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3-DIMENSIONAL HOMOLOGY HANDLES
AND MINIMAL SECOND BETTI NUMBERS
OF 4-MANIFOLDS

YOSHIHISA SATO

(Received March 25, 1997)

1. Introduction

We consider the following problem:

For a given closed 3-manifold \( M \), what is the minimal second Betti number of all compact 4-manifolds bounded by \( M \)?

If we add the condition that 4-manifolds are simply connected, then the answer about the above problem in the topological category can be seen from the Boyer classification theorem [1],[2]. The Boyer classification theorem states that for an oriented, closed, connected 3-manifold \( M \), a symmetric integral bilinear form \((E, \mathcal{L})\) and a presentation \( \mathcal{P} \) of \( H_*(M; \mathbb{Z}) \) by \((E, \mathcal{L})\), there exists an oriented, compact, simply connected, topological 4-manifold with boundary \( M \) whose intersection form is isomorphic over \( \mathbb{Z} \) to \((E, \mathcal{L})\) and which represents \( \mathcal{P} \) geometrically. Furthermore, Boyer gave the result about the uniqueness of such 4-manifolds up to orientation-preserving homeomorphism. Here a presentation \( \mathcal{P} \) of \( H_*(M; \mathbb{Z}) \) by \((E, \mathcal{L})\) is the following short exact sequence with some algebraic data corresponding to the relationship between the linking form of \( M \) and \((E, \mathcal{L})\), spin structures and the Kirby-Siebenmann obstruction:

\[
0 \to H_2(M; \mathbb{Z}) \to E \xrightarrow{ad(\mathcal{L})} E^* \to H_1(M; \mathbb{Z}) \to 0.
\]

Hence, in the topological category, we can calculate algebraically the minimal second Betti number of all simply connected 4-manifolds bounded by \( M \). The key to this classification theorem is the Freedman theorem [4], and in particular the fact that every homology 3-sphere can bound a contractible compact topological 4-manifold. In the topological category, it follows from this that the minimal second Betti number of all simply connected 4-manifolds bounded by a given homology 3-sphere is zero. However, the Roholin theorem and the gauge theory say that in the smooth category, a homology 3-sphere can not always bound a homology 4-ball, and so the minimal second Betti number of all simply connected 4-manifolds bounded by a homology 3-sphere is not always zero in the smooth category.

If we consider the Boyer theorem with the condition that the fundamental groups of 4-manifolds are isomorphic to the infinite cyclic group instead of simply connect-
edness, then the key seems to be orientable closed 3-manifolds $M$ with the same integral homology groups as $S^1 \times S^2$, which are called homology handles [8]. Of course, the situation changes according as the homomorphisms of $\pi_1$ induced from inclusions are trivial or not. In this paper, we consider the case where such homomorphisms $i_t : \pi_1 M \to \mathbb{Z}$ are surjective, and under this condition we consider the above problem.

By $\beta^{\text{TOP}}(M)$ and $\beta^{\text{DIFF}}(M)$, we denote the minimal second Betti number of such 4-manifolds in the topological category and in the smooth category, respectively. For example, it is clear that $\beta^{\text{TOP}}(S^1 \times S^2) = \beta^{\text{DIFF}}(S^1 \times S^2) = 0$. But it does not always hold that $\beta^{\text{TOP}}(M) = 0$, since there is a homology handle which can not bound a compact topological 4-manifold homotopy equivalent to $S^4$ in contrast with the case of homology 3-spheres. In this paper we show that for any positive integer $n$, there exist infinitely many distinct homology handles $\{M_m^{(n)}\}_{m \in \mathbb{N}}$ with $\beta^{\text{TOP}}(M_m^{(n)}) = \beta^{\text{DIFF}}(M_m^{(n)}) = n$, and furthermore that there exists a difference between $\beta^{\text{TOP}}$ and $\beta^{\text{DIFF}}$.

In §2, we introduce two operations on framed links to construct compact smooth 4-manifolds which are bounded by given 3-manifolds and whose fundamental groups are isomorphic to $\mathbb{Z}$. In §§3 and 4, we investigate $\beta^{\text{TOP}}$ and $\beta^{\text{DIFF}}$ of certain homology handles, and in particular homology handles obtained by 0-surgery on knots. In §4, we show that $\beta^{\text{TOP}}$ and $\beta^{\text{DIFF}}$ are functions onto $\mathbb{N} \cup \{0\}$ and there is a difference between $\beta^{\text{TOP}}$ and $\beta^{\text{DIFF}}$.

Through this paper, we suppose that manifolds are connected and oriented, and we denote the closed interval $[0,1]$ by $I$. Furthermore, the symbol $b_i$ stands for the $i$-th Betti number.

