_*^\alpha + 2n^{-\epsilon_1}$, $m_n = [n\tau_n^L]$ and $M_n = [n\tau_n^U]$.

Proof of Theorem 6. (1) Define

$$\mathcal{L}_{1,n}^{(1)} = \frac{1}{\sqrt{d\underline{\tau}_n T}} \max_{1\le k\le m_n} \left| \sum_{i=1}^{k} \hat{\xi}_{1,i} - \frac{k}{[n\underline{\tau}_n]} \sum_{i=1}^{[n\underline{\tau}_n]} \hat{\xi}_{1,i} \right|, \quad \mathcal{U}_{1,n}^{(1)} = \frac{1}{\sqrt{d\underline{\tau}_n T}} \max_{1\le k\le [n\tau_*^\alpha]} \left| \sum_{i=1}^{k} \hat{\xi}_{1,i} - \frac{k}{[n\underline{\tau}_n]} \sum_{i=1}^{[n\underline{\tau}_n]} \hat{\xi}_{1,i} \right|$$

and $D_n = \{n^{\epsilon_1}|\hat{\tau}_n^\alpha - \tau_*^\alpha| \le 1\}$. Note that the probability of D_n converges to one from [E4], and $m_n \le [n\underline{\tau}_n] \le [n\tau_*^\alpha] \le [n\overline{\tau}_n] \le M_n$ on D_n. Since $\mathcal{L}_{1,n}^{(1)} \le \mathcal{T}_{1,n}^{(1)} \le \mathcal{U}_{1,n}^{(1)}$ on D_n, if

$$\mathcal{L}_{1,n}^{(1)} \stackrel{\mathrm{d}}{\to} \sup_{0\le s\le 1} |\mathbb{B}_1^0(s)|, \tag{B.64}$$

$$\mathcal{U}_{1,n}^{(1)} \stackrel{\mathrm{d}}{\to} \sup_{0\le s\le 1} |\mathbb{B}_1^0(s)|, \tag{B.65}$$

then it follows that for any $x \in \mathbb{R}$,

$$\lim_{n\to\infty} \mathbf{P}(\mathcal{T}_{1,n}^{(1)} \le x, D_n) = \mathbf{P}\left(\sup_{0\le s\le 1} |\mathbb{B}_1^0(s)| \le x \right). \tag{B.66}$$

Hence, we obtain from (B.66) and $\mathbf{P}(D_n) \to 1$ the result that the desired result. From the above, it suffices to show (B.64) and (B.65).

We first show (B.64). It can be expressed as

$$\begin{aligned} &\sum_{i=1}^{k} \hat{\xi}_{1,i} - \frac{k}{[n\underline{\tau}_n]} \sum_{i=1}^{[n\underline{\tau}_n]} \hat{\xi}_{1,i} \\ &= \sum_{i=1}^{k} \hat{\xi}_{1,i} - \frac{k}{m_n} \sum_{i=1}^{m_n} \hat{\xi}_{1,i} + \frac{k}{m_n}\left(1 - \frac{m_n}{[n\underline{\tau}_n]}\right) \sum_{i=1}^{m_n} \hat{\xi}_{1,i} - \frac{k}{[n\underline{\tau}_n]} \sum_{i=m_n+1}^{[n\underline{\tau}_n]} \hat{\xi}_{1,i}. \end{aligned} \tag{B.67}$$

Let $\xi_{k,i} = 1_d^\top a_{i-1}^{-1}(\alpha_k^*)(\Delta_i X - h_n b_{i-1}(\beta_k))$, $\mathcal{M}_{k,i} = \xi_{k,i} - \mathbb{E}_{\theta_k}[\xi_{k,i}|\mathscr{G}_{i-1}^n]$. Note that

$$\hat{\xi}_{k,i} = \xi_{k,i} + R_{i-1}\left(\sqrt{\frac{h_n}{n}} \right) = \mathcal{M}_{k,i} + R_{i-1}\left(\sqrt{\frac{h_n}{n}} \right) \tag{B.68}$$

under [E1] and $[\mathrm{B2}']_1$. Since

$$\begin{aligned} &\frac{1}{\sqrt{T}} \max_{1\le k\le m_n} \left| \frac{k}{m_n}\left(1 - \frac{m_n}{[n\underline{\tau}_n]}\right) \sum_{i=1}^{m_n} \hat{\xi}_{1,i} \right| \\ &\le \frac{m_n}{[n\underline{\tau}_n]} n^{\epsilon_1 - 1} |[n\underline{\tau}_n] - m_n| \left(\frac{n^{1-\epsilon_1}}{\sqrt{T} m_n} \left| \sum_{i=1}^{m_n} \mathcal{M}_{1,i} \right| + \frac{n^{1-\epsilon_1}}{\sqrt{T}} O_{\mathbf{P}}\left(\sqrt{\frac{h_n}{n}} \right) \right), \end{aligned}$$

$\frac{m_n}{[n\underline{\tau}_n]} n^{\epsilon_1 - 1} |[n\underline{\tau}_n] - m_n| = O_{\mathbf{P}}(1)$, $\frac{n^{1-\epsilon_1}}{\sqrt{T}} \sqrt{\frac{h_n}{n}} = n^{-\epsilon_1} \to 0$, $\mathbb{E}_{\theta_1}[\mathcal{M}_{1,i}^2] \lesssim h_n$ and

$$\frac{n^{2-2\epsilon_1} m_n h_n}{T m_n^2} = O(n^{-2\epsilon_1}) = o(1),$$

we have from Lemma 11 the result that

$$\frac{1}{\sqrt{T}} \max_{1\le k\le m_n} \left| \frac{k}{m_n}\left(1 - \frac{m_n}{[n\underline{\tau}_n]}\right) \sum_{i=1}^{m_n} \hat{\xi}_{1,i} \right| = o_{\mathbf{P}}(1). \tag{B.69}$$

In the same way, we have

$$\frac{1}{\sqrt{T}} \max_{1\le k\le m_n} \left| \frac{k}{[n\underline{\tau}_n]} \sum_{i=m_n+1}^{[n\underline{\tau}_n]} \hat{\xi}_{1,i} \right| = o_{\mathbf{P}}(1). \tag{B.70}$$

Therefore, we obtain from (B.67), (B.69), (B.70) and (1) of Theorem 3 the result that

$$\mathcal{L}_{1,n}^{(1)} = \frac{1}{\sqrt{d\underline{\tau}_n T}} \max_{1\le k\le m_n} \left| \sum_{i=1}^{k} \hat{\xi}_{1,i} - \frac{k}{m_n} \sum_{i=1}^{m_n} \hat{\xi}_{1,i} \right| + o_{\mathbf{P}}(1) \xrightarrow{\mathrm{d}} \sup_{0\le s\le 1} |\mathbb{B}_1^0(s)|,$$

which concludes the proof of (B.64). Similarly, (B.65) can be shown.

