Celikbas, E., Laxmi, J. and Weyman, J.
Osaka J. Math.
60 (2023), 903–931

SPINOR STRUCTURES ON FREE RESOLUTIONS OF CODIMENSION FOUR GORENSTEIN IDEALS

Ela CELIKBAS, Jai LAXMI and Jerzy WEYMAN

(Received August 23, 2021, revised October 6, 2022)

Abstract
We analyze the structure of spinor coordinates on resolutions of Gorenstein ideals of codimension four. As an application we produce a family of such ideals with seven generators which are not specializations of the Kustin-Miller model.

1. Introduction

In this paper we investigate spinor structures on free resolutions of Gorenstein ideals of codimension 4. Such structures were first described by Reid in [21].

Let R be a Gorenstein local ring in which 2 is a unit and $I \subset R$ be a Gorenstein ideal of codimension 4. Let

$$\mathbb{F} : 0 \to F_4 \xrightarrow{d_4} F_3 \xrightarrow{d_3} F_2 \xrightarrow{d_2} F_1 \xrightarrow{d_1} R \tag{1}$$

be a minimal free resolution of R/I. For the minimal free resolution $\mathbb{F}$ of a Gorenstein ideal I of codimension 4, there is a quadratic form on F_2 of $\mathbb{F}$; see [17, Theorem 2.4]. It is also shown that this quadratic form on F_2 can be reduced to a hyperbolic form; see [17, Theorem 5.3].

In this setting, we show in our main result (see Theorem 4.2) that there is a spinor structure on $\mathbb{F}$. We give an explicit relation between spinor coordinates of $\mathrm{im}(d_3)$ and the Buchsbaum-Eisenbud multipliers. In particular, the spinor coordinates are square roots of some special Buchsbaum-Eisenbud multipliers; see Remark 4.3.

We calculate the spinor coordinates for some well-known examples of Gorenstein ideals of codimension 4 with few generators; see Section 5. For ideals with 7 generators, Kustin and Miller constructed a family of ideals associated to a 3×4 matrix, a 4-vector, and a variable. This family is also known as the Kustin-Miller model (KMM); see [16, 18]. We discuss a generic doubling of an almost complete intersection of codimension 3 that leads to a specialization of the KMM; see Subsection 6.3.

Reid asks if every case of 7×12 resolution is the known KMM in [21, page 29]. In this paper, we construct a new family in Subsection 7.2 using a resolution of a perfect ideal with 5 generators, of Cohen-Macaulay type 2 which was described in [7]. As an application of our main result (Theorem 4.2), we show that the new family given in Subsection 7.2 is not a specialization of the Kustin-Miller model; see Theorem 7.2. This answers Reid's question.

Our calculations uncover a nice structure of Buchsbaum-Eisenbud multipliers and also reveal an interesting pattern. Assume we look at a resolution of a Gorenstein ideal I of

2020 Mathematics Subject Classification. 13H05, 13H10, 15A63, 15A66, 16W22.

codimension 4 in a local ring $(R, \mathfrak{m})$. In all the examples we know, the spinor coordinates belong to the ideal I; see Remark 4.6. Furthermore, for all known examples of ideals I with 6, 7, and 8 generators, some spinor coordinates are not in $\mathfrak{m}I$, and hence they can be taken as minimal generators of I. However, for 9 or more generators, we find an example when all spinor coordinates are in $\mathfrak{m}I$. This suggests that Gorenstein ideals of codimension 4 with up to 8 generators are easier to classify than those with more than 8 generators; see Remark 7.3.

This paper is organized as follows. As the intended audience are commutative algebraists, we include an extended Section 2 on representations of general linear groups and special orthogonal groups. In Subsection 2.2, while working with orthogonal spin groups, we first deal with characteristic zero case, and then indicate which results stay true in arbitrary characteristic different from 2. In Section 3 we recall the Buchsbaum-Eisenbud First Structure Theorem for finite free resolutions and the results of Kustin-Miller on the resolutions of Gorenstein ideals of codimension 4.

In Section 4 we prove the existence of spinor structures on resolutions of Gorenstein ideals of codimension 4. We also apply results from Section 2.3 to explicitly calculate the relation between the Buchsbaum-Eisenbud multipliers and the spinor coordinates.

Section 5 contains the computations of the spinor coordinates for complete intersections, for hyperplane sections of codimension 3 Gorenstein ideals, and for the KMM with 7 generators.

In Section 6 we analyze the resolutions of "doublings" of almost complete intersection ideals of codimension 3. We show that these ideals are closely related to the KMM model. Finally, in Section 7.1, we show that a Gorenstein ideal of codimension 4 which is a doubling of a resolution of a perfect ideal of codimension 3 with 5 generators of Cohen-Macaulay type 2 is not a specialization of the KMM.

2. Background in representation theory

In this section we give all the representation theory framework used in the paper.

2.1. Representation Theory of GL(V). Let V be a vector space over a field K (or a free module over a ring R). We will use the following notation for the representations of the group $GL_n(V)$. For the dominant integral weight $(a_1, \cdots, a_n)$ where $a_i \in \mathbb{Z}$ and $a_1 \geq a_2 \geq \cdots \geq a_n$, $S_{(a_1,\cdots,a_n)}V$ denotes the corresponding Schur module. We denote the Lie algebras of $GL(V)$ and $SL(V)$ as $\underline{gl}(V)$ and $\underline{sl}(V)$, respectively.

Next we recall the definition of a perfect pairing of a bilinear map.

Definition 2.1. A bilinear map $Q : V \otimes_K V \to K$ is called a perfect pairing if it is a symmetric bilinear map such that the induced map $\tilde{Q} : V \to V^*$ defined by $f \mapsto Q(-, f)$ is an isomorphism. A perfect pairing is in the hyperbolic form if rank(V) is even and we can write it as a direct sum of hyperbolic two-dimensional pairings with matrices of the form

$$\begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix}.$$

In this case, the corresponding basis of V is called a hyperbolic basis.

2.2. Preliminaries on representations of the spin groups. In this section we collect the material involving representation theory of the spin group. We will work over a field K of characteristic different from 2. However, for the convenience of the reader we first recall the basic facts over a field of characteristic zero, and then indicate what needs to be modified in finite characteristics different than 2.

2.2.1. **Representations of the spin group over an algebraically closed field of characteristic zero.** We work over an algebraically closed field K of characteristic zero. Most of the material in this section can be found in [11, Lectures 18–20]. Other references are [13, Chapters 2,3,6], [14, Section 2.15], and [10, Chapter 2]. Let K be an algebraically closed field of characteristic zero and let V be an orthogonal space of dimension $2m$ over K. We put the symmetric bilinear map Q in the hyperbolic form. More precisely, let W be an isotropic space in V of dimension m. We can identify V with $W \oplus W^*$ and the symmetric bilinear map Q with the duality

$$Q : W \otimes W^* \to K,$$

also requiring W and W^* being isotropic.

Throughout we deal with the representations of the special orthogonal Lie algebra $\underline{so}(V)$, as it is well known that the categories of rational representations of the spin group $\mathrm{Spin}(2m)$ and of $\underline{so}(V)$ are equivalent. The maximal toral subalgebra in the Lie algebra $\underline{so}(V)$ is the maximal toral subalgebra of diagonal matrices in $\underline{gl}(W)$. It consists of matrices

$$\begin{pmatrix} A & 0 \\ 0 & -A \end{pmatrix}$$

where A is an $m \times m$ diagonal matrix. We denote the basis of V as follows. Vectors $\{e_1, \ldots, e_m\}$ form a basis of W, and $\{e_{-1} = e_1^*, \ldots, e_{-m} = e_m^*\}$ form the dual basis in W^*. Their weights are ε_i and $-\varepsilon_i$ for $1 \le i \le m$, respectively.

Since the symmetric bilinear map Q is in hyperbolic form in any characteristics different from 2, we can use the representation of $\mathrm{Spin}(2m)$ as well. For the representation of the spin group, we use the following notation. In this case, a maximal torus T of $\mathrm{Spin}(2m)$ and the Lie algebra of T, denoted $\mathfrak{t}$, are

$$T = \{\mathrm{diag}[x_1, \cdots, x_m, x_m^{-1}, \cdots, x_1^{-1}] : x_i \in K \setminus \{0\}\},$$

$$\mathfrak{t} = \{\mathrm{diag}[a_1, \cdots, a_m, -a_m, \cdots, -a_1] : a_i \in K\}.$$

For $i = 1, \ldots, m$, define $\langle \varepsilon_i, D \rangle = a_i$, where $D = \mathrm{diag}[a_1, \cdots, a_m, -a_m, \cdots, -a_1]$ is in $\mathfrak{t}$. Then $\{\varepsilon_1, \ldots, \varepsilon_m\}$ is a basis for $\mathfrak{t}^*$. All representations of $\mathrm{Spin}(2m)$ are restricted to T so they decompose into weights with respect to T. The simple roots are given by $\varepsilon_i - \varepsilon_{i+1}$ for $i < m$ along with the element $\varepsilon_{m-1} + \varepsilon_m$.

Let us recall that when K is an algebraically closed field of characteristic zero, then irreducible representations of $\underline{so}(V)$ are parametrized by dominant integral weights

$$\lambda = \sum_{i-1}^{m} \lambda_i \omega_i,$$

where

$$\omega_i = \varepsilon_1 + \cdots + \varepsilon_i$$

for $1 \leq i \leq m-2$, $\omega_{m-1} = \frac{1}{2}(\varepsilon_1 + \cdots + \varepsilon_{m-1} + \varepsilon_m)$, and $\omega_m = \frac{1}{2}(\varepsilon_1 + \cdots + \varepsilon_{m-1} - \varepsilon_m)$ are so-called fundamental weights, and $\lambda_i \in \mathbf{Z}_{\geq 0}$.

We denote $V(\lambda)$ the irreducible representation corresponding to the highest weight λ.

The fundamental representations of Spin$(2m)$ are $V(\omega_i) = \bigwedge^i V$ for $1 \leq i \leq m-2$, and the fundamental representations are the half-spinor representations for $i = m-1$ and $i = m$. To define them, we need a Clifford algebra

$$C(V, Q) = T(V)/I(V),$$

where $T(V)$ is a tensor algebra of V and $I(V)$ is the two-sided ideal in $T(V)$ generated by the elements

$$v_1 \otimes v_2 + v_2 \otimes v_1 - 2Q(v_1, v_2)$$

for $v_1, v_2 \in V$. Note that since the ideal $I(V)$ has generators with components in the 0th and 2nd graded component of $T(V)$, the Clifford algebra decomposes into its even part $C(V)_+$ and its odd part $C(V)_-$. Additively we have decompositions

$$C(V)_+ = \bigoplus_{i \text{ even}} \bigwedge^i V,$$

$$C(V)_- = \bigoplus_{i \text{ odd}} \bigwedge^i V.$$

Let $f = e_1 \wedge \ldots \wedge e_m$.

We have the following result (see [11, Lecture 20] for more details, note that our convention interchanges W and W^*).

Proposition 2.2. *The left ideal*

$$S = C(Q)^{\cdot} f$$

is additively isomorphic to the exterior algebra $\bigwedge^\bullet W^*$. *It is therefore a representation of* $\underline{so}(V)$. *It decomposes into even and odd parts* $S_+ := \bigwedge^{even} W^*$ *and* $S_- := \bigwedge^{odd} W^*$. *Here the left ideal* S *is called a Clifford module, and* S_+ *and* S_- *are called half-spinor modules.*

We also have

$$V(\omega_{m-1}) = S_+ \text{ and } V(\omega_m) = S_-.$$

Both half-spinor representations$V(\omega_{m-1})$ and $V(\omega_m)$ have dimension 2^{m-1}. Let $\mathcal{L} \subset [1, m]$ be a subset, and let $\mathcal{L}^c$ be its complement. We denote a coset of the tensor $\wedge_{i \in \mathcal{L}} w_{-i}$ by $u_{\mathcal{L}}$. This is a weight vector of weight $\frac{1}{2}(-\sum_{i \in \mathcal{L}} \varepsilon_i + \sum_{i \in \mathcal{L}^c} \varepsilon_i)$.

Note that Clifford algebra is then identified with $\mathrm{End}_{C(Q)}(S)$ and the spin group appears as certain subset of invertible elements of $C(Q)$. However, we do not need this description, so we refer the reader to [11, Lecture 20].

For the convenience of the reader we describe the action of the Lie algebra $\underline{so}(V)$ on half-spinor representations. Strictly speaking, it will not be needed but it explains weight decompositions of half-spinor representations.

For $a, b \in V$, define $R_{a,b} \in \mathrm{End}(V)$ as $R_{a,b}v = Q(b, v)a - Q(a, v)b$. By [10, Section 2.4], $R_{a,b}$ spans $\underline{so}(V)$ for $a, b \in V$. Then $R_{e_i,e_j} = e_{-i,j} - e_{-j,i}$ where $e_{i,j}$ be an elementary transformation on V that carries e_i to e_j and others to 0.

For $y^* \in W^*$, the exterior product $\boldsymbol{\epsilon}(y^*)$ and the interior product operator $\mathfrak{i}(y)$ on $\bigwedge W$ are defined as $\boldsymbol{\epsilon}(y^*)x^* = y^* \wedge x^*$ and

$$\mathfrak{i}(y)(y_1^* \wedge \cdots \wedge y_k^*) = \sum_{j=1}^{k} (-1)^{j-1} Q(y, y_j^*) y_1^* \wedge \cdots \wedge \widehat{y_j^*} \wedge \cdots \wedge y_k^*,$$

where $y_i^* \in W^*$, $x^* \in \bigwedge W^*$ and $\widehat{y_j^*}$ means to omit y_j^*.

Define linear maps $\boldsymbol{\gamma} : V \to \mathrm{End}(\bigwedge W^*)$ as $\boldsymbol{\gamma}(y + y^*) = \mathfrak{i}(y) + \boldsymbol{\epsilon}(y^*)$ for $y \in W$ and $y^* \in W^*$, and $\boldsymbol{\varphi} : \underline{so}(V) \to \mathrm{Cliff}_2(V, Q)$ as $\boldsymbol{\varphi}(R_{a,b}) = \frac{1}{2}[\boldsymbol{\gamma}(a), \boldsymbol{\gamma}(b)]$ for $a, b \in V$ where $[\boldsymbol{\gamma}(a), \boldsymbol{\gamma}(b)] = \boldsymbol{\gamma}(a)\boldsymbol{\gamma}(b) - \boldsymbol{\gamma}(b)\boldsymbol{\gamma}(a)$. By [10, Chapter 2], $\boldsymbol{\varphi}$ is injective, and the Lie algebra of $\mathrm{Spin}(V)$ is $\boldsymbol{\varphi}(\underline{so}(V))$.