### 2. Two kinds of 2-handle attachings

For a positive integer $p$, let $\rho : S^3 \to S^3$ be the $(2\pi/p)$-rotation around the $z$-axis and $B^3_j (j = 0, 1, \ldots, p-1)$ small 3-balls in $S^3$ with $\rho(B^3_j) = B^3_{j+1}$ ($j = 0, 1, \ldots, p-2$) and $\rho(B^3_{p-1}) = B^3_0$. Moreover, let $D_p = (S^3 - \bigcup_{j=0}^{p-1} \text{int}B^3_j) \times \rho S^1$ be the mapping torus with monodromy $\rho$. The compact smooth 4-manifold $D_p$ is bounded by $S^1 \times S^2$ and has the fundamental group $\pi_1 D_p$ isomorphic to $\mathbb{Z}$. The homomorphism $i_t : \pi_1(S^1 \times S^2) \to \pi_1 D_p$ has index $p$, where $i : S^1 \times S^2 \to D_p$ is the inclusion.

Let $M$ be an oriented closed 3-manifold. If $M$ bounds an oriented compact 4-manifold $V$ such that the fundamental group $\pi_1 V$ is isomorphic to $\mathbb{Z}$ and the homomorphism of $\pi_1$ induced from the inclusion $i : M \to V$ is not trivial, then the first Betti number of $M$ is positive. In this section we shall show that for any given 3-manifold $M$ with $b_1(M) \geq 1$, $M$ bounds an oriented compact smooth 4-manifold $V$ such that $\pi_1 V$ is isomorphic to $\mathbb{Z}$ and the inclusion $i_t : \pi_1 M \to \pi_1 V \cong \mathbb{Z}$ is not trivial. To show this, we need the following two operations. Every closed 3-manifold is obtained from $S^3$ by an integral surgery on a link in $S^3$. Let $M$ be obtained by a framed link $L$.

**Operation 1.** Let $K$ be a component of $L$ with framing $n$ and $c$ a crossing on
a diagram of $K \subset L$. Add a trivial knot $O$ with framing 0 to $L$ at $c$ so that the linking number $lk(O, K)$ between $O$ and $K$ is zero. See Fig. 1. Let $K'$ be a knot obtained from $K$ by crossing-change at $c$. Then, by the Kirby calculus (or handle-slide), the resultant 3-manifold obtained by this new framed link $L \cup O$ is orientation-preserving homeomorphic to the 3-manifold obtained by a framed link $L'$ containing a new component $O$ with framing 0 and the component $K'$ with framing $n$ instead of $K$ with framing $n$. See Fig. 2.

**Operation 2.** Let $K$ and $L$ be two components of $L$ with framing $m$ and $n$, respectively. Let $c$ be a crossing of $K$ and $L$ on a diagram of $L$. Give the framing 0 to a meridional curve $O$ of $L$. See Fig. 3. Then, by the Kirby calculus (or handle-slide), the resultant 3-manifold obtained by this new framed link $L \cup O$ is orientation-
preserving homeomorphic to the 3-manifold obtained by a framed link $L'$ which contains a new component $O$ with framing 0 and which has an opposite crossing at $c$. See Fig. 4. Note that this operation leaves the knot type of $K$ invariant, since $O$ is trivial.

We use Operations 1 and 2 to make a knot trivial and to split geometrically a component of a link from other components, respectively.
Proposition 1. For any positive integer \( p \) and for any given 3-manifold \( M \) with \( b_1(M) \geq 1 \), there exists an oriented compact smooth 4-manifold \( V \) bounded by \( M \) such that

(1) \( \pi_1 V \) is isomorphic to \( \mathbb{Z} \), and
(2) the index, \((\pi_1 V : \text{Im} \iota)\), of \( \text{Im}\{i_\sharp : \pi_1 M \to \pi_1 V\} \) in \( \pi_1 V \) is \( p \).

Every oriented 3-manifold is obtained from \( S^3 \) by an integral surgery on a link in \( S^3 \), but this link is not always an algebraically split link. Here, we say that a link \( L = K_1 \cup K_2 \cup \cdots \cup K_\mu \) is an algebraically split link if for each pair of distinct components \( K_i, K_j (i \neq j) \) of \( L \), the linking number \( \text{lk}(K_i, K_j) \) is zero.

We use the following lemma.

Lemma 1 ([13]). Any integral symmetric matrix is made diagonalizable over \( \mathbb{Z} \) by taking block sums of some \( 1 \times 1 \)-matrices \( (p_j) \).

We can translate Lemma 1 into geometric terms: Let \( M \) be an oriented closed 3-manifold. Then, there are some lens spaces \( L(p_j, 1) \) \((j = 1, 2, \cdots, k)\) such that after taking connected sums of \( L(p_j, 1) \) \((j = 1, 2, \cdots, k)\), the 3-manifold \( M_1 = M \sqcup L(p_1, 1) \sqcup L(p_2, 1) \sqcup \cdots \sqcup L(p_k, 1) \) has a surgery description by a framed algebraically split link.

