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SPECTRAL BOUNDS FOR NON-SMOOTH PERTURBATIONS OF THE LANDAU HAMILTONIAN

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Abstract

In this paper we study spectral estimates for perturbations of the Landau Hamiltonian by a pseudo-differential operator with non-smooth Weyl symbol in a modulation space. We obtain an upper bound for the counting function of the eigenvalues in a spectral gap.

1. Introduction

1.1. Perturbed Landau Hamiltonian. We recall first some classical results. Let H_0 be the self-adjoint realization in $L^2(\mathbf{R}^2)$ of the second order partial differential operator

$$\left(\frac{1}{i}\frac{\partial}{\partial x}+\frac{1}{2}y\right)^2+\left(\frac{1}{i}\frac{\partial}{\partial y}-\frac{1}{2}x\right)^2$$

initially defined on $\mathcal{S}(\mathbf{R}^2)$; this operator is usually called Landau Hamiltonian. The spectrum of H_0 consists of eigenvalues $\Lambda_q := 2q+1$ of infinite multiplicity, the Landau levels (see 4.1).

Let $V = \mathrm{Op}^w(b)$ be a bounded selfadjoint pseudo-differential operator (ΨDO). Under the general assumption that VH_0^{-1} is a compact operator, the Kato-Rellich theorem and the Weyl perturbation theorem imply that the operator $H_0 - V$ is selfadjoint and that

$$\sigma_{ess}(H_0 - V) = \sigma_{ess}(H_0) = \sigma(H_0)\ .$$

Hence the discrete spectrum of $H_0 - V$ consists of eigenvalues of finite multiplicity; these eigenvalues can accumulate only to the Landau levels.

Suppose furthermore that the operator V is non-negative. Fix $q \geq 1$ and let E' be a fixed real number in the gap $(\Lambda_{q-1}, \Lambda_q)$ and E a positive real number such that $\Lambda_{q-1} < E' < \Lambda_q - E < \Lambda_q$. Denote by $N(E', \Lambda_q - E; H_0 - V)$ the number of eigenvalues of the perturbed operator $H_0 - V$ lying in the open interval $(E', \Lambda_q - E)$; for brevity we set

$$N_q(E) := N(E', \Lambda_q - E; H_0 - V). \tag{1.1}$$

Similarly for $q = 0$, we put $N_0(E) = N(-\infty, \Lambda_0 - E; H_0 - V)$. The behaviour of the function N_q has been extensively studied under various assumptions on the perturbation V. The most precise results have been obtained when V is a local smooth potential with derivatives decaying polynomially or exponentially (see [13],[15]). Recently one of the main results proved in [4] concerns the case where the Weyl symbol of V is smooth and belongs to

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a Hörmander-Shubin class $\Gamma_\rho^m(\mathbf{R}^4)$ of negative order. The authors obtain Weyl's law type asymptotics when $E \downarrow 0^+$.

1.2. Main results. The aim of this paper is to investigate the behaviour of the function N_q defined in (1.1) when the perturbation V is a bounded ΨDO. In particular we are interested in perturbation by an integral operator whose kernel is square-integrable. It is then convenient to consider the scale of Sobolev-Shubin classes $Q^s(\mathbf{R}^4)$ (see 2.5) for $s \geq 0$ as spaces of symbols or kernels. These classes are particular cases of weighted modulation spaces.

Denote by $\mathbb{P}_q$, $q \in \mathbf{Z}_+$, the spectral projection of H_0 corresponding to the eigenvalue Λ_q. A common feature of the papers cited above is the important fact that the spectral properties of the eigenvalues of $H_0 - V$ accumulating near the Landau level Λ_q is governed (in a sense which will be stated more precisely, see section 5) by the compact operator $\mathbb{P}_q V \mathbb{P}_q$; this operator is called the effective Hamiltonian associated with the Landau level Λ_q.

Assume V is a ΨDO with Weyl symbol $b \in L^2(\mathbf{R}^4)$. Evidently $\mathbb{P}_q V \mathbb{P}_q$ is a Hilbert-Schmidt operator. But this property is not sharp. Our first result proves that, under additional assumptions on the symbol of V, the effective Hamiltonian belongs to a Schatten class $\mathbf{S}_p$ with $p \leq 2$. Let us denote by $L^2_s(\mathbf{R}^4)$ the L^2-space with weight $w_s(z, w) = \left(1 + |z|^2 + |w|^2\right)^{s/2}$ and by $\mathcal{F}_2$ the partial Fourier transform on $L^2(\mathbf{R}^{2d})$ (see precise definition in subsection 1.4).

Theorem 1.1. *Let $b \in L^2_s(\mathbf{R}^4)$ $(s \geq 0)$. Suppose furthermore that $\mathcal{F}_2 b \in L^2_s(\mathbf{R}^4)$. Then the effective Hamiltonian $\mathbb{P}_q V \mathbb{P}_q$ belongs to the Schatten class $\mathbf{S}_p(L^2(\mathbf{R}^2))$ for $p > 2/(s + 1)$. In particular it is a trace-class operator for $s > 1$.*

The second result concerns an upper bound for the counting function N_q defined in (1.1).

Theorem 1.2. *We suppose that b satisfies the same assumptions as in* Theorem 1.1 *and that furthermore $V \geq 0$. Then*

$$N_q(E) = O(E^{-\frac{2}{s+1}}) \quad (E \downarrow 0^+) . \tag{1.2}$$

For these two results we use earlier results concerning eigenvalues of ΨDO with symbol in Sobolev-Shubin classes (see [11]).

The paper is organized as follows. Section 2 contains the backgrounds of phase space analysis. We list here some important properties of the Fourier-Wigner and Wigner transforms. Section 3 is devoted to some results of phase space analysis needed in the following section. In Section 4 we apply the results proved in Section 3 to the Landau Hamiltonian and its perturbations. We define in particular the effective Hamiltonian which is the principal object of our work. Section 5 contains the proof of the main theorem on the number of bound states in a gap of the essential spectrum.

1.3. Comments. The technics used in [13] and [4] cannot be used here because there is no symbolic calculus with symbols in Sobolev-Shubin classes. Instead we must use phase-space analysis technics (see Section 3). Similarly the precise asymptotics for ΨDO's with negative order smooth symbols in Hörmander-Shubin classes proved in [5] and used in [13] and [4] cannot be used. Instead we use the results for eigenvalues of ΨDO with symbol in Sobolev-Shubin classes in [11].

Moreover it can be noted that the operator H_0 is unitary equivalent to the operator L selfadjoint realization of the partial differential operator

$$\left(\frac{1}{i}\frac{\partial}{\partial x}-\frac{1}{2}y\right)^2+\left(\frac{1}{i}\frac{\partial}{\partial y}+\frac{1}{2}x\right)^2 .$$

This operator is called twisted Laplacian and plays a major role in harmonic analysis on the Heisenberg group. The spectral analysis of the operators L and H_0 is easily performed with the special Hermite functions (see Section 2.4 for the definitions).

1.4. Notations. For $f, g \in L^2(\mathbf{R}^d)$, (f, g) denotes the standard scalar product in $L^2(\mathbf{R}^d)$; it is linear in the first argument.

The standard symplectic form on $\mathbf{R}^{2d}$ is defined by $\sigma(x,\xi;y,\eta) := \xi.y - x.\eta$. Let J be the linear map on $\mathbf{R}^{2d}$ defined by $J(x,\xi) := (\xi,-x)$; J is a symplectic map and $\sigma(x,\xi;y,\eta) = J(x,\xi).(y,\eta)$ where $u.v$ denote the standard scalar product on $\mathbf{R}^d$. For $z = (x,\xi) \in \mathbf{R}^{2d}$ we set $\overline{z} := (x,-\xi)$.

Let us recall some standard notations in spectral theory. For a bounded operator T on a Hilbert space with range $R(T) := T(\mathbf{H})$, rank T is the dimension of $R(T)$, $N(T)$ is the kernel of T, i.e. $\{x \in \mathbf{H} : Tx = 0\}$, $\|T\|$ is the standard norm on $\mathcal{L}(\mathbf{H})$ and T^* is the adjoint of T. Let A be a selfadjoint operator acting in a Hilbert space $\mathbb{H}$. The sets $\rho(A)$, $\sigma(A)$, $\sigma_{ess}(A)$, $\sigma_{disc}(A)$ are respectively the resolvent set, the spectrum, the essential spectrum and the discrete spectrum of A. For $\Omega \in \mathfrak{B}(\mathbf{R})$, the Borel σ-algebra on $\mathbf{R}$, $E_\Omega(A)$ is the spectral projection corresponding to the Borel set Ω. If Ω is a relatively compact Borel subset inside an open gap of the essential spectrum of A, then rank $E_\Omega(A)$ is the (finite) number of eigenvalues, counted with multiplicity, of A lying in Ω. Let T be a compact self-adjoint operator in $\mathbb{H}$, $\lambda_j^+(T)$ be the positive eigenvalues of T, arranged in descending order, counting multiplicity. For $s > 0$ we set

$$n_+(s,T) := \text{card}\,\{j \in \mathbb{Z}_+ : \lambda_j^+(T) > s\} .$$

For any compact operator T we define the singular values of T by

$$s_j(T) := \lambda_j^+(T^*T)^{1/2},\ j \geq 1 .$$

For $p \geq 1$ the compact operator T belongs to the Schatten class $\mathbf{S}_p(\mathbf{H})$ if

$$\|T\|_p := \left(\sum_{j=1}^{+\infty} s_j(T)^p\right)^{1/p}$$

is finite.