Let us also look at other exterior powers of V. We have

$$\bigwedge^{m-1} V = V(\omega_{m-1} + \omega_m),$$

$$\bigwedge^{m} V = V(2\omega_{m-1}) \bigoplus V(2\omega_m).$$

To see the decomposition in the second formula, we proceed as follows. Let $\tilde{Q} : V \to V^*$ be an $\underline{so}(V)$-equivariant isomorphism defined by the formula

$$\tilde{Q}(v_1)(v_2) := Q(v_1, v_2).$$

This isomorphism defines a similar $\underline{so}(V)$-equivariant isomorphism

$$\bigwedge^{m} \tilde{Q} : \bigwedge^{m} V \to \bigwedge^{m} V^*.$$

We also have an $\underline{sl}(V)$-equivariant isomorphism

$$\phi : \bigwedge^{m} V^* \to \bigwedge^{m} V,$$

using $e_1^* \wedge \ldots \wedge e_m^* \wedge e_1 \wedge \ldots \wedge e_m$ as a volume form. We define an $\underline{so}(V)$-equivariant isomorphism

$$\tau = \phi \circ (\bigwedge^{m} \tilde{Q}) : \bigwedge^{m} V \to \bigwedge^{m} V.$$

One proves easily that $\tau^2 = 1$. The representation $V(2\omega_{m-1})$ can be identified with the 1-eigenspace of τ and $V(2\omega_m)$ can be identified with the -1-eigenspace of τ. Thus the operators $\frac{1}{2}(\tau - 1)$ and $\frac{1}{2}(\tau + 1)$ are the projections on both direct summands.

Let us also mention the tensor product decompositions that will be useful. They go back at least to 1967 Cartan's book [6], but, for our purposes, we refer to [1] and [10].

Proposition 2.3 ([1, Theorem 4.6]). *Let K be an algebraically closed field of characteristic zero.*

$$\bigwedge^2 V(\omega_1) = V(\omega_2) \tag{1}$$

$$S_2 V(\omega_1) = V(2\omega_1) \bigoplus K \tag{2}$$

$$\bigwedge^2 V(\omega_{m-1}) = \bigoplus_i V(\omega_{m-2-4i}) \tag{3}$$

$$S_2 V(\omega_{m-1}) = V(2\omega_{m-1}) \bigoplus \bigoplus_i V(\omega_{m-4i}) \tag{4}$$

$$\bigwedge^2 V(\omega_m) = \bigoplus_i V(\omega_{m-2-4i}) \tag{5}$$

$$S_2 V(\omega_m) = V(2\omega_m) \bigoplus \bigoplus_i V(\omega_{m-4i}) \tag{6}$$

with the convention that $V(\omega_0) = K$.

2.2.2. **Representations of the spin group over an algebraically closed field of arbitrary characteristic.** Let K be an algebraically closed field of characteristic different than 2.

In what follows we apply the results of [20] to the fundamental weights ω_{m-1} and ω_m of the root system D_m. The most relevant part of [20] is the Appendix (sections A.2, A.4, A.5 and A.9), and references therein.

Let (V, Q) be a quadratic space where $\dim(V) = 2m$ and Q is in the hyperbolic form. Consider the isotropic Grassmannian $\mathrm{IGrass}(m, 2m)$. It consists of two connected components $\mathrm{IGrass}(m, 2m)_+$ and $\mathrm{IGrass}(m, 2m)_-$. The homogeneous coordinate rings of these varieties are respectively

$$K[\mathrm{IGrass}(m, 2m)_+] = \bigoplus_{d\geq 0} V(d\omega_{m-1}), \quad K[\mathrm{IGrass}(m, 2m)_-] = \bigoplus_{d\geq 0} V(d\omega_m). \tag{2}$$

For the root system D_m, both fundamental weights ω_{m-1} and ω_m are minuscule, so the results from [20] apply and all admissible pairs defined in [20, A.2, A.5] are trivial, as the irreducible module $V(\omega_{m-1})$ (respectively $V(\omega_m)$) has only extremal weights. Therefore it has only one $\mathbb{Z}$-form, i.e. its Schur module and Weyl module are the same (see [20, A.2, A.5]). So there are modules $V_{\mathbb{Z}}(\omega_{m-1})$ and $V_{\mathbb{Z}}(\omega_m)$ defined over $\mathbb{Z}$ such that after tensoring with any field we get over K the simple modules over the spin group, which are also Schur and Weyl modules.

Our definition of $V(d\omega_{m-1})$ (respectively $V(d\omega_m)$) is as the d-th homogeneous components of the rings in equation (2) above, tensored with K. The identification of the coordinate rings in (2) describes two components of $\mathrm{IGrass}(m, 2m)$ as closed subvarieties of projective spaces $\mathbf{P}(V(\omega_{m-1}))$ and $\mathbf{P}(V(\omega_m))$, respectively. We refer to these embeddings as half-spinor embeddings.

The Plücker embedding of the Grassmannian $\mathrm{Grass}(m, 2m)$ restricted to either of the connected components of $\mathrm{IGrass}(m, 2m)$ is a double of the corresponding half-spinor embedding.

In order to make everything explicit, let us choose a hyperbolic basis

$$\{e_1, \ldots, e_m, e_{-m}, \ldots, e_{-1}\}$$

in V. Consider a subspace U in IGrass(m, V) whose Plücker coordinate corresponding to $e_1, \ldots, e_m$ is nonzero (this contains a choice of connected component of IGrass(m, V) in which U is contained). Then the subspace U has a unique basis whose expansions in our hyperbolic basis give rows of a matrix

$$M = \begin{pmatrix} J & X \end{pmatrix} = \begin{pmatrix} 0 & 0 & \ldots & 0 & 1 & x_{1,1} & x_{1,2} & \ldots & x_{1,m} \\ 0 & 0 & \ldots & 1 & 0 & x_{2,1} & x_{2,2} & \ldots & x_{2,m} \\ \ldots & \ldots & \ldots & \ldots & \ldots & \ldots & \ldots & \ldots & \ldots \\ 1 & 0 & \ldots & 0 & 0 & x_{m,1} & x_{m,2} & \ldots & x_{m,m} \end{pmatrix}.$$

The subspace U is isotropic which makes the matrix X skew-symmetric. So the big cell Z in $\mathrm{Spin}(2m)/P_{m-1}$ is isomorphic to the space of skew-symmetric matrices, where P_{m-1} denote maximal parabolic subgroup corresponding to the simple roots $\varepsilon_{m-1} + \varepsilon_m$.

The Plücker coordinates restrict on Z to the maximal minors of the matrix M, and the spinor coordinates restrict to Pfaffians of all sizes of all submatrices of X which are themselves skew symmetric. Each such Pfaffian can be described by even number of columns of X (taking also the same rows), and it is the spinor coordinate (i.e. weight vector in $V(\omega_{m-1})$) whose weight has minus signs precisely at these places. This includes identity which corresponds to empty subset of columns.

So the quadratic relations defining our homogeneous space IGrass$(m, 2m)_+$ are the quadratic identities among Pfaffians of all sizes of the matrix X.

Example 2.4. Let us write explicitly the case of $n = 5$ where we have 10 quadratic relations, going back to Cartan; see [6]. Let $\mathcal{I} \subset [1, m]$ be a subset of even cardinality. We write $pf(\mathcal{I})$ for the Pfaffian of a submatrix of X on the rows and columns from $\mathcal{I}$. The Cartan equations are

$$pf(\emptyset)pf(1234) - pf(12)pf(34) + pf(13)pf(24) - pf(14)pf(23)$$

$$pf(12)pf(1345) - pf(13)pf(1245) + pf(14)pf(1235) - pf(15)pf(1234)$$

and eight others which we get by permuting the numbers $1, 2, 3, 4, 5$.

2.3. Certain Spin(V)-equivariant map P and its properties. Let V be an orthogonal space of rank $2m$ over an algebraically closed field K of characteristic different from 2.

The discussion in subsection 2.2.2 implies the existence of a Spin(V)-equivariant map

$$\mathbf{P} : V(2\omega_{m-1}) \to \bigwedge^m V \quad (\text{or } \mathbf{P} : V(2\omega_m) \to \bigwedge^m V). \tag{3}$$

Notice also that since the weights in our representations are integral combinations of ϵ_i, in fact this is the map of $\mathbf{SO}(V)$-modules.

Remark 2.5. The map $\mathbf{P}$ will be very important in our application as it will give polynomial formula expressing arbitrary Buchsbaum-Eisenbud multipliers by quadratic expressions involving spinor coordinates.

Before we start we need some notation. The signature of a permutation of the set $[1, m]$, denoted by sgn, is a multiplicative map from the group of permutations S_m to ± 1. Permutations with signature +1 are even and those with sign -1 are odd. Also $\mathcal{L}^c$ denotes the complement of a subset $\mathcal{L}$ of $[1,m]$. For a subset $\mathcal{L}$ of $[1,m]$ of even cardinality we denote $u_{\mathcal{L}}$ the weight vector of $V(\omega_{m-1})$ of weight with $\frac{1}{2}$ on coordinates from $\mathcal{L}^c$ and $\frac{-1}{2}$ on the coordinates from $\mathcal{L}$.

Lemma 2.6. *Set* $q = \left\lfloor \frac{m}{2} \right\rfloor$. *There is an equivariant map* $\mathbf{P} : S_2(V(\omega_m)) \to \bigwedge^m V$ *which is defined as*

$$\mathbf{P}(u_{\mathcal{J}_{2k}} u_\phi) = \frac{1}{2^{\ell(\mathcal{J}_{2k})-1}} \sum_{\substack{\mathcal{L}\subset\mathcal{J}_{2k},\\ \ell(\mathcal{J}_{2k})=2\ell(\mathcal{L})}} \operatorname{sgn}(\mathcal{J}_{2k},\mathcal{L}) e_{-\mathcal{L}} \wedge e_{\mathcal{J}_{2k}^c} \wedge e_{\mathcal{L}} \tag{4}$$

such that $\mathbf{P}(u_\phi u_\phi) = e_1 \wedge e_2 \wedge \cdots \wedge e_m$. *Here* $\mathcal{J}_{2k} = \{\gamma_1, \ldots, \gamma_{2k}\}$ *with* $1 \le \gamma_1 < \cdots < \gamma_{2k} \le m$, $1 \le k \le q$,

$$\begin{aligned} e_{-\mathcal{L}} &= \bigwedge_{i\in\mathcal{L}} e_{-l}, \\ u_{\mathcal{J}_{2k}} &= e_{-\gamma_1} \wedge e_{-\gamma_2} \wedge \cdots \wedge e_{-\gamma_{2k}}, \\ e_{\mathcal{L}} &= \bigwedge_{i\in\mathcal{L}} e_l, \end{aligned}$$

$\operatorname{sgn}(\mathcal{J}_{2k},\mathcal{L})$ *is the signature of permutations of* $\mathcal{J}_{2k}$, *and* $\ell(\mathcal{J})$ *is the length of any indexing set* $\mathcal{J} \subset [1,m]$.

Proof. We prove formula (4) by reverse induction on k. For $k = q$,

$$\mathbf{P}(u_{\mathcal{J}_{2q}} u_\phi) = \frac{1}{2^{\ell(\mathcal{J}_{2q})-1}} \sum_{\substack{\mathcal{L}\subset\mathcal{J}_{2q}\\ \ell(\mathcal{J}_{2q})=2\ell(\mathcal{L})}} \operatorname{sgn}(\mathcal{J}_{2q},\mathcal{L}) e_{-\mathcal{L}} \wedge e_{\mathcal{J}_{2q}^c} \wedge e_{\mathcal{L}}.$$

Using the action of internal operator on $V(2\omega_m)$ we see that

$$\mathrm{i}(e_{\gamma_i})\mathrm{i}(e_{\gamma_j})(u_{\mathcal{J}_{2q}} u_\phi) = (-1)^{i+j} u_{\mathcal{J}_{2q}\setminus\{\gamma_i,\gamma_j\}} u_\phi.$$

The map $\mathbf{P}$ is equivariant, and hence, by [13, Lemma 6.2.1], the following diagram

$$\begin{array}{ccc} V(2\omega_m) & \xrightarrow{\mathbf{P}} & \bigwedge^m V \\ {\scriptstyle \mathrm{i}(e_i)\mathrm{i}(e_j)}\big\downarrow & & \big\downarrow{\scriptstyle R_{e_i,e_j}} \\ V(2\omega_m) & \xrightarrow{\mathbf{P}} & \bigwedge^m V \end{array}$$

commutes. Therefore $\mathbf{P}(u_{J_{2q}\setminus\{\gamma_i,\gamma_j\}} u_\phi)$ is of the form

$$\frac{1}{2^{2q-3}} \sum_{\substack{\mathcal{L}\subset\mathcal{J}_{2q}\setminus\{\gamma_i,\gamma_j\}\\ \ell(\mathcal{L})=q-1}} \operatorname{sgn}(\mathcal{J}_{2q}\setminus\{\gamma_i,\gamma_j\},\mathcal{L}) e_{-\mathcal{L}} \wedge e_{(\mathcal{J}_{2q}\setminus\{\gamma_i,\gamma_j\})^c} \wedge e_{\mathcal{L}}.$$

Applying interior operator successively, one gets expression for $k = 1$ as

$$\mathbf{P}(u_{\{\gamma_i,\gamma_j\}} u_\phi) = \frac{\operatorname{sgn}(\{\gamma_i,\gamma_j\},\gamma_i,\gamma_j)}{2} (e_{-\gamma_i} \wedge e_{\{\gamma_i,\gamma_j\}^c} \wedge e_{\gamma_i} + e_{-\gamma_j} \wedge e_{\{\gamma_i,\gamma_j\}^c} \wedge e_{\gamma_j}).$$

Again, by applying internal operator, we obtain

$$\mathbf{P}(u_\phi u_\phi) = e_1 \wedge \cdots \wedge e_m.$$

We extend the map $\mathbf{P} : V(2\omega_m) \to \bigwedge^m V$ to $\mathbf{P} : S_2(V(\omega_m)) \to \bigwedge^m V$ by setting $\mathbf{P}|_{V(\omega_{m-4i})} = 0$ for $i \geq 1$. □

It remains to show that the formula for the map $\mathbf{P}$ is valid in any characteristic different from 2. The reason is that we derive the formula using the action of Lie algebra $\underline{so}(2m)$ but this comes from using the one parameter subgroup corresponding to a given root in $\mathbf{SO}(V)$.