(2) Let

$$\mathcal{L}_{1,n}^{(2)} = \frac{1}{\sqrt{d(1-\overline{\tau}_n)T}} \max_{1\le k\le n-M_n} \left| \sum_{i=[n\overline{\tau}_n]+1}^{[n\overline{\tau}_n]+k} \hat{\xi}_{2,i} - \frac{k}{n-[n\overline{\tau}_n]} \sum_{i=[n\overline{\tau}_n]+1}^{n} \hat{\xi}_{2,i} \right|,$$

$$\mathcal{U}_{1,n}^{(2)} = \frac{1}{\sqrt{d(1-\overline{\tau}_n)T}} \max_{1\le k\le n-[n\tau_*^{\alpha}]} \left| \sum_{i=[n\overline{\tau}_n]+1}^{[n\overline{\tau}_n]+k} \hat{\xi}_{2,i} - \frac{k}{n-[n\overline{\tau}_n]} \sum_{i=[n\overline{\tau}_n]+1}^{n} \hat{\xi}_{2,i} \right|.$$

Since $\mathcal{L}_{1,n}^{(2)} \le \mathcal{T}_{1,n}^{(2)} \le \mathcal{U}_{1,n}^{(2)}$ on D_n, it is enough to show

$$\mathcal{L}_{1,n}^{(2)} \xrightarrow{\mathrm{d}} \sup_{0\le s\le 1} |\mathbb{B}_1^0(s)|, \tag{B.71}$$

$$\mathcal{U}_{1,n}^{(2)} \xrightarrow{\mathrm{d}} \sup_{0\le s\le 1} |\mathbb{B}_1^0(s)|. \tag{B.72}$$

We verify (B.71). It follows that

$$\begin{aligned}
&\sum_{i=[n\overline{\tau}_n]+1}^{[n\overline{\tau}_n]+k} \hat{\xi}_{2,i} - \frac{k}{n-[n\overline{\tau}_n]} \sum_{i=[n\overline{\tau}_n]+1}^{n} \hat{\xi}_{2,i} \\
&= \sum_{i=M_n+1}^{M_n+k} \hat{\xi}_{2,i} - \frac{k}{n-M_n} \sum_{i=M_n+1}^{n} \hat{\xi}_{2,i} - \sum_{i=[n\overline{\tau}_n]+k+1}^{M_n+k} \hat{\xi}_{2,i} \\
&\qquad + \frac{k}{n-M_n}\left(1 - \frac{n-M_n}{n-[n\overline{\tau}_n]}\right) \sum_{i=M_n+1}^{n} \hat{\xi}_{2,i} + \left(1 - \frac{k}{n-[n\overline{\tau}_n]}\right) \sum_{i=[n\overline{\tau}_n]+1}^{M_n} \hat{\xi}_{2,i}.
\end{aligned} \tag{B.73}$$

It also follows from (B.68), $\mathbb{E}_{\theta_2}[\mathcal{M}_{2,i}^2] \lesssim h_n$ and Lemma 11 that

$$\frac{1}{\sqrt{T}} \max_{1\le k\le n-M_n} \left| \frac{k}{n-M_n}\left(1 - \frac{n-M_n}{n-[n\overline{\tau}_n]}\right) \sum_{i=M_n+1}^{n} \hat{\xi}_{2,i} \right| = o_{\mathbf{P}}(1), \tag{B.74}$$

$$\frac{1}{\sqrt{T}} \max_{1\le k\le n-M_n} \left| \left(1 - \frac{k}{n-[n\bar{\tau}_n]}\right) \sum_{i=[n\bar{\tau}_n]+1}^{M_n} \hat{\xi}_{2,i} \right| = o_{\mathbf{P}}(1). \tag{B.75}$$

Furthermore, we have

$$\frac{1}{\sqrt{T}} \max_{1\le k\le n-M_n} \left| \sum_{i=[n\bar{\tau}_n]+k+1}^{M_n+k} \hat{\xi}_{2,i} \right| \le \frac{1}{\sqrt{T}} \max_{1\le k\le n-M_n} \left| \sum_{i=[n\bar{\tau}_n]+k+1}^{M_n+k} \mathcal{M}_{2,i} \right| + o_{\mathbf{P}}(1) =: Q_n + o_{\mathbf{P}}(1). \tag{B.76}$$

For all $\epsilon > 0$,

$$\mathbf{P}(Q_n > 2\epsilon) \le \mathbf{P}(Q_n > 2\epsilon, D_n) + \mathbf{P}(D_n^c). \tag{B.77}$$

Here the first term on the right hand side can be transformed as follows.

$$\begin{aligned}\mathbf{P}(Q_n > 2\epsilon, D_n) &\le \mathbf{P}\left(\frac{1}{\sqrt{T}} \max_{[n\tau_*^\alpha]\le l<M_n} \max_{1\le k\le n-M_n} \left| \sum_{i=l+k+1}^{M_n+k} \mathcal{M}_{2,i} \right| > 2\epsilon, D_n \right) \\ &\le \mathbf{P}\left(\frac{1}{\sqrt{T}} \max_{1\le k\le n-M_n} \left| \sum_{i=[n\tau_*^\alpha]+k+1}^{M_n+k} \mathcal{M}_{2,i} \right| > \epsilon \right) \\ &\quad + \mathbf{P}\left(\frac{1}{\sqrt{T}} \max_{[n\tau_*^\alpha]<l\le M_n} \max_{1\le k\le n-M_n} \left| \sum_{i=[n\tau_*^\alpha]+k+1}^{l+k} \mathcal{M}_{2,i} \right| > \epsilon \right).\end{aligned} \tag{B.78}$$