For f belonging to the Schwartz class $S(\mathbf{R}^d)$, the Fourier transform of f, denoted by $\mathcal{F}f$ or $\widehat{f}$, is defined by

$$\mathcal{F}(f)(\xi) = (2\pi)^{-\frac{d}{2}} \int e^{-i\xi.x} f(x)\,dx .$$

The partial Fourier transforms are defined respectively by

$$\mathcal{F}_1F(\xi,y) := (2\pi)^{-\frac{d}{2}} \int e^{-i\xi.x} F(x,y)dx$$

and

$$\mathcal{F}_2F(x,\eta) := (2\pi)^{-\frac{d}{2}} \int e^{-i\eta.y} F(x,y)dy$$

for F in the Schwartz class $S(\mathbf{R}^{2d})$; $\overline{\mathcal{F}}_1$ and $\overline{\mathcal{F}}_2$ are the inverse partial Fourier transforms.

For a complex-valued function f defined on $\mathbf{R}^d$, we put $f^*(x) := \overline{f(-x)}$.

2. Fourier-Wigner transform

2.1. Basic properties. Let R be the Schrödinger representation of the Heisenberg group defined by

$$R(z,t) = e^{it}\rho(z)$$

with

$$\rho(z).f(y) := e^{\frac{i}{2}x.\xi}e^{ix.y}f(y+\xi), \qquad z=(x,\xi)\in\mathbf{R}^{2d},\ y\in\mathbf{R}^d,\ f\in L^2(\mathbf{R}^d)\ .$$

The composition law for the projective representation ρ is the following:

$$\rho(z_1)\circ\rho(z_2) = e^{\frac{i}{2}\sigma(z_1,z_2)}\rho(z_1+z_2), \qquad z_1,z_2\in\mathbf{R}^{2d}\ . \tag{2.1}$$

It follows that

$$\rho(z)^* = \rho(z)^{-1} = \rho(-z)\ . \tag{2.2}$$

If $f,g\in L^2(\mathbf{R}^d)$ we define the Fourier-Wigner transform $V(f,g)$ by

$$V(f,g)(z) := (2\pi)^{-\frac{d}{2}}(\rho(z)f,g), \quad z\in\mathbf{R}^{2d}\ .$$

Proposition 2.1. (i) *For all $f,g\in L^2(\mathbf{R}^d)$ we have*

$$V(f,g)(x,\xi) = (2\pi)^{-\frac{d}{2}}\int e^{ix.y}f\left(y+\frac{1}{2}\xi\right)\overline{g\left(y-\frac{1}{2}\xi\right)}dy\ .$$

(ii) *Let U_L be the mixing operator defined on $L^2(\mathbf{R}^{2d})$ by*

$$U_LF(x,\xi) := F\left(x+\frac{1}{2}\xi, x-\frac{1}{2}\xi\right)\ .$$

Then, for all $f,g\in L^2(\mathbf{R}^d)$ we have

$$V(f,g) = \overline{\mathcal{F}_1}U_L(f\otimes\bar{g})\ .$$

(iii) *(The Moyal identity) For all $f_1,\ f_2,\ g_1,\ g_2\in L^2(\mathbf{R}^d)$, we have*

$$(V(f_1,g_1),V(f_2,g_2)) = (f_1,f_2)\overline{(g_1,g_2)}\ .$$

(iv)

$$V(g,f) = V(f,g)^* \qquad f,g\in L^2(\mathbf{R}^d)\ .$$

Furthermore if the functions f and g are real-valued, if $z=(x,\xi)\in\mathbf{R}^{2d}$ and $\bar{z}=(x,-\xi)$, then

$$V(g,f)(z) = V(f,g)(\bar{z})\ .$$

Proof. (i) [17] p.10; (ii) follows from (i); (iii) is a consequence of (ii) since U_L and $\mathcal{F}_1$ are unitary operators.

We prove now (iv): by definition and (2.2), we have

$$V(f,g)(-z) = (2\pi)^{-\frac{d}{2}}(\rho(-z)f,g) = (2\pi)^{-\frac{d}{2}}(f,\rho(z)g),$$

hence

$$\overline{V(f,g)(-z)} = (2\pi)^{-\frac{d}{2}}(\rho(z)g,f) = V(g,f)(z),$$

which proves the first part of (iv). Furthermore

$$\overline{V(f,g)(-z)} = (2\pi)^{-\frac{d}{2}}\overline{(\rho(-z)f,g)} = (2\pi)^{-\frac{d}{2}}\int \overline{\rho(-z)f}\; g\, dy\ .$$

A straightforward verification proves that for all $z \in \mathbf{R}^{2d}$ and $f \in L^2(\mathbf{R}^d)$ we have $\overline{\rho(z)f} = \rho(-\bar{z})\bar{f}$. Consequently

$$\overline{V(f,g)(-z)} = (2\pi)^{-\frac{d}{2}}\int \rho(\bar{z})\bar{f}(y)\; g(y)dy\ .$$

If furthermore f and g are real-valued, then

$$\begin{aligned}\overline{V(f,g)(-z)} &= (2\pi)^{-\frac{d}{2}}\int \rho(\bar{z})f(y)\;\bar{g}(y)dy\\ &= (2\pi)^{-\frac{d}{2}}(\rho(\bar{z})f,g))\\ &= V(f,g)(\bar{z})\ .\end{aligned}$$

□

By easy computations taking into account (2.1), we obtain the following

Lemma 2.1. *For all f, $g \in L^2(\mathbf{R}^d)$ and for all z_1, $z_2 \in \mathbf{R}^{2d}$ we have*

$$V(\rho(z_1)f,\rho(z_2)g)(z) = e^{\frac{i}{2}\sigma(z_1,z_2)}e^{i\sigma(z,\frac{1}{2}(z_1+z_2))}V(f,g)(z+(z_1-z_2))\ .$$

From this lemma we deduce the following result which is similar to formula (14.34) in [9].

Proposition 2.2.

$$V(V(f,g),V(\phi_1,\phi_2))(z,\zeta) = V(f,\phi_1)\left(Jz+\frac{1}{2}\zeta\right)\overline{V(g,\phi_1)\left(Jz-\frac{1}{2}\zeta\right)}\ .$$

Proof. By definition of the Fourier-Wigner transform we have

$$V(V(f,g),V(\phi_1,\phi_2))(z,\zeta) := (2\pi)^{-d}\left(\rho(z,\zeta).V(f,g),V(\phi_1,\phi_2)\right)_{L^2(\mathbf{R}^{2d})}$$

$$= (2\pi)^{-d}e^{\frac{i}{2}z.\zeta}\int V(f,g)(w+\zeta)\overline{e^{-iz.w}V(\phi_1,\phi_2)(w)}dw\ .$$

As particular cases of Lemma 2.1 we obtain

$$\begin{cases} V(f,g)(w+\zeta) = V\left(\rho\left(\frac{1}{2}\zeta\right)f,\rho\left(-\frac{1}{2}\zeta\right)g\right)(w)\\[2ex] e^{-iz.w}V(\phi_1,\phi_2)(w) = V(\rho(-Jz)\phi_1,\rho(-Jz)\phi_2)(w)\end{cases}$$

By use of the Moyal identity, we deduce

$$V(V(f,g),V(\phi_1,\phi_2))(z,\zeta) = (2\pi)^{-d}e^{\frac{i}{2}z.\zeta}\left(\rho\left(\frac{1}{2}\zeta\right)f,\rho\left(-Jz\right).\phi_1\right)\overline{\left(\rho\left(-\frac{1}{2}\zeta\right)g,\rho(-Jz).\phi_2\right)}$$

$$= (2\pi)^{-d} e^{\frac{i}{2}z.\zeta} \left(\rho(Jz)\rho\left(\frac{1}{2}\zeta\right) f, \phi_1\right) \overline{\left(\rho(Jz)\rho\left(-\frac{1}{2}\zeta\right) g, \phi_2\right)}$$

and by a new application of (2.1) and a simplification of the complex exponents, we obtain the announced equality. □

We define the Wigner transform of $f,\ g \in L^2(\mathbf{R}^d)$ by:

$$W(f,g)(x,\xi) := (2\pi)^{-\frac{d}{2}} \int e^{-i\xi.y} f\left(x + \frac{1}{2}y\right) \overline{g\left(x - \frac{1}{2}y\right)} dy\ .$$

