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<th><strong>Title</strong></th>
<th>3-dimensional homology handles and minimal second Betti numbers of 4-manifolds</th>
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<td><strong>Author(s)</strong></td>
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<tr>
<td><strong>Citation</strong></td>
<td>Osaka Journal of Mathematics. 35(3) P.509-P.527</td>
</tr>
<tr>
<td><strong>Issue Date</strong></td>
<td>1998</td>
</tr>
<tr>
<td><strong>Text Version</strong></td>
<td>publisher</td>
</tr>
<tr>
<td><strong>URL</strong></td>
<td><a href="https://doi.org/10.18910/9302">https://doi.org/10.18910/9302</a></td>
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<tr>
<td><strong>DOI</strong></td>
<td>10.18910/9302</td>
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Osaka University
1. Introduction

We consider the following problem:
For a given closed 3-manifold $M$, what is the minimal second Betti number of all compact 4-manifolds bounded by $M$?

If we add the condition that 4-manifolds are simply connected, then the answer about the above problem in the topological category can be seen from the Boyer classification theorem [1],[2]. The Boyer classification theorem states that for an oriented, closed, connected 3-manifold $M$, a symmetric integral bilinear form $(E, \mathcal{L})$ and a presentation $P$ of $H_*(M; \mathbb{Z})$ by $(E, \mathcal{L})$, there exists an oriented, compact, simply connected, topological 4-manifold with boundary $M$ whose intersection form is isomorphic over $\mathbb{Z}$ to $(E, \mathcal{L})$ and which represents $P$ geometrically. Furthermore, Boyer gave the result about the uniqueness of such 4-manifolds up to orientation-preserving homeomorphism. Here a presentation $P$ of $H_*(M; \mathbb{Z})$ by $(E, \mathcal{L})$ is the following short exact sequence with some algebraic data corresponding to the relationship between the linking form of $M$ and $(E, \mathcal{L})$, spin structures and the Kirby-Siebenmann obstruction;

\[
\begin{array}{c}
0 \rightarrow H_2(M; \mathbb{Z}) \rightarrow E \xrightarrow{ad(L)} E^* \rightarrow H_1(M; \mathbb{Z}) \rightarrow 0.
\end{array}
\]

Hence, in the topological category, we can calculate algebraically the minimal second Betti number of all simply connected 4-manifolds bounded by $M$. The key to this classification theorem is the Freedman theorem [4], and in particular the fact that every homology 3-sphere can bound a contractible compact topological 4-manifold. In the topological category, it follows from this that the minimal second Betti number of all simply connected 4-manifolds bounded by a given homology 3-sphere is zero. However, the Rohlin theorem and the gauge theory say that in the smooth category, a homology 3-sphere can not always bound a homology 4-ball, and so the minimal second Betti number of all simply connected 4-manifolds bounded by a homology 3-sphere is not always zero in the smooth category.

If we consider the Boyer theorem with the condition that the fundamental groups of 4-manifolds are isomorphic to the infinite cyclic group instead of simply connect-
edness, then the key seems to be orientable closed 3-manifolds $M$ with the same integral homology groups as $S^1 \times S^2$, which are called homology handles [8]. Of course, the situation changes according as the homomorphisms of $\pi_1$ induced from inclusion-s are trivial or not. In this paper, we consider the case where such homomorphisms $i_\#: \pi_1 M \to \mathbb{Z}$ are surjective, and under this condition we consider the above problem.

By $\beta^{\text{TOP}}(M)$ and $\beta^{\text{DIFF}}(M)$, we denote the minimal second Betti number of such 4-manifolds in the topological category and in the smooth category, respectively. For example, it is clear that $\beta^{\text{TOP}}(S^1 \times S^2) = \beta^{\text{DIFF}}(S^1 \times S^2) = 0$. But it does not always hold that $\beta^{\text{TOP}}(M) = 0$, since there is a homology handle which can not bound a compact topological 4-manifold homotopy equivalent to $S^1$ in contrast with the case of homology 3-spheres. In this paper we show that for any positive integer $n$, there exist infinitely many distinct homology handles $\{M^{(n)}_m\}_{m \in \mathbb{N}}$ with $\beta^{\text{TOP}}(M^{(n)}_m) = \beta^{\text{DIFF}}(M^{(n)}_m) = n$, and furthermore that there exists a difference between $\beta^{\text{TOP}}$ and $\beta^{\text{DIFF}}$.

In §2, we introduce two operations on framed links to construct compact smooth 4-manifolds which are bounded by given 3-manifolds and whose fundamental groups are isomorphic to $\mathbb{Z}$. In §§3 and 4, we investigate $\beta^{\text{TOP}}$ and $\beta^{\text{DIFF}}$ of certain homology handles, and in particular homology handles obtained by 0-surgery on knots. In §4, we show that $\beta^{\text{TOP}}$ and $\beta^{\text{DIFF}}$ are functions onto $\mathbb{N} \cup \{0\}$ and there is a difference between $\beta^{\text{TOP}}$ and $\beta^{\text{DIFF}}$.

Through this paper, we suppose that manifolds are connected and oriented, and we denote the closed interval $[0,1]$ by $I$. Furthermore, the symbol $b_i$ stands for the $i$-th Betti number.