REMARK 2.7. Let $\mathcal{L}, \mathcal{M} \subset [1, m]$ of even cardinality. Set $\mathcal{L} \ominus \mathcal{M} = (\mathcal{L} \setminus \mathcal{M}) \cup (\mathcal{M} \setminus \mathcal{L})$. Assume that $\mathcal{L} \ominus \mathcal{M}$ is nonempty. Note that $\mathcal{L} \ominus \mathcal{M}$ is of even cardinality. Using Lemma 2.6, one can evaluate the map $\mathbf{P}$ by permuting indices of the monomial $u_\mathcal{L} u_\mathcal{M}$ as

$$\frac{1}{2^{\ell(\mathcal{L}\ominus\mathcal{M})-1}} \sum_{\substack{\mathcal{N}\subset\mathcal{L}\ominus\mathcal{M}\\ \ell(\mathcal{L}\ominus\mathcal{M})=2\ell(\mathcal{N})}} \operatorname{sgn}(\mathcal{L}\cup\mathcal{M}, \mathcal{N}) e_{-(\mathcal{L}\cap\mathcal{M})} \wedge e_{-\mathcal{N}} \wedge e_{\mathcal{L}^c\cap\mathcal{M}^c} \wedge e_{\mathcal{N}}.$$

Moreover, $\mathbf{P}(u_\mathcal{L} u_\mathcal{L}) = \operatorname{sgn}(\mathcal{L}, \mathcal{L}^c) e_{-\mathcal{L}} \wedge e_{\mathcal{L}^c}$.

Corollary 2.8. *For $\mathcal{Q} \subset [1, m]$, let $\mathcal{L} = \mathcal{Q} \cup \{p\}$ and $\mathcal{M} = \mathcal{Q} \cup \{q\}$ where $p, q \in [1, m]$ and $p \neq q$. Then $\mathbf{P}(u_\mathcal{L} u_\mathcal{M})$ is*

$$\frac{1}{2}(\operatorname{sgn}(\mathcal{Q}\sqcup\{p,q\},\{p\}) e_{-\mathcal{Q}}\wedge e_{-p}\wedge e_{(\mathcal{Q}\sqcup\{p,q\})^c}\wedge e_p + \operatorname{sgn}(\mathcal{Q}\sqcup\{p,q\},\{q\}) e_{-\mathcal{Q}}\wedge e_{-q}\wedge e_{(\mathcal{Q}\sqcup\{p,q\})^c}\wedge e_q).$$

2.4. Arbitrary ring R with 2 invertible in R. The constructions of fundamental representations $\bigwedge^i V$, $V(\omega_{m-1})$ and $V(\omega_m)$ are also valid over any commutative ring R such that 2 is invertible in R. One assumes that V is a free orthogonal R-module, i.e. V has a symmetric bilinear form which in some basis is hyperbolic. Note that Definition 2.1 is still valid if K is replaced by any arbitrary ring R with 2 invertible in R.

REMARK 2.9. The formulas for the map $\mathbf{P}$ remain valid over any commutative ring R with 2 invertible in R and any orthogonal module V in the sense of Definition 2.1. One can just apply the formulas from Lemma 2.6 and they remain true in any hyperbolic basis of V.

3. Background on free resolutions

Throughout the rest of the paper R and S denote Noetherian commutative rings unless otherwise stated, $\mu(I)$ denotes the minimal number of generators of an ideal I of R, and I_n denotes the $n \times n$ identity matrix. For an R-module M, we use M^* to denote $\operatorname{Hom}_R(M, R)$. Moreover, $\ker(f)$ and $\operatorname{im}(f)$ denote the kernel and the image of a ring map $f : R \to S$, respectively.

Buchsbaum and Eisenbud gave a structure theorem (also known as the First Structure Theorem) that describes an arithmetic structure of free resolutions as follows:

Theorem 3.1 ([3, Theorem 3.1]). *(The First Structure Theorem) Let R be a Noetherian ring and let I be an ideal of R. Let*

$$0 \to F_n \xrightarrow{\boldsymbol{d}_n} F_{n-1} \xrightarrow{\boldsymbol{d}_{n-1}} \cdots \xrightarrow{\boldsymbol{d}_3} F_2 \xrightarrow{\boldsymbol{d}_2} F_1 \xrightarrow{\boldsymbol{d}_1} F_0$$

be a free R-resolution of R/I and $r_i = \operatorname{rank}(\boldsymbol{d}_i)$. Then there exists a unique sequence of *homomorphisms* $\boldsymbol{a}_k : R \to \bigwedge^{r_k} F_{k-1}$ *for* $1 \leq k \leq n$ *such that* $\boldsymbol{a}_n := \bigwedge^{r_n} \boldsymbol{d}_n$ *and the following diagram commutes:*

$$\begin{array}{ccc} \bigwedge^{r_k} F_k & \xrightarrow{\bigwedge^{r_k} \boldsymbol{d}_k} & \bigwedge^{r_k} F_{k-1} \\ \simeq \downarrow & & \uparrow \boldsymbol{a}_k \\ \bigwedge^{r_{k+1}} F_k^* & \xrightarrow{\boldsymbol{a}_{k+1}^*} & R \end{array}$$

We refer to maps $\boldsymbol{a}_k$ *in* Theorem 3.1 *as the Buchsbaum-Eisenbud multiplier maps. Their coordinates are called the Buchsbaum-Eisenbud multipliers.*

The next remark reveals the structure of a minimal free resolution of R/I where R is a complete regular local ring and I is a Gorenstein ideal of codimension four.

REMARK 3.2 ([17]). Let R be a Gorenstein local ring in which 2 is a unit and let $I \subset R$ be a Gorenstein ideal of codimension four with $\mu(I) = n$. Let

$$\mathbb{F} : 0 \to F_4 \xrightarrow{\boldsymbol{d}_4} F_3 \xrightarrow{\boldsymbol{d}_3} F_2 \xrightarrow{\boldsymbol{d}_2} F_1 \xrightarrow{\boldsymbol{d}_1} R \tag{5}$$

be a minimal free resolution of R/I. Then we have the following:

(a) By Gorenstein duality, $F_{4-i} \cong F_i^*$.

(b) $\mathrm{rank}(F_1) = n$ and $\mathrm{rank}(F_2) = 2n - 2$.

(c) By [17, Theorem 2.4], for a minimal resolution $\mathbb{F}$ of R/I

$$\mathbb{F} : 0 \to R \xrightarrow{\boldsymbol{d}_1^*} F_1^* \xrightarrow{\boldsymbol{d}_3} F_2 \xrightarrow{\boldsymbol{d}_2} F_1 \xrightarrow{\boldsymbol{d}_1} R,$$

there exist a symmetric isomorphism $\tilde{Q}$ and an isomorphism $\rho : \mathbb{F} \to \mathbb{F}^*$ of the form:

$$\begin{array}{ccccccccccc} 0 & \longrightarrow & R & \xrightarrow{\boldsymbol{d}_1^*} & F_1^* & \xrightarrow{\boldsymbol{d}_3} & F_2 & \xrightarrow{\boldsymbol{d}_2} & F_1 & \xrightarrow{\boldsymbol{d}_1} & R \\ & & \| & & \| & & \downarrow \tilde{Q} & & \| & & \| \\ 0 & \longrightarrow & R & \xrightarrow{\boldsymbol{d}_1^*} & F_1^* & \xrightarrow{\boldsymbol{d}_2^*} & F_2^* & \xrightarrow{\boldsymbol{d}_3^*} & F_1 & \xrightarrow{\boldsymbol{d}_1} & R \end{array}$$

(d) If R is a complete regular local ring, then the dualizing matrix $\tilde{Q}$ is of the form

$$\begin{bmatrix} 0 & \mathrm{I_{n-1}} \\ \mathrm{I_{n-1}} & 0 \end{bmatrix},$$

and, by part (c), the resolution of R/I is

$$\mathbb{F} : 0 \to R \xrightarrow{\boldsymbol{d}_1^t} F_1^* \xrightarrow{\tilde{Q}\boldsymbol{d}_2^t} F_2 \xrightarrow{\boldsymbol{d}_2} F_1 \xrightarrow{\boldsymbol{d}_1} R.$$

(e) Let $\langle\ ,\ \rangle : F_1 \otimes_R F_1^* \to R$ be the evaluation map and let $Q : F_2 \otimes_R F_2 \to R$ be the symmetric bilinear map induced by $\tilde{Q}$. Then

$$\langle \boldsymbol{d}_2 x_2, x_3 \rangle = Q(x_2, \boldsymbol{d}_3 x_3) \ \text{ for all } \ x_2 \in F_2,\ x_3 \in F_1^*.$$

(f) $\mathbb{F}$ has a multiplicative structure which makes it an associative differential graded R-algebra.

(g) The module F_2 has a structure of an even orthogonal module of rank $2n-2$ according to Definition 2.1.

Remark 3.3. Let R be a regular local ring or a polynomial ring over a field and let $I \subset R$ be a Gorenstein ideal of codimension four with $\mu(I) = n$. Let

$$\mathbb{F} : 0 \to F_4 \xrightarrow{d_4} F_3 \xrightarrow{d_3} F_2 \xrightarrow{d_2} F_1 \xrightarrow{d_1} R \tag{6}$$

be a minimal free resolution of R/I. Then, by Theorem 3.1, we have $\boldsymbol{a}_4 = \boldsymbol{d}_4$ and there is a map $\boldsymbol{a}_3 : R \to \bigwedge^{n-1} F_2$ such that the following diagram commutes

$$\begin{tikzcd} \bigwedge^{n-1} F_3 \arrow[r, "\bigwedge^{n-1} \boldsymbol{d}_3"] \arrow[d, "\simeq"'] & \bigwedge^{n-1} F_2 \\ F_3^* \arrow[r, "\boldsymbol{d}_4^*"] & R \arrow[u, "\boldsymbol{a}_3"'] \end{tikzcd}$$

3.1. Generic doubling of perfect ideals. Let R be a regular local ring, let J be a perfect ideal of R of codimension 3, and let $S := R/J$. Assume S is generically Gorenstein with a canonical module ω_S. It is known that one can identify ω_S with an ideal of S and S/ω_S is a Gorenstein ring, [2, Proposition 3.3.18].

Let $(\mathbb{F}, \boldsymbol{d})$ be a minimal free resolution of S over R. Then $(\mathbb{F}^*, \boldsymbol{d}^*)$ is a minimal free resolution of ω_S as J is perfect. Take the minimal generators $f_1, \ldots, f_\ell$ of $\operatorname{Hom}_S(\omega_S, S)$, and let $\widetilde{R} := R[\tau_1, \ldots, \tau_\ell]$. Now we consider the injective map $\psi : \omega_{\widetilde{S}} \to \widetilde{S}$ where $\psi = \sum_{i=1}^{\ell} \tau_i f_i$, $\widetilde{S} := \widetilde{R}/J\widetilde{R}$, and $\omega_{\widetilde{S}}$ is a canonical module of $\widetilde{S}$. Then $\omega_{\widetilde{S}} \simeq \operatorname{im}(\psi)$ since ψ is injective. Next ψ lifts to a map of complexes $\phi : \widetilde{\mathbb{F}}^* \to \widetilde{\mathbb{F}}$ which gives us a resolution of a Gorenstein ring $\widetilde{S}/\omega_{\widetilde{S}}$ of codimension 4. In this case, we say that the resolution of $\widetilde{S}/\omega_{\widetilde{S}}$ is obtained by a generic doubling of $\mathbb{F}$.

4. Spinor structures on resolutions of Gorenstein ideals of codimension four

In this section we show the existence of a spinor structure on a length four minimal resolution of a Gorenstein ideal over a complete regular local rings and a polynomial ring over a field; see Theorem 4.2. This is the main result of our paper. Throughout this section, let K be an algebraically closed field.

Definition 4.1. Let $(R, \mathfrak{m}, K)$ be a regular local ring (respectively a polynomial ring over K) in which 2 is a unit and let $I \subset R$ be a Gorenstein ideal (respectively a graded Gorenstein ideal) of codimension four with $\mu(I) = n$. Let

$$\mathbb{F} : 0 \to F_4 \xrightarrow{d_4} F_3 \xrightarrow{d_3} F_2 \xrightarrow{d_2} F_1 \xrightarrow{d_1} R$$

be a minimal free resolution of R/I. We say that the resolution $\mathbb{F}$ has a spinor structure if there exists a map $\tilde{\boldsymbol{a}}_3 : R \to V(\omega_{n-1}) \otimes_K R$ such that the following diagram commutes

$$\begin{tikzcd} R \arrow[r, "S_2(\tilde{\boldsymbol{a}}_3)"] \arrow[dr, "\boldsymbol{a}_3"'] & S_2(V(w_{n-1})) \otimes_K R \arrow[d, "\mathbf{P} \otimes R"] \\ & \bigwedge^{n-1} F_2 \end{tikzcd}$$

where $\boldsymbol{a}_3$ is the map given by the First Structure Theorem of Buchsbaum and Eisenbud in Theorem 3.1 and $\mathbf{P}$ is the map described in Section 2.3.

Now we are ready to show the existence of a spinor structure on a length four minimal resolution of a Gorenstein ideal over a complete regular local rings and a polynomial ring over a field.

Theorem 4.2.

(1) *Let $(R, \mathfrak{m}, K)$ be a regular local ring in which 2 is a unit and K is algebraically closed. Let $I \subset R$ be Gorenstein ideal of codimension 4 and let $\mathbb{F}$ be a minimal free resolution of R/I of the form*

$$\mathbb{F} : 0 \to F_4 \xrightarrow{d_4} F_3 \xrightarrow{d_3} F_2 \xrightarrow{d_2} F_1 \xrightarrow{d_1} R \to 0.$$

Assume that the multiplication $Q : F_2 \otimes F_2 \to F_4$ defined in Remark 3.2.(e) *is in the hyperbolic form. Then there exists a spinor structure on $\mathbb{F}$.*

(2) *The same conclusion holds when R is a polynomial ring over an algebraically closed field K of characteristic different from 2 and I is a homogeneous Gorenstein ideal of codimension 4.*

Before we prove Theorem 4.2, let us explain its meaning more precisely.

REMARK 4.3.

