



Title	High degree anti-integral extensions of Noetherian domains
Author(s)	Oda, Susumu; Junro, Sato; Yoshida, Ken-ichi
Citation	Osaka Journal of Mathematics. 1993, 30(1), p. 119-135
Version Type	VoR
URL	https://doi.org/10.18910/9307
rights	
Note	

The University of Osaka Institutional Knowledge Archive : OUKA

<https://ir.library.osaka-u.ac.jp/>

The University of Osaka

HIGH DEGREE ANTI-INTEGRAL EXTENSIONS OF NOETHERIAN DOMAINS

SUSUMU ODA, JUNRO SATO and KEN-ICHI YOSHIDA

(Received January 30, 1991)

Introduction. Let R be a Noetherian integral domain and $R[X]$ a polynomial ring. Let α be an element of an algebraic field extension L of the quotient field K of R and let $\pi: R[X] \rightarrow R[\alpha]$ be the R -algebra homomorphism sending X to α . Let $\varphi_\alpha(X)$ be the monic minimal polynomial of α over K with $\deg \varphi_\alpha(X) = d$ and write $\varphi_\alpha(X) = X^d + \eta_1 X^{d-1} + \cdots + \eta_d$. Let $I_{[\alpha]} := \bigcap_{i=1}^d (R :_R \eta_i)$. For $f(X) \in R[X]$, let $C(f(X))$ denote the ideal generated by the coefficients of $f(X)$. Let $J_{[\alpha]} := I_{[\alpha]} C(\varphi_\alpha(X))$, which is an ideal of R and contains $I_{[\alpha]}$. The element α is called an anti-integral element of degree d over R if $\text{Ker } \pi = I_{[\alpha]} \varphi_\alpha(X) R[X]$. When α is an anti-integral element over R , $R[\alpha]$ is called an anti-integral extension of R . In the case $K(\alpha) = K$, an anti-integral element α is the same as an anti-integral element (i.e., $R = R[\alpha] \cap R[1/\alpha]$) defined in [5]. The element α is called a super-primitive element of degree d over R if $J_{[\alpha]} \not\subset \mathfrak{p}$ for all primes \mathfrak{p} of depth one.

For $\mathfrak{p} \in \text{Spec}(R)$, $k(\mathfrak{p})$ denotes the residue field $R_{\mathfrak{p}}/\mathfrak{p}R_{\mathfrak{p}}$ and $\text{rank}_{k(\mathfrak{p})} R[\alpha] \otimes_R k(\mathfrak{p})$ denotes the dimension as a vector space over $k(\mathfrak{p})$. We are interested in characterizing the flatness and the integrality of an anti-integral extension $R[\alpha]$ of R . Indeed, among others we obtain the following results:

- (i) $R[\alpha]$ is flat over R if and only if $\text{rank}_{k(\mathfrak{p})} R[\alpha] \otimes_R k(\mathfrak{p}) \leq d$ for all $\mathfrak{p} \in \text{Spec}(R)$,
- (ii) $R[\alpha]$ is integral over R if and only if $\text{rank}_{k(\mathfrak{p})} R[\alpha] \otimes_R k(\mathfrak{p}) = d$ for all $\mathfrak{p} \in \text{Spec}(R)$.

Thus if an anti-integral extension $R[\alpha]$ is integral over R , then $R[\alpha]$ is flat over R . Concerning a super-primitive element, we obtain that if R is a Krull domain and α is an algebraic element over R , then α is a super-primitive element. We also obtain that a super-primitive element is an anti-integral element. More precisely, α is super-primitive over R if and only if α is anti-integral over R and $R[\alpha]_{\mathfrak{p}}$ is flat over $R_{\mathfrak{p}}$ for any prime ideal \mathfrak{p} of depth one.

Using these results, we obtain the following:

Let $\Delta(S)$ denote the set $\{\mathfrak{p} \in \text{Spec}(R) \mid \text{rank}_{k(\mathfrak{p})} S \otimes_R k(\mathfrak{p}) = d\}$, where S is an extension of R of degree d and let $Dp_1(R)$ denote the set of all prime ideals of R of depth one. Assume that $[L:K] = d$, and that $\alpha_1, \dots, \alpha_n \in L$ are anti-integral elements of degree d , and let $A = R[\alpha_1, \dots, \alpha_n]$. If $\Delta(R[\alpha_i]) \supset Dp_1(R)$ ($1 \leq i \leq n$)

and $Ur(R[\alpha_i]) \supset Dp_1(R)$, where $Ur(A)$ denotes the set $\{p \in \text{Spec}(R) \mid A_p \text{ is unramified over } R_p\}$, then A is integral over R , and A_p is etale over R_p for $p \in \Delta(A)$. If $\Delta(A) = \text{Spec}(R)$ in addition to the above assumptions, then A is integral and etale over R .

Notations and Conventions. Throughout this paper, we use the following notations unless otherwise specified.

R : a Noetherian integral domain,
 $K := K(R)$: the quotient field of R ,
 L : an algebraic field extension of K ,
 α : a non-zero element of L ,
 $d = [K(\alpha) : K]$,
 $\varphi_\alpha(X) = X^d + \eta_1 X^{d-1} + \cdots + \eta_d$, the minimal polynomial of α over K .

Let $\pi: R[X] \rightarrow R[\alpha]$ be an R -algebra homomorphism defined by $X \rightarrow \alpha$ and let $A_{[\alpha]} := \text{Ker } \pi$. Then $A_{[\alpha]}$ is a prime ideal of $R[X]$ with $A_{[\alpha]} \cap R = (0)$. By definition, $A_{[\alpha]} = \{\psi(X) \in R[X] \mid \psi(\alpha) = 0\}$.

Let $I_{[\alpha]} := \bigcap_{i=1}^d (R :_R \eta_i)$, which is an ideal of R .

For $f(X) \in K[X]$,

$C(f(X)) :=$ the ideal generated by all coefficients of $f(X)$,
that is, $C(f(X))$ is the content ideal of $f(X)$.

Let $J_{[\alpha]} := I_{[\alpha]} C(\varphi_\alpha(X))$, which is an ideal of R and contains $I_{[\alpha]}$.

We also use the following standard notations:

$k(p) :=$ the residue field R_p/pR_p for $p \in \text{Spec}(R)$,
 $Dp_1(R) := \{p \in \text{Spec}(R) \mid \text{depth } R_p = 1\}$,
 $Ht_1(R) := \{p \in \text{Spec}(R) \mid \text{ht } p = 1\}$.

Throughout this paper, all fields, rings and algebras are assumed to be commutative with unity. Our special notations are indicated above and our general reference for unexplained technical terms is [3].

1. Anti-Integral Elements and Super-Primitive Elements

We start with the following definition.

DEFINITION 1.1. Let I be an ideal of $R[X]$ with $I \cap R = (0)$ and let $f(X) = a_0 X^n + a_1 X^{n-1} + \cdots + a_n$ be a polynomial in $R[X]$. We say that $f(X)$ is a *Sharma polynomial* in I if there does not exist $t \in R$ with $t \notin a_0 R$ such that $ta_i \in a_0 R$ for $1 \leq i \leq n$.

We give an equivalent condition for a polynomial to be a Sharma polynomial in the following proposition.