2. Two kinds of 2-handle attachings

For a positive integer $p$, let $\rho: S^3 \to S^3$ be the $(2\pi/p)$-rotation around the $z$-axis and $B^3_j (j = 0, 1, \ldots, p-1)$ small 3-balls in $S^3$ with $\rho(B^3_j) = B^3_{j+1}$ ($j = 0, 1, \ldots, p-2$) and $\rho(B^3_{p-1}) = B^3_0$. Moreover, let $D_p = (S^3 - \bigcup_{j=0}^{p-1} \text{int} B^3_j) \times \rho S^1$ be the mapping torus with monodromy $\rho$. The compact smooth 4-manifold $D_p$ is bounded by $S^1 \times S^2$ and has the fundamental group $\pi_1 D_p$ isomorphic to $\mathbb{Z}$. The homomorphism $i_\#: \pi_1(S^1 \times S^2) \to \pi_1 D_p$ has index $p$, where $i: S^1 \times S^2 \to D_p$ is the inclusion.

Let $M$ be an oriented closed 3-manifold. If $M$ bounds an oriented compact 4-manifold $V$ such that the fundamental group $\pi_1 V$ is isomorphic to $\mathbb{Z}$ and the homomorphism of $\pi_1$ induced from the inclusion $i: M \to V$ is not trivial, then the first Betti number of $M$ is positive. In this section we shall show that for any given 3-manifold $M$ with $b_1(M) \geq 1$, $M$ bounds an oriented compact smooth 4-manifold $V$ such that $\pi_1 V$ is isomorphic to $\mathbb{Z}$ and $i_\#: \pi_1 M \to \pi_1 V \cong \mathbb{Z}$ is not trivial. To show this, we need the following two operations. Every closed 3-manifold is obtained from $S^3$ by an integral surgery on a link in $S^3$. Let $M$ be obtained by a framed link $L$.

Operation 1. Let $K$ be a component of $L$ with framing $n$ and $c$ a crossing on
a diagram of $K \subset L$. Add a trivial knot $O$ with framing 0 to $L$ at $c$ so that the linking number $\text{lk}(O, K)$ between $O$ and $K$ is zero. See Fig. 1. Let $K'$ be a knot obtained from $K$ by crossing-change at $c$. Then, by the Kirby calculus (or handle-slide), the resultant 3-manifold obtained by this new framed link $L \cup O$ is orientation-preserving homeomorphic to the 3-manifold obtained by a framed link $L'$ containing a new component $O$ with framing 0 and the component $K'$ with framing $n$ instead of $K$ with framing $n$. See Fig. 2.

**Operation 2.** Let $K$ and $L$ be two components of $L$ with framing $m$ and $n$, respectively. Let $c$ be a crossing of $K$ and $L$ on a diagram of $L$. Give the framing 0 to a meridional curve $O$ of $L$. See Fig. 3. Then, by the Kirby calculus (or handle-slide), the resultant 3-manifold obtained by this new framed link $L \cup O$ is orientation-
preserving homeomorphic to the 3-manifold obtained by a framed link \( L' \) which contains a new component \( O \) with framing 0 and which has an opposite crossing at \( c \). See Fig. 4. Note that this operation leaves the knot type of \( K \) invariant, since \( O \) is trivial.

We use Operations 1 and 2 to make a knot trivial and to split geometrically a component of a link from other components, respectively.
Proposition 1. For any positive integer $p$ and for any given $3$-manifold $M$ with $b_1(M) \geq 1$, there exists an oriented compact smooth $4$-manifold $V$ bounded by $M$ such that

1. $\pi_1V$ is isomorphic to $\mathbb{Z}$, and
2. the index, $(\pi_1V : \text{Im} i_\sharp)$, of $\text{Im}\{i_\sharp : \pi_1M \to \pi_1V\}$ in $\pi_1V$ is $p$.

Every oriented $3$-manifold is obtained from $S^3$ by an integral surgery on a link in $S^3$, but this link is not always an algebraically split link. Here, we say that a link $L = K_1 \cup K_2 \cup \cdots \cup K_\mu$ is an algebraically split link if for each pair of distinct components $K_i, K_j (i \neq j)$ of $L$, the linking number $lk(K_i, K_j)$ is zero.

We use the following lemma.

Lemma 1 ([13]). Any integral symmetric matrix is made diagonalizable over $\mathbb{Z}$ by taking block sums of some $1 \times 1$-matrices $(p_j)$.

We can translate Lemma 1 into geometric terms: Let $M$ be an oriented closed $3$-manifold. Then, there are some lens spaces $L(p_j, 1)$ ($j = 1, 2, \cdots, k$) such that after taking connected sums of $L(p_j, 1)$ ($j = 1, 2, \cdots, k$), the $3$-manifold $M \amalg L(p_1, 1) \amalg L(p_2, 1) \amalg \cdots \amalg L(p_k, 1)$ has a surgery description by a framed algebraically split link.