We choose $r > \frac{2-\epsilon_1}{\epsilon_1}$. Noting that $\epsilon_1 r > 2 - \epsilon_1 > 1$, we see from Theorem 2.11 of Hall and Heyde (1980), the convex inequality and $\mathbb{E}_{\theta_2}[\mathcal{M}_{2,i}^{2r}] \lesssim h_n^r$ that

$$\begin{aligned}\mathbf{P}\left(\frac{1}{\sqrt{T}} \max_{1\le k\le n-M_n} \left| \sum_{i=[n\tau_*^\alpha]+k+1}^{M_n+k} \mathcal{M}_{2,i} \right| > \epsilon \right) &\le \sum_{k=1}^{n-M_n} \frac{1}{T^r\epsilon^{2r}} \mathbb{E}_{\theta_2}\left[\left| \sum_{i=[n\tau_*^\alpha]+k+1}^{M_n+k} \mathcal{M}_{2,i} \right|^{2r} \right] \\ &\lesssim \sum_{k=1}^{n-M_n} \frac{1}{T^r\epsilon^{2r}} \mathbb{E}_{\theta_2}\left[\left(\sum_{i=[n\tau_*^\alpha]+k+1}^{M_n+k} \mathcal{M}_{2,i}^2 \right)^r \right] \\ &\lesssim \sum_{k=1}^{n-M_n} \frac{[n^{1-\epsilon_1}]^r}{T^r\epsilon^{2r}} \frac{1}{[n^{1-\epsilon_1}]} \sum_{i=[n\tau_*^\alpha]+k+1}^{M_n+k} \mathbb{E}_{\theta_2}[\mathcal{M}_{2,i}^{2r}] \\ &= O(n^{1-\epsilon_1 r}) = o(1)\end{aligned} \tag{B.79}$$

and

$$\begin{aligned}\mathbf{P}\left(\frac{1}{\sqrt{T}} \max_{[n\tau_*^\alpha]<l\le M_n} \max_{1\le k\le n-M_n} \left| \sum_{i=[n\tau_*^\alpha]+k+1}^{l+k} \mathcal{M}_{2,i} \right| > \epsilon \right) &\lesssim \sum_{l=[n\tau_*^\alpha]+1}^{M_n} \sum_{k=1}^{n-M_n} \frac{1}{T^r\epsilon^{2r}} [n^{1-\epsilon_1}]^r h_n^r \\ &= O(n^{2-\epsilon_1-\epsilon_1 r}) = o(1).\end{aligned} \tag{B.80}$$

According to (B.77)-(B.80) and $\mathbf{P}(D_n^c) \to 0$, we have $Q_n = o_{\mathbf{P}}(1)$. Therefore, it follows from (B.76) that

$$\frac{1}{\sqrt{T}} \max_{1\le k\le n-M_n} \left| \sum_{i=[n\bar{\tau}_n]+k+1}^{M_n+k} \hat{\xi}_{2,i} \right| = o_{\mathbf{P}}(1). \tag{B.81}$$

Moreover, it follows from the proofs of (B.32) and (B.34) that under [B4],

$$\frac{1}{\sqrt{d(1-\tau_n^U)T}} \max_{1\le k\le n-M_n} \left| \sum_{i=M_n+1}^{M_n+k} \hat{\xi}_{2,i} - \frac{k}{n-M_n} \sum_{i=M_n+1}^{n} \hat{\xi}_{2,i} \right| \xrightarrow{\mathrm{d}} \sup_{0\le s\le 1} |\mathbb{B}_1^0(s)|. \tag{B.82}$$

Hence we obtain from (B.73)-(B.75), (B.81) and (B.82) the result that

$$\mathcal{L}_{1,n}^{(2)} = \frac{1}{\sqrt{d(1-\bar{\tau}_n)T}} \max_{1\le k\le n-M_n} \left| \sum_{i=M_n+1}^{M_n+k} \hat{\xi}_{2,i} - \frac{k}{n-M_n} \sum_{i=M_n+1}^{n} \hat{\xi}_{2,i} \right| + o_{\mathbf{P}}(1) \xrightarrow{\mathrm{d}} \sup_{0\le s\le 1} |\mathbb{B}_1^0(s)|.$$

Similarly, (B.72) can be shown.

(3) We prove that $\mathbf{P}(\mathcal{T}_{1,n}^{(1)} > w_1(\epsilon))$ converges to one as $n \to \infty$ under $H_1^{(1)}$.

(a) If we prove

$$\frac{1}{\tau_*^\beta T} \sum_{i=1}^{[n\tau_*^\beta]} \hat{\xi}_{1,i} \xrightarrow{\mathbf{P}} \mathcal{G}(\alpha_1^{(0)}, \beta_{1,1}^*, \beta_1'), \tag{B.83}$$

$$\frac{1}{(\underline{\tau}_n - \tau_*^\beta)T} \sum_{i=[n\tau_*^\beta]+1}^{[n\underline{\tau}_n]} \hat{\xi}_{1,i} \xrightarrow{\mathbf{P}} \mathcal{G}(\alpha_1^{(0)}, \beta_{1,2}^*, \beta_1'), \tag{B.84}$$

then it follows that

$$\frac{1}{\underline{\tau}_n T} \sum_{i=1}^{[n\underline{\tau}_n]} \hat{\xi}_{1,i} \xrightarrow{\mathbf{P}} \frac{\tau_*^\beta}{\tau_*^\alpha} \mathcal{G}(\alpha_1^{(0)}, \beta_{1,1}^*, \beta_1') + \left(1 - \frac{\tau_*^\beta}{\tau_*^\alpha}\right) \mathcal{G}(\alpha_1^{(0)}, \beta_{1,2}^*, \beta_1'),$$

and from $[\mathrm{H2}']_1$ that

$$\frac{1}{\underline{\tau}_n T} \sum_{i=1}^{[n\tau_*^\beta]} \hat{\xi}_{1,i} - \frac{[n\tau_*^\beta]}{[n\underline{\tau}_n]} \frac{1}{\underline{\tau}_n T} \sum_{i=1}^{[n\underline{\tau}_n]} \hat{\xi}_{1,i} \xrightarrow{\mathbf{P}} \frac{\tau_*^\beta}{\tau_*^\alpha}\left(1 - \frac{\tau_*^\beta}{\tau_*^\alpha}\right) (\mathcal{G}(\alpha_1^{(0)}, \beta_{1,1}^*, \beta_1') - \mathcal{G}(\alpha_1^{(0)}, \beta_{1,2}^*, \beta_1')) \neq 0.$$