Proof of Proposition 1. By Lemma 1, there are some lens spaces \( L(p_j, 1) \) \((j = 1, 2, \cdots, k)\) such that the 3-manifold \( M' = M \sqcup L(p_1, 1) \sqcup L(p_2, 1) \sqcup \cdots \sqcup L(p_k, 1) \) is obtained by an integral surgery on an algebraically split link \( L \). Let \( r(\geq 1) \) be the first Betti number of \( M \). Then, the linking matrix of \( L \) is an \((r + n) \times (r + n)\)-matrix

\[
\begin{pmatrix}
0 & \cdots & 0 & \cdots & 0 \\
\vdots & \ddots & \vdots & \ddots & \vdots \\
0 & \cdots & 0 & \cdots & 0 \\
0 & \cdots & 0 & m_1 & \\
\vdots & \ddots & \ddots & \ddots & \\
0 & \cdots & 0 & \cdots & m_n
\end{pmatrix}
\]

where \( |m_1 m_2 \cdots m_n| \) is not zero and the order of the torsion part of \( H_1(M'; \mathbb{Z}) \). Generators of \( H_1(M'; \mathbb{Z}) \) are given by meridional curves of the components of \( L \). Let \( K_i \) \((i = 1, 2, \cdots, r)\) be the components of \( L \) with framing 0 and \( L_j \) \((j = 1, 2, \cdots, n)\) the other components of \( L \). The 3-manifold \( L(p_1, 1) \sqcup L(p_2, 1) \sqcup \cdots \sqcup L(p_k, 1) \) bounds an oriented simply connected compact smooth 4-manifold \( W \), for example the \( \mathbb{Z} \)-sum of \( k D^2 \)-bundles over \( S^2 \). Then the smooth 4-manifold \( (M \times I) \sqcup (-W) \) is bounded by \( M \sqcup (-M') \). We shall make \( K_1 \) a trivial knot which is split geometrically from the other components of \( L \).
Step 1. If $K_1$ is not trivial, then we can make $K_1$ a trivial knot $K'_1$ by a finite sequence of Operation 1 at some crossings of $K_1$. Then the framed link $L$ changes into another framed link $L'$, which is algebraically split. The trivial knot $K'_1$ has framing 0.

In general, $K'_1$ is not split geometrically from the other components of $L'$.

Step 2. By a finite sequence of Operation 2, we can split geometrically $K'_1$ from the other components of $L'$ keeping $K'_1$ trivial and without changing the framing of $K'_1$. By $L''$ we denote the framed link obtained by the operations as above. Let $L''_2$ be the link consisting of the other components of $L''$ except $K'_1$, that is, $L'' = K'_1 \cup L''_2$. Then the 3-manifold given by the framed link $L''$ is $S^1 \times S^2 \# N$, where $N$ is the 3-manifold given by $L''_2$.

Hence it follows that by attaching 2-handles to $M' \times \{1\} \subset M' \times I$ in ways corresponding to Steps 1 and 2, we get an oriented compact smooth 4-manifold $X$ whose boundary is $M' \bigsqcup(-(S^1 \times S^2 \# N))$. Set $Y = ((M \times I) \natural (-W)) \cup_{M'} X$. Let $W'$ be an oriented simply connected compact smooth 4-manifold bounded by $N$, for example, the 4-manifold consisting of one 0-handle and some 2-handles given by the
framed link $L''$. Then $Z = Y \cup ((S^1 \times S^2) \times I \cup W')$ is an oriented compact smooth 4-manifold with boundary $\partial Z = M \bigsqcup (-S^1 \times S^2)$. See Schema 1. Now let $V$ be the 4-manifold $Z \cup \partial D_p$, which is an oriented compact smooth 4-manifold with boundary $\partial V = M$. By van Kampen’s theorem, $\pi_1 V$ is isomorphic to $Z$. If we let $t$ be a generator of $\pi_1 D_p$, then a loop coming from a meridional curve of $K_1$ represents $t^\pm p$ in $\pi_1 D_p$, and so $(\pi_1 V : \text{Im} t) = p$.

**Example 1.** Let $m$ be an integer. Let $M(m)$ be the homology handle given by the following framed link $K_1 \cup K_2$ in Fig. 5. The link $K_1 \cup K_2$ is an algebraically split link. Let $\tilde{M}(m)$ be the universal abelian covering of $M(m)$, that is, the infinite cyclic covering of $M(m)$ associated to the kernel of the Hurewicz homomorphism $\alpha : \pi_1 M(m) \to H_1(M(m); \mathbb{Z}) \cong \mathbb{Z}$. Then $\tilde{M}(m)$ is obtained from the universal covering $q : \mathbb{R} \times S^2 \to S^1 \times S^2$ by 1-surgeries on the preimage of $K_2$ via $q$ as in Fig. 6. See [14]. By $\Lambda = \mathbb{Z}(t)$ we denote the ring of Laurent polynomials with integer coefficients. Thus $H_1(\tilde{M}(m); \mathbb{Z})$ has a $\Lambda$-module structure by the group of deck transformations and is isomorphic to $\Lambda/(mt^{-1} - (2m-1) + mt)$ as $\Lambda$-modules. Here $(f(t))$ stands for the principal ideal generated by $f(t) \in \Lambda$. Now attach one 2-handle $h^{(2)}$ to $M(m) \times I$ so that the attaching circle of $h^{(2)}$ is a meridional curve of $K_2$ and the framing of $h^{(2)}$ is zero. Let $W$ be the resultant 4-manifold. By Op-
Fig. 6.

Fig. 6.