Proof of Proposition 1. By Lemma 1, there are some lens spaces $L(p_j, 1)$ ($j = 1, 2, \cdots, k$) such that the $3$-manifold $M' = M \amalg L(p_1, 1) \amalg L(p_2, 1) \amalg \cdots \amalg L(p_k, 1)$ is obtained by an integral surgery on an algebraically split link $L$. Let $r(\geq 1)$ be the first Betti number of $M$. Then, the linking matrix of $L$ is an $(r + n) \times (r + n)$-matrix

$$
\begin{pmatrix}
0 & \cdots & 0 & \cdots & 0 \\
\vdots & \ddots & \vdots & \ddots & \vdots \\
0 & \cdots & 0 & \cdots & 0 \\
0 & \cdots & 0 & m_1 \\
\vdots & \ddots & \vdots & \ddots & \vdots \\
0 & \cdots & 0 & \cdots & m_n
\end{pmatrix}
$$

where $|m_1m_2 \cdots m_n|$ is not zero and the order of the torsion part of $H_1(M'; \mathbb{Z})$. Generators of $H_1(M'; \mathbb{Z})$ are given by meridional curves of the components of $L$. Let $K_i$ ($i = 1, 2, \cdots, r$) be the components of $L$ with framing $0$ and $L_j$ ($j = 1, 2, \cdots, n$) the other components of $L$. The $3$-manifold $L(p_1, 1) \amalg L(p_2, 1) \amalg \cdots \amalg L(p_k, 1)$ bounds an oriented simply connected compact smooth $4$-manifold $W$, for example the $\mathbb{Z}$-sum of $k$ $D^2$-bundles over $S^2$. Then the smooth $4$-manifold $(M \times I)\amalg (-W)$ is bounded by $M \amalg (-M')$. We shall make $K_1$ a trivial knot which is split geometrically from the other components of $L$. 
Step 1. If $K_1$ is not trivial, then we can make $K_1$ a trivial knot $K'_1$ by a finite sequence of Operation 1 at some crossings of $K_1$. Then the framed link $L$ changes into another framed link $L'$, which is algebraically split. The trivial knot $K'_1$ has framing 0.

In general, $K'_1$ is not split geometrically from the other components of $L'$.

Step 2. By a finite sequence of Operation 2, we can split geometrically $K'_1$ from the other components of $L'$ keeping $K'_1$ trivial and without changing the framing of $K'_1$. By $L''$ we denote the framed link obtained by the operations as above. Let $L''_2$ be the link consisting of the other components of $L''$ except $K'_1$, that is, $L'' = K'_1 \cup L''_2$. Then the 3-manifold given by the framed link $L''$ is $S^1 \times S^2 \# N$, where $N$ is the 3-manifold given by $L''_2$.

Hence it follows that by attaching 2-handles to $M' \times \{1\} \subset M' \times I$ in ways corresponding to Steps 1 and 2, we get an oriented compact smooth 4-manifold $X$ whose boundary is $M' \coprod(-(S^1 \times S^2 \# N))$. Set $Y = ((M \times I) \# (-W)) \cup_{M'} X$. Let $W'$ be an oriented simply connected compact smooth 4-manifold bounded by $N$, for example, the 4-manifold consisting of one 0-handle and some 2-handles given by the
framed link $L''$. Then $Z = Y \cup ((S^1 \times S^2) \times I \cup W')$ is an oriented compact smooth 4-manifold with boundary $\partial Z = M \bigsqcup (-S^1 \times S^2)$. See Schema 1. Now let $V$ be the 4-manifold $Z \cup_D p$, which is an oriented compact smooth 4-manifold with boundary $\partial V = M$. By van Kampen's theorem, $\pi_1 V$ is isomorphic to $Z$. If we let $t$ be a generator of $\pi_1 D_p$, then a loop coming from a meridional curve of $K_1$ represents $t^\pm p$ in $\pi_1 D_p$, and so $(\pi_1 V : \text{Im} t) = p$.

**Example 1.** Let $m$ be an integer. Let $M(m)$ be the homology handle given by the following framed link $K_1 \cup K_2$ in Fig. 5. The link $K_1 \cup K_2$ is an algebraically split link. Let $\tilde{M}(m)$ be the universal abelian covering of $M(m)$, that is, the infinite cyclic covering of $M(m)$ associated to the kernel of the Hurewitz homomorphism $\alpha : \pi_1 M(m) \to H_1(M(m); \mathbb{Z}) \cong \mathbb{Z}$. Then $\tilde{M}(m)$ is obtained from the universal covering $q : \mathbb{R} \times S^2 \to S^1 \times S^2$ by 1-surgeries on the preimage of $K_2$ via $q$ as in Fig. 6. See [14]. By $\Lambda = \mathbb{Z}(t)$ we denote the ring of Laurent polynomials with integer coefficients. Thus $H_1(\tilde{M}(m); \mathbb{Z})$ has a $\Lambda$-module structure by the group of deck transformations and is isomorphic to $\Lambda/(mt^{-1} - (2m-1) + mt)$ as $\Lambda$-modules. Here $(f(t))$ stands for the principal ideal generated by $f(t) \in \Lambda$. Now attach one 2-handle $h^{(2)}$ to $M(m) \times I$ so that the attaching circle of $h^{(2)}$ is a meridional curve of $K_2$ and the framiing of $h^{(2)}$ is zero. Let $W$ be the resultant 4-manifold. By Op-
Fig. 6.