Proposition 1.2. *Let $f(X)$ be a polynomial in $R[X]$. Then $f(X)$ is a Sharma polynomial if and only if $C(f(X)) \not\subset p$ for any $p \in Dp_1(R)$.*

Proof. Let $f(X) = a_0 X^n + \dots + a_n (a_i \in R)$.

(\Rightarrow) Suppose that $C(f(X)) \subset p$ for some $p \in Dp_1(R)$. Then $a_0 \in p$, and there exists $t \in a_0 R$ such that $p = (a_0 R :_R t)$. In this case, $a_i \in p$ implies that $a_i t \in a_0 R$ ($1 \leq i \leq n$), which asserts that $f(X)$ is not a Sharma polynomial.

(\Leftarrow) Suppose that $f(X)$ is not a Sharma polynomial. Then there exists $t \in R$ such that $t \notin a_0 R$, $ta_i \in a_0 R$ ($1 \leq i \leq n$). Since there exists $p \in Dp_1(R)$ such that $(a_0 R :_R t) \subset p$, we have $a_i \in (a_0 R :_R t) \subset p$ ($1 \leq i \leq n$) and obviously $a_0 \in p$. So $C(f(X)) = (a_0, \dots, a_n) \subset p$, a contradiction. Q.E.D.

Proposition 1.3. *The following statements are equivalent:*

- (i) $A_{[\alpha]}$ is a principal ideal of $R[X]$,
- (ii) $I_{[\alpha]}$ is a principal ideal of R ,
- (iii) there exists a Sharma polynomial in $A_{[\alpha]}$ of degree d .

If one of the above conditions holds, then $A_{[\alpha]}$ is generated by a Sharma polynomial.

Proof. (iii) \Rightarrow (i): Let $f(X)$ be a Sharma polynomial in $A_{[\alpha]}$ of degree d . Since $\deg \varphi_\alpha(X) = d$, this Sharma polynomial has the least degree. So by [6], $A_{[\alpha]}$ is principal.

(i) \Rightarrow (ii): Let $A_{[\alpha]} = f(X)R[X]$. Then $f(X)R[X] \supset I_{[\alpha]} \varphi_\alpha(X)R[X]$. Note that $A_{[\alpha]} \otimes_R K = f(X)K[X] = \varphi_\alpha(X)K[X]$ and hence $\deg f(X) = \deg \varphi_\alpha(X) = d$. Take $a \in I_{[\alpha]}$. Then $a \varphi_\alpha(X) = bf(X)$. Let $f(X) = a_0 X^d + \dots + a_d$ with $a_i \in R$. Then $a = ba_0$, so that $I_{[\alpha]} \supset a_0 R$ for some $b \in R$. Since $ba_0 \eta_i = a \eta_i = ba_i$ ($1 \leq i \leq d$), we have $a_0 \eta_i = a_i \in R$. Hence $a_0 \in I_{[\alpha]}$, which implies that $I_{[\alpha]} = a_0 R$.

(ii) \Rightarrow (iii): Let $I_{[\alpha]} = bR$. Then $I_{[\alpha]} \varphi_\alpha(X)R[X] = b \varphi_\alpha(X)R[X] \subset A_{[\alpha]}$ and $b \eta_i \in R$ ($1 \leq i \leq d$). Suppose that there exists $t \notin bR$ with $t b \eta_i \in bR$ ($1 \leq i \leq d$). Then $t \eta_i \in R$ and hence $t \in I_{[\alpha]} = bR$, a contradiction. Thus $b \varphi_\alpha(X) \in R[X]$ is a Sharma polynomial of degree d . Q.E.D.

For later use, we quote the following.

Lemma 1.4 ([6, Cor. 3]). *Let R be an integral domain and I a non-zero ideal of a polynomial ring $R[X]$ such that $I \cap R = (0)$. If there exists a polynomial $f(X) \in I$ such that $f(X)$ is of the least positive degree in I and $C(f(X)) = R$, then I is generated by the polynomial $f(X)$.*

DEFINITION 1.5. i) $\alpha \in L$ is called an *anti-integral element* of degree d over R if $A_{[\alpha]} = I_{[\alpha]} \varphi_\alpha(X)R[X]$. When α is an anti-integral element, we say that $R[\alpha]$ is an *anti-integral extension* of R .

ii) $\alpha \in L$ is called a *super-primitive element* of degree d over R if $J_{[\alpha]} \not\subset p$ for all $p \in Dp_1(R)$. When α is a super-primitive element, we say that $R[\alpha]$ is a *super-primitive extension* of R .

REMARK 1.6. i) In [5], we studied the anti-integrality which is defined as follows: An element $\alpha \in K$ is called anti-integral over R if $R = R[\alpha] \cap R[1/\alpha]$ ($:= R(\alpha)$). We knew that α is anti-integral over R in this sense if and only if $A_{[\alpha]}$ has a linear basis, that is,

$$A_{[\alpha]} = \sum (c_i X - d_i) R[X]$$

with $d_i/c_i = \alpha$ [5, Proof of (1.9)]. The last condition is equivalent to $A_{[\alpha]} = I_{[\alpha]} \varphi_\alpha(X) R[X]$, where $\varphi_\alpha(X) = X - \alpha$. So $\alpha \in K$ is anti-integral over R in this sense if and only if α is an anti-integral element of degree one over R in the sense of Definition 1.5, that is, the anti-integrality defined in [5] is equivalent to the one defined in (1.5) in the case of degree one.

ii) It is immediate that $\alpha \in L$ is a super-primitive element of degree d over R if and only if α is a super-primitive element of degree d over R_p for any $p \in \text{Spec}(R)$. Thus $R[\alpha]$ is a super-primitive extension of R if and only if $R[\alpha]_p$ is a super-primitive extension of R_p for all $p \in \text{Spec}(R)$, where $R[\alpha]_p$ denotes the localization $S^{-1} R[\alpha]$ with $S = R \setminus p$.

Lemma 1.7. *Let $f(X)$ be an element of a polynomial ring $R[X]$ and let $p \in \text{Spec}(R)$. Then $p \supset C(f(X))$ if and only if $R_p[X]/f(X) R_p[X]$ is not flat over R_p .*

Proof. The implication (\Leftarrow) follows from [3, (20.F)].

(\Rightarrow) Since $C(f(X)) \subset p$, $pR[X]$ contains $f(X)$, and hence $Q = pR[X]/f(X) R[X]$ is a prime ideal of $B := R[X]/f(X) R[X]$. Suppose that $B_p = R_p[X]/f(X) R_p[X]$ is flat over R_p . Then B_Q is obtained from B_p by localizing at QB_p . So $\text{depth } B_Q \geq \text{depth } B_p$, and hence $\text{depth } B_Q \geq \text{depth } R_p$. It is easy to see that $\text{depth } B_{pB} = \text{depth } B_Q$ and $B_{pB} = R[X]_{pB[X]}/f(X) R[X]_{pB[X]}$. Since R is an integral domain, we have $\text{depth } B_{pB} = \text{depth } R[X]_{pR[X]} - 1 = \text{depth } R_p - 1$, which is a contradiction.

Q.E.D.

Our almost all main results are based on the following theorem.