Therefore, we have

$$\begin{aligned}
\mathcal{T}_{1,n}^{(1)} &\ge \frac{1}{\sqrt{d\underline{\tau}_n T}} \left| \sum_{i=1}^{[n\tau_*^\beta]} \hat{\xi}_{1,i} - \frac{[n\tau_*^\beta]}{[n\underline{\tau}_n]} \sum_{i=1}^{[n\underline{\tau}_n]} \hat{\xi}_{1,i} \right| \\
&= \sqrt{\frac{\underline{\tau}_n T}{d}} \left| \frac{1}{\underline{\tau}_n T} \sum_{i=1}^{[n\tau_*^\beta]} \hat{\xi}_{1,i} - \frac{[n\tau_*^\beta]}{[n\underline{\tau}_n]} \frac{1}{\underline{\tau}_n T} \sum_{i=1}^{[n\underline{\tau}_n]} \hat{\xi}_{1,i} \right| \xrightarrow{\mathbf{P}} \infty,
\end{aligned}$$

which implies $\mathbf{P}(\mathcal{T}^{(1)}_{1,n} > w_1(\epsilon)) \to 1$. (B.83) can be shown similarly to the first part of (B.40).

We show (B.84). It can be proved that

$$\frac{1}{(\tau^\alpha_* - \tau^\beta_*)T} \sum_{i=[n\tau^\beta_*]+1}^{[n\tau^\alpha_*]} \hat{\xi}_{1,i} \xrightarrow{\mathbf{P}} \mathcal{G}(\alpha^{(0)}_1, \beta^*_{1,2}, \beta'_1) \tag{B.85}$$

with the same argument as the second part of (B.40). We have

$$\begin{aligned}
\Delta_n &= \left| \frac{1}{(\underline{\tau}_n - \tau^\beta_*)T} \sum_{i=[n\tau^\beta_*]+1}^{[n\underline{\tau}_n]} \hat{\xi}_{1,i} - \frac{1}{(\tau^\alpha_* - \tau^\beta_*)T} \sum_{i=[n\tau^\beta_*]+1}^{[n\tau^\alpha_*]} \hat{\xi}_{1,i} \right| \\
&\le \left| \frac{n^{\epsilon_1}(\tau^\alpha_* - \underline{\tau}_n)}{(\underline{\tau}_n - \tau^\beta_*)(\tau^\alpha_* - \tau^\beta_*)} \right| \frac{n^{-\epsilon_1}}{T} \left| \sum_{i=[n\tau^\beta_*]+1}^{[n\underline{\tau}_n]} \hat{\xi}_{1,i} \right| + \frac{1}{(\tau^\alpha_* - \tau^\beta_*)T} \left| \sum_{i=[n\tau^\beta_*]+1}^{[n\underline{\tau}_n]} \hat{\xi}_{1,i} - \sum_{i=[n\tau^\beta_*]+1}^{[n\tau^\alpha_*]} \hat{\xi}_{1,i} \right| \\
&=: \left| \frac{n^{\epsilon_1}(\tau^\alpha_* - \underline{\tau}_n)}{(\underline{\tau}_n - \tau^\beta_*)(\tau^\alpha_* - \tau^\beta_*)} \right| \check{\mathcal{S}}_n + \frac{1}{\tau^\alpha_* - \tau^\beta_*} \check{\mathcal{Q}}_n.
\end{aligned}$$

If we show $\check{\mathcal{S}}_n \xrightarrow{\mathbf{P}} 0$ and $\check{\mathcal{Q}}_n \xrightarrow{\mathbf{P}} 0$, then we have from $n^{\epsilon_1}(\tau^\alpha_* - \underline{\tau}_n) = O_\mathbf{P}(1)$ and (B.85) the result that $\Delta_n \xrightarrow{\mathbf{P}} 0$ and (B.84). In the following, we prove them.

Set $\mathcal{Y}_{k,i} = \kappa^{-1}_{i-1}(\alpha^*_k)\Delta_i X$, $\mathcal{M}_{k,i} = \mathcal{Y}_{k,i} - \mathbb{E}_{\theta_k}[\mathcal{Y}_{k,i}|\mathscr{G}^n_{i-1}]$. We see from $\hat{\xi}_{k,i} = \mathcal{M}_{k,i} + R_{i-1}(h_n)$ that

$$\check{\mathcal{S}}_n = \frac{n^{-\epsilon_1}}{T} \left| \sum_{i=[n\tau^\beta_*]+1}^{[n\underline{\tau}_n]} \mathcal{M}_{1,i} \right| + O_\mathbf{P}\left(\frac{n^{1-\epsilon_1} h_n}{T} \right) =: \mathcal{S}_n + o_\mathbf{P}(1),$$

and

$$\check{\mathcal{Q}}_n = \frac{1}{T} \left| \sum_{i=[n\underline{\tau}_n]+1}^{[n\tau^\alpha_*]} \hat{\xi}_{1,i} \right| = \frac{1}{T} \left| \sum_{i=[n\underline{\tau}_n]+1}^{[n\tau^\alpha_*]} \mathcal{M}_{1,i} \right| + O_\mathbf{P}(n^{-\epsilon_1}) =: \mathcal{Q}_n + o_\mathbf{P}(1)$$

on D_n. We have from Lemma 11 the result that $\mathcal{S}_n \xrightarrow{\mathbf{P}} 0$ and $\mathcal{Q}_n \xrightarrow{\mathbf{P}} 0$. Hence, we obtain the desired results.