eration 1, it is seen that $W$ is bounded by $M(m) \amalg (-S^1 \times S^2)$. See Fig. 7. Thus $V = W \cup_{S^1 \times S^2} D_p$ is an oriented compact smooth 4-manifold bounded by $M(m)$ with $\pi_1 V \cong \mathbb{Z}$, $(\pi_1 V : \text{Im} i_\#) = p$, and $H_2(V; \mathbb{Z}) \cong \mathbb{Z} \oplus \mathbb{Z}_p$. In §§3 and 4 we show that in the case of $p = 1$ this 4-manifold $V$ gives the minimal second Betti number of all oriented compact topological 4-manifolds $X$ bounded by $M(m)$ with $\pi_1 X \cong \mathbb{Z}$ and $(\pi_1 X : \text{Im} i_\#) = 1$.

We have the following proposition for a 3-manifold $M$ such that $H_1(M; \mathbb{Z})$ has a torsion subgroup.

**Proposition 2.** Let $p$ be any positive integer and $L = K_1 \cup K_2$ a 2-component
framed link such that

1. $K_1$ is a trivial knot,
2. the linking number $lk(K_1, K_2)$ is zero, and
3. the framings of $K_1$ and $K_2$ is $0$ and $n$, respectively.

Let $M$ be the resultant 3-manifold obtained by surgery on the framed link $L$. If $|n| > 1$, then the smooth 4-manifold $V$ constructed in the manner of Example 1 gives the minimal second Betti number of all oriented compact topological 4-manifolds $X$ bounded by $M$ with $\pi_1 X \cong \mathbb{Z}$ and $(\pi_1 X : \text{Im} i) = p$. Note that $H_2(V; \mathbb{Z}) \cong \mathbb{Z} \oplus \mathbb{Z}_p$.

Proof. Suppose that $b_2(V) = 1$ is not minimal. Namely, there is an oriented compact topological 4-manifold $X$ as above with $b_2(X) = 0$. By considering the homology exact sequence of the pair $(X, M)$, we have the following short exact se-
sequence;

\[ 0 \to \mathbb{Z} \to H_2(M; \mathbb{Z}) \to \mathbb{Z}_p \to 0 \to H_1(M; \mathbb{Z}) \to \mathbb{Z} \to \mathbb{Z}_p \to 0. \]

Because of \(|n| > 1\), \(H_1(M; \mathbb{Z})\) has a torsion subgroup. This contradicts that \(H_1(M; \mathbb{Z}) \to \mathbb{Z}\) is injective.

\[ \square \]

3. Minimal second Betti numbers for homology handles

Through §§3 and 4, we consider the case of \(p = 1\), namely, the case where the homomorphisms on \(\pi_1\) induced from inclusions are surjective. If \(M\) is an oriented closed 3-manifold with \(H_*(M; \mathbb{Z}) \cong H_*(S^1 \times S^2; \mathbb{Z})\), then we call \(M\) a homology handle. See [8]. Since a homology handle \(M\) has \(H^1(M; \mathbb{Z}_2) \cong \mathbb{Z}_2\), \(M\) admits two spin structures \(\tau_0\) and \(\tau_1\). By \(\mu(M, \tau)\) we denote the Roholin invariant of \(M\) with respect to a spin structure \(\tau\).

Proposition 3. Let \(M\) be a homology handle with spin structures \(\tau_0\) and \(\tau_1\). Suppose that \(\mu(M, \tau_0) = 0\) and \(\mu(M, \tau_1) = 1\). Then, there is no orientable compact topological spin 4-manifold \(V\) bounded by \(M\) such that \(\pi_1V \cong \mathbb{Z}\) and the homomorphism \(i_\sharp : \pi_1M \to \pi_1V \cong \mathbb{Z}\) is surjective.

Proof. Suppose that there would be such a 4-manifold \(V\). Because of \(\pi_1V \cong \mathbb{Z}\), \(V\) admits two spin structures \(\sigma_0\) and \(\sigma_1\). Since \(i_\sharp : \pi_1M \to \pi_1V \cong \mathbb{Z}\) is surjective, \(\pi_1(V, M) = 0\) and so \(H^1(E(\tau_V), E(\tau_M); \mathbb{Z}_2) = 0\). Here \(E(\tau_M)\) and \(E(\tau_V)\) are the total spaces of the principal \(\text{STop}(3)\)-bundle and the principal \(\text{STop}(4)\)-bundle associated with stable topological tangent bundles over \(M\) and \(V\), respectively. From the following cohomology exact sequence of the pair \((E(\tau_V), E(\tau_M))\),

\[ 0 = H^1(E(\tau_V), E(\tau_M); \mathbb{Z}_2) \to H^1(E(\tau_V); \mathbb{Z}_2) \to H^1(E(\tau_M); \mathbb{Z}_2) \to \delta, \]

if follows that the restrictions of \(\sigma_0\) and \(\sigma_1\) to \(M\) are \(\tau_0\) and \(\tau_1\), say \(\sigma_0|_M = \tau_0\) and \(\sigma_1|_M = \tau_1\). By [5, Chapter 10], we can calculate the Kirby-Siebenmann obstruction \(ks(V) \in H^4(V, M; \mathbb{Z}_2)\) of \(V\) from \((V, \sigma_0)\) and we have that

\[ 8ks(V) \equiv \text{signature}(V) + \mu(M, \tau_0) \pmod{16} \]

\[ \equiv \text{signature}(V) \pmod{16}. \]

From \((V, \sigma_1)\) it follows that

\[ 8ks(V) \equiv \text{signature}(V) + 1 \pmod{16}, \]

and this equation contradicts that one.
For any given homology handle $M$, we would like to investigate the minimal second Betti number of 4-manifolds bounded by $M$.