In Fig. 1, it is seen that $W$ is bounded by $M(m) \coprod (-S^1 \times S^2)$. See Fig. 7. Thus $V = W \cup_{S^1 \times S^2} D_p$ is an oriented compact smooth 4-manifold bounded by $M(m)$ with $\pi_1 V \cong \mathbb{Z}$, $(\pi_1 V : \text{Im} \tilde{i}_\delta) = p$, and $H_2(V; \mathbb{Z}) \cong \mathbb{Z} \oplus \mathbb{Z}_{p}$. In §§3 and 4 we show that in the case of $p = 1$ this 4-manifold $V$ gives the minimal second Betti number of all oriented compact topological 4-manifolds $X$ bounded by $M(m)$ with $\pi_1 X \cong \mathbb{Z}$ and $(\pi_1 X : \text{Im} \tilde{i}_\delta) = 1$.

We have the following proposition for a 3-manifold $M$ such that $H_1(M; \mathbb{Z})$ has a torsion subgroup.

**Proposition 2.** Let $p$ be any positive integer and $L = K_1 \cup K_2$ a 2-component
framed link such that
(1) $K_1$ is a trivial knot,
(2) the linking number $\text{lk}(K_1, K_2)$ is zero, and
(3) the framings of $K_1$ and $K_2$ is 0 and $n$, respectively.

Let $M$ be the resultant 3-manifold obtained by surgery on the framed link $L$. If $|n| > 1$, then the smooth 4-manifold $V$ constructed in the manner of Example 1 gives the minimal second Betti number of all oriented compact topological 4-manifolds $X$ bounded by $M$ with $\pi_1 X \cong \mathbb{Z}$ and $(\pi_1 X : \text{Im} i) = p$. Note that $H_2(V; \mathbb{Z}) \cong \mathbb{Z} \oplus \mathbb{Z}_p$.

Proof. Suppose that $b_2(V) = 1$ is not minimal. Namely, there is an oriented compact topological 4-manifold $X$ as above with $b_2(X) = 0$. By considering the homology exact sequence of the pair $(X, M)$, we have the following short exact sequence...
Because of $|n| > 1$, $H_1(M;\mathbb{Z})$ has a torsion subgroup. This contradicts that $H_1(M;\mathbb{Z}) \rightarrow \mathbb{Z}$ is injective. 

3. Minimal second Betti numbers for homology handles

Through §§3 and 4, we consider the case of $p = 1$, namely, the case where the homomorphisms on $\pi_1$ induced from inclusions are surjective. If $M$ is an oriented closed 3-manifold with $H_*(M;\mathbb{Z}) \cong H_*(S^1 \times S^2;\mathbb{Z})$, then we call $M$ a homology handle. See [8]. Since a homology handle $M$ has $H^1(M;\mathbb{Z}_2) \cong \mathbb{Z}_2$, $M$ admits two spin structures $\tau_0$ and $\tau_1$. By $\mu(M, \tau)$ we denote the Roholin invariant of $M$ with respect to a spin structure $\tau$.

Proposition 3. Let $M$ be a homology handle with spin structures $\tau_0$ and $\tau_1$. Suppose that $\mu(M, \tau_0) = 0$ and $\mu(M, \tau_1) = 1$. Then, there is no orientable compact topological spin 4-manifold $V$ bounded by $M$ such that $\pi_1 V \cong \mathbb{Z}$ and the homomorphism $i^*: \pi_1 M \rightarrow \pi_1 V \cong \mathbb{Z}$ is surjective.

Proof. Suppose that there would be such a 4-manifold $V$. Because of $\pi_1 V \cong \mathbb{Z}$, $V$ admits two spin structures $\sigma_0$ and $\sigma_1$. Since $i^*: \pi_1 M \rightarrow \pi_1 V \cong \mathbb{Z}$ is surjective, $\pi_1(V, M) = 0$ and so $H^1(E(\tau_V), E(\tau_M);\mathbb{Z}_2) = 0$. Here $E(\tau_M)$ and $E(\tau_V)$ are the total spaces of the principal $\text{STop}(3)$-bundle and the principal $\text{STop}(4)$-bundle associated with stable topological tangent bundles over $M$ and $V$, respectively. From the following cohomology exact sequence of the pair $(E(\tau_V), E(\tau_M))$,

$$0 = H^1(E(\tau_V), E(\tau_M);\mathbb{Z}_2) \rightarrow H^1(E(\tau_V);\mathbb{Z}_2) \rightarrow H^1(E(\tau_M);\mathbb{Z}_2) \delta,$$

if follows that the restrictions of $\sigma_0$ and $\sigma_1$ to $M$ are $\tau_0$ and $\tau_1$, say $\sigma_0|_M = \tau_0$ and $\sigma_1|_M = \tau_1$. By [5, Chapter 10], we can calculate the Kirby-Siebenmann obstruction $ks(V) \in H^4(V, M;\mathbb{Z}_2)$ of $V$ from $(V, \sigma_0)$ and we have that

$$8ks(V) \equiv \text{signature}(V) + \mu(M, \tau_0) \pmod{16}$$

$$\equiv \text{signature}(V) \pmod{16}.$$

From $(V, \sigma_1)$ it follows that

$$8ks(V) \equiv \text{signature}(V) + 1 \pmod{16},$$

and this equation contradicts that one.
For any given homology handle $M$, we would like to investigate the minimal second Betti number of 4-manifolds bounded by $M$.