Proposition 2.3. *For all $f,\ g \in L^2(\mathbf{R}^d)$, we have*

$$W(f,g) = \mathcal{F}_2 U_L(f \otimes \bar{g}) = \mathcal{F}(V(f,g))\ .$$

Proof. For the first equality, see [9]. The second equality results of Proposition 2.1 since $\mathcal{F} = \mathcal{F}_2\mathcal{F}_1$. □

Proposition 2.4.

$$W(V(f,g), V(\phi_1,\phi_2))(z,\zeta) = W(f,\phi_1)\left(\zeta + \frac{1}{2}Jz\right) \overline{W(g,\phi_2)\left(\zeta - \frac{1}{2}Jz\right)}$$

Proof. By the last proposition we get

$$W(V(f,g), V(\phi_1,\phi_2))(z,\zeta) = (2\pi)^{-2d} \int e^{-i(z,\zeta).(v,w)} V(V(f,g), V(\phi_1,\phi_2))(v,w) dv dw$$

$$= (2\pi)^{-2d} \int e^{-i(z.v+\zeta.w)} V(f,\phi_1)\left(Jv + \frac{1}{2}w\right) \overline{V(g,\phi_1)\left(Jv - \frac{1}{2}w\right)} dv dw\ .$$

We consider the change of variables defined by

$$\begin{cases} Jv + \frac{1}{2}w &= v' \\ Jv - \frac{1}{2}w &= w' \end{cases}$$

Then

$$z.v + \zeta.w = (\zeta + \frac{1}{2}Jz).v' + (-\zeta + \frac{1}{2}Jz).w'.$$

We deduce that

$$W(V(f,g), V(\phi_1,\phi_2))(z,\zeta) =$$

$$(2\pi)^{-d} \int e^{-i(\zeta+\frac{1}{2}Jz).v'} V(f,\phi_1)(v') dv'\ (2\pi)^{-d} \int e^{i(\zeta-\frac{1}{2}Jz).w'} V(g,\phi_2)(w') dw'$$

$$= W(f,\phi_1)\left(\zeta + \frac{1}{2}Jz\right) \overline{W(g,\phi_2)\left(\zeta - \frac{1}{2}Jz\right)}\ .$$ □

2.2. Weyl transform. We summarize the main properties of the Weyl quantization we will use. We omit certain proofs for classical results (see [17]).

Theorem 2.1. *There exists a unique bounded operator* $W : L^2(\mathbf{R}^{2d}) \mapsto \mathcal{L}(L^2(\mathbf{R}^d))$ *with the following properties :*
(i) *Denoting* $W(a) = Op^w(a)$*, then for all* $a \in L^2(\mathbf{R}^{2d})$*, for all* $f, g \in L^2(\mathbf{R}^d)$ *we have*

$$(Op^w(a)f, g) = (2\pi)^{-d} \int \widehat{a}(z)\, V(f, g)(z)dz$$

and

$$\|Op^w(a)\| \leq (2\pi)^{-\frac{d}{2}} \|a\| \ .$$

(ii) *Furthermore*

$$(Op^w(a)f, g) = (2\pi)^{-\frac{d}{2}} \int a(z)\, W(f, g)(z)dz = (2\pi)^{-\frac{d}{2}} (a, W(g, f))$$

and

$$(Op^w(\widehat{a})f, g) = (2\pi)^{-\frac{d}{2}} (a, V(g, f))$$

(iii) *For all* $a \in L^2(\mathbf{R}^{2d})$ *the operator* $Op^w(a)$ *is a Hilbert-Schmidt operator. Let* k *be the kernel of this operator; then*

$$a = (2\pi)^{\frac{d}{2}} \mathcal{F}_2 U_L\, k \ .$$

In particular

$$\|Op^w(a)\|_{S_2} = \|k\|_{L^2} = (2\pi)^{-\frac{d}{2}} \|a\|_{L^2} \ .$$

Remark Let $\phi, \psi \in L^2(\mathbf{R}^d)$ and $f \in L^2(\mathbf{R}^d)$. We put

$$\Pi_{\psi,\phi} f := (f, \phi)\psi \ .$$

Then $\Pi_{\psi,\phi} = S_k = \mathrm{Op}^w(a)$ with $a = W(\psi, \phi)$ and $k = \psi \otimes \bar{\phi}$.

The following proposition will be particularly useful in the next section.

Proposition 2.5. *Let* $a \in L^2(\mathbf{R}^{2d})$ *and* $f, g \in L^2(\mathbf{R}^d)$*. Then*

$$V(Op^w(a)f, g) = (2\pi)^{\frac{d}{2}} Op^w(\widetilde{a}).V(f, g)$$

where $\widetilde{a}(x, \xi) := a(\xi + \frac{1}{2}Jx)$*.*

Proof. The subspace generated by the Fourier-Wigner transforms $V(\phi, \psi)$ when $\phi, \psi \in L^2(\mathbf{R}^d)$ is dense in $L^2(\mathbf{R}^{2d})$. Thus it suffices to prove that for all $\phi, \psi \in L^2(\mathbf{R}^d)$

$$(V(\mathrm{Op}^w(a)f, g), V(\phi, \psi)) = (2\pi)^{\frac{d}{2}} (\mathrm{Op}^w(\widetilde{a}).V(f, g), V(\phi, \psi)) \ .$$

By Theorem 2.1 (ii), Proposition 2.4 and a linear change of variables, we obtain

$$(\mathrm{Op}^w(\widehat{a}).V(f, g), V(\phi, \psi)) = (2\pi)^{-d} (\widetilde{a}, W(V(\phi, \psi), V(f, g)))$$

$$= (2\pi)^{-d} \iint a\Big(\xi + \frac{1}{2}Jx\Big) \overline{W(\phi, f)\Big(\xi + \frac{1}{2}Jx\Big)} W(\psi, g)\Big(\xi - \frac{1}{2}Jx\Big) dx d\xi$$

$$= (2\pi)^{-d} \iint a(u)\overline{W(\phi, f)(u)}W(\psi, g)(v)dudv = (2\pi)^{-d}(a, W(\phi, f)) \int W(\psi, g)(v)dv \, .$$

But

$$\int W(\psi, g)(v)dv = \int 1(v)W(\psi, g)(v)dv = (\psi, g) = \overline{(g, \psi)} \, .$$

Therefore by Proposition 2.1(ii) and the Moyal identity we get

$$\begin{aligned} (\mathrm{Op}^w(\widetilde{a})V(f, g), V(\phi, \psi)) &= (2\pi)^{-d}(a, W(\phi, f))\overline{(g, \psi)} \\ &= (2\pi)^{-\frac{d}{2}}(\mathrm{Op}^w(a)f, \phi)\overline{(g, \psi)} \\ &= \left((2\pi)^{-\frac{d}{2}} V\left(\mathrm{Op}^w(a)f, g\right), V(\phi, \psi)\right) \, . \end{aligned}$$

□

2.3. Twisted convolution. Let F and G be measurable functions defined on $\mathbf{R}^{2d}$.We define the twisted convolution $F \underset{\sigma}{*} G$ of F and G by

$$(F \underset{\sigma}{*} G)(z) := \int F(z - w)G(w)e^{\frac{i}{2}\sigma(z,w)}dw, \qquad z \in \mathbf{R}^{2d} \tag{2.3}$$

$$= \int F(w)G(z - w)e^{-\frac{i}{2}\sigma(z,w)}dw \tag{2.4}$$

whenever the function of w in the integral is integrable. The notation $F\natural G$ is also used. The following property is classical ([6]).

Proposition 2.6. *If $F, G \in L^2(\mathbf{R}^{2d})$, then so is $F \underset{\sigma}{*} G$ and*

$$\|F \underset{\sigma}{*} G\|_{L^2(\mathbf{R}^{2d})} \leq \|F\|_{L^2(\mathbf{R}^{2d})} \, \|G\|_{L^2(\mathbf{R}^{2d})} \, .$$

Before stating the next proposition we must recall some classical notations. Let $\omega : \mathbf{R}^{2d} \to (0, \infty)$ be a submultiplicative weight defined on $\mathbf{R}^{2d}$, which means that for all $z, w \in \mathbf{R}^{2d}$

$$\omega(z + w) \leq \omega(z)\, \omega(w) \, .$$

Let $s \in \mathbf{R}$; the standard weight ω_s is defined on $\mathbf{R}^{2d}$ by

$$\omega_s(x, \xi) = \omega_s(z) = (1 + |z|)^s$$

is submultiplicative for $s \geq 0$.

A weight v is said to be ω-moderate if for all $z \in \mathbf{R}^{2d}$

$$v(z + w) \leq Cv(z)\omega(w) \, .$$

We frequently use the weight v_s defined by

$$v_s(z) := (1 + |z|^2)^{s/2} \, .$$

Let $1 \leq p, q \leq +\infty$. The weighted mixed Lebesgue space $L^{p,q}_\omega(\mathbf{R}^{2d})$ is the set of all measurable functions F for which

$$\|F\|_{L^{p,q}_\omega(\mathbf{R}^{2d})} := \left(\int \left(\int |F(x, \xi)|^p \omega(x, \xi)^p dx\right)^{q/p} d\xi\right)^{1/q} \, .$$

is finite. We note L^p_ω in the case where $p = q$.