Let $M$ be a homology handle. By $\beta^{\text{TOP}}(M)$ we denote the minimal second Betti number of all oriented compact topological 4-manifolds $V$ bounded by $M$ such that $\pi_1V$ is isomorphic to $\mathbb{Z}$ and the homomorphism $i_* : \pi_1M \to \pi_1V$ is surjective. Furthermore, we denote by $\beta^{\text{DIFF}}(M)$ the minimal second Betti number of all oriented compact smooth 4-manifolds as above. Then it is clear that $\beta^{\text{DIFF}}(M) \geq \beta^{\text{TOP}}(M) \geq 0$.

**REMARK.** If we define $\beta^{\text{TOP}}(M)$ and $\beta^{\text{DIFF}}(M)$ for a general 3-manifold $M$ in the same manner, then it follows from the homology exact sequence of the pair $(V, M)$ that $\beta^{\text{DIFF}}(M) > \beta^{\text{TOP}}(M) > \text{rank}_2 \pi_1(M; \mathbb{Z}) - 1$.

**Corollary 1.** Let $M$ be a homology handle as in Proposition 3. Then, $\beta^{\text{TOP}}(M) \geq 1$.

**Corollary 2.** Let $\mathbb{L} = K_1 \cup K_2$ be a 2-component framed link such that

1. $K_1$ is a trivial knot,
2. the linking number $\text{lk}(K_1, K_2)$ is 0, and
3. the framings of $K_1$ and $K_2$ is 0 and $\pm 1$, respectively.

Let $M$ be the homology handle obtained by surgery on $\mathbb{L}$. If $M$ admits two spin structures $\tau_0$ and $\tau_1$ with $\mu(M, \tau_0) = 0$ and $\mu(M, \tau_1) = 1$, then $\beta^{\text{DIFF}}(M) = \beta^{\text{TOP}}(M) = 1$.

Proof. We can construct a smooth 4-manifold $V$ bounded by $M$ with $H_2(V; \mathbb{Z}) \cong \mathbb{Z}$ in the same manner as Example 1. Hence, it follows from Corollary 1 that $\beta^{\text{DIFF}}(M) = \beta^{\text{TOP}}(M) = 1$. \hfill $\square$

**Example 2.** Let $M(m)$ be the homology handle in Example 1. If $m$ is odd, then $M(m)$ admits two spin structures $\tau_0$ and $\tau_1$ with $\mu(M, \tau_0) = 0$ and $\mu(M, \tau_1) = 1$. If $m$ is even, then $M(m)$ admits two spin structures $\tau_0$ and $\tau_1$ with $\mu(M, \tau_0) = \mu(M, \tau_1) = 0$, hence, if $m$ is odd, then $\beta^{\text{DIFF}}(M(m)) = \beta^{\text{TOP}}(M(m)) = 1$.

For what homology handle $M$ does it hold that $\beta^{\text{TOP}}(M) = 0$ or $\beta^{\text{DIFF}}(M) = 0$? Note that $\beta^{\text{TOP}}(M) = 0$ if and only if $M$ bounds an oriented compact topological 4-manifold homotopy equivalent to $S^1$. Freedman and Quinn give a necessary and sufficient condition to hold that $\beta^{\text{TOP}}(M) = 0$ in [5, Proposition 11.6A and 11.6C].

**Theorem 2 ([5]).** Let $M$ be a homology handle. Let $C = [\pi_1M, \pi_1M]$ be the commutator subgroup of $\pi_1M$. Then, $\beta^{\text{TOP}}(M) = 0$ if and only if $C$ is perfect.
Since the universal abelian covering $\tilde{M}$ of a homology handle $M$ is the infinite cyclic covering associated to the kernel of the Hurewicz homomorphism $\pi_1 M \to H_1(M; \mathbb{Z}) \cong \mathbb{Z}$, Theorem 2 implies that $\beta^{\text{TOP}}(M) = 0$ if and only if $H_1(\tilde{M}; \mathbb{Z}) = 0$. Furthermore, the group of deck transformation of $\tilde{M}$ gives a $\Lambda$-modules structure to $H_1(M; \Lambda)$ as $\Lambda$-modules. So, one can define the Alexander polynomials $\Delta_M(t) \in \Lambda$ for homology handles $M$ as well as for knots. Kawauchi gave in [8, 9] a characterization of the Alexander polynomials of homology handles and how to calculate the Alexander polynomials. Thus $H_1(\tilde{M}; \mathbb{Z}) = 0$, that is, $\beta^{\text{TOP}}(M) = 0$ if and only if the Alexander polynomial $\Delta_M(t)$ of $M$ is trivial, that is, a unit of $\Lambda$.