Theorem 1.8. *Assume that α is an anti-integral element of degree d over R . Then for $p \in \text{Spec}(R)$, the following are equivalent:*

- (i) $\text{rank}_{k(p)} R[\alpha] \otimes_R k(p) \leq d$,
- (ii) $\text{rank}_{k(p)} R[\alpha] \otimes_R k(p) < \infty$,
- (iii) $R[\alpha] \otimes_R k(p)$ is not isomorphic to a polynomial ring $k(p)[T]$,
- (iv) $J_{[\alpha]} \not\subset p$,
- (v) $pR[X] \not\supset A_{[\alpha]}$,
- (vi) $R[\alpha]_p$ is flat over R_p .

Proof. Since α is anti-integral, $A_{[\alpha]} = I_{[\alpha]} \varphi_\alpha(X) R[X]$.

(iv) \Rightarrow (vi): Since $R_p = (J_{[\alpha]})_p = (I_{[\alpha]})_p C(\varphi_\alpha(X))_p$, $(I_{[\alpha]})_p$ is a principal ideal bR_p ,

for some $b \in I_{[\alpha]}$. So $(A_{[\alpha]})_p = b\varphi_\alpha(X) R_p[X]$. It follows that $R[\alpha]_p \simeq R_p[X]/(A_{[\alpha]})_p = R_p[X]/b\varphi_\alpha(X) R_p[X]$. Thus $R[\alpha]_p$ is flat over R_p by Lemma 1.7 because $R_p = (J_{[\alpha]})_p = C(b\varphi_\alpha(X))_p$.

(iv) \Rightarrow (i): By the same argument as above, we have $R[\alpha]_p \simeq R_p[X]/(A_{[\alpha]})_p = R_p[X]/b\varphi_\alpha(X) R_p[X]$. Since $R_p = (J_{[\alpha]})_p = C(b\varphi_\alpha(X))_p$, there exists i ($0 \leq i \leq d$) such that $b\eta_i \notin pR_p[X]$. We take i minimal among such ones. Then $b\varphi_\alpha(X) = bX^d + b\eta_1 X^{d-1} + \dots + b\eta_d \equiv b\eta_i X^{d-1} + \dots + b\eta_d \equiv 0 \pmod{pR_p[X]}$, which means that $\text{rank}_{k(p)} R[\alpha] \otimes_R k(p) \leq d - i \leq d$.

(i) \Rightarrow (ii) is trivial.

(ii) \Rightarrow (iv): Note that $R[\alpha]_p/pR[\alpha]_p \simeq R_p[X]/(pR[X] + A_{[\alpha]})_p$. Since $\text{rank}_{k(p)} R[\alpha] \otimes_R k(p) < \infty$, $(pR[X] + A_{[\alpha]})_p$ contains an element $f(X) \in R[X]$ such that $C(f(X))_p = R_p$. Indeed, if not, we conclude that $R[\alpha] \otimes_R k(p) \simeq k(p)[T]$, a polynomial ring, a contradiction. We may assume that $f(X) \in A_{[\alpha]}$. So the equality $(A_{[\alpha]})_p = I_{[\alpha]} \varphi_\alpha(X) R_p[X]$ yields that $(J_{[\alpha]})_p = (I_{[\alpha]})_p C(\varphi_\alpha(X))_p = R_p$.

(vi) \Rightarrow (iv): Suppose that $J_{[\alpha]} \subset p$. Localizing at p , we may assume that R is a local ring (R, m) . Consider the exact sequence:

$$0 \rightarrow A_{[\alpha]} \rightarrow R[X] \rightarrow R[\alpha] \rightarrow 0.$$

Then $A_{[\alpha]}$ is flat over R because $R[X]$ and $R[\alpha]$ are flat over R . The isomorphism $A_{[\alpha]} = I_{[\alpha]} \varphi_\alpha(X) R[X] \simeq I_{[\alpha]} R[X]$ yields that $I_{[\alpha]} R[X]$ is flat over $R[X]$ and hence $I_{[\alpha]}$ is flat over R . Since R is local, $I_{[\alpha]} = bR$ for some $b \in I_{[\alpha]}$. So $J_{[\alpha]} = bC(\varphi_\alpha(X))$ and $A_{[\alpha]} = b\varphi_\alpha(X) R[X]$. So $C(b\varphi_\alpha(X)) \subset m$, and hence $R[\alpha]$ is not flat over R by Lemma 1.7.

(iv) \Rightarrow (v): Since $J_{[\alpha]} = I_{[\alpha]} C(\varphi_\alpha(X)) \subset p$, there exists $a \in I_{[\alpha]}$ such that $aC(\varphi_\alpha(X)) = C(a\varphi_\alpha(X)) \subset p$. Thus $a\varphi_\alpha(X) \notin pR[X]$ and hence $A_{[\alpha]} \not\subset pR[X]$.

(v) \Rightarrow (iv): Since $A_{[\alpha]} = I_{[\alpha]} \varphi_\alpha(X) R[X]$, there exists $a \in I_{[\alpha]}$ such that $C(a\varphi_\alpha(X)) \subset p$. So $J_{[\alpha]} = J_{[\alpha]} C(\varphi_\alpha(X)) \subset p$.

(v) \Rightarrow (iii): There exists $f(X) \in A_{[\alpha]}$ with $f(X) \notin pR[X]$. So $R[\alpha]/pR[\alpha] = (R/p)[\alpha']$, where α' denotes the residue class of α in $R[\alpha]/pR[\alpha]$, and $f(\alpha') = 0$. Thus α' is algebraic over R/p .

(iii) \Rightarrow (v): Suppose that $A_{[\alpha]} \subset pR[X]$. Then $R[\alpha]/pR[\alpha] = (R[X]/A_{[\alpha]})/p(R[X]/A_{[\alpha]}) = R[X]/pR[X] = (R/p)[X]$, which is a polynomial ring over R/p .

Q.E.D.

After the definition in [5], we employ the following.

DEFINITION 1.9. Let A be an extension of R and let $p \in \text{Spec}(R)$. We say that A is a *blowing-up at p* or p is a *blowing-up point* of A/R if the following two conditions are satisfied:

- (i) $pA_p \cap R_p = pR_p$ (equivalently $pA \cap R = p$),
- (ii) A_p/pA_p is isomorphic to a polynomial ring $(R_p/pR_p)[T]$.

Making use of the above definition, we get the following corollary to The-

orem 1.8.

Corollary 1.10. *When α is an anti-integral element over R , the blowing-up locus $\{p \in \text{Spec}(R) \mid p \text{ is not a blowing-up point of } R[\alpha]\}$ is given by $V(J_{[\alpha]})$, and is the same as the non-flat locus $\{p \in \text{Spec}(R) \mid R[\alpha]_p \text{ is not flat over } R_p\}$.*

Proof. This follows from Theorem 1.8 and Lemma 1.7.

The next proposition gives rise to the relation between Sharma polynomials and the ideal $A_{[\alpha]}$.

Proposition 1.11.