The following property is known ([9]); we recall it for further explicit references.

Proposition 2.7. *Let p satisfying* $1 \leq p < +\infty$*, v a submultiplicative weight and ω a v-moderate weight. Let* $F \in L^p_\omega(\mathbf{R}^{2d})$*,* $G \in L^1_v(\mathbf{R}^{2d})$ *and* $z, w \in \mathbf{R}^d$*. Denote by* T_z *the translation operator defined by* $T_zF(w) := F(w - z)$ *for* $F \in L^2(\mathbf{R}^d)$*. Then*
(i) $T_z(F) \in L^p_\omega(\mathbf{R}^{2d})$ *and*

$$\|T_z(F)\|_{L^p_\omega(\mathbf{R}^{2d})} \leq Cv(z)\|F\|_{L^p_\omega(\mathbf{R}^{2d})} .$$

(ii) *The function* $F * G$ *belongs to* $L^p_\omega(\mathbf{R}^{2d})$ *and*

$$\|F * G\|_{L^p_\omega(\mathbf{R}^{2d})} \leq C\|F\|_{L^p_\omega(\mathbf{R}^{2d})}\|G\|_{L^1_v(\mathbf{R}^{2d})} .$$

(iii) *The same result is true when convolution is replaced by twisted convolution in* (ii)*: the function* $F \underset{\sigma}{*} G$ *belongs to* $L^p_\omega(\mathbf{R}^{2d})$ *and*

$$\|F \underset{\sigma}{*} G\|_{L^p_\omega(\mathbf{R}^{2d})} \leq C\|F\|_{L^p_\omega(\mathbf{R}^{2d})}\|G\|_{L^1_v(\mathbf{R}^{2d})} .$$

Proof. (i) Since the weight ω is v-moderate

$$\begin{aligned}
\int |T_z(F)|^p\omega(w)^p dw &= \int |F(w - z)|^p\omega(w)^p dw \\
&\leq \int |F(w)|^p\omega(z + w)^p dw \\
&\leq C^p v(z)^p \int |F(w)|^p\omega(w)^p dw .
\end{aligned}$$

(ii) Since $L^p_\omega(\mathbf{R}^{2d})$ is invariant by translation, we can define a vector-valued map $\phi : \mathbf{R}^{2d} \mapsto L^p_\omega(\mathbf{R}^{2d})$ by setting

$$\phi(w) := T_w(F)G(w) .$$

An application of (i) yields to the following inequality

$$\|\phi(w)\|_{L^p_\omega(\mathbf{R}^{2d})} \leq C\|F\|_{L^p_\omega(\mathbf{R}^{2d})}|G(w)|v(w) .$$

Hence the function ϕ is Bochner integrable and

$$\Big\| \int \phi(w)dw \Big\|_{L^p_\omega(\mathbf{R}^{2d})} \leq \int \|\phi(w)\|_{L^p_\omega(\mathbf{R}^{2d})}dw \leq C\|F\|_{L^p_\omega(\mathbf{R}^{2d})}\|G\|_{L^1_v(\mathbf{R}^{2d})} .$$

Since the convolution can be rewrited as the Bochner integral

$$F * G = \int \phi(w)dw$$

we obtain (ii).

(iii) For all z, $w \in \mathbf{R}^{2d}$ we have

$$|(F \underset{\sigma}{*} G)(z)| \leq (|F| * |G|)(z) .$$

Again with (ii) we obtain that $F \underset{\sigma}{*} G \in L^p_\omega(\mathbf{R}^{2d})$ and

$$\|F \underset{\sigma}{*} G\|_{L^p_\omega(\mathbf{R}^{2d})} \leq \||F|*|G|\|_{L^p_\omega(\mathbf{R}^{2d})}$$

which proves (iii). □

By use of the twisted convolution, we get a result similar to Proposition 2.5.

Proposition 2.8. *Let $a \in L^2(\mathbf{R}^d)$ and $f, g \in L^2(\mathbf{R}^d)$. Then*

$$V(Op^w(\widehat{a})f, g) = (2\pi)^{-d}\ a \underset{\sigma}{*} V(f,g)\ .$$

Proof. By Proposition 2.1 (ii), we have

$$\begin{aligned} V(\mathrm{Op}^w(\widehat{a})f, g)(z) &= (2\pi)^{-\frac{d}{2}}(\rho(z)\mathrm{Op}^w(\widehat{a})f, g) \\ &= (2\pi)^{-\frac{d}{2}}(\mathrm{Op}^w(\widehat{a})f, \rho(-z)g) \\ &= (2\pi)^{-d}(a, V(\rho(-z)g, f))\ . \end{aligned}$$

But by Lemma 2.1

$$V(\rho(-z)g, f)(w) = e^{-\frac{i}{2}\sigma(w,z)}V(g, f)(w - z) = e^{\frac{i}{2}\sigma(z,w)}\overline{V(f, g)(z - w)},$$

therefore

$$V(\mathrm{Op}^w(\widehat{a})f, g) = (2\pi)^{-d}\int a(w)e^{-\frac{i}{2}\sigma(z,w)}V(f, g)(z - w)dw = (2\pi)^{-d}a \underset{\sigma}{*} V(f, g)\ .$$ □

2.4. Special Hermite functions. Let e_n be the Hermite function of order n defined by

$$e_n(x) = \pi^{-\frac{1}{4}}2^{-\frac{n}{2}}(n!)^{-\frac{1}{2}}H_n(x)e^{-\frac{x^2}{2}}$$

where H_n is the Hermite polynomial of order n.

We define the special Hermite functions by setting

$$e_{i,j} := V(e_i, e_j)$$

for $i,\ j \in \mathbf{Z}_+$.

Proposition 2.9. (i) *The system $(e_{i,j})_{i,j}$ is an orthonormal basis of $L^2(\mathbf{R}^2)$.*
(ii) *Let be $q \in \mathbb{N}$. Then we have*

$$e_{q,q}(z) = (2\pi)^{-\frac{1}{2}}L_q\left(\frac{1}{2}|z|^2\right)e^{-\frac{1}{4}|z|^2}$$

where L_q is the Laguerre polynomial of degree q and order 0.
(iii) *For all $z \in \mathbf{R}^2$ and $j, k \in \mathbf{Z}_+$*

$$e_{k,j}(z) = e_{j,k}(\bar{z})\ .$$

Proof. For (i) and (ii), see [17] and (iii) results of Proposition 2.1 (iv). □

2.5. Modulation spaces. As explained in the introduction, we will be concerned with pseudo-differential operators with symbols in modulation spaces rather than in Hörmander-Shubin classes. We state the definitions and results we need in the following.

First at all, we recall the definition of the Short Time Fourier Transform (STFT). Let $g \in S(\mathbf{R}^d) \setminus \{0\}$. For $f \in L^2(\mathbf{R}^d)$ we define the STFT $V_g f$ by

$$V_g f(x,\xi) := (2\pi)^{-d/2} \int f(t)\overline{g(t-x)}e^{-it.\xi}dt \ .$$

The modulation space $M^{p,q}_{\omega}(\mathbf{R}^d)$ is the set of all tempered distributions $f \in \mathcal{S}'(\mathbf{R}^d)$ for which the short-time Fourier transform $V_g f \in L^{p,q}_{\omega}(\mathbf{R}^{2d})$. This space equipped with the norm

$$\|f\|_{M^{p,q}_{\omega}(\mathbf{R}^d)} := \left(\int \left(\int |V_g f(x,\xi)|^p \omega(x,\xi)^p dx\right)^{q/p} d\xi\right)^{1/q}$$

is a Banach space. We note L^p_{ω}, resp. M^p_{ω}, in the case where $p = q$. The definition of $M^{p,q}_{\omega}(\mathbf{R}^d)$ is independent of the choice of the window $g \in S(\mathbf{R}^d) \setminus \{0\}$ and the norms corresponding to different choices of the window are equivalent. In particular we will use the following spaces

$$L^{2,2}_{v_s}(\mathbf{R}^{2d}),\ M^{2,2}_{v_s}(\mathbf{R}^d),\ L^{1,1}_{\omega_s}(\mathbf{R}^{2d})$$

abbreviated respectively by L^2_s, M^2_s and L^1_s (the weights v_s and ω_s are defined in section 2.3).