(b) According to

$$\mathcal{T}^{(1)}_{1,n} \ge \frac{1}{\sqrt{d\underline{\tau}_n T}} \left| \sum_{i=1}^{[n\tau^\beta_*]} \hat{\xi}_{1,i} - \frac{[n\tau^\beta_*]}{[n\underline{\tau}_n]} \sum_{i=1}^{[n\underline{\tau}_n]} \hat{\xi}_{1,i} \right| = \sqrt{\frac{T\vartheta^2_{\beta_1}}{d\underline{\tau}_n}} \left| \frac{1}{T\vartheta_{\beta_1}} \sum_{i=1}^{[n\tau^\beta_*]} \hat{\xi}_{1,i} - \frac{[n\tau^\beta_*]}{[n\underline{\tau}_n]} \frac{1}{T\vartheta_{\beta_1}} \sum_{i=1}^{[n\underline{\tau}_n]} \hat{\xi}_{1,i} \right|,$$

it is enough to prove that there exists $c \neq 0$ such that

$$\mathcal{K}^{(1)}_n = \frac{1}{T\vartheta_{\beta_1}} \sum_{i=1}^{[n\tau^\beta_*]} \hat{\xi}_{1,i} - \frac{[n\tau^\beta_*]}{[n\underline{\tau}_n]} \frac{1}{T\vartheta_{\beta_1}} \sum_{i=1}^{[n\underline{\tau}_n]} \hat{\xi}_{1,i} \xrightarrow{\mathbf{P}} c. \tag{B.86}$$

Note that there exists $c' \neq 0$ such that

$$\mathcal{K}_n^{(2)} = \frac{1}{T\vartheta_{\beta_1}} \sum_{i=1}^{[n\tau_*^\beta]} \hat{\xi}_{1,i} - \frac{[n\tau_*^\beta]}{[n\tau_*^\alpha]} \frac{1}{T\vartheta_{\beta_1}} \sum_{i=1}^{[n\tau_*^\alpha]} \hat{\xi}_{1,i} \xrightarrow{\mathbf{P}} c'$$

in the same way as the proof of (2) of Theorem 3 under $[\mathrm{I4}']_1$. Meanwhile, we see

$$\begin{aligned}
\Delta_n &= |\mathcal{K}_n^{(1)} - \mathcal{K}_n^{(2)}| \\
&\leq \frac{[n\tau_*^\beta]}{[n\underline{\tau}_n]} \frac{1}{T\vartheta_{\beta_1}} \left| \sum_{i=1}^{[n\underline{\tau}_n]} \hat{\xi}_{1,i} - \sum_{i=1}^{[n\tau_*^\alpha]} \hat{\xi}_{1,i} \right| + n^{\epsilon_1} \left| \frac{[n\tau_*^\beta]}{[n\underline{\tau}_n]} - \frac{[n\tau_*^\beta]}{[n\tau_*^\alpha]} \right| \frac{n^{-\epsilon_1}}{T\vartheta_{\beta_1}} \left| \sum_{i=1}^{[n\tau_*^\alpha]} \hat{\xi}_{1,i} \right| \\
&=: \frac{[n\tau_*^\beta]}{[n\underline{\tau}_n]} \check{\mathcal{Q}}_n + n^{\epsilon_1} \left| \frac{[n\tau_*^\beta]}{[n\underline{\tau}_n]} - \frac{[n\tau_*^\beta]}{[n\tau_*^\alpha]} \right| \check{\mathcal{S}}_n.
\end{aligned}$$

Here, it follows from $[\mathrm{I6}]_1$ that

$$\check{\mathcal{Q}}_n = \frac{1}{T\vartheta_{\beta_1}} \left| \sum_{i=[n\underline{\tau}_n]+1}^{[n\tau_*^\alpha]} \hat{\xi}_{1,i} \right| = \frac{1}{T\vartheta_{\beta_1}} \left| \sum_{i=[n\underline{\tau}_n]+1}^{[n\tau_*^\alpha]} \mathcal{M}_{1,i} \right| + O_{\mathbf{P}}\left(\frac{1}{n^{\epsilon_1}\vartheta_{\beta_1}} \right) =: \mathcal{Q}_n + o_{\mathbf{P}}(1)$$

and

$$\check{\mathcal{S}}_n = \frac{n^{-\epsilon_1}}{T\vartheta_{\beta_1}} \left| \sum_{i=1}^{[n\tau_*^\alpha]} \mathcal{M}_{1,i} \right| + O_{\mathbf{P}}\left(\frac{1}{n^{\epsilon_1}\vartheta_{\beta_1}} \right) =: \mathcal{S}_n + o_{\mathbf{P}}(1)$$

on D_n. Applying Lemma 11, we obtain $\mathcal{S}_n = o_{\mathbf{P}}(1)$ and $\mathcal{Q}_n = o_{\mathbf{P}}(1)$, that is, $\check{\mathcal{S}}_n = o_{\mathbf{P}}(1)$ and $\check{\mathcal{Q}}_n = o_{\mathbf{P}}(1)$. Consequently, we have from $n^{\epsilon_1}\left| \frac{[n\tau_*^\beta]}{[n\underline{\tau}_n]} - \frac{[n\tau_*^\beta]}{[n\tau_*^\alpha]} \right| = O_{\mathbf{P}}(1)$ the result that $\Delta_n \xrightarrow{\mathbf{P}} 0$ and thus (B.86).