(1) Assume the hypothesis of Theorem 4.2 and let $\mu(I) = n$. The Buchsbaum-Eisenbud multiplier $\boldsymbol{a}_{3,\mathcal{K}}$ is the square of the spinor coordinate $\widetilde{\boldsymbol{a}}_{3,\mathcal{J}}$ for $\mathcal{K} = -\mathcal{J} \cup \mathcal{J}^c$, where $\mathcal{J} \subset [1, n-1]$ with $\ell(\mathcal{J})$ even, i.e when the multiindex $\mathcal{K} \subset \{\pm 1, \dots, \pm(n-1)\}$ of cardinality $n-1$ contains all the numbers $1, 2, \dots, n-1$ (with arbitrary signs). The Buchsbaum-Eisenbud multipliers for $\mathcal{K} = -\mathcal{J} \cup \mathcal{J}^c$, where $\mathcal{J} \subset [1, n-1]$ with $\ell(\mathcal{J})$ odd are all zero. Note that we already made a choice that $\operatorname{im}(d_3)$ is in the connected component $\operatorname{Spin}(2(n-1))/P_{n-2}$. If we make another choice, then the opposite happens: the multiplier $\boldsymbol{a}_{3,\mathcal{K}}$ is the square of the spinor coordinate $\widetilde{\boldsymbol{a}}_{3,\mathcal{J}}$ for $\mathcal{K} = -\mathcal{J} \cup \mathcal{J}^c$, where $\mathcal{J} \subset [1, n-1]$ with $\ell(\mathcal{J})$ odd, i.e when the multi-index $\mathcal{K} \subset \{\pm 1, \dots, \pm(n-1)\}$ of cardinality $n-1$ contains all the numbers $1, 2, \dots, n-1$ (with arbitrary signs). The Buchsbaum-Eisenbud multipliers for $\mathcal{K} = -\mathcal{J} \cup \mathcal{J}^c$, where $\mathcal{J} \subset [1, n-1]$ with $\ell(\mathcal{J})$ even are all zero.

(2) For other indices $\mathcal{K}$ (i.e. those where some numbers $\pm i$ are missing for some $1 \leq i \leq n-1$), the multiplier $\boldsymbol{a}_{3,\mathcal{K}}$ is given by the expression from Lemma 2.6.

(3) As a consequence of Corollary 2.8 and Theorem 4.2, we see that

$$\boldsymbol{a}_{3,\mathcal{K}} = \widetilde{\boldsymbol{a}}_{3,Q\cup\{p\}}\widetilde{\boldsymbol{a}}_{3,Q\cup\{q\}}$$

where $\mathcal{K} = -Q \cup \{\pm p\} \cup (Q \cup \{p, q\})^c$ or $\mathcal{K} = -Q \cup \{\pm q\} \cup (Q \cup \{p, q\})^c$.

We need the following lemma for the proof of Theorem 4.2.

Lemma 4.4. *Let R be a polynomial ring over an algebraically closed field K of characteristic different than 2 and let I be a codimension four homogeneous Gorenstein ideal of R with $\mu(I) = n$. Let*

$$\mathbb{F} : 0 \to R(-d) \xrightarrow{d_4} F \xrightarrow{d_3} G \simeq G^* \xrightarrow{d_2} F^* \xrightarrow{d_1} R$$

be a graded free resolution of R/I. Then G has a hyperbolic basis.

Proof. Let

$$G = (\bigoplus_{i=1}^{n-1} R(-a_i)) \bigoplus (\bigoplus_{i=1}^{n-1} R(-a_i + d))$$

where $a_1 \leq \cdots \leq a_{n-1} \leq d - a_{n-1} \leq \cdots \leq d - a_1$. Let $\tilde{Q} : G \to G^*$ be a symmetric isomorphism that induces a bilinear map $Q : G \otimes G \to R$. Now choose a basis element e_1 of highest degree $d - a_1$ of G. Then there exists a complementary e_1' of lowest degree a_1 such that $Q(e_1, e_1') = 1$ since Q is non-degenerate; and, for any basis element e_j in G, we have $Q(e_1, e_j)$ is a constant. Let $W_1 = Re_1 + Re_1'$, and let $W_1^\perp$ denote the orthogonal complement of W_1 in G.

If $d - a_1 = a_1$, then the entries of the matrix with respect to the map $\tilde{Q}$ belong to K. Then, by the change of basis, one can transform $\tilde{Q}$ into a hyperbolic form. In the case that $d - a_1 \neq a_1$, there exists a 2×2 submatrix of $\tilde{Q}$ with respect to the basis $\{e_1, e_1'\}$ of the form

$$\tilde{Q}_1 = \begin{bmatrix} 0 & 1 \\ 1 & z \end{bmatrix}, \quad \text{where } z \in R \text{ is of positive degree.}$$

Next choose a 2×2 matrix $A = \begin{bmatrix} 1 & -\frac{1}{2}z \\ 0 & 1 \end{bmatrix}$ such that $A^T\tilde{Q}_1 A = \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix}$. Then $A^T\tilde{Q}_1 A$ restricted to W_1 is in the hyperbolic form.

If the basis elements of $W_1^\perp$ are of the same degree, then we are done. Otherwise, repeating the argument above for $\tilde{Q}_1$, we can always construct an hyperbolic pair using the highest and the lowest degree basis elements of $W_1^\perp$.

Continuing in this way, we get $W_k = W_{k-1} \oplus Re_k + Re_k'$ for some $k \leq n - 1$ such that the lowest and highest degree basis elements of W_k can be transformed into a hyperbolic pair and the basis elements of $W_k^\perp$ are of the same degree. Then the entries of the matrix with respect to the map φ restricted to $W_k^\perp$ belong to K, and, by the change of basis, one can transform this matrix into the hyperbolic form. This finishes the proof of the lemma. □

Proof of Theorem 4.2. We first prove part (1). Note that, by the hypothesis in part (1) and Remark 3.2.(d), a minimal free resolution of R/I is of the form

$$\mathbb{F} : 0 \to R \xrightarrow{\boldsymbol{d}_1^t} F_1^* \xrightarrow{\tilde{\boldsymbol{Q}}\boldsymbol{d}_2^t} F_2 \xrightarrow{\boldsymbol{d}_2} F_1 \xrightarrow{\boldsymbol{d}_1} R.$$

Let $\mu(I) = n$ and let $\{g_1, \ldots, g_n\}$ be a basis of F_1. Let $\{e_1, \ldots, e_{n-1}, e_{-n+1}, \ldots, e_{-1}\}$ be a hyperbolic basis of F_2. By the Leibniz formula, we see that the image of $\boldsymbol{d}_3 = \tilde{\boldsymbol{Q}}\boldsymbol{d}_2^t$ is an isotropic submodule.

Let $R_{(0)}$ be the field of fractions of R. The complex $\mathbb{F}\otimes_R R_{(0)}$ is split exact. We can choose a hyperbolic basis $\{e_1', \ldots, e_{n-1}', e_{-n+1}', \ldots, e_{-1}'\}$ of $F_2 \otimes_R R_{(0)}$ such that $\mathrm{im}(\boldsymbol{d}_3 \otimes_R R_{(0)})$ is the span of $e_1', \ldots, e_{n-1}'$. This subspace is in the connected component of the isotropic Grassmannian $\mathrm{IGrass}(n-1, F_2 \otimes_R R_{(0)})$ corresponding to $\mathrm{Spin}(2n-2)/P_{n-2}$. The subspace $\mathrm{im}(\boldsymbol{d}_3 \otimes_R R_{(0)})$ is also isotropic, and it has spinor coordinates in $V(\omega_{n-2})$.

The Plücker coordinates of $\mathrm{im}(\boldsymbol{d}_3 \otimes_R R_{(0)})$ are Buchsbaum-Eisenbud multipliers, i.e., the coordinates of the map a_3. The Buchsbaum-Eisenbud multipliers must therefore have expressions given by Lemma 2.6 in terms of spinor coordinates. By equivariance, such relations have to be satisfied for every choice of hyperbolic basis of $F_2 \otimes_R R_{(0)}$, and for every choice of basis in $F_1 \otimes R_{(0)}$.

The only remaining thing is to check that in our original bases $\{g_1, \ldots, g_n\}$ and $\{e_1, \ldots, e_{n-1}, e_{-1}, \ldots, e_{-n+1}\}$ the spinor coordinates are not just in $R_{(0)}$ but in R. But for each of them its square is the appropriate Buchsbaum-Eisenbud multiplier which is in R. Since R is normal, we conclude that all spinor coordinates are in R. This means all the relations from Lemma 2.6 are satisfied in R since they hold in $R_{(0)}$. This proves part (1).

In case of polynomial rings over fields, F_2 has a hyperbolic basis by Lemma 4.4. Thus, the proof of part (2) of the theorem follows the same as in part (1). □

As a consequence of Remark 3.2.(d) and Theorem 4.2, we get the following corollary.

Corollary 4.5. *If R is a complete regular local ring in which 2 is a unit and I is a Gorenstein ideal of codimension 4, then there exists a spinor structure on a minimal free resolution of R/I.*

REMARK 4.6.

(1) The hypotheses of R being a domain could probably be dropped. It would follow by localizing at the set of non-zero divisors in R, but probably requires some additional work on half-spinor representations over commutative rings.
(2) The normality assumption also might not be necessary.
(3) As the examples in the next section show, often spinor structures exist even without characteristic different from 2 assumptions. However this involves finding a hyperbolic basis for F_2 which was established by Kustin and Miller only under this assumption.
(4) Spinor coordinates are in the radical of the ideal I. If I is a radical ideal, then spinor coordinates are in I. We do not know any Gorenstein ideal I of codimension 4 for which the spinor coordinates are not in the ideal I.

5. Examples of spinor coordinates on resolutions of codimension four Gorenstein ideals

In this section, we give explicit calculations of spinor coordinates on resolutions of well-known Gorenstein ideals with 4, 6, and 9 generators. In some of these examples, we can find the spinor coordinates even under weaker assumptions on R than claimed in Theorem 4.2. The first two examples are also discussed in Reid's paper, see [21].

EXAMPLE 5.1. Let R be an arbitrary commutative ring and $\mathcal{K}(x_1, x_2, x_3, x_4; R)_\bullet$ be the Koszul complex resolving a complete intersection in codimension 4 on elements x_1, x_2, x_3, x_4 from R. Let $F = R^4$ and $R = \mathrm{Sym}(F)$. Then $\mathcal{K}(x_1, x_2, x_3, x_4; R)$ is a resolution of R/I of the form

$$\mathcal{K}(x_1, x_2, x_3, x_4; R) : 0 \to \bigwedge^4 F \xrightarrow{d_1^t} \bigwedge^3 F \xrightarrow{\tilde{Q}d_2^t} \bigwedge^2 F \xrightarrow{d_2} \bigwedge^1 F \xrightarrow{d_1} \bigwedge^0 F$$

with

$$\boldsymbol{d}_1 = [x_1 \quad x_2 \quad x_3 \quad x_4], \quad \boldsymbol{d}_2 = \begin{bmatrix} -x_4 & 0 & 0 & 0 & x_3 & -x_2 \\ 0 & -x_4 & 0 & -x_3 & 0 & x_1 \\ 0 & 0 & -x_4 & x_2 & -x_1 & 0 \\ x_1 & x_2 & x_3 & 0 & 0 & 0 \end{bmatrix}.$$

Our calculation of the matrices $\boldsymbol{d}_1$ and $\boldsymbol{d}_2$ gives us the map $\tilde{Q} : \bigwedge^2 F \to \bigwedge^2 F^*$ in the hyperbolic form, that is, $\tilde{Q} = \left[\begin{smallmatrix} 0 & \mathrm{I}_3 \\ \mathrm{I}_3 & 0 \end{smallmatrix}\right]$. Note that the quadratic form $Q : \bigwedge^2 F \otimes_R \bigwedge^2 F \to \bigwedge^4 F$ is just the exterior multiplication.

For the integral weight $(3, 1, 1, 1)$, Schur module is $S_{(3,1,1,1)}F = \bigwedge^4 F \otimes S_2F$ as $\operatorname{rank}(F) = 4$. Then the following diagram

$$\begin{tikzcd}
\bigwedge^4 F \otimes S_2F \arrow[rr, twoheadrightarrow] \arrow[d, "\Delta\otimes\Delta"'] & & \bigwedge^3(\bigwedge^2 F) \\
F \otimes \bigwedge^3 F \otimes F \otimes F \arrow[d, "\Lambda"'] & & \\
\bigwedge^2 F \otimes \bigwedge^2 F \otimes \bigwedge^2 F \arrow[uurr, bend right] & &
\end{tikzcd}$$

commutes where $\Delta : \bigwedge^i F \to F \otimes \bigwedge^{i-1} F$ is the diagonal map and Λ is the composition of interchange of the first and second, diagonalization of the second, and then multiplication of the second and third components.

Since $F = R^4$, the maps $\bigwedge^4 F \otimes \bigwedge^2 F \to \bigwedge^3(\bigwedge^2 F)$, $\bigwedge^5 F \otimes F \to \bigwedge^3(\bigwedge^2 F)$, and $\bigwedge^6 F \to \bigwedge^3(\bigwedge^2 F)$ are zero. Thus $S_{(2,2,2,0)}(F) = S_2F \otimes \bigwedge^3 F \otimes \bigwedge^3 F$. For $\operatorname{char}(K) \neq 2$,

$$\bigwedge^3(\bigwedge^2 F) \cong S_{(2,2,2,0)}(F) \oplus S_{(3,1,1,1)}(F).$$

Note that $R = \bigwedge^4 F \otimes \bigwedge^4 F$ and there is a map $(\boldsymbol{m}_{13} \otimes 1 \otimes 1) \circ (\Delta \otimes \Delta) : R \to S_2F \otimes \bigwedge^3 F \otimes \bigwedge^3 F$ where $\boldsymbol{m}_{13} : F \otimes F \to S_2F$ is a multiplication map (for details; see [22, Section 1.1.1]). We use the diagonal map

$$\Delta \otimes \Delta : \bigwedge^3 F \otimes \bigwedge^3 F \to F \otimes \bigwedge^2 F \otimes F \otimes \bigwedge^2 F$$

and the map $\Omega : F \otimes \bigwedge^2 F \otimes F \otimes \bigwedge^2 F \to \bigwedge^2 F \otimes \bigwedge^2 F \otimes \bigwedge^2 F$ which interchanges the first and second, and then multiplies the second and third components to get the map

$$\boldsymbol{p}_{24} : \bigwedge^3 F \otimes \bigwedge^3 F \twoheadrightarrow \bigwedge^3(\bigwedge^2 F)$$

by the commutativity of the following diagram

$$\begin{tikzcd}
\bigwedge^3 F \otimes \bigwedge^3 F \arrow[d, "\Delta\otimes\Delta"'] \arrow[rr, two heads, "p_{24}"] & & \bigwedge^3(\bigwedge^2 F) \\
F \otimes \bigwedge^2 F \otimes F \otimes \bigwedge^2 F \arrow[d, "\Omega"'] & & \\
\bigwedge^2 F \otimes \bigwedge^2 F \otimes \bigwedge^2 F \arrow[rruu, bend right] & &
\end{tikzcd}$$

Thus there is a map $(1 \otimes \boldsymbol{p}_{24}) \circ \sigma : R \to S_2F \otimes \bigwedge^3(\bigwedge^2 F)$, where $\sigma = (\boldsymbol{m}_{13} \otimes 1 \otimes 1) \circ (\Delta \otimes \Delta)$. Now we set $\boldsymbol{a}_3 = (1 \otimes \boldsymbol{p}_{24}) \circ \sigma$.

