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# 3-DIMENSIONAL HOMOLOGY HANDLES AND MINIMAL SECOND BETTI NUMBERS OF 4-MANIFOLDS

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### 1. Introduction

We consider the following problem:

For a given closed 3-manifold M, what is the minimal second Betti number of all compact 4-manifolds bounded by M?

If we add the condition that 4-manifolds are simply connected, then the answer about the above problem in the topological category can be seen from the Boyer classification theorem [1],[2]. The Boyer classification theorem states that for an oriented, closed, connected 3-manifold M, a symmetric integral bilinear form  $(E, \mathcal{L})$  and a presentation  $\mathcal{P}$  of  $H_*(M;\mathbb{Z})$  by  $(E,\mathcal{L})$ , there exists an oriented, compact, simply connected, topological 4-manifold with boundary M whose intersection form is isomorphic over  $\mathbb{Z}$  to  $(E,\mathcal{L})$  and which represents  $\mathcal{P}$  geometrically. Furthermore, Boyer gave the result about the uniqueness of such 4-manifolds up to orientation-preserving homemorphism. Here a presentation  $\mathcal{P}$  of  $H_*(M;\mathbb{Z})$  by  $(E,\mathcal{L})$  is the following short exact sequence with some algebraic data corresponding to the relationship between the linking form of M and  $(E,\mathcal{L})$ , spin structures and the Kirby-Siebenmann obstruction;

$$0 \longrightarrow H_2(M; \mathbb{Z}) \longrightarrow E \xrightarrow{ad(\mathcal{L})} E^* \longrightarrow H_1(M; \mathbb{Z}) \longrightarrow 0.$$

Hence, in the topological category, we can calculate algebraically the minimal second Betti number of all simply connected 4-manifolds bounded by M. The key to this classification theorem is the Freedman theorem [4], and in particular the fact that every homology 3-sphere can bound a contractible compact topological 4-manifold. In the topological category, it follows from this that the minimal second Betti number of all simply connected 4-manifolds bounded by a given homology 3-sphere is zero. However, the Roholin theorem and the gauge theory say that in the smooth category, a homology 3-sphere can not always bound a homology 4-ball, and so the minimal second Betti number of all simply connected 4-manifolds bounded by a probability of the smooth category.

If we consider the Boyer theorem with the condition that the fundamental groups of 4-manifolds are isomorphic to the infinite cyclic group instead of simply connectedness, then the key seems to be orientable closed 3-manifolds M with the same integral homology groups as  $S^1 \times S^2$ , which are called *homology handles* [8]. Of course, the situation changes according as the homomorphisms of  $\pi_1$  induced from inclusions are trivial or not. In this paper, we consider the case where such homomorphisms  $i_{\text{ff}}: \pi_1 M \to \mathbb{Z}$  are surjective, and under this condition we consider the above problem.

By  $\beta^{TOP}(M)$  and  $\beta^{DIFF}(M)$ , we denote the minimal second Betti number of such 4-manifolds in the topological category and in the smooth category, respectively. For example, it is clear that  $\beta^{TOP}(S^1 \times S^2) = \beta^{DIFF}(S^1 \times S^2) = 0$ . But it does not always hold that  $\beta^{TOP}(M) = 0$ , since there is a homology handle which can not bound a compact topological 4-manifold homotopy equivalent to  $S^1$  in contrast with the case of homology 3-spheres. In this paper we show that for any positive integer n, there exist infinitely many distinct homology handles  $\{M_m^{(n)}\}_{m \in \mathbb{N}}$  with  $\beta^{TOP}(M_m^{(n)}) = \beta^{DIFF}(M_m^{(n)}) = n$ , and furthermore that there exists a difference between  $\beta^{TOP}$  and  $\beta^{DIFF}$ .

In §2, we introduce two operations on framed links to construct compact smooth 4-manifolds which are bounded by given 3-manifolds and whose fundamental groups are isomorphic to  $\mathbb{Z}$ . In §§3 and 4, we investigate  $\beta^{TOP}$  and  $\beta^{DIFF}$  of certain homology handles, and in particular homology handles obtained by 0-surgery on knots. In §4, we show that  $\beta^{TOP}$  and  $\beta^{DIFF}$  are functions onto  $\mathbb{N} \cup \{0\}$  and there is a difference between  $\beta^{TOP}$  and  $\beta^{DIFF}$ .

Through this paper, we suppose that manifolds are connected and oriented, and we denote the closed interval [0,1] by *I*. Furthermore, the symbol  $b_i$  stands for the *i*-th Betti number.

#### 2. Two kinds of 2-handle attachings

For a positive integer p, let  $\rho: S^3 \to S^3$  be the  $(2\pi/p)$ -rotation around the z-axis and  $B_j^3(j=0,1,\ldots,p-1)$  small 3-balls in  $S^3$  with  $\rho(B_j^3) = B_{j+1}^3$   $(j=0,1,\ldots,p-2)$  and  $\rho(B_{p-1}^3) = B_0^3$ . Moreover, let  $D_p = (S^3 - \bigcup_{j=0}^{p-1} \operatorname{int} B_j^3) \times \rho S^1$  be the mapping torus with monodromy  $\rho$ . The compact smooth 4-manifold  $D_p$  is bounded by  $S^1 \times S^2$  and has the fundamental group  $\pi_1 D_p$  isomorphic to  $\mathbb{Z}$ . The homomorphism  $i_{\sharp}: \pi_1(S^1 \times S^2) \to \pi_1 D_p$  has index p, where  $i: S^1 \times S^2 \to D_p$  is the inclusion.