- (a) *$R[\alpha]$ is not a blowing-up at any point in $Dp_1(R)$ if and only if $A_{[\alpha]}$ contains a Sharma polynomial.*
- (b) *$R[\alpha]$ is not a blowing-up at any point in $\text{Spec}(R)$ if and only if there exists a polynomial $f(X)$ in $A_{[\alpha]}$ such that $C(f(X))=R$.*

Proof. (a) Take $g_0(X) \in A_{[\alpha]} \setminus (0)$. If $g_0(X)$ is a Sharma polynomial, then we are done. Suppose that $g_0(X)$ is not a Sharma polynomial. Let $\{p_1, \dots, p_t\}$ be the set of all elements in $Dp_1(R)$ satisfying $C(g_0(X)) \subset p_i$. Such p_i exists by Proposition 1.2. Since $A_{[\alpha]} \not\subset pR[X]$ for any $p \in Dp_1(R)$, there are $g_i(X) \in A_{[\alpha]}$ such that $C(g_i(X)) \not\subset p_i$ ($1 \leq i \leq t$). Put $N(0) := \deg(g_0(X))$ and $N(i) := N(i-1) + \deg(g_i(X)) + 1$ inductively. Let $f(X) := \sum g_i(X) X^{N(i)}$. Then $C(f(X)) = C(g_0(X)) + \dots + C(g_t(X))$. By the choice of p_i , there does not exist $p \in Dp_1(R)$ such that $C(f(X)) \subset p$. Hence $f(X)$ is a Sharma polynomial. Assume that $A_{[\alpha]}$ contains a Sharma polynomial. Then $A_{[\alpha]} \not\subset pR[X]$ for any $p \in Dp_1(R)$ by Proposition 1.2. So a blowing-up does not occur for $R[\alpha]/R$ on $Dp_1(R)$.

(b) Let $A_{[\alpha]} = (f_1(X), \dots, f_n(X))R[X]$. Take $p \in \text{Spec}(R)$. Then $A_{[\alpha]} \not\subset pR[X]$. So there exists i such that $C(f_i(X)) \not\subset p$. Put $N(0) = 0$ and $N(i) = N(i-1) + \deg(f_i(X)) + 1$, and let $f(X) = \sum f_i(X) X^{N(i)}$. Then $C(f(X)) = C(f_i(X)) + \dots + C(f_n(X)) = R$. The converse is obvious. Q.E.D.

By the following theorem, we see that a super-primitive element is an anti-integral element.

Theorem 1.12. *Under the above notations, the following statements are equivalent:*

- (i) *α is a super-primitive element of degree d ,*
- (ii) *α is an anti-integral element of degree d over R and $R_p[\alpha]$ is flat over R_p for all $p \in Dp_1(R)$,*
- (iii) *α is an anti-integral element of degree d over R and $pR[X] \not\supset A_{[\alpha]}$ for all $p \in Dp_1(R)$,*
- (iv) *α is an anti-integral element of degree d over R and there exists a Sharma polynomial in $A_{[\alpha]}$,*

(v) $J_{[\alpha]}^{-1} = R$, where $J_{[\alpha]}^{-1} := (R :_K J_{[\alpha]})$.

Proof. (i) \Rightarrow (ii): It is clear that $I_{[\alpha]} \varphi_\alpha(X) R[X] \subset A_{[\alpha]}$, and hence $I_{[\alpha]} R[X] \subset \varphi_\alpha(X)^{-1} A_{[\alpha]}$. Put $J = \varphi_\alpha(X)^{-1} A_{[\alpha]}$. Let $I_{[\alpha]} R[X] = Q_1 \cap \dots \cap Q_n$ be an irredundant primary decomposition of the ideal $I_{[\alpha]} R[X]$ and let $P_i = \sqrt{Q_i}$ ($1 \leq i \leq n$). Assume that Q (resp. P) represents some Q_i (resp. P_i). Since $I_{[\alpha]}$ is a divisorial ideal of R , $I_{[\alpha]} R[X]$ is a divisorial ideal of $R[X]$, and hence $\text{depth } R[X]_p = 1$. Put $p = P \cap R$. As $p \supset I_{[\alpha]}$, we see that $p \neq (0)$. Thus we have $P = pR[X]$ and $\text{depth}(R_p) = 1$. Since α is a super-primitive element, $J_{[\alpha]} \not\subset p$ by definition. Therefore there exists an element $a \in I_{[\alpha]}$ such that $(A_{[\alpha]})_p = a\varphi_\alpha(X) R_p[X]$. Hence we have $J_p = aR_p[X] \subset I_{[\alpha]} R_p[X] \subset QR_p[X]$. Thus we get $J \subset R[X] \cap QR_p[X] = Q$, that is, $J \subset I_{[\alpha]} R[X]$ because Q (resp. P, p) is any Q_i (resp. $P_i, p_i := P_i \cap R$) for $1 \leq i \leq n$. This implies that α is an anti-integral element. Hence the assertion follows from Theorem 1.8.

(ii) \Leftrightarrow (iii) \Leftrightarrow (iv): It is immediate from Theorem 1.8 and Proposition 1.11.

(iv) \Rightarrow (i): Since α is an anti-integral element, $A_{[\alpha]} = I_{[\alpha]} \varphi_\alpha(X) R[X]$. By Proposition 1.11(a), $A_{[\alpha]} \not\subset pR[X]$ for all $p \in Dp_1(R)$. Hence there exists an element $a(p) \in I_{[\alpha]}$ such that $f(X) = a(p) \varphi_\alpha(X)$ and $C(f(X)) \not\subset p$. Thus $J_{[\alpha]} \not\subset p$ for any $p \in Dp_1(R)$. Therefore α is a super-primitive element.

(i) \Rightarrow (v): Assume that $J_{[\alpha]} \not\subset p$ for any $p \in Dp_1(R)$. Then $(J_{[\alpha]}^{-1})_p = (R :_K J_{[\alpha]})_p = (R_p :_K (J_{[\alpha]})_p) = (R_p :_K R_p) = R_p$ for any $p \in Dp_1(R)$. Since $J_{[\alpha]}^{-1}$ is a divisorial ideal of R , we have $R = \bigcap R_p = \bigcap (J_{[\alpha]}^{-1})_p \supset J_{[\alpha]}^{-1}$, where p ranges over prime ideals of depth one. Thus $R = J_{[\alpha]}^{-1}$. Conversely, suppose that $R = J_{[\alpha]}^{-1}$ and $J_{[\alpha]} \subset p$ for some $p \in Dp_1(R)$. Then $J_{[\alpha]}^{-1} \supset p^{-1}$ and hence $R = (J_{[\alpha]}^{-1})^{-1} \subset (p^{-1})^{-1} = p$, a contradiction. Q.E.D.

More equivalent conditions will be seen in the section 2.

By the following result, we see that a super-primitive element is not so special.

Theorem 1.13. *Assume that R is a Krull domain, then any element α which is algebraic over R is a super-primitive element over R .*

Proof. Since R is a Krull domain, $Dp_1(R) = \text{Ht}_1(R)$. Take $p \in \text{Ht}_1(R)$. Then R_p is a DVR. Let v denote the valuation corresponding to R_p . Let $\varphi_\alpha(X) = X^d + \eta_1 X^{d-1} + \dots + \eta_d$ be the minimal polynomial of α . Put $\eta_0 = 1$. Then there exists j such that $v(\eta_j) \leq v(\eta_i)$ for all i . Thus $\eta_i/\eta_j = a_i/b \in R_p$, where $b \in R \setminus p, a_i \in R$. In particular, $a_j = b \notin p$. Hence

$$\varphi_\alpha(X) = \eta_j(a_0/\eta_j) X^d + \dots + \eta_j(a_d/\eta_j) \eta_d.$$

Hence $f(X) := (b/\eta_j) \varphi_\alpha(X) = a_0 X^d + \dots + a_d \in \varphi_\alpha(X) K[X]$. Since $a_j = b \notin p$, we have $C(f(X)) \not\subset p$. Since $\deg f(X) = d$, we conclude that α is a super-primitive element over R by Theorem 1.10. Q.E.D.