Therefore $\boldsymbol{a}_3$ goes to the summand $S_{(2,2,2,0)}(F)$ as in the diagram below:

$$\begin{tikzcd}
\bigwedge^3 F \otimes \bigwedge^3 F \arrow[r] \arrow[d] \arrow[dr, two heads, "p_{24}"] & \bigwedge^2 F \otimes \bigwedge^2 F \otimes \bigwedge^2 F \arrow[d] \\
S_{(2,2,2,0)}(F) \arrow[r] & \bigwedge^3(\bigwedge^2 F)
\end{tikzcd}$$

In fact, $\boldsymbol{a}_3$ is the second symmetric power of the map

$$\widetilde{\boldsymbol{a}}_3 : R \to \bigwedge^3 F$$

sending 1 to $x_1e_2 \wedge e_3 \wedge e_4 - x_2e_1 \wedge e_3 \wedge e_4 + x_3e_1 \wedge e_2 \wedge e_4 - x_4e_1 \wedge e_2 \wedge e_3$. This last map $\widetilde{\boldsymbol{a}}_3$ gives us the spinor structure.

Let us interpret this in terms of the root systems. Here we deal with a root system D_3 which is just A_3. So the vector representation G of rank 6 can be considered as the second fundamental representation $\bigwedge^2 H$ where H is the 4-dimensional space. Finding the structure map $\boldsymbol{a}_3$, we see that it is given by $R \to \bigwedge^3(\bigwedge^2 H)$. The map $\widetilde{\boldsymbol{a}}_3$ is just the map from R to H and it allows us to identify H and F.

Next we look at a hyperplane section of a codimension three Gorenstein ideal of Pfaffians of a skew-symmetric matrix.

EXAMPLE 5.2. Let R be a polynomial ring over $\mathbb{Z}$ on the entries of $(2n + 1) \times (2n + 1)$ skew-symmetric matrix $X = (x_{ij})$ with $x_{ij} = -x_{ji}$, and an additional variable y. Consider the resolution given by

$$\mathbb{F} : (0 \to R \xrightarrow{y} R \to 0) \otimes (0 \to R \xrightarrow{\boldsymbol{\partial}_1^*} R^{2n+1} \xrightarrow{\boldsymbol{\partial}_2} R^{2n+1} \xrightarrow{\boldsymbol{\partial}_1} R \to 0) \tag{7}$$

where $\boldsymbol{\partial}_1 = \left[(-1)^i \mathrm{Pf}([1, 2n + 1] \setminus \{i\}, X)\right]$ where $[1, 2n + 1] = \{1, \dots, 2n + 1\}$ and $\boldsymbol{\partial}_2 = X$. Here $\mathrm{Pf}(\mathcal{I}, X)$ denotes the Pfaffian of the submatrix of X on rows and columns from $\mathcal{I}$ where $\mathcal{I}$ be an index set of $[1, 2n + 1]$. Note that the resolution $\mathbb{F}$ in (7) is a hyperplane section of a codimension 3 Gorenstein ideal of Pfaffians of a skew-symmetric matrix X.

The matrix of the differential $\boldsymbol{d}_2 : R^{2n+1} \oplus R^{2n+1} \to R \oplus R^{2n+1}$ of $\mathbb{F}$ is given by

$$\boldsymbol{d}_2 = \begin{bmatrix} \boldsymbol{\partial}_1 & 0 \\ -y\mathrm{I}_{2n+1} & X \end{bmatrix}.$$

Our calculation directly gives

$$\tilde{Q} : R^{2n+1} \oplus R^{2n+1} \to (R^{2n+1} \oplus R^{2n+1})^*$$

which is in the hyperbolic form up to permutation, that is, $\tilde{Q} = \left[\begin{smallmatrix} 0 & \mathrm{I}_{2n+1} \\ \mathrm{I}_{2n+1} & 0 \end{smallmatrix}\right]$. Note that, by Theorem 4.2, a spinor structure exists on $\mathbb{F}$.

Denote the ith column of $\boldsymbol{d}_2$ by e_i. Set $e_{-i} = e_{2n+1+i}$. Hence the associated hyperbolic basis of the middle module $R^{2n+1} \oplus R^{2n+1}$ of $\mathbb{F}$ is $\{e_1, \dots, e_{2n+1}, e_{-1}, \dots, e_{-(2n+1)}\}$.

Let $\mathcal{N} = -\mathcal{I} \cup ([1, 2n+1] \setminus \mathcal{I})$. Let $\mathcal{J}$ be an index set of $[1, 2n+2] = \{1, \dots, 2n+2\}$ where the cardinality of $\mathcal{J}$ is equal to $2n+1$, and let $(\boldsymbol{d}_2)_{\mathcal{J},\mathcal{N}}$ denote the $(2n+1) \times (2n+1)$ minors $\boldsymbol{d}_2$ on rows $\mathcal{J}$ and columns $\mathcal{N}$. Note that $(2n+1) \times (2n+1)$ minors of the matrix $\boldsymbol{d}_2$ corresponding to $\mathcal{J}$ rows and $\mathcal{N}$ columns are of the form:

$$(\boldsymbol{d}_2)_{\mathcal{J},\mathcal{N}} = \begin{cases} y^{2n+1-\ell(\mathcal{I})}(\mathrm{Pf}(\mathcal{I}, X))^2, & \text{if } \mathcal{J} = [1, 2n+2] \setminus \{1\}, \\ \mathrm{Pf}([1, 2n+1] \setminus \{i\}, X) y^{2n-\ell(\mathcal{I})}(\mathrm{Pf}(\mathcal{I}, X))^2, & \text{if } \mathcal{J} = [1, 2n+2] \setminus \{i\}. \end{cases}$$

By Remark 3.3, we have $\boldsymbol{a}_{3,\,\mathcal{N}} = y^{2n-\ell(\mathcal{I})}(\mathrm{Pf}(\mathcal{I}, X))^2$. Then the spinor coordinates $\widetilde{\boldsymbol{a}}_{3,\,\mathcal{I}}$ corresponding to $\mathcal{I}$ are $\pm y^{n-\ell(\mathcal{I})/2}\mathrm{Pf}(\mathcal{I}, X)$ by Remark 4.3.(1). If the cardinality of $\mathcal{I}$ is $2n$, then $\mathrm{Pf}(\mathcal{I}, X)$ is the spinor coordinate.

EXAMPLE 5.3. In the 9-generator case, we have two examples of resolutions where none of the minimal generators are spinor coordinates. Note that we cannot explicitly get a hyperbolic basis over $\mathbb{Q}$, but, over $\mathbb{C}$, we can by Theorem 4.2.(2).

(1) The ring R is a polynomial ring in 9 variables on the entries of 3×3 generic matrix X over a field K of characteristic different than 2. The ideal I is generated by 2×2-minors of the matrix X.

(2) The ring R is a polynomial ring in 8 variables over a field K of characteristic different from 2 and I is the ideal of the generated by the equation of the Segre embedding $\mathbf{P}^1 \times \mathbf{P}^1 \times \mathbf{P}^1$ into $\mathbf{P}^7$.

We observe that degree of all spinor coordinates is 3 whereas minimal generators of I are of degree 2. Hence none of the minimal generators of I are spinor coordinates.

6. A generic doubling of an almost complete intersection and Kustin-Miller model

In this section, we discuss a generic doubling of an almost complete intersection of codimension 3 which leads to a specialization of Kustin-Miller model given in Section 6.1. Throughout all polynomial rings are over a field K of characteristic different from two.

6.1. Kustin-Miller model (KMM). We recall the well-known Kustin-Miller family of ideals associated to a 3×4 matrix, a 4-vector, and a variable. For details, see [16, 18].
Let R be a polynomial ring over K with indeterminates v, x_i, and a_{mn} where $i = 1, \dots, 4$, $m = 1, \dots, 3$, and $n = 1, \dots, 4$.

Let

$$X = \begin{bmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \end{bmatrix} \quad \text{and} \quad M = \begin{bmatrix} a_{11} & a_{12} & a_{13} & a_{14} \\ a_{21} & a_{22} & a_{23} & a_{24} \\ a_{31} & a_{32} & a_{33} & a_{34} \end{bmatrix}. \tag{8}$$

We set $q_i = \sum_{j=1}^{4} a_{ij}x_j = a_{i1}x_1 + a_{i2}x_2 + a_{i3}x_3 + a_{i4}x_4$ and

$$I = \langle q_1, q_2, q_3, x_1v + M_{123;234}, x_2v - M_{123;134}, x_3v + M_{123;124}, x_4v - M_{123;123}\rangle, \tag{9}$$

where $M_{\mathcal{K};\mathcal{L}}$ is the minor of the submatrix of M involving $\mathcal{K}$ rows and $\mathcal{L}$ columns. Let $\boldsymbol{s}$ be a 12×12 exchange matrix with entries of the form

$$\boldsymbol{s}_{ij} = \begin{cases} 1, & j = 12 - i + 1 \\ 0, & j \neq 12 - i + 1. \end{cases}$$

Note that $\boldsymbol{s}$ can be put in the form $\begin{bmatrix} 0 & \mathrm{I}_6 \\ \mathrm{I}_6 & 0 \end{bmatrix}$ up to permutation of columns. Then a minimal free resolution for I is given by

$$0 \to R \xrightarrow{\boldsymbol{d}_1^t} R^7 \xrightarrow{\boldsymbol{s}\boldsymbol{D}^t} R^{12} \xrightarrow{\boldsymbol{D}} R^7 \xrightarrow{\boldsymbol{d}_1} R \to R/I \to 0 \tag{10}$$

where

$$\boldsymbol{d}_1 = \begin{bmatrix} q_1 & q_2 & q_3 & x_1v + M_{123;234} & x_2v - M_{123;134} & x_3v + M_{123;124} & x_4v - M_{123;123} \end{bmatrix},$$

and the matrix $\boldsymbol{D}$ is

$$\begin{bmatrix}
-q_2 & -q_3 & 0 & M_{23;34} & M_{23;24} & M_{23;23} & M_{23;14} & M_{23;13} & M_{23;12} & -v & 0 & 0 \\
q_1 & 0 & -q_3 & -M_{13;34} & -M_{13;24} & -M_{13;23} & -M_{13;14} & -M_{13;13} & -M_{13;12} & 0 & -v & 0 \\
0 & q_1 & q_2 & M_{12;34} & M_{12;24} & M_{12;23} & M_{12;14} & M_{12;13} & M_{12;12} & 0 & 0 & -v \\
0 & 0 & 0 & -x_2 & x_3 & -x_4 & 0 & 0 & 0 & a_{11} & a_{21} & a_{31} \\
0 & 0 & 0 & x_1 & 0 & 0 & -x_3 & x_4 & 0 & a_{12} & a_{22} & a_{32} \\
0 & 0 & 0 & 0 & -x_1 & 0 & x_2 & 0 & -x_4 & a_{13} & a_{23} & a_{33} \\
0 & 0 & 0 & 0 & 0 & x_1 & 0 & -x_2 & x_3 & a_{14} & a_{24} & a_{34}
\end{bmatrix}.$$

6.2. Spinor coordinates of the Kustin-Miller model. Next we calculate the spinor coordinates on the Kustin-Miller model. Let us assume the notation in Section 6.1 and denote the ith column of the matrix $\boldsymbol{D}$ by e_i and $e_{-i} = e_{6+i}$.

REMARK 6.1. By Theorem 4.2, a spinor structure exists on the resolution (10) and hence we get spinor coordinates. Let us discuss the computation of spinor coordinates displayed in Table 1: We first find the Buchsbaum-Eisenbud map $\boldsymbol{a}_3$ with the help of Remark 3.3. For $\mathcal{J} = \{1\}$, $\boldsymbol{a}_{3,\mathcal{K}} = -x_1^2q_3^2$ where $\mathcal{K} = -\mathcal{J} \cup \mathcal{J}^c$. By Remark 4.3(1), we get $\widetilde{a}_{3,\{1\}} = \iota x_1 q_3$ where $\iota \in \mathbb{C}$ denotes the imaginary part of a complex number. Let us consider $\mathcal{K} = \{\pm 1, 3, 4, 5, 6\}$. Then $\boldsymbol{a}_{3,\mathcal{K}} = -x_1^2q_2q_3$. By using Remark 4.3(3), we have $\widetilde{\boldsymbol{a}}_{3,\{\pm 1,3,4,5,6\}} = \widetilde{\boldsymbol{a}}_{3,\{1\}}\widetilde{\boldsymbol{a}}_{3,\{2\}}$. Then $\widetilde{\boldsymbol{a}}_{3,\{2\}} = \iota x_1 q_2$. A repeated application of Remark 4.3(3) gives

$$-x_1x_2q_3^2 = \boldsymbol{a}_{3,\{-1,2,3,4,\pm 5\}} = \widetilde{\boldsymbol{a}}_{3,\{1\}}\widetilde{\boldsymbol{a}}_{3,\{1,5,6\}}.$$

Hence we get $\widetilde{\boldsymbol{a}}_{3,\{1,5,6\}} = \iota x_2 q_3$. Using the relations in Remark 4.3(3), one can similarly calculate the rest of the nonzero spinor coordinates given in Table 1. Also we observe that four minimal generators of the ideal I in (9) are among the spinor coordinates in Table 1.

6.3. Generic doubling of an almost complete intersection. In this subsection, we recall a resolution of an almost complete intersection ideal of codimension 3 given in [9]. We start with a generic doubling of an almost complete intersection of codimension 3. Next in Theorem 6.3 we see that such doubling is a specialization of the KMM. In Theorem 6.4 we get a resolution of the deformation of the ideal given in (17). Finally, in Corollary 6.5, we conclude that resolutions of such deformed ideals are specializations of the KMM given in Section 6.1. For generic doubling computations, we use Macaulay2 software [12].