Let M be an oriented closed 3-manifold. If M bounds an oriented compact 4manifold V such that the fundamental group  $\pi_1 V$  is isomorphic to  $\mathbb{Z}$  and the homomorphism of  $\pi_1$  induced from the inclusion  $i: M \to V$  is not trivial, then the first Betti number of M is positive. In this section we shall show that for any given 3manifold M with  $b_1(M) \ge 1$ , M bounds an oriented compact smooth 4-manifold Vsuch that  $\pi_1 V$  is isomorphic to  $\mathbb{Z}$  and  $i_{\sharp}: \pi_1 M \to \pi_1 V \cong \mathbb{Z}$  is not trivial. To show this, we need the following two operations. Every closed 3-manifold is obtained from  $S^3$  by an integral surgery on a link in  $S^3$ . Let M be obtained by a framed link  $\mathbb{L}$ .

**Operation 1.** Let K be a component of  $\mathbb{L}$  with framing n and c a crossing on



a diagram of  $K \subset \mathbb{L}$ . Add a trivial knot O with framing 0 to  $\mathbb{L}$  at c so that the linking number lk(O, K) between O and K is zero. See Fig. 1. Let K' be a knot obtained from K by crossing-change at c. Then, by the Kirby calculus (or handle-slide), the resultant 3-manifold obtained by this new framed link  $\mathbb{L} \cup O$  is orientation-preserving homeomorphic to the 3-manifold obtained by a framed link  $\mathbb{L}'$  containing a new component O with framing 0 and the component K' with framing n instead of K with framing n. See Fig. 2.

**Operation 2.** Let K and L be two components of L with framing m and n, respectively. Let c be a crossing of K and L on a diagram of L. Give the framing 0 to a meridional curve O of L. See Fig. 3. Then, by the Kirby calculus (or handle-slide), the resultant 3-manifold obtained by this new framed link  $\mathbb{L} \cup O$  is orientation-





Fig. 4.

preserving homeomorphic to the 3-manifold obtained by a framed link  $\mathbb{L}'$  which contains a new component O with framing 0 and which has an opposite crossing at c. See Fig. 4. Note that this operation leaves the knot type of K invariant, since O is trivial.

We use Operations 1 and 2 to make a knot trivial and to split geometrically a component of a link from other components, respectively.

**Proposition 1.** For any positive integer p and for any given 3-manifold M with  $b_1(M) \ge 1$ , there exists an oriented compact smooth 4-manifold V bounded by M such that

(1)  $\pi_1 V$  is isomorphic to  $\mathbb{Z}$ , and

(2) the index,  $(\pi_1 V : \operatorname{Im} i_{\sharp})$ , of  $\operatorname{Im} \{i_{\sharp} : \pi_1 M \to \pi_1 V\}$  in  $\pi_1 V$  is p.

Every oriented 3-manifold is obtained from  $S^3$  by an integral surgery on a link in  $S^3$ , but this link is not always an algebraically split link. Here, we say that a link  $\mathbb{L} = K_1 \cup K_2 \cup \cdots \cup K_{\mu}$  is an algebraically split link if for each pair of distinct components  $K_i, K_j (i \neq j)$  of  $\mathbb{L}$ , the linking number  $lk(K_i, K_j)$  is zero.

We use the following lemma.

**Lemma 1** ([13]). Any integral symmetric matrix is made diagonalizable over  $\mathbb{Z}$  by taking block sums of some  $1 \times 1$ -matrices  $(p_j)$ .

We can translate Lemma 1 into geometric terms : Let M be an oriented closed 3-manifold. Then, there are some lens spaces  $L(p_j, 1)$   $(j = 1, 2, \dots, k)$  such that after taking connected sums of  $L(p_j, 1)$   $(j = 1, 2, \dots, k)$ , the 3-manifold  $M \# L(p_1, 1) \# L(p_2, 1) \# \dots \# L(p_k, 1)$  has a surgery description by a framed algebraically split link.