Once we find one super-primitive element, we can get many such elements. Indeed we obtain the following.

Proposition 1.14. *Assume that a is a super-primitive element of degree d over R . Then for any unit u of R and any element $b \in R$, $\beta = u\alpha + b$ is a super-primitive element of degree d over R .*

Proof. We may assume that $u=1$. It is clear that $\varphi_\beta(X) = \varphi_\alpha(X-b)$ because $K(\beta) = K(\alpha)$, $d = \deg \varphi_\alpha(X-b)$ and $\varphi_\alpha(X-b)$ is monic in $K[X]$. We see that $I_{[\alpha]} \subset I_{[\beta]}$ and $C(\varphi_\alpha(X)) = C(\varphi_\alpha(X-b)) = C(\varphi_\beta(X))$. Since $(J_{[\alpha]})_p = (I_{[\alpha]})_p$, $C(\varphi_\alpha(X))_p = R_p$ for any $p \in Dp_1(R)$ by Theorem 1.12, $R_p = (J_{[\alpha]})_p \subset (J_{[\beta]})_p$ and hence $(J_{[\beta]})_p = R_p$ for any $p \in Dp_1(R)$. Thus β is a super-primitive element of degree d over R by Theorem 1.12. Q.E.D.

Proposition 1.15. *Assume that R is a local ring containing an infinite field k and that $J_{[\alpha]} = R$. Then there exists an element $\lambda \in k$ which satisfies that*

- (a) $1/(\alpha - \lambda)$ belongs to $R[\alpha]$,
- (b) $1/(\alpha - \lambda)$ is a super-primitive element of degree d over R ,
- (c) $1/(\alpha - \lambda)$ is integral over R .

Proof. Since R is local, there exists an element λ in k such that $I_{[\alpha]} \varphi_\alpha(X + \lambda)$ contains a degree d polynomial $g(X)$ in $R[X]$ of which constant term is 1. Put $\beta = \alpha - \lambda$. Then $g(\beta) = 0$. Let $h(X) = X^d g(1/X) \in R[X]$. Then $h(1/\beta) = (1/\beta)^d g(\beta) = 0$. So $1/\beta$ is integral over R . Since $[K(\alpha) : K] = [K(\beta) : K] = d$, we conclude that $\varphi_{1/\beta}(X) = h(X) \in R[X]$. Thus $I_{[1/\beta]} = R$ and hence $J_{[1/\beta]} = I_{[1/\beta]} C(\varphi_{1/\beta}(X)) = R$. In particular, $1/\beta$ is a super-primitive element of degree d over R by Theorem 1.12. Q.E.D.

2. Integrality and Flatness of Anti-Integral Extensions

The following result asserts that the integrality of an extension of R is determined by localizing at prime ideals in $Dp_1(R)$.

Proposition 2.1. *Let A be an integral domain containing R . Then A is integral over R if and only if $A_p (= A \otimes_R R_p)$ is integral over R_p for any $p \in Dp_1(R)$.*

Proof. The implication (\Rightarrow) is trivial. Consider the converse and assume that A_p is integral over R_p for any $p \in Dp_1(R)$. We have only to show that α is integral over R . Let R' be the integral closure of R in K . Then R' is a Krull domain [3, p.144]. It suffices to show that α is integral over R' . Let R'' be the integral closure of R in $K(A)$ and let $C = R'' :_{R''} \alpha$, a denominator ideal of R'' . Then $K(R'') = K(A)$ and C is a divisorial ideal of R'' . There exists $P \in Dp_1(R'') = Ht_1(R'')$ such that $C \subset P$. Since R''/R' is integral and R' is integrally closed in K , the Going-Down Theorem holds for R''/R' . Thus $P \cap R' \in$

$Ht_1(R')=Dp_1(R')$. In particular, $P \cap R'$ is a divisorial ideal of R' . So $R'' :_{R'} \alpha = C \cap R' \subset P \cap R' \in Dp_1(R')$. By [2, (4.6)], $(P \cap R') \cap R$ is a divisorial ideal of R . Hence $R'' :_R \alpha = (C \cap R') \cap R \subset (P \cap R') \cap R \in Dp_1(R)$. Put $p = (P \cap R') \cap R$. Then we have $p \in Dp_1(R)$ and $R'' :_R \alpha \subset p$, which is a contradiction. Q.E.D.₃

The integrality of anti-integral extensions is characterized as follows:

Theorem 2.2. *Assume that α is an anti-integral element of degree d over R . Then the following are equivalent:*

- (i) $R[\alpha]$ is integral over R ,
- (ii) $\varphi_\alpha(X) \in R[X]$,
- (iii) $I_{[\alpha]} = R$,
- (iv) $\text{rank}_{k(q)} R[\alpha] \otimes_R k(q) = d$ for any $q \in Dp_1(R)$,
- (v) $\text{rank}_{k(q)} R[\alpha] \otimes_R k(q) = d$ for any $q \in \text{Spec}(R)$.

Proof. Since α is anti-integral, $A_{[\alpha]} = I_{[\alpha]} \varphi_\alpha(X) R[X]$. So the equivalence of (i), (ii) and (iii) are immediate because $R[X]/A_{[\alpha]} \simeq R[\alpha]$, and implications (ii) \Rightarrow (v) \Rightarrow (iv) are obvious.

(iv) \Rightarrow (ii): Suppose that $I_{[\alpha]} \subset p$ for some $p \in Dp_1(R)$. Since $J_{[\alpha]} = I_{[\alpha]} C(\varphi_\alpha(X)) \not\subset p$ by Theorem 1.8, $(I_{[\alpha]})_p$ is an invertible ideal of R_p and hence $(I_{[\alpha]})_p$ is a principal ideal bR_p of R_p for some b . So $(A_{[\alpha]})_p = (I_{[\alpha]})_p \varphi_\alpha(X) R_p[X] = (b\varphi_\alpha(X)) R_p[X]$. Since $I_{[\alpha]} \subset p$, $b\varphi_\alpha(X) \in R_p[X]$ is not monic. Hence either $R[\alpha] \otimes_R k(p) \simeq k(p)[T]$, a polynomial ring or $\text{rank}_{k(p)} R[\alpha] \otimes_R k(p) < d$, a contradiction.

Q.E.D.

By the above theorem, we see that the obstruction of integrality of anti-integral extensions is given by $I_{[\alpha]}$. Namely, we obtain the following.

Corollary 2.3. *Assume that α is an anti-integral element over R . Then $V(I_{[\alpha]}) = \{p \in \text{Spec}(R) \mid R[\alpha]_p \text{ is not integral over } R_p\}$.*

Proof. The integrality is a local-global property. So our conclusion follows from Theorem 2.2. Q.E.D.