Lemma 2.2. *$f \in M^2_s$ if and only if $V(f,g) \in L^2_s$ and for all $f \in M^2_s$ we have*

$$\|f\|_{M^2_s} = \|V(f,g)\|_{L^2_s} \ .$$

Proof. The Fourier-Wigner $V(f,g)$ and the STFT $V_g f$ are connected by the relation

$$V(f,g)(x,\xi) = e^{-ix.\xi} V_g f(\xi,-x) \ .$$

Furthermore $v_s(\xi,-x) = v_s(x,\xi)$; then by using the (symplectic) change of variables $(x,\xi) \mapsto (\xi,-x)$ we obtain

$$\|f\|^2_{M^2_s} = \|V_g f\|^2_{L^2_s} = \iint |V_g f(\xi,-x)|^2 v_s(\xi,-x)^2 dx d\xi$$

$$= \iint |V_g f(x,\xi)|^2 v_s(x,\xi)^2 dx d\xi = \|V(f,g)\|^2_{L^2_s} \ . \qquad \square$$

It happens that the modulation space $M^2_s(\mathbf{R}^d)$ coincide with the Sobolev-Shubin space $Q^s(\mathbf{R}^d)$ initially defined in [16] as a Sobolev space corresponding to pseudo-differential calculus with symbols in Hörmander-Shubin classes (see the remark below). This space is defined for $s \geq 0$ by

$$Q^s(\mathbf{R}^d) := \{f \in L^2(\mathbf{R}^d) : \mathrm{Op}^{aw}_{\psi_0}(v_s) f \in L^2(\mathbf{R}^d)\}$$

where $\mathrm{Op}^{aw}_{\psi_0}(v_s)$ is the Anti-Wick operator with symbol v_s and Gaussian window defined by

$$\psi_0(x) := \pi^{-d/4} \exp\Big(-\frac{|x|^2}{2}\Big) \ .$$

Lemma 2.3 (2, lemma 2.3)**.** *For all $s \geq 0$ $M^2_s(\mathbf{R}^d)$ coincide with $Q^s(\mathbf{R}^d)$ and the norms are equivalent.*

Combining the two preceding lemma, we obtain immediatly the following result.

Proposition 2.10. *Let $f \in L^2(\mathbf{R}^d)$ and $s \geq 0$. Then $f \in Q^s(\mathbf{R}^d)$ if and only if $V(f,g) \in L^2_s(\mathbf{R}^{2d})$. Furthermore, there exists $C_s > 0$ such that for all $f \in Q^s(\mathbf{R}^d)$*

$$C_s^{-1}\|V(f,g)\|_{L^2_s} \le \|f\|_{Q^s(\mathbf{R}^d)} \le C_s\|V(f,g)\|_{L^2_s} .$$

Remark For $s = m \in \mathbb{Z}_+$, the Sobolev-Shubin space is defined by

$$Q^m(\mathbf{R}^d) = \{u \in \mathcal{S}'(\mathbf{R}^d);\ x^\alpha D^\beta u \in L^2(\mathbf{R}^d),\ |\alpha| + |\beta| \le m\} .$$

Another explicit characterization of $M^2_s(\mathbf{R}^d)$ is (see [9], [11])

$$M^2_s(\mathbf{R}^d) = L^2_s(\mathbf{R}^d) \cap H^s(\mathbf{R}^d) .$$

3. Some results of phase-space analysis

Let U be an isometry from a Hilbert space $\mathbb{H}_1$ into an other $\mathbb{H}_2$; denote by G its range $G := U(\mathbb{H}_1)$. It is well known that G is a closed subspace of $\mathbb{H}_2$ and that, if P is the orthogonal projection on G, then $U^*U = I_{\mathbb{H}_1}$ and $UU^* = P$.

Let $g \in S(\mathbf{R}^d)$ satisfying $\|g\|_2 = 1$. For $f \in L^2(\mathbf{R}^d)$ define $\mathcal{V}_g(f) := V(f,g)$.

Proposition 3.1. *The map $\mathcal{V}_g$ is an isometry from $L^2(\mathbf{R}^d)$ on a closed subspace of $L^2(\mathbf{R}^{2d})$ and if P_g is the orthogonal projection on this closed subspace, then $\mathcal{V}_g^*\mathcal{V}_g = I_{L^2(\mathbf{R}^d)}$ and $\mathcal{V}_g\mathcal{V}_g^* = P_g$.*

Proof. $\mathcal{V}_g$ is an isometry by the Moyal identity (Proposition 2.1 (iii)) and the assumption on g. We apply then the result above to the isometry $U = \mathcal{V}_g$. □

The following results are inspired by [7] and [8] Chapter 18; the isometry $\mathcal{V}_g$ plays the role of the windowed wavepacket transform W_ϕ in [8] p. 299.

Proposition 3.2. *Let be $F \in L^2(\mathbf{R}^{2d})$ and $g \in L^2(\mathbf{R}^d)$. Then*

$$\mathcal{V}_g^*(F) = (2\pi)^{\frac{d}{2}} Op^w(\mathcal{F}F).g .$$

Proof. Let $h \in L^2(\mathbf{R}^d)$. By (ii) of Theorem 2.1 and Proposition 2.3 we get

$$(\mathrm{Op}^w(\mathcal{F}F).g, h) = (2\pi)^{-\frac{d}{2}}(\mathcal{F}F, W(h,g)) = (2\pi)^{-\frac{d}{2}}(F, V(h,g)) .$$

On the other hand

$$(\mathcal{V}_g^*(F), h) = (F, \mathcal{V}_g(h)) = (F, V(h,g)) .$$ □

Corollary 3.1. *Under the same assumption on F and g, we have $\mathcal{V}_g^*(F) = S_k\, g$ where S_k is the Hilbert-Schmidt operator with kernel*

$$k(x,y) := (2\pi)^{-\frac{d}{2}} \mathcal{F}_1 F\left(\frac{1}{2}(x+y), x-y\right) .$$

Proof. We apply Theorem 2.1 (iii) :

$$\mathrm{Op}^w(\mathcal{F}F) = S_k$$

with

$$k = (2\pi)^{-\frac{d}{2}} U_L^{-1}\overline{\mathcal{F}_2}(\mathcal{F}F) = (2\pi)^{-\frac{d}{2}} U_L^{-1}(\mathcal{F}_1(F))$$

and

$$U_L^{-1}(\mathcal{F}_1(F))(x,y) = \mathcal{F}_1 F\left(\frac{1}{2}(x+y), x-y\right) . \qquad \square$$

We deduce in particular

Corollary 3.2. *Under the same assumptions as in* Corollary 3.1 *we have*

$$(2\pi)^{-\frac{d}{2}} \int F\left(\frac{1}{2}(x+y), x-y\right) g(y)dy = \mathcal{V}_g^*(\overline{\mathcal{F}_1} F)(x) \quad (a.e.\ in\ \mathbf{R}^d) .$$

Proposition 3.3. *Let be $F \in L_s^2(\mathbf{R}^{2d})$ and $g \in \mathcal{S}(\mathbf{R}^d)$. Then $\mathcal{V}_g^*(F)$ belongs to $Q^s(\mathbf{R}^d)$ and there is $C_s > 0$ such that*

$$\|\mathcal{V}_g^*(F)\|_{Q^s(\mathbf{R}^d)} \le (2\pi)^{-\frac{d}{2}} C_s \|V(g,g)\|_{L_s^1} \|F\|_{L_s^2} .$$

Proof. Put $f := \mathcal{V}_g^*(F) \in L^2(\mathbf{R}^d)$. Applying Proposition 3.2 and Proposition 2.8, we obtain

$$\begin{aligned}\mathcal{V}_g(f) &= \mathcal{V}_g(\mathcal{V}_g^*(F)) = (2\pi)^{\frac{d}{2}} \mathcal{V}_g(\mathrm{Op}^w(\mathcal{F}F).g) = (2\pi)^{\frac{d}{2}} V(\mathrm{Op}^w(\mathcal{F}F).g, g) \\ &= (2\pi)^{-\frac{d}{2}} F \underset{\sigma}{*} V(g,g) .\end{aligned}$$

And now we conclude from Proposition 2.7 (iii) that $\mathcal{V}_g(f) \in L_s^2(\mathbf{R}^{2d})$ and

$$\|\mathcal{V}_g(f)\|_{L_s^2} \le (2\pi)^{-\frac{d}{2}} \|V(g,g)\|_{L_s^1} \|F\|_{L_s^2},$$

or equivalently by Proposition 2.10 that $f \in Q^s(\mathbf{R}^d)$ and

$$\|f\|_{Q^s(\mathbf{R}^d)} \le C_s \|\mathcal{V}_g(f)\|_{L_s^2} \le (2\pi)^{-\frac{d}{2}} C_s \|V(g,g)\|_{L_s^1} \|F\|_{L_s^2} . \qquad \square$$

The next result is a consequence of Proposition 2.5; we use the same notations.