Let $M$ be a homology handle. By $\beta^{\text{TOP}}(M)$ we denote the minimal second Betti number of all oriented compact topological 4-manifolds $V$ bounded by $M$ such that $\pi_1 V$ is isomorphic to $\mathbb{Z}$ and the homomorphism $i_* : \pi_1 M \to \pi_1 V$ is surjective. Furthermore, we denote by $\beta^{\text{DIFF}}(M)$ the minimal second Betti number of all oriented compact smooth 4-manifolds as above. Then it is clear that $\beta^{\text{DIFF}}(M) \geq \beta^{\text{TOP}}(M) \geq 0$.

**Remark.** If we define $\beta^{\text{TOP}}(M)$ and $\beta^{\text{DIFF}}(M)$ for a general 3-manifold $M$ in the same manner, then it follows from the homology exact sequence of the pair $(V, M)$ that $\beta^{\text{DIFF}}(M) > \beta^{\text{TOP}}(M) > \text{rank}_\mathbb{Z} H_1(M; \mathbb{Z}) - 1$.

**Corollary 1.** Let $M$ be a homology handle as in Proposition 3. Then, $\beta^{\text{TOP}}(M) \geq 1$.

**Corollary 2.** Let $L = K_1 \cup K_2$ be a 2-component framed link such that
1. $K_1$ is a trivial knot,
2. the linking number $\text{lk}(K_1, K_2)$ is 0, and
3. the framings of $K_1$ and $K_2$ is 0 and $\pm 1$, respectively.

Let $M$ be the homology handle obtained by surgery on $L$. If $M$ admits two spin structures $\tau_0$ and $\tau_1$ with $\mu(M, \tau_0) = 0$ and $\mu(M, \tau_1) = 1$, then $\beta^{\text{DIFF}}(M) = \beta^{\text{TOP}}(M) = 1$.

**Proof.** We can construct a smooth 4-manifold $V$ bounded by $M$ with $H_2(V; \mathbb{Z}) \cong \mathbb{Z}$ in the same manner as Example 1. Hence, it follows from Corollary 1 that $\beta^{\text{DIFF}}(M) = \beta^{\text{TOP}}(M) = 1$. □

**Example 2.** Let $M(m)$ be the homology handle in Example 1. If $m$ is odd, then $M(m)$ admits two spin structures $\tau_0$ and $\tau_1$ with $\mu(M, \tau_0) = 0$ and $\mu(M, \tau_1) = 1$. If $m$ is even, then $M(m)$ admits two spin structures $\tau_0$ and $\tau_1$ with $\mu(M, \tau_0) = \mu(M, \tau_1) = 0$, Hence, if $m$ is odd, then $\beta^{\text{DIFF}}(M(m)) = \beta^{\text{TOP}}(M(m)) = 1$.

For what homology handle $M$ does it hold that $\beta^{\text{TOP}}(M) = 0$ or $\beta^{\text{DIFF}}(M) = 0$? Note that $\beta^{\text{TOP}}(M) = 0$ if and only if $M$ bounds an oriented compact topological 4-manifold homotopy equivalent to $S^1$. Freedman and Quinn give a necessary and sufficient condition to hold that $\beta^{\text{TOP}}(M) = 0$ in [5, Proposition 11.6A and 11.6C].

**Theorem 2 ([5]).** Let $M$ be a homology handle. Let $C = [\pi_1 M, \pi_1 M]$ be the commutator subgroup of $\pi_1 M$. Then, $\beta^{\text{TOP}}(M) = 0$ if and only if $C$ is perfect.
Since the universal abelian covering $\tilde{M}$ of a homology handle $M$ is the infinite cyclic covering associated to the kernel of the Hurewicz homomorphism $\pi_1 M \rightarrow H_1(M;\mathbb{Z}) \cong \mathbb{Z}$, $H_1(\tilde{M};\mathbb{Z})$ is isomorphic to $C/[C,C]$. Theorem 2 implies that $\beta^{\text{TOP}}(M) = 0$ if and only if $H_1(\tilde{M};\mathbb{Z}) = 0$. Furthermore, the group of deck transformation of $\tilde{M}$ gives a $\Lambda$-modules structure to $H_1(\tilde{M};\mathbb{Z})$, which is isomorphic to $H_1(M;\Lambda)$ as $\Lambda$-modules. So, one can define the Alexander polynomials $\Delta_M(t) \in \Lambda$ for homology handles $M$ as well as for knots. Kawauchi gave in [8, 9] a characterization of the Alexander polynomials of homology handles and how to calculate the Alexander polynomials. Thus $H_1(\tilde{M};\mathbb{Z}) = 0$, that is, $\beta^{\text{TOP}}(M) = 0$ if and only if the Alexander polynomial $\Delta_M(t)$ of $M$ is trivial, that is, a unit of $\Lambda$.