Similarly, the consistency of the test $\mathcal{T}_{1,n}^{(2)}$ can be shown. □

Proof of Theorem 7. (1) Define

$$\begin{aligned}
\mathcal{L}_{2,n}^{(1)} &= \frac{1}{\sqrt{\underline{\tau}_n T}} \max_{1 \leq k \leq m_n} \left| \mathcal{I}_{1,n}^{-1/2} \left(\sum_{i=1}^{k} \hat{\zeta}_{1,i} - \frac{k}{[n\underline{\tau}_n]} \sum_{i=1}^{[n\underline{\tau}_n]} \hat{\zeta}_{1,i} \right) \right|, \\
\mathcal{Z}_{k,i}^{[j]} &= \frac{1}{j!} \partial_\beta^j \Big(\partial_\beta b_{i-1}(\beta)^\top A_{i-1}^{-1}(\alpha_k^*)(\Delta_i X - h b_{i-1}(\beta_k)) \Big)\Big|_{\beta=\beta_k}, \\
\mathcal{N}_{k,i}^{[j]} &= \mathcal{Z}_{k,i}^{[j]} - \mathbb{E}_{\theta_k}[\mathcal{Z}_{k,i}^{[j]} | \mathscr{G}_{i-1}^n].
\end{aligned}$$

Since $\mathbb{E}_{\theta_1}[|\mathcal{N}_{1,i}^{[j]}|^2] \lesssim h_n$ and

$$\hat{\zeta}_{k,i} = \sum_{j=0}^{\mathrm{m}_1 - 1} \frac{1}{T^{j/2}} \mathcal{N}_{k,i}^{[j]} \otimes \left(\sqrt{T}(\hat{\beta}_k - \beta_k) \right)^{\otimes j} + R_{i-1}\left(\sqrt{\frac{h_n}{n}} \right)$$

under [B3], it can be shown in the same way as (B.71) that

$$\mathcal{L}_{2,n}^{(1)} = \frac{1}{\sqrt{\underline{\tau}_n T}} \max_{1\le k\le m_n} \left| \mathcal{I}_{1,n}^{-1/2} \left(\sum_{i=1}^{k} \hat{\zeta}_{1,i} - \frac{k}{m_n} \sum_{i=1}^{m_n} \hat{\zeta}_{1,i} \right) \right| + o_{\mathbf{P}}(1) \xrightarrow{\mathrm{d}} \sup_{0\le s\le 1} |\mathbb{B}_q^0(s)|.$$

(2) Let

$$\mathcal{L}_{2,n}^{(2)} = \frac{1}{\sqrt{(1-\overline{\tau}_n)T}} \max_{1\le k\le n-M_n} \left| \mathcal{I}_{2,n}^{-1/2} \left(\sum_{i=[n\overline{\tau}_n]+1}^{[n\overline{\tau}_n]+k} \hat{\zeta}_{2,i} - \frac{k}{n-[n\overline{\tau}_n]} \sum_{i=[n\overline{\tau}_n]+1}^{n} \hat{\zeta}_{2,i} \right) \right|.$$

According to the proofs of (B.43), (B.45) and (B.47), it follows that under [B5],

$$\frac{1}{\sqrt{(1-\tau_n^U)T}} \max_{1\le k\le n-M_n} \left| \mathcal{I}_{2,n}^{-1/2} \left(\sum_{i=M_n+1}^{M_n+k} \hat{\zeta}_{2,i} - \frac{k}{n-M_n} \sum_{i=M_n+1}^{n} \hat{\zeta}_{2,i} \right) \right| \xrightarrow{\mathrm{d}} \sup_{0\le s\le 1} |\mathbb{B}_q^0(s)|.$$

Therefore, it can be shown in the same way as (B.71) that $\mathcal{L}_{2,n}^{(2)} \xrightarrow{\mathrm{d}} \sup_{0\le s\le 1} |\mathbb{B}_q^0(s)|$.

(3) We show that $\mathbf{P}(\mathcal{T}_{2,n}^{(1)} > w_1(\epsilon)) \to 1$ under $H_1^{(1)}$.

(a) If we prove

$$\frac{1}{\tau_*^{\beta} T} \sum_{i=1}^{[n\tau_*^{\beta}]} \hat{\zeta}_{1,i} \xrightarrow{\mathbf{P}} \mathcal{H}(\alpha_1^{(0)}, \beta_{1,1}^*, \beta_1'), \quad \frac{1}{(\underline{\tau}_n - \tau_*^{\beta})T} \sum_{i=[n\tau_*^{\beta}]+1}^{[n\underline{\tau}_n]} \hat{\zeta}_{1,i} \xrightarrow{\mathbf{P}} \mathcal{H}(\alpha_1^{(0)}, \beta_{1,2}^*, \beta_1'), \tag{B.87}$$

then it follows from $[\mathrm{H4}']_1$ that

$$\frac{1}{\underline{\tau}_n T} \sum_{i=1}^{[n\tau_*^{\beta}]} \hat{\zeta}_{1,i} - \frac{[n\tau_*^{\beta}]}{[n\underline{\tau}_n]} \frac{1}{\underline{\tau}_n T} \sum_{i=1}^{[n\underline{\tau}_n]} \hat{\zeta}_{1,i} \xrightarrow{\mathbf{P}} \frac{\tau_*^{\beta}}{\tau_*^{\alpha}} \left(1 - \frac{\tau_*^{\beta}}{\tau_*^{\alpha}}\right) (\mathcal{H}(\alpha_1^{(0)}, \beta_{1,1}^*, \beta_1') - \mathcal{H}(\alpha_1^{(0)}, \beta_{1,2}^*, \beta_1')) \neq 0,$$

and

$$\mathcal{T}_{2,n}^{(1)} \ge \sqrt{\underline{\tau}_n T} \left| \mathcal{I}_{1,n}^{-1/2} \left(\frac{1}{\underline{\tau}_n T} \sum_{i=1}^{[n\tau_*^{\beta}]} \hat{\zeta}_{1,i} - \frac{[n\tau_*^{\beta}]}{[n\underline{\tau}_n]} \frac{1}{\underline{\tau}_n T} \sum_{i=1}^{[n\underline{\tau}_n]} \hat{\zeta}_{1,i} \right) \right| \xrightarrow{\mathbf{P}} \infty.$$

The first part of (B.87) can be shown similarly to (B.54).