Proposition 3.4. *We suppose that the symbol a is real-valued and therefore that the operator $Op^w(a)$ is self-adjoint. The operators $Op^w(a)$ and $Op^w(\widetilde{a})$ have the same eigenvalues. If f is an eigenfunction of $Op^w(a)$ corresponding to the eigenvalue λ, then for all $g \in S(\mathbf{R}^d)$ the function $F := \mathcal{V}_g(f)$ is an eigenfunction of $Op^w(\widetilde{a})$ corresponding to the same eigenvalue.*

Proof. By Proposition 2.5 we have

$$\mathcal{V}_g \circ \mathrm{Op}^w(a) = \mathrm{Op}^w(\widetilde{a}) \circ \mathcal{V}_g . \tag{3.1}$$

Let λ be an eigenvalue of $\mathrm{Op}^w(a)$, let f be in $L^2(\mathbf{R}) \setminus \{0\}$ such that $\mathrm{Op}^w(a)f = \lambda f$ and $F := \mathcal{V}_g(f)$. We have $\mathrm{Op}^w(\widetilde{a}).F = \lambda F$, and since $\|F\| = \|f\| > 0$ we deduce that λ is an eigenvalue of $\mathrm{Op}^w(\widetilde{a})$ and F is an eigenfunction of $\mathrm{Op}^w(\widetilde{a})$.

Let now λ be an eigenvalue of $\mathrm{Op}^w(\widetilde{a})$) and let $F \neq 0 \in L^2(\mathbf{R}^d)$ be an eigenfunction. Taking adjoint operators in (3.1), it results that $\mathcal{V}_g^* \circ \mathrm{Op}^w(\widetilde{a}) = \mathrm{Op}^w(a) \circ \mathcal{V}_g^*$. Since $\mathrm{Op}^w(\widetilde{a})F = \lambda F$, we get that $\mathrm{Op}^w(a)\mathcal{V}_g^* F = \lambda \mathcal{V}_g^* F$ for all $g \in L^2(\mathbf{R}^d)$. Suppose that $\mathcal{V}_g^* F = 0$ for all $g \in L^2(\mathbf{R}^d)$; then $F \in \mathrm{N}(\mathcal{V}_g^*) = \mathrm{R}(\mathcal{V}_g)^\perp$, hence $(F, V(f,g)) = 0$ for all $f, g \in L^2(\mathbf{R}^d)$; in particular $(F, V(e_j, e_k)) = (F, e_{j,k}) = 0$. Since $(e_{j,k})$ is an orthonormal basis of $L^2(\mathbf{R}^d)$, we deduce that $F = 0$ which is not possible. Therefore there exists $g \in L^2(\mathbf{R}^d)$ such that $\mathcal{V}_g^*(F)$ is not the null function and this proves that λ is an eigenvalue of $\mathrm{Op}^w(a)$ and that F is a corresponding eigenfunction. $\square$

4. Pseudo-differential perturbation: the Hilbert-Schmidt case

We will now apply the results proved in the previous section to the Landau Hamiltonian perturbed by a Hilbert-Schmidt ΨDO operator.

4.1. The Landau Hamiltonian. We suppose $d = 1$ and $a(q,p) = q^2 + p^2$ for $(q,p) \in \mathbf{R}^2$. The DO $\mathrm{Op}^w(a)$, initially defined on $S(\mathbf{R})$, is essentially self-adjoint and its unique realization as unbounded operator on $L^2(\mathbf{R})$ is the harmonic oscillator h_0. It is well known that $\sigma(h_0) = \{2q+1; q \in \mathbb{N}\}$, $h_0(e_q) = (2q+1)e_q$ and $\mathrm{Ker}(h_0 - (2q+1)I) = \mathrm{Vect}\,(e_q)$, where e_q is the Hermite function of order q.

With the notations of section 3, the symbol $\widetilde{a}$ associated to a is defined by $\widetilde{a}(x,\xi) = a(\xi + \frac{1}{2}Jx)$; more precisely here

$$\widetilde{a}(x_1, x_2, \xi_1, \xi_2) = a\left(\xi_1 + \frac{1}{2}x_2,\, \xi_2 - \frac{1}{2}x_1\right) = \left(\xi_1 + \frac{1}{2}x_2\right)^2 + \left(\xi_2 - \frac{1}{2}x_1\right)^2 .$$

Therefore

$$\mathrm{Op}^w(\widetilde{a}) = \left(\frac{1}{i}\frac{\partial}{\partial x} + \frac{1}{2}y\right)^2 + \left(\frac{1}{i}\frac{\partial}{\partial y} - \frac{1}{2}x\right)^2 .$$

We deduce from Proposition (3.4) that

$$\sigma(H_0) = \sigma(h_0) = \{2q+1; q \in \mathbb{N}\} .$$

Let E_q^0 be the eigenspace of H_0 corresponding to the eigenvalue $\Lambda_q = 2q+1$:

$$E_q^0 := \mathrm{Ker}(H_0 - \Lambda_q I) .$$

We know again from Proposition 3.4 that, since e_q is an eigenfunction of h_0 with respect to the eigenvalue Λ_q, then $\mathcal{V}_{e_j}(e_q) = e_{q,j}$ is an eigenfunction of H_0 for the same eigenvalue:

$$H_0\, e_{q,j} = (2q+1)e_{q,j}, \qquad j = 0, 1, \dots .$$

Proposition 4.1. *Let $q \in \mathbf{Z}_+$ be fixed and let* $\mathrm{Vect}(\{e_{q,j}; j \in \mathbf{Z}_+\})$ *be the subspace of $L^2(\mathbf{R}^2)$ spanned by the special Hermite functions $e_{q,j}$ for $j \in \mathbf{Z}_+$. Then*

$$E_q^0 = \overline{\mathrm{Vect}(\{e_{q,j}; j \in \mathbf{Z}_+\})} .$$

Proof. It is sufficient to prove the inclusion $E_q^0 \subseteq \overline{\mathrm{Vect}(\{e_{q,j}; j \in \mathbb{N}\})}$. Let be $F \in E_q^0$; we have $\mathcal{V}_g^* \circ H_0 = h_0 \circ \mathcal{V}_g^*$ and $H_0 F = (2q+1)F$, from which we deduce $(2q+1)\mathcal{V}_g^* F = h_0 \mathcal{V}_g^* F$, thereby $\mathcal{V}_g^* F \in \mathrm{Ker}(h_0 - (2q+1)I) = \mathrm{Vect}\,(e_q)$. In particular $\mathcal{V}_{e_j}^* F \in \mathrm{Vect}\,(e_q)$. But for all $k \in \mathbb{N}$

$$(\mathcal{V}_{e_j}^* F, e_k) = (F, V(e_k, e_j)) = (F, e_{k,j}) .$$

Since $\mathcal{V}_{e_j}^*(F) \in \mathrm{Vect}\,(e_q)$, we deduce that $(F, e_{k,j}) = 0$ for $k \neq q$ and then

$$F = \sum_k \sum_j (F, e_{k,j}) e_{k,j} = \sum_j (F, e_{k,j}) e_{k,j}$$

which proves that $F \in \mathrm{Vect}(\{e_{q,j}; j \in \mathbb{N}\})$. □

Next we wish to express E_q^0 with $\mathcal{V}_{e_q}$ and factorise $\mathbb{P}_q$ the orthogonal projection on E_q^0.

Recall that $R(\mathcal{V}_{e_q}) = \text{Vect}(\{e_{j,q};\ j \in \mathbb{N}\}) := E_q$. Let be U the unitary involutive operator defined on $L^2(\mathbf{R}^2)$ by

$$U\,F(z) := F(\bar{z}), \qquad F \in L^2(\mathbf{R}^2),\ z \in \mathbf{R}^2$$

or equivalently $U\,F(x_1, x_2) = F(x_1, -x_2)$.

By Proposition 2.1, for all $j, k \in \mathbf{Z}_+$ we have $e_{j,k}(\bar{z}) = e_{k,j}(z)$ or $U(e_{j,q}) = e_{q,j}$. We deduce that $U(E_q) = E_q^0$. Let P_q be the orthogonal projection on E_q; we have $P_q = \mathcal{V}_{e_q}\mathcal{V}_{e_q}^*$; therefore $UP_qU^* = \mathbb{P}_q$. Define now

$$\mathbb{V}_{e_q} := U\,\mathcal{V}_{e_q}\ . \tag{4.1}$$

Then

$$\mathbb{P}_q = \mathbb{V}_{e_q}\mathbb{V}_{e_q}^*\ . \tag{4.2}$$

4.2. Hilbert-Schmidt perturbation. Let be $b \in L^2(\mathbf{R}^4)$ a real-valued symbol and $V := \text{Op}^w(b)$ the ΨDO with Weyl symbol b. Since b is square-integrable, V is an Hilbert-Schmidt operator. Hence the operators VH_0^{-1} and

$$T_q := \mathbb{P}_q\,V\,\mathbb{P}_q$$

are also Hilbert-Schmidt operators on $L^2(\mathbf{R}^2)$. The last operator is the effective Hamiltonian corresponding to the perturbed operator $H_0 - V$ and to the Landau level Λ_q as we will see in the next section.

Our first goal is to show that the operator T_q has the same spectrum as a ΨDO operator S_q on $L^2(\mathbf{R})$ easier to study. We can first give an abstract result.