REMARK 6.2. Let $R = K[c_{ij}, u_{kl}]$ be a polynomial ring over K where the variables c_{ij} are skew-symmetric in i, j and variables u_{kl} are generic variables for $1 \leq k, l \leq 3$. Consider a 3×3 generic skew-symmetric matrix $C = (c_{ij})$ and a generic matrix N as

$$N = \begin{bmatrix} -u_{11} & u_{12} & -u_{13} \\ -u_{21} & u_{22} & -u_{23} \\ -u_{31} & u_{32} & -u_{33} \end{bmatrix}.$$

Let $J = \langle q_1, q_2, q_3, -N_{123;123} \rangle$ be an ideal of R where $q_1 = c_{23}u_{11} - c_{13}u_{12} + c_{12}u_{13}$, $q_2 = c_{23}u_{21} - c_{13}u_{22} + c_{12}u_{23}$, $q_3 = c_{23}u_{31} - c_{13}u_{32} + c_{12}u_{33}$, and $N_{\mathcal{J};\mathcal{K}}$ is the submatrix of N involving $\mathcal{J}$ rows and $\mathcal{K}$ columns. By [9, Proposition 2.4], a minimal free resolution of R/J is

$$\mathbb{F} : 0 \to R^3 \xrightarrow{d_3} R^6 \xrightarrow{d_2} R^4 \xrightarrow{d_1} R \to R/J \to 0, \tag{11}$$

where

$$\boldsymbol{d}_1 = \begin{bmatrix} q_1 & q_2 & q_3 & -N_{123;123} \end{bmatrix},$$

$$\boldsymbol{d}_2 = \begin{bmatrix} -q_2 & -q_3 & 0 & N_{23;12} & N_{23;13} & N_{23;23} \\ q_1 & 0 & -q_3 & -N_{13;12} & -N_{13;13} & -N_{13;23} \\ 0 & q_1 & q_2 & N_{12;12} & N_{12;13} & N_{12;23} \\ 0 & 0 & 0 & -c_{12} & c_{13} & -c_{23} \end{bmatrix},$$

$$\boldsymbol{d}_3 = \begin{bmatrix} 0 & -c_{12} & c_{13} \\ c_{12} & 0 & -c_{23} \\ -c_{13} & c_{23} & 0 \\ -u_{11} & u_{12} & -u_{13} \\ -u_{21} & u_{22} & -u_{23} \\ -u_{31} & u_{32} & -u_{33} \end{bmatrix}.$$

Next we study a generic doubling of the resolution $\mathbb{F}$ given in (20) above. Applying $\mathrm{Hom}_R(-, R)$ to $\mathbb{F}$, one gets an acyclic complex

$$\mathbb{F}^* : 0 \to R \xrightarrow{d_1^*} R^4 \xrightarrow{d_2^*} R^6 \xrightarrow{d_3^*} R^3 \to \omega_{R/J} \to 0,$$

where $\boldsymbol{d}_3^* = -\boldsymbol{d}_3^T$, $\boldsymbol{d}_2^* = -\boldsymbol{d}_2^T$ and $\boldsymbol{d}_1^* = -\boldsymbol{d}_1^T$. By using Macaulay2 software [12], we compute $\mathrm{Hom}_{R/J}(\omega_{R/J}, R/J)$ which is generated by the image of the following matrix

$$\mathcal{H} = \begin{bmatrix} -c_{23} & N_{23;23} & N_{13;23} & N_{12;23} \\ -c_{13} & -N_{23;13} & -N_{13;13} & -N_{12;13} \\ -c_{12} & N_{23;12} & N_{13;12} & N_{12;12} \end{bmatrix}.$$

Let $\widetilde{R} = R[\tau_1, \tau_2, \tau_3, \tau_4]$. Note that

$$\widetilde{\mathbb{F}} : 0 \to \widetilde{R}^3 \xrightarrow{d_3} \widetilde{R}^6 \xrightarrow{d_2} \widetilde{R}^4 \xrightarrow{d_1} \widetilde{R} \to \widetilde{R}/J\widetilde{R} \to 0, \tag{12}$$

and

$$\widetilde{\mathbb{F}}^* : 0 \to \widetilde{R} \xrightarrow{d_1^*} \widetilde{R}^4 \xrightarrow{d_2^*} \widetilde{R}^6 \xrightarrow{d_3^*} \widetilde{R}^3 \to \omega_{\widetilde{R}/J\widetilde{R}} \to 0 \tag{13}$$

are minimal free resolutions of $\widetilde{R}/J\widetilde{R}$ and $\omega_{\widetilde{R}/J\widetilde{R}}$, respectively.

We set

$$\begin{aligned} s_1 &= \tau_4 c_{23} + \tau_1 N_{23;23} + \tau_2 N_{13;23} + \tau_3 N_{12;23}, \\ s_2 &= \tau_4 c_{13} - \tau_1 N_{23;13} - \tau_2 N_{13;13} - \tau_3 N_{12;13}, \\ s_3 &= \tau_4 c_{12} + \tau_1 N_{23;12} + \tau_2 N_{13;12} + \tau_3 N_{12;12}. \end{aligned}$$

Let

$$M' = \begin{bmatrix} -u_{11} & u_{12} & -u_{13} & \tau_1 \\ -u_{21} & u_{22} & -u_{23} & -\tau_2 \\ -u_{31} & u_{32} & -u_{33} & \tau_3 \end{bmatrix}. \tag{14}$$

Take $\boldsymbol{\psi}_1 = \begin{bmatrix} s_1 & s_2 & s_3 \end{bmatrix}$, and

$$\boldsymbol{\psi}_2 = \begin{bmatrix} M'_{23;14} & M'_{23;24} & M'_{23;34} & -\tau_4 & 0 & 0 \\ -M'_{13;14} & -M'_{13;24} & -M'_{13;34} & 0 & -\tau_4 & 0 \\ M'_{12;14} & M'_{12;24} & M'_{12;34} & 0 & 0 & -\tau_4 \\ 0 & 0 & 0 & -\tau_1 & \tau_2 & -\tau_3 \end{bmatrix},$$

where $M'_{\mathcal{K};\mathcal{L}}$ denotes the minor of M' involving $\mathcal{K}$ rows and $\mathcal{L}$ columns.

Set $\boldsymbol{\psi}_3 = -\boldsymbol{\psi}_2^T$ and $\boldsymbol{\psi}_4 = -\boldsymbol{\psi}_1^T$. Then $\boldsymbol{\psi}_1 : \widetilde{R}^3 \to \widetilde{R}$ lifts to the chain map $\boldsymbol{\psi} : \widetilde{\mathbb{F}}^* \to \widetilde{\mathbb{F}}$ of complexes as follows:

$$\begin{tikzcd} \widetilde{\mathbb{F}} : 0 \arrow[r] & \widetilde{R}^3 \arrow[r, "d_3"] & \widetilde{R}^6 \arrow[r, "d_2"] & \widetilde{R}^4 \arrow[r, "d_1"] & \widetilde{R} \\ \widetilde{\mathbb{F}}^* : 0 \arrow[r] & \widetilde{R} \arrow[r, "d_1^*"] \arrow[u, "\psi_4"] & \widetilde{R}^4 \arrow[r, "d_2^*"] \arrow[u, "\psi_3"] & \widetilde{R}^6 \arrow[r, "d_3^*"] \arrow[u, "\psi_2"] & \widetilde{R}^3 \arrow[u, "\psi_1"] \end{tikzcd}$$

Let

$$I = J\widetilde{R} + \langle s_1, s_2, s_3 \rangle. \tag{15}$$

Then the mapping cone with respect to $\boldsymbol{\psi}$ gives us a complex of the form

$$C(\boldsymbol{\psi}) : 0 \to \widetilde{R} \xrightarrow{\delta_4} \widetilde{R}^7 \xrightarrow{\delta_3} \widetilde{R}^{12} \xrightarrow{\delta_2} \widetilde{R}^7 \xrightarrow{\delta_1} \widetilde{R} \to \widetilde{R}/I \to 0 \tag{16}$$

where

$$\delta_1 = \begin{bmatrix} \boldsymbol{d}_1 & \boldsymbol{\psi}_1 \end{bmatrix}, \quad \delta_2 = \begin{bmatrix} \boldsymbol{d}_2 & \boldsymbol{\psi}_2 \\ \boldsymbol{0} & -\boldsymbol{d}_3^T \end{bmatrix}, \quad \delta_3 = \begin{bmatrix} \boldsymbol{d}_3 & -\boldsymbol{\psi}_2^T \\ 0 & -\boldsymbol{d}_2^T \end{bmatrix}, \quad \delta_4 = \begin{bmatrix} -\boldsymbol{\psi}_1^T \\ -\boldsymbol{d}_1^T \end{bmatrix}.$$

Theorem 6.3. *The resolution given in* (16) *is a specialization of the KMM after substituting $x_1 = -c_{23}$, $x_2 = -c_{13}$, $x_3 = -c_{23}$, $x_4 = 0$, $v = \tau_4$, and then sending M in* (8) *to M' in* (14).

In the next theorem, we give a minimal free resolution of the following deformed ideal

$$I(t) = \langle q_1 + t\tau_1, q_2 - \tau_2 t, q_3 + \tau_3 t, -\det(N) - \tau_4 t, s_1, s_2, s_3 \rangle \tag{17}$$

in the bigger polynomial ring $S = \widetilde{R}[t]$. Further, we show that the resolution of $I(t)$ is a KMM.

Theorem 6.4. *The ideal $I(t)$ of S in* (17) *above is a Gorenstein ideal of codimension* 4. *Moreover, a minimal resolution of $S/I(t)$ is*

$$0 \to S \xrightarrow{\delta_4(t)} S^7 \xrightarrow{\delta_3(t)} S^{12} \xrightarrow{\delta_2(t)} S^7 \xrightarrow{\delta_1(t)} S \to S/I(t) \to 0. \tag{18}$$

Proof. Set $\lambda = \begin{bmatrix}\tau_1 & -\tau_2 & \tau_3 & -\tau_4 & 0 & 0 & 0\end{bmatrix}$. Then the deformation of the ideal I, given in (15), along λ is $I(t) = \operatorname{im}(\boldsymbol{\delta}_1(t))$, where

$$\boldsymbol{\delta}_1(t) = \boldsymbol{\delta}_1 + t\lambda,\ \ \boldsymbol{\delta}_2(t) = \begin{bmatrix}\boldsymbol{d}_2 & \boldsymbol{\psi}_2\\ \boldsymbol{\phi}_2 & -\boldsymbol{d}_3^T\end{bmatrix},\ \boldsymbol{\delta}_3(t) = \begin{bmatrix}\boldsymbol{d}_3 & -\boldsymbol{\psi}_2^T\\ -\boldsymbol{\phi}_2^T & -\boldsymbol{d}_2^T\end{bmatrix},$$

$$\boldsymbol{\delta}_4(t) = \begin{bmatrix}-\boldsymbol{\psi}_1^T\\ -\boldsymbol{d}_1^T - t\lambda^T\end{bmatrix},\ \text{and}\ \boldsymbol{\phi}_2 = \begin{bmatrix}0 & 0 & 0 & -t & 0 & 0\\ 0 & 0 & 0 & 0 & t & 0\\ 0 & 0 & 0 & 0 & 0 & -t\end{bmatrix}.$$

By computation in Macaulay2, we see that $\operatorname{im}([\boldsymbol{\delta}_1(t)]^T) = \ker([\boldsymbol{\delta}_2(t)]^T)$. Now we use Buchsbaum-Eisenbud exactness criteria given in [5]. The rank condition is immediately satisfied. We claim that $\operatorname{depth}(I(\boldsymbol{\delta}_i(t))) \geq 4$ where $I(\boldsymbol{\delta}_i(t))$ denotes the ideal generated by $r_i \times r_i$ minors of $\delta_i(t)$ for $1 \leq i \leq 4$. By construction, we see that the ideals $(I(\boldsymbol{\delta}_i(t)), t)$ and $(I(\boldsymbol{\delta}_i), t)$ are the same. Thus depth of $I(\boldsymbol{\delta}_i(t), t)$ is at least 5 and the claim follows. □

As an application of Theorem 6.4, we have the following result.

Corollary 6.5. *The resolution* (18) *in* Theorem 6.4 *is a specialization of the KMM after substituting $x_1 = -c_{23}$, $x_2 = -c_{13}$, $x_3 = -c_{23}$, $x_4 = t$, $v = \tau_4$, and then sending M in* (8) *to M' in* (14).

7. Generic doubling of resolutions of the format (1,5,6,2)

In our last section, we use the resolution of the format $(1, 5, 6, 2)$ studied in [7, Section 3] to construct a generic doubling of a resolution (22) of such format. In Theorem 7.2, we show that the resolution (22) is not a specialization of the Kustin-Miller family.

7.1. Resolution of the format $(1, 5, 6, 2)$**.** We recall a minimal free resolution of a perfect ideal of codimension 3 with 5 generators of Cohen-Macaulay type 2. This subsection is from [7, Section 3].

Let K be a field of characteristics different from two. Let R be a polynomial ring over K with variables $x_{i,j}$, $y_{i,j}$ $(1 \leq i < j \leq 4)$, and $z_{i,j,k}$ $(1 \leq i < j < k \leq 4)$.

We use $\Delta(ij, kl)$ to denote the 2×2 minors of the matrix

$$\begin{bmatrix}x_{1,2} & x_{1,3} & x_{1,4} & x_{2,3} & x_{2,4} & x_{3,4}\\ y_{1,2} & y_{1,3} & y_{1,4} & y_{2,3} & y_{2,4} & y_{3,4}\end{bmatrix}$$

corresponding to the columns labeled by (i, j) and (k, l). The cubic generators are $u_{1,2,3}$, $u_{1,2,4}$, $u_{1,3,4}$ and $u_{2,3,4}$, where

$$u_{1,2,3} = -2z_{2,3,4}\Delta(12, 13) + 2z_{1,3,4}\Delta(12, 23) - 2z_{1,2,4}\Delta(13, 23) + z_{1,2,3}(\Delta(13, 24) - \Delta(12, 34) + \Delta(14, 23)),$$

$$u_{1,2,4} = 2z_{2,3,4}\Delta(12, 14) - 2z_{1,3,4}\Delta(12, 24) + z_{1,2,4}(\Delta(12, 34) + \Delta(13, 24) + \Delta(14, 23)) - 2z_{1,2,3}\Delta(14, 24),$$

$$u_{1,3,4} = 2z_{2,3,4}\Delta(13,14) + z_{1,3,4}(-\Delta(12,34) - \Delta(13,24) + \Delta(14,23)) + 2z_{1,2,4}\Delta(13,34) - 2z_{1,2,3}\Delta(14,34),$$
$$u_{2,3,4} = z_{2,3,4}(-\Delta(12,34) - \Delta(13,24) - \Delta(14,23)) - 2z_{1,3,4}\Delta(23,24) + 2z_{1,2,4}\Delta(23,34) - 2z_{1,2,3}\Delta(24,34).$$

Let $u = b^2 - 4ac$, where

$$\begin{aligned} a &= x_{1,2}x_{3,4} - x_{1,3}x_{2,4} + x_{1,4}x_{2,3}, \\ b &= x_{1,2}y_{3,4} - x_{1,3}y_{2,4} + x_{1,4}y_{2,3} + x_{3,4}y_{1,2} - x_{2,4}y_{1,3} + x_{2,3}y_{1,4}, \\ c &= y_{1,2}y_{3,4} - y_{1,3}y_{2,4} + y_{1,4}y_{2,3}. \end{aligned}$$