Proof of Proposition 1. By Lemma 1, there are some lens spaces  $L(p_j, 1)$   $(j = 1, 2, \dots, k)$  such that the 3-manifold  $M' = M \# L(p_1, 1) \# L(p_2, 1) \# \dots \# L(p_k, 1)$  is obtained by an integral surgery on an algebraically split link  $\mathbb{L}$ . Let  $r(\geq 1)$  be the first Betti number of M. Then, the linking matrix of  $\mathbb{L}$  is an  $(r+n) \times (r+n)$ -matrix

/0	• • •	0	0	• • •	0 \
1:	۰.	÷	÷	·	:
0	•••	0	0		0
0	•••	0	$m_1$		
:	۰.	÷		۰.	
٥/		0			$m_n$ /

where  $|m_1m_2\cdots m_n|$  is not zero and the order of the torsion part of  $H_1(M';\mathbb{Z})$ . Generators of  $H_1(M';\mathbb{Z})$  are given by meridional curves of the components of  $\mathbb{L}$ . Let  $K_i$  $(i = 1, 2, \dots, r)$  be the components of  $\mathbb{L}$  with framing 0 and  $L_j$   $(j = 1, 2, \dots, n)$ the other components of  $\mathbb{L}$ . The 3-manifold  $L(p_1, 1) \sharp L(p_2, 1) \sharp \cdots \sharp L(p_k, 1)$  bounds an oriented simply connected compact smooth 4-manifold W, for example the  $\natural$ -sum of  $k \ D^2$ -bundles over  $S^2$ . Then the smooth 4-manifold  $(M \times I) \natural (-W)$  is bounded by  $M \coprod (-M')$ . We shall make  $K_1$  a trivial knot which is split geometrically from the other components of  $\mathbb{L}$ .





Schema 1.

**Step 1.** If  $K_1$  is not trivial, then we can make  $K_1$  a trivial knot  $K'_1$  by a finite sequence of Operation 1 at some crossings of  $K_1$ . Then the framed link  $\mathbb{L}$  changes into another framed link  $\mathbb{L}'$ , which is algebraically split. The trivial knot  $K'_1$  has framing 0.

In general,  $K'_1$  is not split geometrically from the other components of  $\mathbb{L}'$ .

**Step 2.** By a finite sequence of Operation 2, we can split geometrically  $K'_1$  from the other components of  $\mathbb{L}'$  keeping  $K'_1$  trivial and without changing the framing of  $K'_1$ . By  $\mathbb{L}''$  we denote the framed link obtained by the operations as above. Let  $\mathbb{L}''_2$  be the link consisting of the other components of  $\mathbb{L}''$  except  $K'_1$ , that is,  $\mathbb{L}'' = K'_1 \cup \mathbb{L}''_2$ . Then the 3-manifold given by the framed link  $\mathbb{L}''$  is  $S^1 \times S^2 \sharp N$ , where N is the 3-manifold given by  $\mathbb{L}''_2$ .

Hence it follows that by attaching 2-handles to  $M' \times \{1\} \subset M' \times I$  in ways corresponding to Steps 1 and 2, we get an oriented compact smooth 4-manifold X whose boundary is  $M' \coprod (-(S^1 \times S^2 \sharp N))$ . Set  $Y = ((M \times I) \natural (-W)) \bigcup_{M'} X$ . Let W' be an oriented simply connected compact smooth 4-manifold bounded by N, for example, the 4-manifold consisting of one 0-handle and some 2-handles given by the



Fig. 5.

framed link  $\mathbb{L}_2''$ . Then  $Z = Y \bigcup ((S^1 \times S^2) \times I 
mit W')$  is an oriented compact smooth 4-manifold with boundary  $\partial Z = M \coprod (-S^1 \times S^2)$ . See Schema 1. Now let V be the 4-manifold  $Z \cup_{\partial} D_p$ , which is an oriented compact smooth 4-manifold with boundary  $\partial V = M$ . By van Kampen's theorem,  $\pi_1 V$  is isomorphic to Z. If we let t be a generator of  $\pi_1 D_p$ , then a loop coming from a meridional curve of  $K_1$  represents  $t^{\pm p}$  in  $\pi_1 D_p$ , and so  $(\pi_1 V : \operatorname{Imi}_{\sharp}) = p$ .

EXAMPLE 1. Let m be an integer. Let M(m) be the homology handle given by the following framed link  $K_1 \cup K_2$  in Fig. 5. The link  $K_1 \cup K_2$  is an algebraically split link. Let  $\tilde{M}(m)$  be the universal abelian covering of M(m), that is, the infinite cyclic covering of M(m) associated to the kernel of the Hurewitz homomorphism  $\alpha : \pi_1 M(m) \to H_1(M(m); \mathbb{Z}) \cong \mathbb{Z}$ . Then  $\tilde{M}(m)$  is obtained from the universal covering  $q : \mathbb{R} \times S^2 \to S^1 \times S^2$  by 1-surgeries on the preimage of  $K_2$  via qas in Fig. 6. See [14]. By  $\Lambda = \mathbb{Z}\langle t \rangle$  we denote the ring of Laurent polynomials with integer coffecients. Thus  $H_1(\tilde{M}(m); \mathbb{Z})$  has a  $\Lambda$ -module structure by the group of deck transformations and is isomorphic to  $\Lambda/(mt^{-1} - (2m-1) + mt)$  as  $\Lambda$ -modules. Here (f(t)) stands for the principal ideal generated by  $f(t) \in \Lambda$ . Now attach one 2handle  $h^{(2)}$  to  $M(m) \times I$  so that the attaching circle of  $h^{(2)}$  is a meridional curve of  $K_2$  and the framing of  $h^{(2)}$  is zero. Let W be the resultant 4-manifold. By Op-