We show the second part of (B.87). It can be proved that

$$\frac{1}{(\tau_*^{\alpha} - \tau_*^{\beta})T} \sum_{i=[n\tau_*^{\beta}]+1}^{[n\tau_*^{\alpha}]} \hat{\zeta}_{1,i} \xrightarrow{\mathbf{P}} \mathcal{H}(\alpha_1^{(0)}, \beta_{1,2}^*, \beta_1').$$

We have

$$\Delta_n = \left| \frac{1}{(\underline{\tau}_n - \tau_*^{\beta})T} \sum_{i=[n\tau_*^{\beta}]+1}^{[n\underline{\tau}_n]} \hat{\zeta}_{1,i} - \frac{1}{(\tau_*^{\alpha} - \tau_*^{\beta})T} \sum_{i=[n\tau_*^{\beta}]+1}^{[n\tau_*^{\alpha}]} \hat{\zeta}_{1,i} \right|$$

$$\leq \left|\frac{n^{\epsilon_1}(\tau_*^\alpha - \underline{\tau}_n)}{(\underline{\tau}_n - \tau_*^\beta)(\tau_*^\alpha - \tau_*^\beta)}\right| \frac{n^{-\epsilon_1}}{T}\left|\sum_{i=[n\tau_*^\beta]+1}^{[n\underline{\tau}_n]} \hat{\zeta}_{1,i}\right| + \frac{1}{(\tau_*^\alpha - \tau_*^\beta)T}\left|\sum_{i=[n\tau_*^\beta]+1}^{[n\underline{\tau}_n]} \hat{\zeta}_{1,i} - \sum_{i=[n\tau_*^\beta]+1}^{[n\tau_*^\alpha]} \hat{\zeta}_{1,i}\right|$$

$$=: \left|\frac{n^{\epsilon_1}(\tau_*^\alpha - \underline{\tau}_n)}{(\underline{\tau}_n - \tau_*^\beta)(\tau_*^\alpha - \tau_*^\beta)}\right| \check{\mathcal{R}}_n + \frac{1}{\tau_*^\alpha - \tau_*^\beta}\check{\mathcal{V}}_n.$$

To show the second part of (B.87), we verify $\check{\mathcal{R}}_n \xrightarrow{\mathbf{P}} 0$ and $\check{\mathcal{V}}_n \xrightarrow{\mathbf{P}} 0$.

Let

$$\mathcal{Z}_{k,i}^{[j]} = \frac{1}{j!}\partial_\beta^j\Big(\partial_\beta b_{i-1}(\beta)^\top A_{i-1}^{-1}(\alpha_k^*)\Delta_i X\Big)\Big|_{\beta=\beta_k'}, \quad \mathcal{N}_{k,i}^{[j]} = \mathcal{Z}_{k,i}^{[j]} - \mathbb{E}_{\theta_k}[\mathcal{Z}_{k,i}^{[j]}|\mathcal{G}_{i-1}^n],$$

$$\mathcal{R}_n^{[j]} = \frac{n^{-\epsilon_1}}{T}\left|\sum_{i=[n\tau_*^\beta]+1}^{[n\underline{\tau}_n]} \mathcal{N}_{1,i}^{[j]}\right|, \quad \mathcal{V}_n^{[j]} = \frac{1}{T}\left|\sum_{i=[n\underline{\tau}_n]+1}^{[n\tau_*^\alpha]} \mathcal{N}_{1,i}^{[j]}\right|.$$

Since

$$\hat{\zeta}_{k,i} = \sum_{j=0}^{\mathrm{m}_1-1} \mathcal{N}_{k,i}^{[j]} \otimes (\hat{\beta}_k - \beta_k')^{\otimes j} + R_{i-1}(h_n)$$

under [B3] and $[\mathrm{H}3']_1$, we have

$$\check{\mathcal{R}}_n \leq \sum_{j=0}^{\mathrm{m}_1-1} \mathcal{R}_n^{[j]}|\hat{\beta}_1 - \beta_1'|^j + O_{\mathbf{P}}\left(\frac{n^{1-\epsilon_1}h_n}{T}\right) = \sum_{j=0}^{\mathrm{m}_1-1} \mathcal{R}_n^{[j]}|\hat{\beta}_1 - \beta_1'|^j + o_{\mathbf{P}}(1),$$

$$\check{\mathcal{V}}_n = \frac{1}{T}\left|\sum_{i=[n\underline{\tau}_n]+1}^{[n\tau_*^\alpha]} \hat{\zeta}_{1,i}\right| \leq \sum_{j=0}^{\mathrm{m}_1-1} \mathcal{V}_n^{[j]}|\hat{\beta}_1 - \beta_1'|^j + O_{\mathbf{P}}(n^{-\epsilon_1}) = \sum_{j=0}^{\mathrm{m}_1-1} \mathcal{V}_n^{[j]}|\hat{\beta}_1 - \beta_1'|^j + o_{\mathbf{P}}(1)$$

on D_n. Let $E_n = \{|\hat{\beta}_1 - \beta_1'| \leq 1\}$. Noting that $\mathbf{P}(E_n^c) \to 0$ from $[\mathrm{H}1']_1$, it follows that for all $\epsilon > 0$,

$$\mathbf{P}(\check{\mathcal{R}}_n > (\mathrm{m}_1 + 1)\epsilon) \leq \mathbf{P}\left(\sum_{j=0}^{\mathrm{m}_1-1} \mathcal{R}_n^{[j]} > \mathrm{m}_1\epsilon, D_n \cap E_n\right) + o(1) \leq \frac{1}{\epsilon^2}\sum_{j=0}^{\mathrm{m}_1-1} \mathbb{E}_{\theta_1}[(\mathcal{R}_n^{[j]})^2 : D_n] + o(1),$$

$$\mathbf{P}(\check{\mathcal{V}}_n > (\mathrm{m}_1 + 1)\epsilon) \leq \mathbf{P}\left(\sum_{j=0}^{\mathrm{m}_1-1} \mathcal{V}_n^{[j]} > \mathrm{m}_1\epsilon, D_n \cap E_n\right) + o(1) \leq \frac{1}{\epsilon^2}\sum_{j=0}^{\mathrm{m}_1-1} \mathbb{E}_{\theta_1}[(\mathcal{V}_n^{[j]})^2 : D_n] + o(1).$$

Since $\mathbb{E}_{\theta_1}[|\mathcal{N}_{1,i}^{[j]}|^2] \lesssim h_n$, we have $\mathbb{E}_{\theta_1}[(\mathcal{R}_n^{[j]})^2 : D_n] = o(1)$ and $\mathbb{E}_{\theta_1}[(\mathcal{V}_n^{[j]})^2 : D_n] = o(1)$ for $0 \leq j \leq \mathrm{m}_1 - 1$ as in Lemma 11. Hence, we get the desired results.