REMARK 2.4. Let R be a Noetherian normal domain and let α be an element in a field L containing R . If α is integral over R , then it is a super-primitive element over R . Indeed, when $\varphi_\alpha(X) \in K[X]$ denotes the minimal polynomial of α over R , it is known that α is integral over R if and only if $\varphi_\alpha(X)$ belongs to $R[X]$ ([4, (9.2)]. Since R is normal, $p \in Dp_1(R) \Rightarrow ht(p) = 1 \Rightarrow R_p$ is a DVR. As $R[\alpha]$ is a finite R -module, $R[\alpha]_p$ is free over R_p for any $p \in Dp_1(R)$. By Theorem 1.10, α is a super-primitive element over R . Moreover $R[\alpha]$ is flat over R by Theorems 1.8 and 3.2 because $R[\alpha]/R$ is super-primitive, integral and flat.

Summing up the results in the preceding argument, we obtain the following:

Assume that α is an anti-integral element of degree d . Let \mathfrak{p} be a prime ideal of R . Then

- (1) $R[\alpha]_{\mathfrak{p}}$ is flat over $R_{\mathfrak{p}}$ if and only if $\text{rank}_{k(\mathfrak{p})} R[\alpha] \otimes_R k(\mathfrak{p}) \leq d$,
- (2) $R[\alpha]_{\mathfrak{p}}$ is integral over $R_{\mathfrak{p}}$ if and only if $\text{rank}_{k(\mathfrak{p})} R[\alpha] \otimes_R k(\mathfrak{p}) = d$.

In particular, we conclude:

Corollary 2.5. *Assume that α is an anti-integral element of degree d . If $R[\alpha]$ is integral over R , then $R[\alpha]$ is flat over R .*

In view of Proposition 1.11, we extend Theorem 1.8 to the following.

Proposition 2.6. *Assume that α is an anti-integral element of degree d over R . Then the following are equivalent:*

- (i) $R[\alpha]$ is flat over R ,
- (ii) $J_{[\alpha]} = R$,
- (iii) $\text{rank}_{k(\mathfrak{p})} R[\alpha] \otimes_R k(\mathfrak{p}) < \infty$ for any $\mathfrak{p} \in \text{Spec}(R)$,
- (iv) $\text{rank}_{k(\mathfrak{p})} R[\alpha] \otimes_R k(\mathfrak{p}) \leq d$ for any $\mathfrak{p} \in \text{Spec}(R)$,
- (v) $R[\alpha]$ is not a blowing-up at any point in $\text{Spec}(R)$,
- (vi) $R[\alpha]$ is quasi-finite over R ,
- (vii) $A_{[\alpha]}$ contains a polynomial $f(X)$ with $C(f(X)) = R$.

Proof. The proof follows from Theorem 1.8 and Proposition 1.11 (b).

REMARK 2.7. Let A be over-ring of R (i.e., $R \subset A$ and $K(A) = K$). If A is integral and flat over R on $D\mathfrak{p}_1(R)$, then $A = R$. Indeed, it is known that $R = \bigcap_{\mathfrak{p} \in D\mathfrak{p}_1(R)} R_{\mathfrak{p}}$. For $\mathfrak{p} \in D\mathfrak{p}_1(R)$, $A_{\mathfrak{p}}$ is integral, flat over $R_{\mathfrak{p}}$ by the assumption. So $A_{\mathfrak{p}}$ is a free $R_{\mathfrak{p}}$ -module of rank one. Thus $A_{\mathfrak{p}} = R_{\mathfrak{p}}$ and hence $R = \bigcap_{\mathfrak{p} \in D\mathfrak{p}_1(R)} R_{\mathfrak{p}} \supset A$.

Relating to this remark, we have the following.

Theorem 2.8. *Let α be an algebraic element over R . If $R[\alpha]$ is integral and flat at any point in $D\mathfrak{p}_1(R)$, then $R[\alpha]$ is a free R -module and α is a super-primitive element over R .*

Proof. First, we shall show that $I_{[\alpha]} = R$. Suppose that $I_{[\alpha]} \neq R$. Since $I_{[\alpha]}$ is a divisorial ideal of R , there exists $\mathfrak{p} \in D\mathfrak{p}_1(R)$ such that $I_{[\alpha]} \subset \mathfrak{p}$. Since $R[\alpha]_{\mathfrak{p}}$ is integral over $R_{\mathfrak{p}}$ by assumption, $R[\alpha]_{\mathfrak{p}}$ is a flat extension of $R_{\mathfrak{p}}$. As $R[\alpha]_{\mathfrak{p}}$ is flat over $R_{\mathfrak{p}}$, $R[\alpha]_{\mathfrak{p}}$ is a free $R_{\mathfrak{p}}$ -module of rank d . We want to show that $R[\alpha]_{\mathfrak{p}} = R_{\mathfrak{p}} + R_{\mathfrak{p}}\alpha + \cdots + R_{\mathfrak{p}}\alpha^{d-1}$. For this purpose, we have only to show that $1', \alpha', \dots, \alpha'^{d-1} \in R[\alpha]_{\mathfrak{p}}/\mathfrak{p}R[\alpha]_{\mathfrak{p}}$ are linearly independent over $k(\mathfrak{p})$, where α' denotes its residue class in $R[\alpha]_{\mathfrak{p}}/\mathfrak{p}R[\alpha]_{\mathfrak{p}}$. Suppose the contrary. Then $R[\alpha]_{\mathfrak{p}}/\mathfrak{p}R[\alpha]_{\mathfrak{p}} = k(\mathfrak{p})[\alpha'] = k(\mathfrak{p}) + k(\mathfrak{p})\alpha' + \cdots + k(\mathfrak{p})\alpha'^s$ for some $s < d$. But $R[\alpha]_{\mathfrak{p}}$ is a free $R_{\mathfrak{p}}$ -module of rank d , which asserts that $\text{rank}_{k(\mathfrak{p})} R[\alpha]_{\mathfrak{p}}/\mathfrak{p}R[\alpha]_{\mathfrak{p}} = d$,

a contradiction. Thus we have shown that $R[\alpha]_p = R_p + R_p \alpha + \dots + R_p \alpha^{d-1}$. So we have a relation: $\alpha^d = \lambda_0 + \lambda_1 \alpha + \dots + \lambda_{d-1} \alpha^{d-1}$ ($\lambda_i \in R_p$). Since the minimal polynomial $\varphi_\alpha(X)$ of α is unique, we have $\varphi_\alpha(X) = X^d - \lambda_{d-1} X^{d-1} - \dots - \lambda_0$. So $I_{[\alpha]} \not\subset p$, a contradiction. Thus $\varphi_\alpha(X) \in R[X]$, which implies that $A_{[\alpha]} = \varphi_\alpha(X) R[X]$ and $R[\alpha]$ is a free R -module. Since $C(\varphi_\alpha(X)) = R$, we conclude that $J_{[\alpha]} = R$. By Theorem 1.12, α is a super-primitive element over R . Q.E.D.

Now we consider a certain over-ring of R which is seen in [5].

DEFINITION 2.9. Let J be a fractional ideal of R . Let $\mathcal{R}(J) := J :_K J$, which is an over-ring of R .