4. Minimal second Betti numbers for homology handles obtained by 0-surgery on knots

Consider a homology handle $M$ obtained by 0-surgery on a knot $K$ in $S^3$. Note that the class $\ell \in \pi_1(S^3 - K)$ represented by the preferred longitude for $K$ belongs to the commutator subgroup $[\pi_1(S^3 - K), \pi_1(S^3 - K)]$ of $\pi_1(S^3 - K)$ and that $\pi_1 M$ is isomorphic to $\pi_1(S^3 - K)/\langle \ell \rangle$, where $\langle \ell \rangle$ is the smallest normal subgroup generated by $\ell$. Thus we have the following.

**Lemma 2.** Let $K$ be a knot with exterior $E(K)$, and $\widetilde{E(K)}$ the universal abelian covering of $E(K)$. Let $M$ be the homology handle obtained by 0-surgery on $K$. Then, $H_1(M; \mathbb{Z})$ is isomorphic to $H_1(\widetilde{E(K)}; \mathbb{Z})$ as $\Lambda$-modules. In particular, the Alexander polynomial $\Delta_M(t)$ of $M$ is equal to the Alexander polynomial $\Delta_K(t)$ of $K$ (See Lemma 2.6-(III) in [8].).

Hence, we have the following.

**Corollary 3.** Let $M$ be the homology handle obtained by 0-surgery on a knot $K$. The minimal second Betti number $\beta^{\text{TOP}}(M) = 0$ if and only if the Alexander polynomial $\Delta_K(t)$ of $K$ is trivial.

**Example 3.** Let $M(m)$ be the homology handle in Example 1. In Example 1 we see that $H_1(\tilde{M}(m); \mathbb{Z})$ is isomorphic to $\Lambda/(mt^{-1} - (2m - 1) + mt)$ as $\Lambda$-modules. In fact, it follows from the Kirby calculus that $M(m)$ is also obtained by 0-surgery on the following knot in Fig. 8. Thus the Alexander polynomial for $M(m)$ is $mt^{-1} - (2m - 1) + mt$ and $\beta^{\text{TOP}}(M(m)) \neq 0$. Therefore, in the case when $m$ is even, it also holds that $\beta^{\text{TOP}}(M(m)) = \beta^{\text{DIFF}}(M(m)) = 1$, since we can construct a required 4-manifold in the same manner as Example 1. See Example 2.

We can estimate $\beta^{\text{DIFF}}(M)$ by the unknotted number $u(K)$ of a knot $K$. 
Proposition 4. Let $M$ be the homology handle obtained by 0-surgery on a knot $K$ with unknotting number $u(K)$. Then, $u(K) \geq \beta^\text{DIFF}(M)$. 

Fig. 8.

Fig. 9.
Proof. Note that by the Kirby calculus the 3-manifolds in Fig. 9. are homeomorphic. Let \( u \) be the unknotting number of \( K \). Then after taking cross-changing at certain \( u \) crossings of a diagram of \( K \), \( K \) becomes a trivial knot \( L_0 \). Hence, \( M \) has a surgery description by a framed link \( L = L_0 \cup L_1 \cup \cdots \cup L_u \) such that all \( L_j(j = 0, 1, \cdots, u) \) are trivial knots, the framing of \( L_0 \) is zero and the framings of \( L_j(j = 1, 2, \cdots, u) \) are \( \pm 1 \). See Fig. 10. If we apply Operation 2 to each \( L_j(j = 1, 2, \cdots, u) \), then we get a new framed link \( L' \). See Fig. 11. The 3-manifold given by \( L' \) is \( S^1 \times S^2 \). By attaching \( u \) 2-handles \( h_j^{(2)}(j = 1, 2, \cdots, u) \) as above to \( M \times I \) and identifying one component of the boundary of the resultant smooth 4-manifold with the boundary of \( S^1 \times B^3 \), we get a 4-manifold \( V \) with second Betti number \( u \) and with boundary \( M \) such that \( \pi_1 V \) is isomorphic to \( \mathbb{Z} \) and the homomor-
phism \( i_2 : \pi_1 M \to \pi_1 V \) is surjective. Hence, \( \beta^{\text{DIFF}}(M) \leq u \).

For example, the knots \( K_m \) in Fig. 8 are unknotting number 1 knots. Hence, 
\( 1 = u(K_m) \geq \beta^{\text{DIFF}}(M(m)) \geq \beta^{\text{TOP}}(M(m)) \geq 1 \), and so \( \beta^{\text{TOP}}(M(m)) = \beta^{\text{DIFF}}(M(m)) = 1 \).

We generalize Examples 2 and 3 as follows.

**Theorem 3.** For any positive integer \( n \), there exist infinitely many distinct homology handles \( \{M^{(n)}_m\}_{m \geq 1} \) with \( \beta^{\text{TOP}}(M^{(n)}_m) = \beta^{\text{DIFF}}(M^{(n)}_m) = n \).