4. Minimal second Betti numbers for homology handles obtained by 0-surgery on knots

Consider a homology handle $M$ obtained by 0-surgery on a knot $K$ in $S^3$. Note that the class $\ell \in \pi_1(S^3 - K)$ represented by the preferred longitude for $K$ belongs to the commutator subgroup $[\pi_1(S^3 - K), \pi_1(S^3 - K)]$ of $\pi_1(S^3 - K)$ and that $\pi_1 M$ is isomorphic to $\pi_1(S^3 - K)/\langle \ell \rangle$, where $\langle \ell \rangle$ is the smallest normal subgroup generated by $\ell$. Thus we have the following.

**Lemma 2.** Let $K$ be a knot with exterior $E(K)$, and $E(\tilde{K})$ the universal abelian covering of $E(K)$. Let $M$ be the homology handle obtained by 0-surgery on $K$. Then, $H_1(M;\mathbb{Z})$ is isomorphic to $H_1(E(\tilde{K});\mathbb{Z})$ as $\Lambda$-modules. In particular, the Alexander polynomial $\Delta_M(t)$ of $M$ is equal to the Alexander polynomial $\Delta_K(t)$ of $K$ (See Lemma 2.6-(III) in [8].).

Hence, we have the following.

**Corollary 3.** Let $M$ be the homology handle obtained by 0-surgery on a knot $K$. The minimal second Betti number $\beta^{\text{TOP}}(M) = 0$ if and only if the Alexander polynomial $\Delta_K(t)$ of $K$ is trivial.

**Example 3.** Let $M(m)$ be the homology handle in Example 1. In Example 1 we see that $H_1(\tilde{M}(m);\mathbb{Z})$ is isomorphic to $\Lambda/(mt^{-1} - (2m - 1) + mt)$ as $\Lambda$-modules. In fact, it follows from the Kirby calculus that $M(m)$ is also obtained by 0-surgery on the following knot in Fig. 8. Thus the Alexander polynomial for $M(m)$ is $mt^{-1} - (2m - 1) + mt$ and $\beta^{\text{TOP}}(M(m)) \neq 0$. Therefore, in the case when $m$ is even, it also holds that $\beta^{\text{TOP}}(M(m)) = \beta^{\text{DIFF}}(M(m)) = 1$, since we can construct a required 4-manifold in the same manner as Example 1. See Example 2.

We can estimate $\beta^{\text{DIFF}}(M)$ by the unknotting number $u(K)$ of a knot $K$. 

Proposition 4. Let $M$ be the homology handle obtained by 0-surgery on a knot $K$ with unknotting number $u(K)$. Then, $u(K) \geq \beta_{\text{DIFF}}(M)$. 
Proof. Note that by the Kirby calculus the 3-manifolds in Fig. 9. are homeomorphic. Let \( u \) be the unknotting number of \( K \). Then after taking cross-changing at certain \( u \) crossings of a diagram of \( K \), \( K \) becomes a trivial knot \( L_0 \). Hence, \( M \) has a surgery description by a framed link \( L = L_0 \cup L_1 \cup \cdots \cup L_u \) such that all \( L_j(j = 0, 1, \cdots, u) \) are trivial knots, the framing of \( L_0 \) is zero and the framings of \( L_j(j = 1, 2, \cdots, u) \) are ±1. See Fig. 10. If we apply Operation 2 to each \( L_j(j = 1, 2, \cdots, u) \), then we get a new framed link \( L' \). See Fig. 11. The 3-manifold given by \( L' \) is \( S^1 \times S^2 \). By attaching \( u \) 2-handles \( h_j^{(2)}(j = 1, 2, \cdots, u) \) as above to \( M \times I \) and identifying one component of the boundary of the resultant smooth 4-manifold with the boundary of \( S^1 \times B^3 \), we get a 4-manifold \( V \) with second Betti number \( u \) and with boundary \( M \) such that \( \pi_1 V \) is isomorphic to \( \mathbb{Z} \) and the homomor-
Fig. 11.

For example, the knots $K_m$ in Fig. 8 are unknotting number 1 knots. Hence, $1 = u(K_m) \geq \beta_{\text{DIFF}}(M(m)) \geq \beta_{\text{TOP}}(M(m)) \geq 1$, and so $\beta_{\text{TOP}}(M(m)) = \beta_{\text{DIFF}}(M(m)) = 1$.

We generalize Examples 2 and 3 as follows.

**Theorem 3.** For any positive integer $n$, there exist infinitely many distinct homology handles $\{M_m^{(n)}\}_{m \geq 1}$ with $\beta_{\text{TOP}}(M_m^{(n)}) = \beta_{\text{DIFF}}(M_m^{(n)}) = n$.