Lemma 2.10. *Let J be a divisorial ideal of R . Then $\mathcal{R}(J) = R$ if and only if $\mathcal{R}(J^{-1}) = R$.*

Proof. Since J is divisorial, $(J^{-1})^{-1} = J$. So we have only to prove one of the implications. Assume that $\mathcal{R}(J) = R$. The implication $\mathcal{R}(J^{-1}) \supset R$ is obvious. Take $\lambda \in \mathcal{R}(J^{-1})$. Then $\lambda J^{-1} \subset J^{-1}$. Thus $R : \lambda J^{-1} \supset R : J^{-1} = (J^{-1})^{-1} = J$. On the other hand, we have $R : \lambda J^{-1} = \lambda^{-1} R : J^{-1} = \lambda^{-1} (R : J^{-1}) = \lambda^{-1} (J^{-1})^{-1} = \lambda^{-1} J$. Thus $\lambda^{-1} J \supset J$, which shows that $J \supset \lambda J$, and hence $\lambda \in \mathcal{R}(J) = R$. Q.E.D.

By these arguments, we extend Theorem 1.12 to the following.

Theorem 2.11. *The following conditions are equivalent:*

- (i) α is a super-primitive element over R ,
- (ii) for each $p \in Dp_1(R)$, there exists $f(X) \in A_{[\alpha]}$ with $(A_{[\alpha]})_p = f(X) R_p[X]$,
- (iii) for each $p \in Dp_1(R)$, there exists $a \in I_{[\alpha]}$ with $(I_{[\alpha]})_p = aR_p$,
- (iv) $\mathcal{R}(I_{[\alpha]}) = R$.

Proof. Denote the degree of α by d .

(i) \Rightarrow (ii): Since $J_{[\alpha]} = I_{[\alpha]} C(\varphi_\alpha(X)) \not\subset p$ for any $p \in Dp_1(R)$, there exists $a \in I_{[\alpha]}$ with $f(X) := a\varphi_\alpha(X) \in pR[X]$. Note that $(A_{[\alpha]})_K \cap R_p[X] = (A_{[\alpha]})_p$ and $f(X) \in (A_{[\alpha]})_p$. By Proposition 1.2, $f(X)$ is a Sharma polynomial of degree d in $R_p[X]$. So $(A_{[\alpha]})_p = f(X) R_p[X]$.

(ii) \Rightarrow (iii): Suppose that $(A_{[\alpha]})_p = f(X) R_p[X]$. Then $\deg f(X) = d$. Let a be the leading coefficient of $f(X)$. Then $\varphi_\alpha(X) = (1/a)f(X)$ by the uniqueness of the minimal polynomial of α . So $f(X) = a\varphi_\alpha(X) R[X]$, and hence $a \in I_{[\alpha]}$. Since $(A_{[\alpha]})_p = f(X) R_p[X]$, $(I_{[\alpha]})_p = aR_p$.

(iii) \Leftrightarrow (iv): We know that $\mathcal{R}(I_{[\alpha]}) = R$ if and only if $\mathcal{R}(I_{[\alpha]}^{-1}) = R$ by Lemma 2.10. So apply a result of [5, (3.2)] and we conclude that (iii) and (iv) are equivalent.

(iii) \Rightarrow (i): Since $(I_{[\alpha]})_p$ is a principal ideal of R_p for any $p \in Dp_1(R)$, there exists $f(X) \in A_{[\alpha]}$ such that $\deg f(X) = d$ and $(A_{[\alpha]})_p = f(X) R_p[X]$. Since $f(X)$ is a

Sharma polynomial in $R_p[X]$ by Proposition 1.2 and $\text{depth } R_p = 1$, $C(f(X)) \not\subset p$. Thus $J_{[\alpha]} \not\subset p$ for any $p \in Dp_1(R)$ and hence α is a super-primitive element over R by definition. Q.E.D.

3. Vanishing Points and Blowing-Up Points

Assume that α is an anti-integral element over R . For $p \in \text{Spec}(R)$, $\text{rank}_{k(p)} R[\alpha] \otimes_R k(p) < \infty$ if and only if $R[\alpha]_p$ is flat over R_p by Theorem 2.2. So it may be natural to ask when $\text{rank}_{k(p)} R[\alpha] \otimes_R k(p)$ is infinite or zero.

Let α be an element which is algebraic over R . Recall that $\varphi_\alpha(X) = X^d + \eta_1 X^{d-1} + \cdots + \eta_d$ is the minimal polynomial of α over K , where $d = [K(\alpha) : K]$ and $J_{[\alpha]} := I_{[\alpha]} C(\varphi_\alpha(X)) = I_{[\alpha]} + I_{[\alpha]} \eta_1 + \cdots + I_{[\alpha]} \eta_d$. Define $B_{[\alpha]} := I_{[\alpha]} + I_{[\alpha]} \eta_1 + \cdots + I_{[\alpha]} \eta_{d-1}$, which is an ideal of R .

We use this notation throughout §3.

Lemma 3.1. *Assume that α is an anti-integral element over R and let $A = R[\alpha]$. For $q \in \text{Spec}(R)$, the following are equivalent:*

- i) $qA_q = A_q$,
- ii) $qA \cap R \not\subset q$,
- iii) $q \supset B_{[\alpha]}$ and $q \not\supset I_{[\alpha]} \eta_d$.

Proof. (i) \Rightarrow (ii): Since $qA_q = A_q$, there exist $a_i \in q, \beta_i \in A$ and $s_i \in R \setminus q$ such that $1 = \sum a_i \beta_i / s_i$. Put $s = \prod s_i$. Then $s = \sum a_i \beta_i b_i \in qA \cap R$ with $s \notin q$, where $s \beta_i / s_i = b_i \in A$. Thus $qA \cap R \not\subset q$.

(ii) \Rightarrow (i): Take $s \in qA \cap R$ with $s \notin q$. Then $s \in qA_q$ and s is invertible in A_q . Thus $qA_q = A_q$.

(iii) \Rightarrow (ii): Take $a \in I_{[\alpha]}$ with $a \eta_d \notin q$. Put $f(X) = a \varphi_\alpha(X)$ and $a \eta_i = b_i, a = b_j$, so that $f(X) = b_0 X^d + b_1 X^{d-1} + \cdots + b_d$. Since $f(\alpha) = 0, b_0 \alpha^d + b_1 \alpha^{d-1} + \cdots + b_d = 0$. Noting that $b_d \notin q, b_d$ is a unit in A_q . Since $b_0, \dots, b_{d-1} \in q, b_d \in qA \subset qA_q$. Thus $qA_q = A_q$.

(ii) \Rightarrow (iii): Since $qA_q = A_q, 1 = b_0 + b_1 \alpha + \cdots + b_n \alpha^n$ for some $b_i \in qR_q$. Put $f(x) = b_n X^n + \cdots + b_1 X + b_0 - 1$. Then $f(\alpha) = 0$ and $b_0 - 1$ is a unit in R_q . The kernel of $R_q[X] \rightarrow R[\alpha]_q$ is $(I_{[\alpha]})_q \varphi_\alpha(X) R_q[X]$. So $f(X) \in (I_{[\alpha]})_q \varphi_\alpha(X) R_q[X]$ and $C(f(X))_q = R_q$. Thus it follows that $(J_{[\alpha]})_q = (I_{[\alpha]})_q C(\varphi_\alpha(X))_q = R_q$, which means that $R[\alpha]_q$ is flat over R_q by Theorem 1.8. So $(I_{[\alpha]})_q \varphi_\alpha(X) R_q[X]$ is an invertible ideal of $R_q[X]$. Hence $(I_{[\alpha]})_q$ is a principal ideal of R_q . Let $(I_{[\alpha]})_q = aR_q$. We shall show that all of $a, a\eta_1, \dots, a\eta_{d-1}$ belong to qR_q . Note that $f(X) \in a\varphi_\alpha(X) R_q[X]$ because $f(\alpha) = 0$. So there exists $h(X) \in R_q[X]$ such that $f(X) = a\varphi_\alpha(X) h(X)$. We have $-1 \equiv a\varphi_\alpha(X) h(X) \pmod{qR_q[X]}$. Thus $a\eta_i, a \in qR_q$, for $1 \leq i \leq d-1$ and $a\eta_d \notin qR_q$. Therefore $I_{[\alpha]}, I_{[\alpha]} \eta_1, \dots, I_{[\alpha]} \eta_{d-1} \subset q$ and $I_{[\alpha]} \eta_d \not\subset q$. Q.E.D.