To show Theorem 3, we use the local signatures of homology handles, which are introduced by Kawauchi [8] and defined by generalizing local signatures of knots. See also [12]. In [9], Kawauchi considered the embedding problem of 3-manifolds into 4-manifolds. In particular, he gave an estimation of second Betti numbers and signatures of 4-manifolds by local signatures of their boundaries: Let \( M \) be a homology handle.
and $X$ a compact topological 4-manifold bounded by $M$. Then, he showed that for any $a \in [-1, 1]$,

$$(4.1) \quad \left| \Sigma_{x \in (a,1]} \sigma_x(M) \right| \leq b_2(X) + |\text{signature}(X)|.$$ 

Here $\sigma_x(M)$ is a local signature of $M$. Since $b_2(X) + |\text{signature}(X)| \leq 2b_2(X)$, we have

$$(4.2) \quad \left| \Sigma_{x \in (a,1]} \sigma_x(M) \right| \leq 2b_2(X) \quad \text{for any } a \in [-1, 1],$$

and so

$$(4.3) \quad \left| \Sigma_{x \in (a,1]} \sigma_x(M) \right| \leq 2\beta^{\text{TOP}}(M) \quad \text{for any } a \in [-1, 1].$$

Proof of Theorem 3. For each positive integer $m$, let $K_m$ be a knot in Fig. 8. Then, the Alexander polynomial $\Delta_{K_m}(t)$ of $K_m$ is $mt^2 - (2m - 1)t + m$ up to units in $\Lambda$ and the unknotting number $u(K_m)$ of $K_m$ is 1. Because of $\Delta_{K_m}(t)/m = t^2 - 2((2m-1)/(2m))t + 1$, it follows from Assertion 11 in [12] that the signature $\sigma(K_m)$ of $K_m$ is $\pm 2$. Hence, it follows that for the local signature $\sigma_x(K_m)(x \in [-1, 1])$,

$$\sigma_x(K_m) = \begin{cases} \pm 2, & \text{if } x = (2m - 1)/(2m), \\ 0, & \text{if } x \neq (2m - 1)/(2m). \end{cases}$$

Let $K_m^{(n)}$ be the connected sum of $n$ copies of $K_m$, that is, $K_m^{(n)} = K_m \# K_m \# \cdots \# K_m$. Let $M_m^{(n)}$ be the homology handle obtained by 0-surgery on $K_m^{(n)}$. Since $\Delta_{K_m^{(n)}}(t) = (\Delta_{K_m}(t))^n \neq (\Delta_{K_m}(t))^{m'} = \Delta_{K_m^{(m')}}(t)$ ($m \neq m'$), $M_m^{(n)}$ and $M_m^{(n')}$ ($m \neq m'$) are not homeomorphic. Noting that the quadratic form of the universal abelian covering $\tilde{M_m^{(n)}}$ is the orthogonal sum of $n$ copies of the quadratic form of $K_m$, it follows that for the local signature $\sigma_x(M_m^{(n)})(x \in [-1, 1])$,

$$\sigma_x(M_m^{(n)}) = \begin{cases} \pm 2n, & \text{if } x = (2m - 1)/(2m), \\ 0, & \text{if } x \neq (2m - 1)/(2m). \end{cases}$$

Hence, we have

$$\left| \Sigma_{x \in (0,1]} \sigma_x(M_m^{(n)}) \right| = \left| \sigma_{(2m-1)/(2m)}(M_m^{(n)}) \right| = 2n.$$ 

Thus, by the inequality (4.3) we have

$$n = \frac{1}{2} \left| \Sigma_{x \in (0,1]} \sigma_x(M_m^{(n)}) \right| \leq \beta^{\text{TOP}}(M_m^{(n)}).$$
By noting that $u(K_m^{(n)}) \leq n$ because of $u(K_m) = 1$, it follows from Proposition 4 that $\beta^{\text{DIFF}}(M_m^{(n)}) \leq u(K_m^{(n)}) \leq n$. Therefore, $n \leq \beta^{\text{TOP}}(M_m^{(n)}) \leq \beta^{\text{DIFF}}(M_m^{(n)}) \leq n$, and so $\beta^{\text{TOP}}(M_m^{(n)}) = \beta^{\text{DIFF}}(M_m^{(n)}) = n$. \[\square\]

**Remark.**
(1) The unknotting number $u(K_m^{(n)})$ is $n$ because of $n = |\sigma(K_m^{(n)})|/2 \leq u(K_m^{(n)}) \leq n$.

(2) Consider a short exact sequence of $\Lambda$-modules

$$0 \rightarrow E \rightarrow F \rightarrow \Lambda/(f_1) \oplus \Lambda/(f_2) \oplus \cdots \oplus \Lambda/(f_n) \rightarrow 0,$$

where $E$ and $F$ are free $\Lambda$-modules of the same rank. If each $f_{i+1}$ can be divided by $f_i$, then $\text{rank}_\Lambda E \geq n$. Let $V$ be an oriented compact 4-manifold bounded by $M_m^{(n)}$ such that $\pi_1 V \cong \mathbb{Z}$ and the homomorphism $i_1 : \pi_1 M_m^{(n)} \rightarrow \pi_1 V$ is surjective. Then we have the following homology exact sequence with local coefficient $\Lambda$,

$$0 \rightarrow H_2(V; \Lambda) \rightarrow H_2(V, M_m^{(n)}; \Lambda) \rightarrow H_1(M_m^{(n)}, \Lambda) \rightarrow 0.$$ 