eration 1, it is seen that W is bounded by  $M(m) \coprod (-S^1 \times S^2)$ . See Fig. 7. Thus  $V = W \cup_{S^1 \times S^2} D_p$  is an oriented compact smooth 4-manifold bounded by M(m) with  $\pi_1 V \cong \mathbb{Z}$ ,  $(\pi_1 V : \operatorname{Im}_{i\sharp}) = p$ , and  $H_2(V;\mathbb{Z}) \cong \mathbb{Z} \oplus \mathbb{Z}_p$ . In §§3 and 4 we show that in the case of p = 1 this 4-manifold V gives the minimal second Betti number of all oriented compact topological 4-manifolds X bounded by M(m) with  $\pi_1 X \cong \mathbb{Z}$  and  $(\pi_1 X : \operatorname{Im}_{i\sharp}) = 1$ .

We have the following proposition for a 3-manifold M such that  $H_1(M;\mathbb{Z})$  has a torsion subgroup.

**Proposition 2.** Let p be any positive integer and  $\mathbb{L} = K_1 \cup K_2$  a 2-component



Fig. 7.

framed link such that

(1)  $K_1$  is a trivial knot,

(2) the linking number  $lk(K_1, K_2)$  is zero, and

(3) the framings of  $K_1$  and  $K_2$  is 0 and n, respectively.

Let M be the resultant 3-manifold obtained by surgery on the framed link  $\mathbb{L}$ . If |n| > 1, then the smooth 4-manifold V constructed in the manner of Example 1 gives the minimal second Betti number of all oriented compact topological 4-manifolds X bounded by M with  $\pi_1 X \cong \mathbb{Z}$  and  $(\pi_1 X : \operatorname{Imi}_{\sharp}) = p$ . Note that  $H_2(V; \mathbb{Z}) \cong \mathbb{Z} \oplus \mathbb{Z}_p$ .

Proof. Suppose that  $b_2(V) = 1$  is not minimal. Namely, there is an oriented compact topological 4-manifold X as above with  $b_2(X) = 0$ . By considering the homology exact sequence of the pair (X, M), we have the following short exact seY. SATO

quence;

$$0 \to \mathbb{Z} \to H_2(M;\mathbb{Z}) \to \mathbb{Z}_p \to 0 \xrightarrow{\partial} H_1(M;\mathbb{Z}) \to \mathbb{Z} \to \mathbb{Z}_p \to 0.$$

Because of |n| > 1,  $H_1(M; \mathbb{Z})$  has a torsion subgroup. This contradicts that  $H_1(M; \mathbb{Z}) \to \mathbb{Z}$  is injective.

#### 3. Minimal second Betti numbers for homology handles

Through §§3 and 4, we consider the case of p = 1, namely, the case where the homomorphisms on  $\pi_1$  induced from inclusions are surjective. If M is an oriented closed 3-manifold with  $H_*(M;\mathbb{Z}) \cong H_*(S^1 \times S^2;\mathbb{Z})$ , then we call M a homology handle. See [8]. Since a homology handle M has  $H^1(M;\mathbb{Z}_2) \cong \mathbb{Z}_2$ , M admits two spin structures  $\tau_0$  and  $\tau_1$ . By  $\mu(M, \tau)$  we denote the Roholin invariant of M with respect to a spin structure  $\tau$ .

**Proposition 3.** Let M be a homology handle with spin structures  $\tau_0$  and  $\tau_1$ . Suppose that  $\mu(M, \tau_0) = 0$  and  $\mu(M, \tau_1) = 1$ . Then, there is no orientable compact topological spin 4-manifold V bounded by M such that  $\pi_1 V \cong \mathbb{Z}$  and the homomorphism  $i_{\sharp} : \pi_1 M \to \pi_1 V \cong \mathbb{Z}$  is surjective.

Proof. Suppose that there would be such a 4-manifold V. Because of  $\pi_1 V \cong \mathbb{Z}$ , V admits two spin structures  $\sigma_0$  and  $\sigma_1$ . Since  $i_{\sharp} : \pi_1 M \to \pi_1 V \cong \mathbb{Z}$  is surjective,  $\pi_1(V, M) = 0$  and so  $H^1(E(\tau_V), E(\tau_M); \mathbb{Z}_2) = 0$ . Here  $E(\tau_M)$  and  $E(\tau_V)$  are the total spaces of the principal STop(3)-bundle and the principal STop(4)-bundle associated with stable topological tangent bundles over M and V, respectively. From the following cohomology exact sequence of the pair  $(E(\tau_V), E(\tau_M))$ ,

$$0 = H^1(E(\tau_V), E(\tau_M); \mathbb{Z}_2) \to H^1(E(\tau_V); \mathbb{Z}_2) \to H^1(E(\tau_M); \mathbb{Z}_2) \xrightarrow{\delta},$$

if follows that the restrictions of  $\sigma_0$  and  $\sigma_1$  to M are  $\tau_0$  and  $\tau_1$ , say  $\sigma_0|_M = \tau_0$  and  $\sigma_1|_M = \tau_1$ . By [5, Chapter 10], we can calculate the Kirby-Siebenmann obstruction  $ks(V) \in H^4(V, M; \mathbb{Z}_2)$  of V from  $(V, \sigma_0)$  and we have that

$$\begin{aligned} 8ks(V) &\equiv signature(V) + \mu(M, \tau_0) \pmod{16} \\ &\equiv signature(V) \pmod{16}. \end{aligned}$$

From  $(V, \sigma_1)$  it follows that

$$8ks(V) \equiv signature(V) + 1 \pmod{16},$$

and this equation contradicts that one.