(b) According to

$$\mathcal{T}_{2,n}^{(1)} \geq \frac{1}{\sqrt{\underline{\tau}_n T}}\left|\mathcal{I}_{1,n}^{-1/2}\left(\sum_{i=1}^{[n\tau_*^\beta]} \hat{\zeta}_{1,i} - \frac{[n\tau_*^\beta]}{[n\underline{\tau}_n]}\sum_{i=1}^{[n\underline{\tau}_n]} \hat{\zeta}_{1,i}\right)\right|$$

$$= \sqrt{\frac{T\vartheta_{\beta_1}^2}{\underline{\tau}_n}} \left| \mathcal{I}_{1,n}^{-1/2} \left(\frac{1}{T\vartheta_{\beta_1}} \sum_{i=1}^{[n\tau_*^\beta]} \hat{\zeta}_{1,i} - \frac{[n\tau_*^\beta]}{[n\underline{\tau}_n]} \frac{1}{T\vartheta_{\beta_1}} \sum_{i=1}^{[n\underline{\tau}_n]} \hat{\zeta}_{1,i} \right) \right|,$$

it is sufficient to verify that there exists $c \neq 0$ such that

$$\mathcal{K}_n^{(1)} = \frac{1}{T\vartheta_{\beta_1}} \sum_{i=1}^{[n\tau_*^\beta]} \hat{\zeta}_{1,i} - \frac{[n\tau_*^\beta]}{[n\underline{\tau}_n]} \frac{1}{T\vartheta_{\beta_1}} \sum_{i=1}^{[n\underline{\tau}_n]} \hat{\zeta}_{1,i} \xrightarrow{\mathbf{P}} c. \tag{B.88}$$

Notice that there exists $c' \neq 0$ such that

$$\mathcal{K}_n^{(2)} = \frac{1}{T\vartheta_{\beta_1}} \sum_{i=1}^{[n\tau_*^\beta]} \hat{\zeta}_{1,i} - \frac{[n\tau_*^\beta]}{[n\tau_*^\alpha]} \frac{1}{T\vartheta_{\beta_1}} \sum_{i=1}^{[n\tau_*^\alpha]} \hat{\zeta}_{1,i} \xrightarrow{\mathbf{P}} c' \tag{B.89}$$

in the same way as the proof of (2) of Theorem 4. Meanwhile, it follows from $[\text{I3}']_1$ and $[\text{I5}']_1$ that

$$\hat{\zeta}_{1,i} = \sum_{j=0}^{\mathrm{m}_2'-1} \mathcal{N}_{1,i}^{[j]} \otimes (\hat{\beta}_1 - \beta_1')^{\otimes j} + R_{i-1}(h_n)$$

and it can be shown that $\Delta_n = |\mathcal{K}_n^{(1)} - \mathcal{K}_n^{(2)}| \xrightarrow{\mathbf{P}} 0$ under $[\text{I6}']_1$ in the same way as in (a). Therefore, we have (B.88).

Similarly, it can be shown that $\mathbf{P}(\mathcal{T}_{2,n}^{(2)} > w_1(\epsilon)) \to 1$ under $H_1^{(2)}$. □

Proof of Theorem 8. We have

$$\Psi_{1,n}(\tau : \beta_1, \beta_2|\alpha) - \Psi_{1,n}(\tau_*^\beta : \beta_1, \beta_2|\alpha) = \sum_{i=[n\tau_*^\beta]+1}^{[n\tau]} \Big(G_i(\beta_1|\alpha) - G_i(\beta_2|\alpha) \Big)$$

for $\tau_*^\beta < \tau < \tau_*^\alpha$, and

$$\Psi_{1,n}(\tau : \beta_1, \beta_2|\alpha) - \Psi_{1,n}(\tau_*^\beta : \beta_1, \beta_2|\alpha) = \sum_{i=[n\tau]+1}^{[n\tau_*^\beta]} \Big(G_i(\beta_2|\alpha) - G_i(\beta_1|\alpha) \Big)$$

for $\tau < \tau_*^\beta$. Therefore, in the same way as the proof of Theorem 5, we obtain from [J1], $[\text{J2}']_1$ and [J3] the result that

$$T\vartheta_\beta^2(\hat{\tau}_{1,n}^\beta - \tau_*^\beta) \xrightarrow{\mathrm{d}} \underset{v \in \mathbb{R}}{\operatorname{argmin}}\, \mathbb{G}_1(v : \alpha_1^{(0)})$$

in Case A_β, and from $[\text{K1}']_1$ and [K2] the result that $T(\hat{\tau}_{1,n}^\beta - \tau_*^\beta) = O_{\mathbf{P}}(1)$ in Case B_β. □

Theorem 9 can be shown in the same way as the proof of Theorem 8.

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List of Publications

1. Tonaki, Y., Kaino, Y. and Uchida, M. (2023). Parametric estimation for linear parabolic SPDEs in two space dimensions based on temporal and spatial increments. *arXiv:2304.09441*.

2. Tonaki, Y., Kaino, Y. and Uchida, M. (2022). Parameter estimation for a linear parabolic SPDE model in two space dimensions with a small noise. *arXiv:2206.10363*.

3. Tonaki, Y., Kaino, Y. and Uchida, M. (2023). Parameter estimation for linear parabolic SPDEs in two space dimensions based on high frequency data. *Scandinavian Journal of Statistics*. https://doi.org/10.1111/sjos.12663.

4. Tonaki, Y. and Uchida, M. (2023). Change point inference in ergodic diffusion processes based on high frequency data. *Stochastic Processes and their Applications*, **158**, 1-39.

5. Tonaki, Y., Kaino, Y. and Uchida, M. (2023). Estimation for change point of discretely observed ergodic diffusion processes. *Scandinavian Journal of Statistics*, **50**(1), 142-183.

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