The homology groups $H_2(V; \Lambda)$ and $H_2(V, M_m^{(n)}; \Lambda)$ are free $\Lambda$-modules of the same rank. Since $H_1(M_m^{(n)}; \Lambda) \cong \bigoplus_{i=1}^n (\Lambda/(mt-(2m-1)+mt^{1})_i \cong \Lambda/(mt-(2m-1)+mt^{1}) \oplus \cdots \oplus \Lambda/(mt-(2m-1)+mt^{1})$, $\text{rank}_\Lambda H_2(V; \Lambda) = \text{rank}_\Lambda H_2(V, M_m^{(n)}; \Lambda) \geq n$. Hence it follows that $\beta^{\text{TOP}}(M_m^{(n)}) \geq n$.

Next we give two definitions on sliceness of knots.

**Definition 1.** If a knot $K$ bounds a smooth disk $D$ in the 4-ball $B^4$ such that $(B^4, D) \times I$ is a trivial ball pair, then $K$ is a super slice knot. See [7].

For example, untwisted doubles of slice knots are super slice [7].

**Definition 2.** A knot $K$ is pseudo-slice, if there exists a pair $(W, D)$ for $K$ such that $W$ is a smooth 4-manifold homeomorphic to $B^4$ and $D$ is a smooth disk in $W$ bounded by $K$.

**Proposition 5.** Let $K$ be a super slice knot, and $M$ the homology handle obtained by 0-surgery on $K$. Then, $\beta^{\text{TOP}}(M) = \beta^{\text{DIFF}}(M) = 0$.

Proof. Let $D$ be a slice disk for $K$ such that $(B^4, D) \times I$ is a trivial ball pair. Let $N(D)$ be a closed tubular neighborhood of $D$ in $B^4$. Then, $M$ is the boundary of the smooth 4-manifold $V = B^4 - \text{int} N(D)$. The 4-manifold $V$ is homotopy equivalent to $V \times I = B^4 \times I - \text{int} N(D) \times I$. Since $(B^4, D) \times I$ is trivial, $V$ is homotopy equivalent to $S^1$. Thus $V$ is a required 4-manifold. \[\square\]

Is there a difference between $\beta^{\text{TOP}}$ and $\beta^{\text{DIFF}}$? Now we answer this question.
Theorem 4. Let $K$ be a knot which is not pseudo-slice and whose Alexander polynomial $\Delta_K$ is trivial. Let $M$ be the homology handle obtained by 0-surgery on $K$. Then, $0 = \beta^{TOP}(M) < \beta^{DIFF}(M)$.

Proof. Since $\Delta_K$ is trivial, it follows from Corollary 3 that $\beta^{TOP}(M) = 0$. Suppose that $\beta^{DIFF}(M) = 0$. Then $M$ bounds a smooth 4-manifold $V$ homotopy equivalent to $S^4$. By attaching to $M \times I$ one 2-handle $h^{(2)}$ whose attaching circle is a meridian of $K$ and whose framing is zero, we get the 4-manifold $(M \times I) \cup h^{(2)}$ whose boundary is $M \bigsqcup (-S^3)$. See Operation 1. Furthermore, by identifying $\partial V$ with one component $M$ of the boundary of $(M \times I) \cup h^{(2)}$, we get a compact smooth 4-manifold $W$ bounded by $S^3$. Then, since $W$ is simply-connected and $H_\ast(W;\mathbb{Z}) \cong H_\ast(B^4;\mathbb{Z})$, $W$ is homeomorphic to $B^4$. The co-core of the above 2-handle $h^{(2)}$ gives a smooth disk $D$ in $W$ with $\partial(W,D) = (S^3,K)$. Since $K$ is not pseudo-slice, this is a contradiction.

Example 4. In [3], Cochran and Gompf showed that there are untwisted doubles which are not pseudo-slice. For example, the untwisted double $K$ of the trefoil knot is such a knot. Note that the Alexander polynomials of nontrivial untwisted doubles are trivial and their unknotting numbers are 1. Thus, for the homology handle $M$ obtained by 0-surgery on $K$, $1 = u(K) = \beta^{DIFF}(M) > \beta^{TOP}(M) = 0$, and so $1 = \beta^{DIFF}(M) > \beta^{TOP}(M) = 0$.

Example 5. Let $K(-3,5,7)$ be the pretzel knot of type $(-3,5,7)$. Then $K(-3,5,7)$ has a trivial Alexander polynomial. Furthermore, in [6] Fintushel and Stern showed that $K(-3,5,7)$ is not pseudo-slice. Thus, for the homology handle $M$ obtained by 0-surgery on $K(-3,5,7)$, $\beta^{DIFF}(M) > \beta^{TOP}(M) = 0$.

It follows from [11] that $K(-3,5,7)$ is not an unknotting number 1 knot. One can make $K(-3,5,7)$ a trivial knot by crossing-change at certain 3 crossings. Hence, $2 \leq u(K(-3,5,7)) \leq 3$. Thus it follows that $1 \leq \beta^{DIFF}(M) \leq 3$. What is $\beta^{DIFF}(M)$?

References


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