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For any given homology handle M, we would like to investigate the minimal second Betti number of 4-manifolds bounded by M.

Let M be a homology handle. By  $\beta^{TOP}(M)$  we denote the minimal second Betti number of all oriented compact topological 4-manifolds V bounded by M such that  $\pi_1 V$  is isomorphic to  $\mathbb{Z}$  and the homomorphism  $i_{\sharp} : \pi_1 M \to \pi_1 V$  is surjective. Furthermore, we denote by  $\beta^{DIFF}(M)$  the minimal second Betti number of all oriented compact smooth 4-manifolds as above. Then it is clear that  $\beta^{DIFF}(M) \ge \beta^{TOP}(M) \ge 0$ .

REMARK. If we define  $\beta^{TOP}(M)$  and  $\beta^{DIFF}(M)$  for a general 3-manifold M in the same manner, then it follows from the homology exact sequence of the pair (V, M) that  $\beta^{DIFF}(M) \ge \beta^{TOP}(M) \ge \operatorname{rank}_{\mathbb{Z}} H_1(M;\mathbb{Z}) - 1$ .

**Corollary 1.** Let M be a homology handle as in Proposition 3. Then,  $\beta^{TOP}(M) \ge 1$ .

**Corollary 2.** Let  $\mathbb{L} = K_1 \cup K_2$  be a 2-component framed link such that

(1)  $K_1$  is a trivial knot,

(2) the linking number  $lk(K_1, K_2)$  is 0, and

(3) the framings of  $K_1$  and  $K_2$  is 0 and  $\pm 1$ , respectively.

Let M be the homology handle obtained by surgery on L. If M admits two spin structures  $\tau_0$  and  $\tau_1$  with  $\mu(M, \tau_0) = 0$  and  $\mu(M, \tau_1) = 1$ , then  $\beta^{DIFF}(M) = \beta^{TOP}(M) = 1$ .

Proof. We can construct a smooth 4-manifold V bounded by M with  $H_2(V;\mathbb{Z})$   $\cong \mathbb{Z}$  in the same manner as Example 1. Hence, it follows from Corollary 1 that  $\beta^{DIFF}(M) = \beta^{TOP}(M) = 1.$ 

EXAMPLE 2. Let M(m) be the homology handle in Example 1. If m is odd, then M(m) admits two spin structures  $\tau_0$  and  $\tau_1$  with  $\mu(M, \tau_0) = 0$  and  $\mu(M, \tau_1) = 1$ . If m is even, then M(m) admits two spin structures  $\tau_0$  and  $\tau_1$  with  $\mu(M, \tau_0) = \mu(M, \tau_1) = 0$ , Hence, if m is odd, then  $\beta^{DIFF}(M(m)) = \beta^{TOP}(M(m)) = 1$ .

For what homology handle M does it hold that  $\beta^{TOP}(M) = 0$  or  $\beta^{DIFF}(M) = 0$ ? Note that  $\beta^{TOP}(M) = 0$  if and only if M bounds an oriented compact topological 4manifold homotopy equivalent to  $S^1$ . Freedman and Quinn give a necessary and sufficient condition to hold that  $\beta^{TOP}(M) = 0$  in [5, Proposition 11.6A and 11.6C].

**Theorem 2** ([5]). Let M be a homology handle. Let  $C = [\pi_1 M, \pi_1 M]$  be the commutator subgroup of  $\pi_1 M$ . Then,  $\beta^{TOP}(M) = 0$  if and only if C is perfect.

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Since the universal abelian convering  $\widetilde{M}$  of a homology handle M is the infinite cyclic covering associated to the kernel of the Hurewicz homomorphism  $\pi_1 M \to H_1(M;\mathbb{Z}) \cong \mathbb{Z}$ ,  $H_1(\widetilde{M};\mathbb{Z})$  is isomorphic to C/[C,C]. Theorem 2 implies that  $\beta^{TOP}(M) = 0$  if and only if  $H_1(\widetilde{M};\mathbb{Z}) = 0$ . Furthermore, the group of deck transformation of  $\widetilde{M}$  gives a  $\Lambda$ -modules structure to  $H_1(\widetilde{M};\mathbb{Z})$ , which is isomorphic to  $H_1(M;\Lambda)$  as  $\Lambda$ -modules. So, one can define the Alexander polynomials  $\Delta_M(t) \in \Lambda$ for homology handles M as well as for knots. Kawauchi gave in [8, 9] a characterization of the Alexander polynomials of homology handles and how to calculate the Alexander polynomials. Thus  $H_1(\widetilde{M};\mathbb{Z}) = 0$ , that is,  $\beta^{TOP}(M) = 0$  if and only if the Alexander polynomial  $\Delta_M(t)$  of M is trivial, that is, a unit of  $\Lambda$ .