To show Theorem 3, we use the local signatures of homology handles, which are introduced by Kawauchi [8] and defined by generalizing local signatures of knots. See also [12]. In [9], Kawauchi considered the embedding problem of 3-manifolds into 4-manifolds. In particular, he gave an estimation of second Betti numbers and signatures of 4-manifolds by local signatures of their boundaries: Let $M$ be a homology handle
and $X$ a compact topological 4-manifold bounded by $M$. Then, he showed that for any $a \in [-1, 1]$,

$$\left| \sum_{x \in (a, 1]} \sigma_x(M) \right| \leq b_2(X) + |\text{signature}(X)|.$$  \hfill (4.1)

Here $\sigma_x(M)$ is a local signature of $M$. Since $b_2(X) + |\text{signature}(X)| \leq 2b_2(X)$, we have

$$\left| \sum_{x \in (a, 1]} \sigma_x(M) \right| \leq 2b_2(X) \quad \text{for any} \ a \in [-1, 1],$$  \hfill (4.2)

and so

$$\left| \sum_{x \in (a, 1]} \sigma_x(M) \right| \leq 2\beta^{\text{TOP}}(M) \quad \text{for any} \ a \in [-1, 1].$$  \hfill (4.3)

Proof of Theorem 3. For each positive integer $m$, let $K_m$ be a knot in Fig. 8. Then, the Alexander polynomial $\Delta_{K_m}(t)$ of $K_m$ is $mt^2 - (2m - 1)t + m$ up to units in $\mathbb{Z}$ and the unknotting number $u(K_m)$ of $K_m$ is 1. Because of $\Delta_{K_m}(t)/m = t^2 - 2\{2m - 1/(2m)\}t + 1$, it follows from Assertion 11 in [12] that the signature $\sigma(K_m)$ of $K_m$ is $\pm 2$. Hence, it folds that for the local signature $\sigma_x(K_m)(x \in [-1, 1])$,

$$\sigma_x(K_m) = \begin{cases} \pm 2, & \text{if} \ \ x = (2m - 1)/(2m), \\ 0, & \text{if} \ \ x \neq (2m - 1)/(2m). \end{cases}$$

Let $K_m^{(n)}$ be the connected sum of $n$ copies of $K_m$, that is, $K_m^{(n)} = K_m \# K_m \# \cdots \# K_m$. Let $M_m^{(n)}$ be the homology handle obtained by 0-surgery on $K_m^{(n)}$. Since $\Delta_{K_m^{(n)}}(t) = (\Delta_{K_m}(t))^n \neq (\Delta_{K_m'(t)})^n = \Delta_{K_m'(t)}(t)$ ($m \neq m'$), $M_m^{(n)}$ and $M_{m'}^{(n)}$ ($m \neq m'$) are not homeomorphic. Noting that the quadratic form of the universal abelian covering $\tilde{M}_m^{(n)}$ is the orthogonal sum of $n$ copies of the quadratic form of $K_m$, it follows that for the local signature $\sigma_x(M_m^{(n)})(x \in [-1, 1])$,

$$\sigma_x(M_m^{(n)}) = \begin{cases} \pm 2n, & \text{if} \ \ x = (2m - 1)/(2m), \\ 0, & \text{if} \ \ x \neq (2m - 1)/(2m). \end{cases}$$

Hence, we have

$$\left| \sum_{x \in (0, 1]} \sigma_x(M_m^{(n)}) \right| = \left| \sigma_{(2m - 1)/(2m)}(M_m^{(n)}) \right| = 2n.$$  \hfill (4.3)

Thus, by the inequality (4.3) we have

$$n = \frac{1}{2} \left| \sum_{x \in (0, 1]} \sigma_x(M_m^{(n)}) \right| \leq \beta^{\text{TOP}}(M_m^{(n)}).$$
By noting that $u(K_{m}^{(n)}) \leq n$ because of $u(K_{m}) = 1$, it follows from Proposition 4 that $\beta^{\text{DIFF}}(M_{m}^{(n)}) \leq u(K_{m}^{(n)}) \leq n$. Therefore, $n \leq \beta^{\text{TOP}}(M_{m}^{(n)}) \leq \beta^{\text{DIFF}}(M_{m}^{(n)}) \leq n$, and so $\beta^{\text{TOP}}(M_{m}^{(n)}) = \beta^{\text{DIFF}}(M_{m}^{(n)}) = n$. \hfill \Box

**Remark.**

1. The unknotting number $u(K_{m}^{(n)})$ is $n$ because of $n = |\sigma(K_{m}^{(n)})|/2 \leq u(K_{m}^{(n)}) \leq n$.

2. Consider a short exact sequence of $\Lambda$-modules

$$0 \to E \to F \to \Lambda/(f_{1}) \oplus \Lambda/(f_{2}) \oplus \cdots \oplus \Lambda/(f_{n}) \to 0,$$

where $E$ and $F$ are free $\Lambda$-modules of the same rank. If each $f_{i+1}$ can be divided by $f_{i}$, then rank$_{\Lambda}E \geq n$. Let $V$ be an oriented compact 4-manifold bounded by $M_{m}^{(n)}$ such that $\pi_{1}V \cong \mathbb{Z}$ and the homomorphism $i_{V} : \pi_{1}M_{m}^{(n)} \to \pi_{1}V$ is surjective. Then we have the following homology exact sequence with local coefficient $\Lambda$,

$$0 \to H_{2}(V;\Lambda) \to H_{2}(V,M_{m}^{(n)};\Lambda) \to H_{1}(M_{m}^{(n)};\Lambda) \to 0.$$

The homology groups $H_{2}(V;\Lambda)$ and $H_{2}(V,M_{m}^{(n)};\Lambda)$ are free $\Lambda$-modules of the same rank. Since $H_{1}(M_{m}^{(n)};\Lambda) \cong \bigoplus_{i=1}^{n} (\Lambda/(mt-(2m-1)+mt^{-1}))i = \Lambda/(mt-(2m-1)+mt^{-1}) \oplus \cdots \oplus \Lambda/(mt-(2m-1)+mt^{-1})$, rank$_{\Lambda}H_{2}(V;\Lambda) = \text{rank}_{\Lambda}H_{2}(V,M_{m}^{(n)};\Lambda) \geq n$. Hence it follows that $\beta^{\text{TOP}}(M_{m}^{(n)}) \geq n$.