Consider the ideal

$$J(t) = \langle u_{2,3,4}(t), u_{1,3,4}(t), u_{1,2,4}(t), u_{1,2,3}(t), u(t)\rangle$$

in the bigger polynomial ring $S = R[t]$, where

$$\begin{aligned} u_{1,2,3}(t) &= -u_{1,2,3} + z_{1,2,3}t \\ u_{1,2,4}(t) &= -u_{1,2,4} + z_{1,2,4}t \\ u_{1,3,4}(t) &= -u_{1,3,4} + z_{1,3,4}t \\ u_{2,3,4}(t) &= -u_{2,3,4} + z_{2,3,4}t \\ u(t) &= u - t^2. \end{aligned}$$

It is shown in [7, Theorem 3.1] that $J(t)$ is a perfect ideal and $S/J(t)$ has a minimal free resolution over S of the form

$$\mathbb{F} : 0 \to S^2 \xrightarrow{d_3(t)} S^6 \xrightarrow{d_2(t)} S^5 \xrightarrow{d_1(t)} S \tag{19}$$

with differentials

$$\boldsymbol{d}_2(t) = \begin{bmatrix} v_1 & u_1 & -\delta_1+\delta_2-\delta_3+t & 2\Delta(13,14) & -2\Delta(12,14) & -2\Delta(12,13) \\ -v_2 & -u_2 & -2\Delta(23,24) & -\delta_1-\delta_2+\delta_3+t & 2\Delta(12,24) & 2\Delta(12,23) \\ v_3 & u_3 & 2\Delta(23,34) & 2\Delta(13,34) & -\delta_1-\delta_2-\delta_3-t & -2\Delta(13,23) \\ v_4 & u_4 & 2\Delta(24,34) & 2\Delta(14,34) & -2\Delta(14,24) & \delta_1-\delta_2-\delta_3+t \\ 0 & 0 & -z_{2,3,4} & -z_{1,3,4} & z_{1,2,4} & z_{1,2,3} \end{bmatrix}$$

and

$$\boldsymbol{d}_3(t) = \begin{bmatrix} b+t & 2a \\ -2c & -b+t \\ -v_1 & -u_1 \\ v_2 & u_2 \\ v_3 & u_3 \\ -v_4 & -u_4 \end{bmatrix},$$

where $u_j = \sum_{i\neq j}(-1)^i x_{i,j}z_{\hat{i}}$, $v_j = \sum_{i\neq j}(-1)^i y_{i,j}z_{\hat{i}}$, $\delta_1 = \Delta(12,34)$, $\delta_2 = \Delta(13,24)$, and $\delta_3 = \Delta(14,23)$.

7.2. Generic doubling of resolutions of the format (1,5,6,2). Let us construct a generic doubling of the resolution $\mathbb{F}$ stated in (19) of the format (1,5,6,2).

We apply $\mathrm{Hom}_S(-,S)$ to $\mathbb{F}$ to get the complex

$$\mathbb{F}^* : 0 \to S \xrightarrow{\boldsymbol{d}_1(t)^*} S^5 \xrightarrow{\boldsymbol{d}_2(t)^*} S^6 \xrightarrow{\boldsymbol{d}_3(t)^*} S^2 \to \omega_{S/J(t)} \to 0,$$

where $\boldsymbol{d}_3(t)^* = -\boldsymbol{d}_3(t)^T$, $\boldsymbol{d}_2(t)^* = -\boldsymbol{d}_2(t)^T$, and $\boldsymbol{d}_1(t)^* = -\boldsymbol{d}_1(t)^T$. Here $\mathbb{F}^*$ is acyclic as $J(t)$ is a perfect ideal in S.

Consider the bigger polynomial ring $\widetilde{S} = S[\tau_1, \tau_2, \tau_3, \tau_4, \tau_5, \tau_6]$. Note that

$$\widetilde{\mathbb{F}} : 0 \to \widetilde{S}^2 \xrightarrow{\boldsymbol{d}_3(t)} \widetilde{S}^6 \xrightarrow{\boldsymbol{d}_2(t)} \widetilde{S}^5 \xrightarrow{\boldsymbol{d}_1(t)} \widetilde{S} \to \widetilde{S}/J\widetilde{S} \to 0 \tag{20}$$

$$\widetilde{\mathbb{F}}^* : 0 \to \widetilde{S} \xrightarrow{\boldsymbol{d}_5(t)^*} \widetilde{S}^6 \xrightarrow{\boldsymbol{d}_2(t)^*} \widetilde{S}^6 \xrightarrow{\boldsymbol{d}_1(t)^*} \widetilde{S}^3 \to \omega_{\widetilde{S}/J\widetilde{S}} \to 0 \tag{21}$$

are minimal free resolutions of $\widetilde{S}/J\widetilde{S}$ and $\omega_{\widetilde{S}/J\widetilde{S}}$, respectively.

Then we compute $\operatorname{Hom}_{S/J(t)}(\omega_{S/J(t)}, S/J(t))$ by Macaulay2 [12], which is generated by the image of the matrix

$$\begin{bmatrix} u_4 & u_3 & u_2 & u_1 & b-t & 2a \\ -v_4 & -v_3 & -v_2 & -v_1 & -2c & -b-t \end{bmatrix}.$$

Let $\boldsymbol{\psi}_1(t) = \begin{bmatrix} f_1(t) & f_2(t) \end{bmatrix}$, where

$$f_1(t) = -\tau_1 u_4 - \tau_2 u_3 - \tau_3 u_2 - \tau_4 u_1 + \tau_5 b + 2a\tau_6 - \tau_5 t$$

$$f_2(t) = \tau_1 v_4 + \tau_2 v_3 + \tau_3 v_2 + \tau_4 v_1 - 2c\tau_5 - b\tau_6 - \tau_6 t,$$

and take $\boldsymbol{\psi}_2(t)$ to be the transpose of the matrix given in Figure 1 at the end of this paper. Then $\boldsymbol{\psi}_1(t) : \widetilde{S}^2 \to \widetilde{S}$ lifts to the chain map $\boldsymbol{\psi}(t) : \widetilde{\mathbb{F}}^* \to \widetilde{\mathbb{F}}$ such that

$$\begin{tikzcd}
\widetilde{\mathbb{F}} : 0 \arrow[r] & \widetilde{S}^2 \arrow[r, "\boldsymbol{d}_3(t)"] & \widetilde{S}^6 \arrow[r, "\boldsymbol{d}_2(t)"] & \widetilde{S}^5 \arrow[r, "\boldsymbol{d}_1(t)"] & \widetilde{S} \\
\widetilde{\mathbb{F}}^* : 0 \arrow[r] & \widetilde{S} \arrow[r, "\boldsymbol{d}_1(t)^*"] \arrow[u, "-\boldsymbol{\psi}_1(t)^T"] & \widetilde{S}^5 \arrow[r, "\boldsymbol{d}_2(t)^*"] \arrow[u, "-\boldsymbol{\psi}_2(t)^T"] & \widetilde{S}^6 \arrow[r, "\boldsymbol{d}_3(t)^*"] \arrow[u, "\boldsymbol{\psi}_2(t)"] & \widetilde{S}^2 \arrow[u, "\boldsymbol{\psi}_1(t)"]
\end{tikzcd}$$

Let $I(t) = J(t)\widetilde{S} + \langle f_1(t), f_2(t) \rangle$. Then the mapping cone with respect to $\boldsymbol{\psi}(t)$ gives a complex of the form

$$C(\boldsymbol{\psi}(t)) : 0 \to \widetilde{S} \xrightarrow{\boldsymbol{\delta}_4(t)} \widetilde{S}^7 \xrightarrow{\boldsymbol{\delta}_3(t)} \widetilde{S}^{12} \xrightarrow{\boldsymbol{\delta}_2(t)} \widetilde{S}^7 \xrightarrow{\boldsymbol{\delta}_1(t)} \widetilde{S} \to \widetilde{S}/I(t) \to 0 \tag{22}$$

with differentials

$$\boldsymbol{\delta}_1(t) = \begin{bmatrix} \boldsymbol{d}_1(t) & \boldsymbol{\psi}_1(t) \end{bmatrix}, \quad \boldsymbol{\delta}_2(t) = \begin{bmatrix} \boldsymbol{d}_2(t) & \boldsymbol{\psi}_2(t) \\ \boldsymbol{0} & -\boldsymbol{d}_3(t)^T \end{bmatrix},$$

$$\boldsymbol{\delta}_3(t) = \begin{bmatrix} \boldsymbol{d}_3(t) & -\boldsymbol{\psi}_2(t)^T \\ 0 & -\boldsymbol{d}_2(t)^T \end{bmatrix}, \quad \boldsymbol{\delta}_4(t) = \begin{bmatrix} -\boldsymbol{\psi}_1(t)^T \\ -\boldsymbol{d}_1(t)^T \end{bmatrix}.$$

Note that $\boldsymbol{\delta}_3(t) = \boldsymbol{s}\boldsymbol{\delta}_2(t)^T$ where $\boldsymbol{s}$ is a 12×12 exchange matrix with entries given by

$$\boldsymbol{s}_{ij} = \begin{cases} 1, & j = 12 - i + 1 \\ 0, & j \neq 12 - i + 1, \end{cases}$$

and $\boldsymbol{s}$ can be put in the form $\begin{bmatrix} 0 & \mathrm{I}_6 \\ \mathrm{I}_6 & 0 \end{bmatrix}$ up to permutation of columns.

In the next remark, we provide spinor coordinates of the minimal free resolution $\mathcal{C}(\boldsymbol{\psi}(t))$ stated in (22).

REMARK 7.1. By Theorem 4.2, spinor coordinates of the resolution in (22) exist and we display these coordinates in Table 2. Let us briefly discuss the computation used in Table 2.

By applying column operations on resolution (22), one gets differentials as

$$\boldsymbol{\delta}_1(t) = \begin{bmatrix} \boldsymbol{d}_1(t) & \boldsymbol{\psi}_1(t) \end{bmatrix}, \quad \boldsymbol{\delta}_2(t) = \begin{bmatrix} \boldsymbol{d}_2(t) & \boldsymbol{\psi}_2(t) \\ \mathbf{0} & -\boldsymbol{d}_3(t)^T \end{bmatrix},$$

$$\boldsymbol{\delta}_3(t) = \begin{bmatrix} -\boldsymbol{\psi}_2(t)^T & \boldsymbol{d}_3(t) \\ -\boldsymbol{d}_2(t)^T & 0 \end{bmatrix}, \quad \boldsymbol{\delta}_4(t) = \boldsymbol{\delta}_1(t)^T.$$

We denote the ith column of $\boldsymbol{\delta}_2(t)$ by e_i with $e_{-i} = e_{6+i}$. Then $\{e_1, \dots, e_6, e_{-1}, \dots, e_{-6}\}$ is the associated hyperbolic basis of $\widetilde{S}^{12}$. Computing 6×6 minors of $\boldsymbol{\delta}_2(t)$, we see that the coordinates of $\boldsymbol{a}_3$ corresponding to the multi-index $-\mathcal{J} \cup \mathcal{J}^c$ are squares where $\mathcal{J} \subset [1,6]$ are of even cardinality. Therefore $\boldsymbol{a}(t)_{3,\mathcal{K}} = \widetilde{\boldsymbol{a}}(t)^2_{3,\mathcal{J}}$ by Remark 3.3. For each $\mathcal{J} \subset [1,6]$, we record $\widetilde{\boldsymbol{a}}(t)_{3,\mathcal{J}}$ in Table 2. Note that $\bar{i}$ in Table 2 denotes the column corresponding to e_{-i}.

We conclude with our main application. Using spinor coordinates, we show that the resolution (22), which is a generic doubling of a resolution of the format $(1,5,6,2)$, is not a specialization of the Kustin-Miller family in Section 6.1.

Theorem 7.2. *The resolution given in* (22) *is not a specialization of the Kustin-Miller family in* Section 6.1.

Proof. Suppose the resolution (22) is a specialization of the KMM given in Section 6.1. Then $\boldsymbol{a}_{3,\mathcal{K}}$ for $\mathcal{K} \subset [1,6]$ of the resolution (22) corresponds to $\boldsymbol{a}_{3,\mathcal{L}}$ for some $\mathcal{L} \subset [1,6]$ of the KMM. Therefore spinor coordinates in Table 2 go to spinor coordinates in Table 1 of the KMM. In Table 2, we see that only one of the spinor coordinates is among the minimal generators of the ideal in resolution (22). Then, by specialization, the KMM can have at most one spinor coordinate among minimal generators of I in Section 6.1. This is not possible as Table 1 displays four spinor coordinates among minimal generators of I in Section 6.1. □

REMARK 7.3. Calculations in Examples 5.1, 5.2, Remark 6.1, and Theorem 7.1 show that at least one of the minimal generators of a Gorenstein ideal with 4, 6, or 7 generators are among the spinor coordinates. However, in case of a Gorenstein ideal with 9 generators, we see in Example 5.3 that none of the minimal generators of the ideal comes from spinor coordinates. This suggests that Gorenstein ideals of codimension 4 with up to 8 generators are easier to classify than those with more than 8 generators.

ACKNOWLEDGEMENTS. The authors want to thank anonymous referee for careful reading of the manuscript and many useful remarks. J. Weyman was partially supported by NSF grant DMS 1802067, Sidney Professorial Fund, MAESTRO NCN-UMO-2019/34/A/ST1/00263, and also NAWA POWROTy-PPN/PPo/2018/1/00013/U/00001 grants. J. Laxmi was supported by Fulbright-Nehru fellowship. E. Celikbas and J. Laxmi also acknowledge the partial support of the Sidney Professorial Fund from the University of Connecticut.