## 4. Minimal second Betti numbers for homology handles obtained by 0surgery on knots

Consider a homology handle M obtained by 0-surgery on a knot K in  $S^3$ . Note that the class  $\ell \in \pi_1(S^3 - K)$  represented by the preferred longitude for K belongs to the commutator subgroup  $[\pi_1(S^3 - K), \pi_1(S^3 - K)]$  of  $\pi_1(S^3 - K)$  and that  $\pi_1 M$  is isomorphic to  $\pi_1(S^3 - K)/\langle \ell \rangle$ , where  $\langle \ell \rangle$  is the smallest normal subgroup generated by  $\ell$ . Thus we have the following.

**Lemma 2.** Let K be a knot with exterior E(K), and  $E(\overline{K})$  the universal abelian covering of E(K). Let M be the homology handle obtained by 0-surgery on K. Then,  $H_1(\widetilde{M};\mathbb{Z})$  is isomorphic to  $H_1(\widetilde{E(K)};\mathbb{Z})$  as  $\Lambda$ -modules. In particular, the Alexander polynomial  $\Delta_M(t)$  of M is equal to the Alexander polynomial  $\Delta_K(t)$  of K (See Lemma 2.6-(III) in [8].).

Hence, we have the following.

**Corollary 3.** Let M be the homology handle obtained by 0-surgery on a knot K. The minimal second Betti number  $\beta^{TOP}(M) = 0$  if and only if the Alexander polynomial  $\Delta_K(t)$  of K is trivial.

EXAMPLE 3. Let M(m) be the homology handle in Example 1. In Example 1 we see that  $H_1(\widetilde{M(m)}; \mathbb{Z})$  is isomorphic to  $\Lambda/(mt^{-1} - (2m-1) + mt)$  as  $\Lambda$ -modules. In fact, it follows from the Kirby calculus that M(m) is also obtained by 0-surgery on the following knot in Fig. 8. Thus the Alexander polynomial for M(m) is  $mt^{-1} - (2m-1) + mt$  and  $\beta^{TOP}(M(m)) \neq 0$ . Therefore, in the case when m is even, it also holds that  $\beta^{TOP}(M(m)) = \beta^{DIFF}(M(m)) = 1$ , since we can construct a required 4-manifold in the same manner as Example 1. See Example 2.

We can estimate  $\beta^{DIFF}(M)$  by the unknotting number u(K) of a knot K.



**Proposition 4.** Let M be the homology handle obtained by 0-surgery on a knot K with unknotting number u(K). Then,  $u(K) \ge \beta^{DIFF}(M)$ .



Proof. Note that by the Kirby calculus the 3-manifolds in Fig. 9. are homeomorphic. Let u be the unknotting number of K. Then after taking cross-changing at certain u crossings of a diagram of K, K becomes a trivial knot  $L_0$ . Hence, Mhas a surgery description by a framed link  $\mathbb{L} = L_0 \cup L_1 \cup \cdots \cup L_u$  such that al $l L_j (j = 0, 1, \cdots, u)$  are trivial knots, the framing of  $L_0$  is zero and the framings of  $L_j (j = 1, 2, \cdots, u)$  are  $\pm 1$ . See Fig. 10. If we apply Operation 2 to each  $L_j (j = 1, 2, \cdots, u)$ , then we get a new framed link  $\mathbb{L}'$ . See Fig. 11. The 3-manifold given by  $\mathbb{L}'$  is  $S^1 \times S^2$ . By attaching u 2-handles  $h_j^{(2)}$   $(j = 1, 2, \cdots, u)$  as above to  $M \times I$  and identifying one component of the boundary of the resultant smooth 4manifold with the boundary of  $S^1 \times B^3$ , we get a 4-manifold V with second Betti number u and with boundary M such that  $\pi_1 V$  is isomorphic to  $\mathbb{Z}$  and the homomor-



Fig. 11.

phism  $i_{\sharp}: \pi_1 M \to \pi_1 V$  is surjective. Hence,  $\beta^{DIFF}(M) \leq u$ .

For example, the knots  $K_m$  in Fig. 8 are unknotting number 1 knots. Hence,  $1 = u(K_m) \ge \beta^{DIFF}(M(m)) \ge \beta^{TOP}(M(m)) \ge 1$ , and so  $\beta^{TOP}(M(m)) = \beta^{DIFF}(M(m)) = 1$ .

We generalize Examples 2 and 3 as follows.

**Theorem 3.** For any positive integer n, there exist infinitely many distinct homology handles  $\{M_m^{(n)}\}_{m\geq 1}$  with  $\beta^{TOP}(M_m^{(n)}) = \beta^{DIFF}(M_m^{(n)}) = n$ .