Next we give two definitions on sliceness of knots.

**Definition 1.** If a knot $K$ bounds a smooth disk $D$ in the 4-ball $B^{4}$ such that $(B^{4},D) \times I$ is a trivial ball pair, then $K$ is a super slice knot. See [7].

For example, untwisted doubles of slice knots are super slice [7].

**Definition 2.** A knot $K$ is pseudo-slice, if there exists a pair $(W,D)$ for $K$ such that $W$ is a smooth 4-manifold homeomorphic to $B^{4}$ and $D$ is a smooth disk in $W$ bounded by $K$.

**Proposition 5.** Let $K$ be a super slice knot, and $M$ the homology handle obtained by 0-surgery on $K$. Then, $\beta^{\text{TOP}}(M) = \beta^{\text{DIFF}}(M) = 0$.

Proof. Let $D$ be a slice disk for $K$ such that $(B^{4},D) \times I$ is a trivial ball pair. Let $N(D)$ be a closed tubular neighborhood of $D$ in $B^{4}$. Then, $M$ is the boundary of the smooth 4-manifold $V = B^{4} - \text{int}N(D)$. The 4-manifold $V$ is homotopy equivalent to $V \times I = B^{4} \times I - \text{int}N(D) \times I$. Since $(B^{4},D) \times I$ is trivial, $V$ is homotopy equivalent to $S^{1}$. Thus $V$ is a required 4-manifold. \hfill \Box

Is there a difference between $\beta^{\text{TOP}}$ and $\beta^{\text{DIFF}}$? Now we answer this question.
Theorem 4. Let \( K \) be a knot which is not pseudo-slice and whose Alexander polynomial \( \Delta_K \) is trivial. Let \( M \) be the homology handle obtained by 0-surgery on \( K \). Then, \( 0 = \beta_{\text{TOP}}(M) < \beta_{\text{DIFF}}(M) \).

Proof. Since \( \Delta_K \) is trivial, it follows from Corollary 3 that \( \beta_{\text{TOP}}(M) = 0 \). Suppose that \( \beta_{\text{DIFF}}(M) = 0 \). Then \( M \) bounds a smooth 4-manifold \( V \) homotopy equivalent to \( S^4 \). By attaching to \( M \times I \) one 2-handle \( h(2) \) whose attaching circle is a meridian of \( K \) and whose framing is zero, we get the 4-manifold \( (M \times I) \cup h(2) \) whose boundary is \( M \cup (-S^3) \). See Operation 1. Furthermore, by identifying \( \partial V \) with one component \( M \) of the boundary of \( (M \times I) \cup h(2) \), we get a compact smooth 4-manifold \( W \) bounded by \( S^3 \). Then, since \( W \) is simply-connected and \( H_*(W; \mathbb{Z}) \cong H_*(B^4; \mathbb{Z}) \), \( W \) is homeomorphic to \( B^4 \). The co-core of the above 2-handle \( h(2) \) gives a smooth disk \( D \) in \( W \) with \( \partial(W,D) = (S^3, K) \). Since \( K \) is not pseudo-slice, this is a contradiction.

Example 4. In [3], Cochran and Gompf showed that there are untwisted doubles which are not pseudo-slice. For example, the untwisted double \( K \) of the trefoil knot is such a knot. Note that the Alexander polynomials of nontrivial untwisted doubles are trivial and their unknotting numbers are 1. Thus, for the homology handle \( M \) obtained by 0-surgery on \( K \), \( 1 = u(K) \geq \beta_{\text{DIFF}}(M) > \beta_{\text{TOP}}(M) = 0 \), and so \( 1 = \beta_{\text{DIFF}}(M) > \beta_{\text{TOP}}(M) = 0 \).

Example 5. Let \( K(-3,5,7) \) be the pretzel knot of type \((-3,5,7)\). Then \( K(-3,5,7) \) has a trivial Alexander polynomial. Furthermore, in [6] Fintushel and Stern showed that \( K(-3,5,7) \) is not pseudo-slice. Thus, for the homology handle \( M \) obtained by 0-surgery on \( K(-3,5,7) \), \( \beta_{\text{DIFF}}(M) > \beta_{\text{TOP}}(M) = 0 \).

It follows from [11] that \( K(-3,5,7) \) is not an unknotting number 1 knot. One can make \( K(-3,5,7) \) a trivial knot by crossing-change at certain 3 crossings. Hence, \( 2 \leq u(K(-3,5,7)) \leq 3 \). Thus it follows that \( 1 \leq \beta_{\text{DIFF}}(M) \leq 3 \). What is \( \beta_{\text{DIFF}}(M) \)?

References

3-DIMENSIONAL HOMOLOGY HANDLES


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