Proposition 4.2. *Let $U : \mathbb{H}_1 \mapsto \mathbb{H}_2$ be an isometry, $S = S^* \in \mathcal{L}(\mathbb{H}_1)$ and $T = T^* \in \mathcal{L}(\mathbb{H}_2)$. We suppose that*

$$T := U\,S\,U^*\ .$$

Then the operators S and T have the same non-zero eigenvalues; more precisely for all $\lambda \neq 0$ and for all $u \in \mathbb{H}_1$, u is an eigenvector of S corresponding to the eigenvalue λ iff Uu is an eigenvector of T corresponding to the same eigenvalue.

Proof. Let be $u \in \mathbb{H}_1 \setminus \{0\}$ and $\lambda \neq 0$ such that $Su = \lambda u$. We set $v := Uu$:

$$Tv = (U\,S\,U^*)Uu = USu = \lambda Uu = \lambda v$$

and since v and u have the same norm, then v is non null, and λ is an eigenvalue and v an eigenvecteur of T.

Conversely, suppose $Tv = \lambda v$ with λ and v non null and let be $u := U^*v$. Then, composing on the left hand side by U^*, we obtain that $Su = \lambda u$. Furthermore, if $u = 0$, then $Tv = (US)(u) = 0$ and since $Tv = \lambda v$, we deduce $\lambda v = 0$, hence $v = 0$ since $\lambda \neq 0$, which contradicts the assumption $v \neq 0$. □

We will apply this result to our problem by considering the operator

$$S_q := \mathbb{V}_{e_q}^*\,V\,\mathbb{V}_{e_q}\ .$$

The operator T_q defined above can be rewrited since

$$T_q = \mathbb{P}_q V \mathbb{P}_q = \mathbb{V}_{e_q}\mathbb{V}_{e_q}^* V\,\mathbb{V}_{e_q}\mathbb{V}_{e_q}^* = \mathbb{V}_{e_q} S_q \mathbb{V}_{e_q}^* \,.$$

Furthermore $\mathbb{V}_{e_q} = U\mathcal{V}_{e_q}$ is also an isometry. Applying the latest proposition, we obtain that the operators S_q and T_q have the same non-zero eigenvalues.

We will now prove that the operator S_q is a ΨDO and determine its Weyl-symbol. By definition of S_q and by Theorem 2.1 (ii) we have

$$(S_q f, g) = \frac{1}{2\pi}\iint b(x, y; \xi, \eta) W(V(f, e_q), V(g, e_q))(x, -y; \xi, -\eta)\,dxdyd\xi d\eta \,.$$

By Proposition (2.4) we get

$$W(V(f, e_q), V(g, e_q))(x, -y, \xi, -\eta) = W(f, g)\left(\xi - \frac{1}{2}y, -\eta - \frac{1}{2}x\right) W(e_q, e_q)\left(\xi + \frac{1}{2}y, -\eta + \frac{1}{2}x\right) \,.$$

Making use of a change of variables, we deduce

$$(S_q f, g) =$$

$$\frac{1}{2\pi}\iint \Big[\iint b(\xi - \eta, x - y, \frac{1}{2}(x + y), -\frac{1}{2}(\xi + \eta)) W(e_q, e_q)(x, \xi)\,dxd\xi\Big] W(f, g)(y, \eta)\,dyd\eta$$

therefore S_q is a ΨDO with Weyl symbol γ_q defined by

$$\gamma_q(y, \eta) = (2\pi)^{-\frac{1}{2}}\iint b\left(\xi - \eta, x - y, \frac{1}{2}(x + y), -\frac{1}{2}(\xi + \eta)\right) W(e_q, e_q)(x, \xi)\,dxd\xi. \tag{4.3}$$

Let us define $F \in L^2(\mathbf{R}^4)$ by

$$F(u_1, u_2, v_1, v_2) := b(-v_2, -v_1, u_1, -u_2) \tag{4.4}$$

We put $w = (x, \xi)$ et $z = (y, \eta)$. Then we have

$$F\left(\frac{1}{2}(z + w), z - w\right) = F\left(\frac{1}{2}(x + y), \frac{1}{2}(\xi + \eta), -(x - y), -(\xi - \eta)\right)$$

$$= b\left(\xi - \eta, x - y, \frac{1}{2}(x + y), -\frac{1}{2}(\xi + \eta)\right),$$

hence by (4.3)

$$\gamma_q(z) = (2\pi)^{-\frac{1}{2}}\iint F\left(\frac{1}{2}(z + w), z - w\right) W(e_q, e_q)(w)\,dw$$

and by Corollary 3.2

$$\gamma_q = \mathcal{V}^*_{W(e_q, e_q)}(\overline{\mathcal{F}_1} F) \,.$$

If we set $\Lambda(u_1, u_2, v_1, v_2) := (-v_2, -v_1, -u_1, u_2)$, we obtain easily

$$\gamma_q = \mathcal{V}^*_{W(e_q, e_q)}(\mathcal{F}_2 b \circ \Lambda) \,.$$

Suppose now that $\mathcal{F}_2 b \in L^2_s(\mathbf{R}^4)$. The function space $L^2_s(\mathbf{R}^4)$ is invariant by linear change of variables; thus $\mathcal{F}_2 b \circ \Lambda$ is also in $L^2_s(\mathbf{R}^4)$. According to Proposition 3.3 we deduce that

$$\gamma_q = \mathcal{V}^*_{W(e_q, e_q)}(\mathcal{F}_2 b \circ \Lambda) \tag{4.5}$$

belongs to $Q^s(\mathbf{R}^2)$. We have therefore prove the following result.

Proposition 4.3. *Let $b \in L^2_s(\mathbf{R}^4)$. Suppose that $\mathcal{F}_2 b$ belongs to $L^2_s(\mathbf{R}^4)$. Then $S_q = Op^w(\gamma_q)$ with γ_q, defined by* (4.5)*, belonging to $Q^s(\mathbf{R}^2)$.*

We can now achieve the proof of Theorem 1.1. We recall first the following result about Schatten class properties for ΨDO with symbols in Shubin-Sobolev classes obtained by C. Heil in [11]. Let $s \geq 0$ be given and let $a \in L^2(\mathbf{R}^2)$. Define the operator $L := \mathrm{Op}^w(a)$. By Theorem 2.1 we know that L is a Hilbert-Schmidt operator; let $s_j(L)$ be the singular values of L, arranged in descending order, counting multiplicity.

Proposition 4.4. *If the symbol a lies in $Q^s(\mathbf{R}^2)$, then*

$$s_j(L) = O\left(j^{-\frac{s+1}{2}}\right) .$$

Consequently $L \in S_p(L^2(\mathbf{R}))$ for $p > 2/(s+1)$. In particular L is trace-class if $s > 1$.

From this result we deduce that, with the same hypothesis as in Proposition 4.3, the singular values of $S_q = \mathrm{Op}(\gamma_q)$ verify

$$s_j(S_q) = O\left(j^{-\frac{s+1}{2}}\right) .$$

Therefore we obtain the same estimates for the operator T_q since the two operators S_q and T_q have the same non-zero eigenvalues by Proposition 4.2.

5. Proof of Theorem 1.2

5.1. Reduction to an effective Hamiltonian. The first aim of this section is to prove that the operator $T_q = \mathbb{P}_q V \mathbb{P}_q$ is the effective Hamiltonian for estimating the number of eigenvalues of the perturbed Landau Hamiltonian near the Landau level Λ_q. We follow [15] but some modifications must be precised.

We recall the classical Weyl inequality ([2], chap. 9).

Lemma 5.1. *Let T_1 and T_2 be linear self-adjoint compact operators in a Hilbert space. Then for each $s_1 > 0$ and $s_2 > 0$*

$$n_+(s_1 + s_2, T_1 + T_2) \leq n_+(s_1, T_1) + n_+(s_2, T_2)$$

holds true.

Let H_0 and V be as in Section 1 and let Λ_q be a fixed Landau level, $\mathbb{P}_q$ be the corresponding spectral projection and $\mathbb{Q}_q := I - \mathbb{P}_q$. For $\lambda \in \rho(H_0)$ we set

$$T(\lambda) := V^{1/2}(H_0 - \lambda)^{-1} V^{1/2} .$$

This operator is selfadjoint and compact.

Proposition 5.1. *Assume that the interval $[\lambda_1, \lambda_2]$, $\lambda_1 < \lambda_2$ belongs to the gap $(\Lambda_{q-1}, \Lambda_q)$, then*

$$\operatorname{rank} E_{[\lambda_1, \lambda_2)}(H_0 - V) = n_+(1, T(\lambda_2)) - n_+(1, T(\lambda_1)) .$$

For the proof of this result, we refer to [14, Sections 1 and 3], and to the earlier article [1, Proposition 1.6].