To show Theorem 3, we use the local signatures of homology handles, which are introduced by Kawauchi [8] and defined by generalizing local signatures of knots. See also [12]. In [9], Kawauchi considered the embedding problem of 3-manifolds into 4-manifolds. In particular, he gave an estimation of second Betti numbers and signatures of 4-manifolds by local signatures of their boundaries : Let M be a homology handle

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and X a compact topological 4-manifold bounded by M. Then, he showed that for any  $a \in [-1, 1]$ ,

(4.1) 
$$\left| \Sigma_{x \in (a,1]} \sigma_x(M) \right| \le b_2(X) + |signature(X)|.$$

Here  $\sigma_x(M)$  is a local signature of M. Since  $b_2(X) + |signature(X)| \le 2b_2(X)$ , we have

(4.2) 
$$\left| \Sigma_{x \in (a,1]} \sigma_x(M) \right| \le 2b_2(X) \quad \text{for any } a \in [-1,1],$$

and so

(4.3) 
$$\left| \Sigma_{x \in (a,1]} \sigma_x(M) \right| \leq 2\beta^{TOP}(M) \text{ for any } a \in [-1,1].$$

Proof of Theorem 3. For each positive integer m, let  $K_m$  be a knot in Fig. 8. Then, the Alexander polynomial  $\Delta_{K_m}(t)$  of  $K_m$  is  $mt^2 - (2m-1)t + m$  up to units in  $\Lambda$  and the unknotting number  $u(K_m)$  of  $K_m$  is 1. Because of  $\Delta_{K_m}(t)/m = t^2 - 2\{(2m-1)/(2m)\}t+1$ , it follows from Assertion 11 in [12] that the signature  $\sigma(K_m)$ of  $K_m$  is  $\pm 2$ . Hence, it folds that for the local signature  $\sigma_x(K_m)(x \in [-1, 1])$ ,

$$\sigma_x(K_m) = \begin{cases} \pm 2, & \text{if } x = (2m-1)/(2m), \\ 0 & \text{if } x \neq (2m-1)/(2m). \end{cases}$$

Let  $K_m^{(n)}$  be the connected sum of n copies of  $K_m$ , that is,  $K_m^{(n)} = K_m \# K_m \# \cdots \# K_m$ . Let  $M_m^{(n)}$  be the homology handle obtained by 0-surgery on  $K_m^{(n)}$ . Since  $\Delta_{K_m^{(n)}}(t) = (\Delta_{K_m}(t))^n \neq (\Delta_{K_{m'}}(t))^n = \Delta_{K_{m'}^{(n)}}(t) \ (m \neq m'), \ M_m^{(n)}$  and  $M_{m'}^{(n)}$   $(m \neq m')$  are not homeomorphic. Noting that the quadratic form of the universal abelian covering  $\widetilde{M_m^{(n)}}$  is the orthogonal sum of n copies of the quadratic form of  $K_m$ , it follows that for the local signature  $\sigma_x(M_m^{(n)})(x \in [-1,1])$ ,

$$\sigma_x(M_m^{(n)}) = \begin{cases} \pm 2n, & \text{if } x = (2m-1)/(2m), \\ 0 & \text{if } x \neq (2m-1)/(2m). \end{cases}$$

Hence, we have

$$\left| \Sigma_{x \in (0,1]} \sigma_x(M_m^{(n)}) \right| = \left| \sigma_{(2m-1)/(2m)}(M_m^{(n)}) \right| = 2n.$$

Thus, by the inequality (4.3) we have

$$n = \frac{1}{2} \left| \Sigma_{x \in (0,1]} \sigma_x(M_m^{(n)}) \right| \le \beta^{TOP}(M_m^{(n)}).$$

By noting that  $u(K_m^{(n)}) \leq n$  because of  $u(K_m) = 1$ , it follows from Proposition 4 that  $\beta^{DIFF}(M_m^{(n)}) \leq u(K_m^{(n)}) \leq n$ . Therefore,  $n \leq \beta^{TOP}(M_m^{(n)}) \leq \beta^{DIFF}(M_m^{(n)}) \leq n$ , and so  $\beta^{TOP}(M_m^{(n)}) = \beta^{DIFF}(M_m^{(n)}) = n$ .

REMARK. (1) The unknotting number  $u(K_m^{(n)})$  is *n* because of  $n = |\sigma(K_m^{(n)})|/2 \le u(K_m^{(n)}) \le n$ .