Table 1. Spinor coordinates of the KMM with 7 generators

Cases	$\widetilde{\boldsymbol{a}}_{3,\mathcal{J}}$
$\mathcal{J} = \{i\}$ for $i = 1, 2, 3$	$x_1 q_{4-i}$
$\mathcal{J} = \{5, 6\} \cup \{i\}$ for $i = 1, 2, 3$	$x_2 q_{4-i}$
$\mathcal{J} = \{4, 6\} \cup \{i\}$ for $i = 1, 2, 3$	$x_3 q_{4-i}$
$\mathcal{J} = \{4, 5\} \cup \{i\}$ for $i = 1, 2, 3$	$x_4 q_{4-i}$
$\mathcal{J} = \{1, 2, 3\}$	$x_1 v + M_{123;234}$
$\mathcal{J} = \{1, 2, 3, 5, 6\}$	$x_2 v - M_{123;134}$
$\mathcal{J} = \{1, 2, 3, 4, 6\}$	$x_3 v + M_{123;124}$
$\mathcal{J} = \{1, 2, 3, 4, 5\}$	$x_4 v - M_{123;124}$
$\mathcal{J} = \{1, 2, 4\}$	$a_{22} q_3 - a_{32} q_2$
$\mathcal{J} = \{1, 2, 5\}$	$a_{23} q_3 - a_{33} q_2$
$\mathcal{J} = \{1, 2, 6\}$	$a_{24} q_3 - a_{34} q_2$
$\mathcal{J} = \{1, 3, 4\}$	$a_{12} q_3 - a_{32} q_1$
$\mathcal{J} = \{1, 3, 5\}$	$a_{13} q_3 - a_{33} q_1$
$\mathcal{J} = \{1, 3, 6\}$	$a_{14} q_3 - a_{34} q_1$
$\mathcal{J} = \{2, 3, 4\}$	$a_{22} q_3 - a_{32} q_2$
$\mathcal{J} = \{2, 3, 5\}$	$a_{23} q_3 - a_{33} q_2$
$\mathcal{J} = \{2, 3, 6\}$	$a_{24} q_3 - a_{34} q_2$

$$
\begin{bmatrix}
-x_{14}\tau_4 - x_{24}\tau_3 - x_{34}\tau_2 & x_{13}\tau_4 + x_{23}\tau_3 - x_{34}\tau_1 & -x_{12}\tau_4 + x_{23}\tau_2 + x_{24}\tau_1 & -x_{12}\tau_3 - x_{13}\tau_2 - x_{14}\tau_1 & \tau_5 \\
y_{14}\tau_4 + y_{24}\tau_3 + y_{34}\tau_2 & -y_{13}\tau_4 - y_{23}\tau_3 + y_{34}\tau_1 & y_{12}\tau_4 - y_{23}\tau_2 - y_{24}\tau_1 & y_{12}\tau_3 + y_{13}\tau_2 + y_{14}\tau_1 & \tau_6 \\
x_{14}\tau_6 + y_{14}\tau_5 - \frac{1}{2}z_{124}\tau_3 - \frac{1}{2}z_{134}\tau_2 & -x_{13}\tau_6 - y_{13}\tau_5 + \frac{1}{2}z_{123}\tau_3 - \frac{1}{2}z_{134}\tau_1 & x_{12}\tau_6 + y_{12}\tau_5 + \frac{1}{2}z_{123}\tau_2 + \frac{1}{2}z_{124}\tau_1 & 0 & 0 \\
-x_{24}\tau_6 - y_{24}\tau_5 - \frac{1}{2}z_{124}\tau_4 + \frac{1}{2}z_{234}\tau_2 & x_{23}\tau_6 + y_{23}\tau_5 + \frac{1}{2}z_{123}\tau_4 + \frac{1}{2}z_{234}\tau_1 & 0 & -x_{12}\tau_6 - y_{12}\tau_5 - \frac{1}{2}z_{123}\tau_2 - \frac{1}{2}z_{124}\tau_1 & 0 \\
x_{34}\tau_6 + y_{34}\tau_5 + \frac{1}{2}z_{134}\tau_4 + \frac{1}{2}z_{234}\tau_3 & 0 & -x_{23}\tau_6 - y_{23}\tau_5 - \frac{1}{2}z_{123}\tau_4 - \frac{1}{2}z_{234}\tau_1 & x_{13}\tau_6 + y_{13}\tau_5 - \frac{1}{2}z_{123}\tau_3 + \frac{1}{2}z_{134}\tau_1 & 0 \\
0 & -x_{34}\tau_6 - y_{34}\tau_5 - \frac{1}{2}z_{134}\tau_4 - \frac{1}{2}z_{234}\tau_3 & x_{24}\tau_6 + y_{24}\tau_5 + \frac{1}{2}z_{124}\tau_4 - \frac{1}{2}z_{234}\tau_2 & -x_{14}\tau_6 - y_{14}\tau_5 + \frac{1}{2}z_{124}\tau_3 + \frac{1}{2}z_{134}\tau_2 & 0
\end{bmatrix}^T
$$

Fig. 1. The Matrix $\boldsymbol{\psi}_2(t)$

Table 2. Spinor coordinates of resolution (22)

Cases for $\widetilde{a}(t)_{3,\mathcal{J}}$
$\tilde{a}(t)_{3,\{\phi\}} = 0$
$\tilde{a}(t)_{3,\{1,2\}} = \iota u(t)$
$\tilde{a}(t)_{3,\{1,3\}} = \iota(x_{24}u_{1,3,4}(t) - x_{34}u_{1,2,4}(t) - x_{14}u_{2,3,4}(t))$
$\tilde{a}(t)_{3,\{1,4\}} = \iota(x_{23}u_{1,3,4}(t) - x_{34}u_{1,2,3}(t) - x_{13}u_{2,3,4}(t))$
$\tilde{a}(t)_{3,\{1,5\}} = -x_{24}u_{1,2,3}(t) + x_{23}u_{1,2,4}(t) - x_{12}u_{2,3,4}(t)$
$\tilde{a}(t)_{3,\{1,6\}} = x_{13}u_{1,2,4}(t) - x_{14}u_{1,2,3}(t) - x_{12}u_{1,3,4}(t)$
$\tilde{a}(t)_{3,\{2,3\}} = \iota(y_{34}u_{1,2,4}(t) - y_{24}u_{1,3,4}(t) + y_{14}u_{2,3,4}(t))$
$\tilde{a}(t)_{3,\{2,4\}} = \iota(y_{34}u_{1,2,3}(t) - y_{23}u_{1,3,4}(t) + y_{13}u_{2,3,4}(t))$
$\tilde{a}(t)_{3,\{2,5\}} = -y_{24}u_{1,2,3}(t) + y_{23}u_{1,3,4}(t) - y_{12}u_{2,3,4}(t)$
$\tilde{a}(t)_{3,\{2,6\}} = -y_{14}u_{1,2,3}(t) + y_{13}u_{1,2,4}(t) - y_{12}u_{1,3,4}(t)$
$\tilde{a}(t)_{3,\{3,4\}} = \frac{1}{2}[\iota(z_{2,3,4}u_{1,3,4}(t) - z_{1,3,4}u_{2,3,4}(t))]$
$\tilde{a}(t)_{3,\{3,5\}} = \frac{1}{2}[z_{2,3,4}u_{1,2,4}(t) - z_{1,2,4}u_{2,3,4}(t)]$
$\tilde{a}(t)_{3,\{3,6\}} = \frac{1}{2}[z_{1,3,4}u_{1,2,4}(t) - z_{1,2,4}u_{1,3,4}(t)]$
$\tilde{a}(t)_{3,\{4,5\}} = \frac{1}{2}[z_{2,3,4}u_{1,2,3}(t) - z_{1,2,3}u_{2,3,4}(t)]$
$\tilde{a}(t)_{3,\{4,6\}} = \frac{1}{2}[z_{1,3,4}u_{1,2,3}(t) - z_{1,2,3}u_{1,3,4}(t)]$
$\tilde{a}(t)_{3,\{5,6\}} = \frac{1}{2}[\iota(z_{1,2,4}u_{1,2,3}(t) - z_{1,2,3}u_{1,2,4}(t))]$
$\tilde{a}(t)_{3,\{1,2,3,4\}} = \frac{1}{2}[\iota(\tau_4u_{1,3,4}(t) + \tau_3u_{2,3,4}(t) + 2y_{34}f_1(t) + 2x_{34}f_2(t))]$
$\tilde{a}(t)_{3,\{1,2,3,5\}} = \frac{1}{4}[-\tau_4u_{1,2,4}(t) + \tau_2u(t)_{2,3,4} - 2y_{24}f_1(t) - 2x_{24}f_2(t)]$
$\tilde{a}(t)_{3,\{1,2,3,6\}} = \frac{1}{2}[\tau_3u_{1,2,4}(t) + \tau_2u_{1,3,4}(t) - 2y_{14}f_1(t) - 2x_{14}f_2(t)]$
$\tilde{a}(t)_{3,\{1,2,4,5\}} = \frac{1}{4}[-\tau_4u_{1,2,3}(t) - \tau_1u_{2,3,4}(t) - 2y_{23}f_1(t) - 2x_{23}f_2(t)]$
$\tilde{a}(t)_{3,\{1,2,4,6\}} = \frac{1}{2}[\tau_3u_{1,2,3}(t) - \tau_1u_{1,3,4}(t) - 2y_{13}f_1(t) - 2x_{13}f_2(t)]$
$\tilde{a}(t)_{3,\{1,2,5,6\}} = \frac{1}{4}[\iota(-\tau_2u_{1,2,3}(t) - \tau_1u_{1,2,4}(t) - 2y_{12}f_1(t) - 2x_{12}f_2(t))]$
$\tilde{a}(t)_{3,\{1,3,4,5\}} = \frac{1}{2}[-\tau_5u_{2,3,4}(t) + z_{2,3,4}f_1(t)]$
$\tilde{a}(t)_{3,\{1,3,4,6\}} = \frac{1}{4}[\iota(-\tau_5u_{1,3,4}(t) + z_{1,3,4}f_1(t))]$
$\tilde{a}(t)_{3,\{2,3,4,5\}} = \frac{1}{2}[\tau_6u_{2,3,4}(t) - z_{2,3,4}f_2(t)]$
$\tilde{a}(t)_{3,\{2,3,4,6\}} = \frac{1}{2}[\tau_6u_{1,3,4}(t) - z_{1,3,4}f_2(t)]$
$\tilde{a}(t)_{3,\{2,3,5,6\}} = \frac{1}{2}[\iota(\tau_6u_{1,2,4}(t) - z_{1,2,4}f_2(t))]$
$\tilde{a}(t)_{3,\{2,4,5,6\}} = \frac{1}{2}[\iota(-\tau_6u_{1,2,3}(t) - z_{1,2,3}f_2(t))]$
$\tilde{a}(t)_{3,\{1,3,4,5\}} = \frac{1}{2}[\iota(-\tau_5u_{1,2,3}(t) + z_{1,2,3}f_1(t))]$
$\tilde{a}(t)_{3,\{1,3,5,6\}} = \frac{1}{2}[\iota(-\tau_5u_{1,2,4}(t) + z_{1,2,4}f_1(t))]$
$\tilde{a}(t)_{3,\{1,4,5,6\}} = \frac{1}{2}[\iota(-\tau_5u_{1,2,3}(t) + z_{1,2,3}f_1(t))]$
$\tilde{a}(t)_{3,\{1,2,3,4,5,6\}} = \frac{1}{2}[\iota(\tau_6f_1(t) - \tau_5f_2(t))]$

References

[1] J.F. Adams: Lectures on exceptional Lie groups, Chicago Lectures in Mathematics, University of Chicago Press, Chicago, IL, 1996.

[2] W. Bruns and J. Herzog: Cohen-Macaulay rings, Cambridge Stud. Adv. Math. **39**, Cambridge Univ. Press, Cambridge, 1993.

[3] D.A. Buchsbaum and D. Eisenbud: *Some structure theorems for finite free resolutions*, Advances in Math. **12** (1974), 84–139.

[4] D.A. Buchsbaum and D. Eisenbud: *Algebra structures for finite free resolutions, and some structure theorems for ideals of codimension 3*, Amer. J. Math. **99** (1977), 447–485.

[5] D.A. Buchsbaum and D. Eisenbud: *What makes a complex exact?*, J. Algebra **25** (1973), 259–268.

[6] É. Cartan: The theory of spinors, The M.I.T. Press, Cambridge, Mass., 1967.

[7] E. Celikbas, J. Laxmi, W. Kraś kiewicz and J. Weyman: *The family of perfect ideals of codimension 3, of type 2 with 5 generators*, Proc. Amer. Math. Soc. **148** (2020), 2745–2755.

[8] R. Chirivì and A. Maffei: *Pfaffians and shuffling relations for the spin module*, Algebr. Represent. Theory **16** (2013), 955-978.

[9] L.W. Christensen, O. Veliche and J. Weyman: *Three takes on almost complete intersection ideals of grade 3*; in Commutative Algebra, Springer, Cham, 2021, 219–281.

[10] P. Deligne: *Notes on spinors*; in Quantum fields and strings: a course for mathematicians **1**, **2**, Amer. Math. Soc., Providence, RI, 1999, 99–135.

[11] W. Fulton and J. Harris: Representation theory, A first course, Graduate Texts in Mathematics **129**, Reading in Mathematics, Springer-Verlag, New York, 1991.

[12] R.G Daniel and M.E. Stillman: Macaulay2, a software system for research in algebraic geometry, http://www.math.uiuc.edu/Macaulay2/

[13] R. Goodman and N.R. Wallach: Symmetry, representations, and invariants, Graduate Texts in Mathematics **255**, Springer, New York, 2009.

[14] J.C. Jantzen: Representation of algebraic groups, Second Edition, Mathematical Surveys and Monographs **107**, Amer. Math. Soc., Providence, RI, 2003.

[15] D. Knuth: *Overlapping Pfaffians*, Electron. J. Combin. **3** (1996), 147-159.

[16] A.R. Kustin and M. Miller: *Constructing big Gorenstein ideals from small ones*, J. Algebra **85** (1983), 303–322.

[17] A.R. Kustin and M. Miller: *Algebra structure on minimal resolution of Gorenstein rings of embedding codimension four*, Math. Z. **173** (1980), 171–184.

[18] A.R. Kustin and M. Miller: *Deformation and linkage of Gorenstein algebras*, Trans. Amer. Math. Soc. **284** (1984), 501–534.

[19] A.R. Kustin and B. Ulrich: A family of complexes associated to almost alternating map with applications to residual intersections, Mem. of Amer. Math. Soc. **95** (1992), no. **461**.

[20] V. Lakshmibai and K.N. Raghavan: Standard monomial theory, Invariant theoretic approach, Encyclopaedia of Mathematical Sciences **137**, Springer-Verlag, Berlin, 2008.

[21] M. Reid: *Gorenstein in codimension 4: the general structure theory*; in Algebraic Geometry in East Asia (Taipei 2011), Adv. Stud. Pure Math. **65**, Math. Soc. Japan, Tokyo, 2015, 201–227.

[22] J. Weyman: Cohomology of Vector Bundles and Syzygies, Cambridge Tracts in Mathematics **149**, Cambridge University Press, Cambridge, 2003.

Ela Celikbas
School of Mathematical and Data Sciences, West Virginia University
Morgantown, WV 26506
USA
e-mail: ela.celikbas@math.wvu.edu

Jai Laxmi
School of Mathematics, Tata Institute of Fundamental Research
Mumbai-400005
India
e-mail: laxmiuohyd@gmail.com
jailaxmi@math.tifr.res.in

Jerzy Weyman
Department of Mathematics, University of Connecticut
Storrs, CT 06269
USA
e-mail: jerzy.weyman@uconn.edu
[*Current Address*:]
Instytut Matematyki, Uniwersytet Jagielloński
Kraków 30-348
Poland
e-mail: jerzy.weyman@uj.edu.pl