Lemma 5.2. *Let E', E be positive real numbers satisfying $\Lambda_{q-1} < E' < 2q$ and $0 < E < 1$. Then*

$$N(E', \Lambda_q - E;\ H_0 - V) = n_+\left(1, V^{\frac{1}{2}}(H_0 - \Lambda_q + E)^{-1} V^{\frac{1}{2}}\right) + O(1), \quad E \downarrow 0^+ .$$

Proof. With these assumptions, the interval $[E', \Lambda_q - E]$ is included in the gap $(\Lambda_{q-1}, \Lambda_q)$. Therefore we can apply Proposition 5.1 :

$$\text{rank } E_{[E', \Lambda_q - E)}(H_0 - V) = n_+(1, T(\Lambda_q - E)) - n_+(1, T(E'))$$

or equivalently with the notations of section 1 :

$$N(E', \Lambda_q - E; H_0 - V) = n_+(1, T(\Lambda_q - E)) - n_+(1, T(E')) - \dim[\text{Ker}(H_0 - V - E')] .$$

But the last two terms in the right-hand side are independent of E. □

For brevity, we set

$$T_q(E) := T(\Lambda_q - E) = V^{\frac{1}{2}}(H_0 - \Lambda_q + E)^{-1} V^{\frac{1}{2}} .$$

We then write $T_q(E) = T_{1,q}(E) + T_{q,2}(E)$ with

$$\begin{cases} T_{1,q}(E) & := & V^{\frac{1}{2}}(H_0 - \Lambda_q + E)^{-1} \mathbb{P}_q V^{\frac{1}{2}} \\ T_{2,q}(E) & := & V^{\frac{1}{2}}(H_0 - \Lambda_q + E)^{-1} \mathbb{Q}_q V^{\frac{1}{2}} . \end{cases}$$

First we remark that

$$(H_0 - \Lambda_q + E)^{-1} \mathbb{P}_q = \sum_{l=0}^{+\infty} (\Lambda_l - \Lambda_q + E)^{-1} \mathbb{P}_l \mathbb{P}_q = E^{-1} \mathbb{P}_q,$$

and so

$$T_{1,q}(E) = E^{-1} V^{\frac{1}{2}} \mathbb{P}_q V^{\frac{1}{2}} .$$

The operator $T_{1,q}(E)$ is compact, selfadjoint and positive. The operator $T_{2,q}(E)$ can be rewritten as

$$T_{2,q}(E) = \sum_{l \neq q} (\Lambda_l - \Lambda_q + E)^{-1} V^{\frac{1}{2}} P_l V^{\frac{1}{2}} .$$

Proposition 5.2. *For all $s > 0$ we have*

$$n_+(s, T_{2,q}(E)) \leq 4\Lambda_q^2 s^{-2} \|V^{\frac{1}{2}} H_0^{-1} V^{\frac{1}{2}}\|_{\mathbf{S}_2}^2 .$$

Proof. The operator $T_{2,q}(E)$ is compact, selfadjoint but not positive. We are led to define

$$\begin{cases} T_{2,q}^+(E) & := & \sum_{l>q} (\Lambda_l - \Lambda_q + E)^{-1} V^{\frac{1}{2}} \mathbb{P}_l V^{\frac{1}{2}} \\ T_{2,q}^-(E) & := & -\sum_{l<q} (\Lambda_l - \Lambda_q + E)^{-1} V^{\frac{1}{2}} \mathbb{P}_l V^{\frac{1}{2}} . \end{cases}$$

Since $0 < E < 1$, we have $\Lambda_l - \Lambda_q + E < -1$ if $l < q$, and $\Lambda_l - \Lambda_q + E > 2$ if $l > q$. Consequently the operators $T_{2,q}^+(E)$ and $T_{2,q}^-(E)$ are selfadjoint and positive and

$$T_{2,q}(E) = T_{2,q}^+(E) - T_{2,q}^-(E) .$$

By straightforward inequalities we get

$$0 < (\Lambda_l - \Lambda_q + E)^{-1} \leq \Lambda_{q+1}\Lambda_l^{-1} \quad , \ l > q \ .$$

Thereby it follows that

$$\begin{aligned}(T^+_{2,q}(E)u,u) &\leq \Lambda_{q+1}\sum_{l>q}\Lambda_l^{-1}\left(V^{\frac{1}{2}}P_lV^{\frac{1}{2}}u,u\right) \leq \Lambda_{q+1}\left((V^{\frac{1}{2}}(\sum_{l\neq q}\Lambda_l^{-1}P_l)V^{\frac{1}{2}})u,u\right)\\ &\leq \Lambda_{q+1}\left((V^{\frac{1}{2}}H_0^{-1}V^{\frac{1}{2}})u,u\right) .\end{aligned}$$

Similarly we have

$$(T^-_{2,q}(E)u,u) \leq \Lambda_{q-1}\left((V^{\frac{1}{2}}H_0^{-1}V^{\frac{1}{2}})u,u\right) .$$

By assumption the operator V is a Hilbert-Schmidt operator. Since the operators $V^{\frac{1}{2}}H_0^{-1}V^{\frac{1}{2}}$ and $H_0^{-\frac{1}{2}}VH_0^{-\frac{1}{2}}$ have the same non-zero eigenvalues, we deduce that $V^{\frac{1}{2}}H_0^{-1}V^{\frac{1}{2}}$ is a Hilbert-Schmidt operator. As a consequence of the preceding inequalities, we obtain that $T^+_{2,q}(E)$ and $T^-_{2,q}(E)$ are Hilbert-Schmidt operators and

$$\begin{cases} \|T^+_{2,q}(E)\|_{\mathbf{S}_2} \leq \Lambda_{q+1}\|V^{\frac{1}{2}}H_0^{-1}V^{\frac{1}{2}}\|_{\mathbf{S}_2} \\ \\ \|T^-_{2,q}(E)\|_{\mathbf{S}_2} \leq \Lambda_{q-1}\|V^{\frac{1}{2}}H_0^{-1}V^{\frac{1}{2}}\|_{\mathbf{S}_2} . \end{cases}$$

Since $T_{2,q}(E) = T^+_{2,q}(E) - T^-_{2,q}(E)$, it follows that $T_{2,q}(E) \in \mathbf{S}_2$ and

$$\|T_{2,q}(E)\|_{\mathbf{S}_2} \leq (\Lambda_{q-1} + \Lambda_{q+1})\|V^{\frac{1}{2}}H_0^{-1}V^{\frac{1}{2}}\|_{\mathbf{S}_2} \leq 2\Lambda_q\|V^{\frac{1}{2}}H_0^{-1}V^{\frac{1}{2}}\|_{\mathbf{S}_2},$$

and

$$n_+(s, T_{2,q}(E)) \leq 4\Lambda_q^2 s^{-2}\|V^{\frac{1}{2}}H_0^{-1}V^{\frac{1}{2}}\|^2_{\mathbf{S}_2} .$$

□

5.2. End of the proof of Theorem 1.2. For $0 < \varepsilon < 1$ we deduce from the Weyl inequality that

$$n_+(1, T_q(E)) \leq n_+(1-\varepsilon, T_{1,q}(E)) + n_+(\varepsilon, T_{2,q}(E)) .$$

For the first term of the right-hand side, we have

$$n_+(1-\varepsilon, T_{1,q}(E)) = n_+(1-\varepsilon, E^{-1}V^{\frac{1}{2}}\mathbb{P}_qV^{\frac{1}{2}}) = n_+((1-\varepsilon)E, \mathbb{P}_qV\mathbb{P}_q) .$$

But by Proposition 4.4

$$\lambda_j(\mathbb{P}_qV\mathbb{P}_q) = O\left(j^{-\frac{s+1}{2}}\right)$$

or equivalently there is $C_q > 0$ independent of ε and E such that

$$n_+((1-\varepsilon)E, \mathbb{P}_qV\mathbb{P}_q) \leq C_q(1-\varepsilon)^{-\frac{2}{s+1}}E^{-\frac{2}{s+1}} .$$

For the second term of the right-hand side, we have by Proposition 5.2

$$n_+(\varepsilon, T_{2,q}(E)) \leq 4\Lambda_q^2\varepsilon^{-2}\|V^{\frac{1}{2}}H_0^{-1}V^{\frac{1}{2}}\|^2_{\mathbf{S}_2} .$$

Finally we deduce from the preceding inequalities that

$$E^{\frac{2}{s+1}} n_+(1, T_q(E)) \le C_q(1-\varepsilon)^{-\frac{2}{s+1}} + 4\Lambda_q^2 \varepsilon^{-2} E^{\frac{2}{s+1}} \|V^{\frac{1}{2}} H_0^{-1} V^{\frac{1}{2}}\|_{\mathbf{S}_2}^2 .$$

Consequently

$$\limsup_{E\to 0^+} E^{\frac{2}{s+1}} n_+(1, T_q(E)) \le C_q(1-\varepsilon)^{-\frac{2}{s+1}},$$

and letting ε tend to 0^+ we deduce

$$n_+(1, T_q(E)) \le C_q E^{-\frac{2}{s+1}}$$

or equivalently

$$N(E', \Lambda_q - E;\ H_0 - V) \le C_q E^{-\frac{2}{s+1}} .$$

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