(2) Consider a short exact sequence of  $\Lambda$ -modules

$$0 \to E \to F \to \Lambda/(f_1) \oplus \Lambda/(f_2) \oplus \cdots \oplus \Lambda/(f_n) \to 0,$$

where E and F are free  $\Lambda$ -modules of the same rank. If each  $f_{i+1}$  can be divided by  $f_i$ , then rank  $\Lambda E \geq n$ . Let V be an oriented compact 4-manifold bounded by  $M_m^{(n)}$  such that  $\pi_1 V \cong \mathbb{Z}$  and the homomorphism  $i_{\sharp} : \pi_1 M_m^{(n)} \to \pi_1 V$  is surjective. Then we have the following homology exact sequence with local coefficient  $\Lambda$ ,

$$0 \to H_2(V;\Lambda) \to H_2(V,M_m^{(n)};\Lambda) \to H_1(M_m^{(n)};\Lambda) \to 0.$$

The homology groups  $H_2(V;\Lambda)$  and  $H_2(V, M_m^{(n)};\Lambda)$  are free  $\Lambda$ -modules of the same rank. Since  $H_1(M_m^{(n)};\Lambda) \cong \bigoplus_{i=1}^n (\Lambda/(mt - (2m-1) + mt^{-1}))_i = \Lambda/(mt - (2m-1) + mt^{-1}))_i \oplus (M_m^{(n)};\Lambda) \cong (\Lambda/(mt - (2m-1) + mt^{-1}))_i \oplus (M_m^{(n)};\Lambda) = n$ . Hence it follows that  $\beta^{TOP}(M_m^{(n)}) \ge n$ .

Next we give two definitions on sliceness of knots.

DEFINITION 1. If a knot K bounds a smooth disk D in the 4-ball  $B^4$  such that  $(B^4, D) \times I$  is a trivial ball pair, then K is a super slice knot. See [7].

For example, untwisted doubles of slice knots are super slice [7].

DEFINITION 2. A knot K is *pseudo-slice*, if there exists a pair (W, D) for K such that W is a smooth 4-manifold homemorphic to  $B^4$  and D is a smooth disk in W bounded by K.

**Proposition 5.** Let K be a super slice knot, and M the homology handle obtained by 0-surgery on K. Then,  $\beta^{TOP}(M) = \beta^{DIFF}(M) = 0$ .

Proof. Let D be a slice disk for K such that  $(B^4, D) \times I$  is a trivial ball pair. Let N(D) be a closed tubular neighborhood of D in  $B^4$ . Then, M is the boundary of the smooth 4-manifold  $V = B^4 - intN(D)$ . The 4-manifold V is homotopy equivalent to  $V \times I = B^4 \times I - intN(D) \times I$ . Since  $(B^4, D) \times I$  is trivial, V is homotopy equivalent to  $S^1$ . Thus V is a required 4-manifold.

Is there a difference between  $\beta^{TOP}$  and  $\beta^{DIFF}$ ? Now we answer this question.

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**Theorem 4.** Let K be a knot which is not pseudo-slice and whose Alexander polynomial  $\Delta_K$  is trivial. Let M be the homology handle obtained by 0-surgery on K. Then,  $0 = \beta^{\text{TOP}}(M) < \beta^{\text{DIFF}}(M)$ .

Proof. Since  $\Delta_K$  is trivial, it follows from Corollary 3 that  $\beta^{TOP}(M) = 0$ . Suppose that  $\beta^{DIFF}(M) = 0$ . Then M bounds a smooth 4-manifold V homotopy equivalent to  $S^1$ . By attaching to  $M \times I$  one 2-handle  $h^{(2)}$  whose attaching circle is a meridian of K and whose framing is zero, we get the 4-manifold  $(M \times I) \cup h^{(2)}$  whose boundary is  $M \coprod (-S^3)$ . See Operation 1. Furthermore, by identifying  $\partial V$  with one component M of the boundary of  $(M \times I) \cup h^{(2)}$ , we get a compact smooth 4-manifold W bounded by  $S^3$ . Then, since W is simply-connected and  $H_*(W;\mathbb{Z}) \cong H_*(B^4;\mathbb{Z})$ , W is homeomorphic to  $B^4$ . The co-core of the above 2-handle  $h^{(2)}$  gives a smooth disk D in W with  $\partial(W, D) = (S^3, K)$ . Since K is not pseudo-slice, this is a contradiction.

EXAMPLE 4. In [3], Cochran and Gompf showed that there are untwisted doubles which are not pseudo-slice. For example, the untwisted double K of the trefoil knot is such a knot. Note that the Alexander polynomials of nontrivial untwisted doubles are trivial and their unknotting numbers are 1. Thus, for the homology handle M obtained by 0-surgery on K,  $1 = u(K) \ge \beta^{DIFF}(M) > \beta^{TOP}(M) = 0$ , and so  $1 = \beta^{DIFF}(M) > \beta^{TOP}(M) = 0$ .

EXAMPLE 5. Let K(-3, 5, 7) be the pretzel knot of type (-3, 5, 7). Then K(-3, 5, 7) has a trivial Alexander polynomial. Furthermore, in [6] Fintushel and Stern showed that K(-3, 5, 7) is not pseudo-slice. Thus, for the homology handle M obtained by 0-surgery on K(-3, 5, 7),  $\beta^{DIFF}(M) > \beta^{TOP}(M) = 0$ .

It follows from [11] that K(-3, 5, 7) is not an unknotting number 1 knot. One can make K(-3, 5, 7) a trivial knot by crossing-change at certain 3 crossings. Hence,  $2 \le u(K(-3, 5, 7)) \le 3$ . Thus it follows that  $1 \le \beta^{DIFF}(M) \le 3$ . What is  $\beta^{DIFF}(M)$ ?

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