DEFINITION 3.2. Let A be an extension of R and let $p \in \text{Spec}(R)$. We say

that p is a *vanishing point* of A/R if $pA_p = A_p$.

Recall that A is a *blowing-up* at p or p is a *blowing-up point* of A/R if the following two conditions are satisfied:

- i) $pA_p \cap R_p = pR_p$ (equivalently $pA \cap R = p$, cf. Lemma 3.1),
- ii) A_p/pA_p is isomorphic to a polynomial ring $(R_p/pR_p)[T]$.

By Lemma 3.1, we obtain the following theorem.

Theorem 3.3. *Assume that α is an anti-integral element over R and let $A = R[\alpha]$. Then the set of vanishing points (i.e., $\{q \in \text{Spec}(R) \mid qA_q = A_q\}$) is given by $\bigcap_{i=0}^{d-1} V(I_{[\alpha]} \eta_i) \setminus V(I_{[\alpha]} \eta_d)$, where $\eta_0 = 1$.*

Proposition 3.4. *Assume that α is an anti-integral element of degree d over R and let $A = R[\alpha]$. Consider the following conditions:*

- (i) A is flat over R ,
- (ii) $J_{[\alpha]} = R$,
- (iii) If $pA_p = A_p$ for $p \in \text{Spec}(R)$, then $pA = A$.

Then we have implications (i) \Leftrightarrow (ii) \Rightarrow (iii). If moreover R is a local ring and $\sqrt{B_{[\alpha]}} \not\supset I_{[\alpha]} \eta_d$, then (i), (ii) and (iii) are equivalent to each other.

Proof. (i) \Leftrightarrow (ii) was proved in Proposition 2.6. (ii) \Rightarrow (iii): Take $p \in \text{Spec}(R)$ and assume that $pA_p = A_p$. Then $p \supset B_{[\alpha]} = I_{[\alpha]} + I_{[\alpha]} \eta_1 + \dots + I_{[\alpha]} \eta_{d-1}$ and $p \not\supset I_{[\alpha]} \eta_d$ by Lemma 3.1. Take $a \in I_{[\alpha]}$ and put $f(X) = a\varphi_\alpha(X) = aX^d + a\eta_1 X^{d-1} + \dots + a\eta_d$. Since $f(\alpha) = 0$, we get $a\eta_d \in pA$ and hence $I_{[\alpha]} \eta_d \subset pA$. So $J_{[\alpha]} = B_{[\alpha]} + I_{[\alpha]} \eta_d \subset pA$. Since $J_{[\alpha]} = R$, we conclude that $pA = A$. We will show the last part. Since $\sqrt{B_{[\alpha]}} \not\supset I_{[\alpha]} \eta_d$, there exists $q \in \text{Spec}(R)$ such that $q \supset B_{[\alpha]}$ but $q \not\supset I_{[\alpha]} \eta_d$. Thus $qA_q = A_q$ and so $qA = A$. Let m denote the maximal ideal of R . Suppose that $m \supset J_{[\alpha]}$. Then we have $A/mA \simeq (R/m)[T]$, a polynomial ring (cf. Theorem 1.8). Hence $mA \neq A$. But $q \subset m$ implies that $mA = A$, a contradiction. Thus $J_{[\alpha]} = R$. Q.E.D.

REMARK 3.5. Let the notation be the same as in Proposition 3.4.

- (i) When $d = 1$ (i.e., α is an element of K), then (i), (ii) and (iii) of Proposition 3.4 are equivalent.
- (2) $pA \cap R = p$ if and only if there exists $P \in \text{Spec}(A)$ such that $P \cap R = p$.

REMARK 3.6. Let the notation be the same as in Lemma 3.1. If $B_{[\alpha]} \subset q$, then q is either a vanishing point (i.e., $I_{[\alpha]} \eta_d \not\subset q$) or a blowing-up point (i.e., $I_{[\alpha]} \eta_d \subset q$). So if $\sqrt{J_{[\alpha]}}$ contains $\sqrt{B_{[\alpha]}}$ properly, there exists a vanishing point. Thus $\text{Spec}(A) \rightarrow \text{Spec}(R)$ is not surjective.

Proposition 3.7. *Assume that α is an anti-integral element of degree d over R and let $A = R[\alpha]$. Then $\text{Spec}(A) \rightarrow \text{Spec}(R)$ is surjective if and only if $\sqrt{J_{[\alpha]}} = \sqrt{B_{[\alpha]}}$.*

Proof. (\Rightarrow): Since $J_{[\alpha]} \supset B_{[\alpha]}$, $\sqrt{J_{[\alpha]}} \supset \sqrt{B_{[\alpha]}}$. If $B_{[\alpha]} \subset q$ for some $q \in \text{Spec}(R)$, there exists $Q \in \text{Spec}(A)$ such that $Q \cap R = q$ because $\text{Spec}(A) \rightarrow \text{Spec}(R)$ is surjective. So $qA_q \neq A_q$, which means that q is not a vanishing point. Thus by Remark 3.6, q is a blowing-up point, that is, $q \supset J_{[\alpha]}$. Therefore $\sqrt{J_{[\alpha]}} = \sqrt{B_{[\alpha]}}$. (\Leftarrow): Suppose that $\text{Spec}(A) \rightarrow \text{Spec}(R)$ is not surjective. There exists $q \in \text{Spec}(R)$ such that $qA_q = A_q$. So $q \supset \sqrt{B_{[\alpha]}} = \sqrt{J_{[\alpha]}} \supset J_{[\alpha]} \supset I_{[\alpha]} \eta_d$, a contradiction. Q.E.D.

Proposition 3.8. *Let the notation be the same as in Proposition 3.7 and let $p \in \text{Spec}(R)$ satisfy $pA_p = A_p$. If $q \supset pA \cap R$, then q is a blowing-up point.*

Proof. Since $p \in \text{Spec}(R)$ satisfies $pA_p = A_p$, we have $p \supset B_{[\alpha]}$. Thus $\eta_d I_{[\alpha]} \subset \alpha^d I_{[\alpha]} + \cdots + \eta_{d-1} \alpha I_{[\alpha]} \subset B_{[\alpha]} A \subset pA$. So $q \supset pA \cap R \supset B_{[\alpha]} + I_{[\alpha]} \eta_d = J_{[\alpha]}$, which means that q is a blowing-up point. Q.E.D.

REMARK 3.9. Let k be a field, a, b indeterminates and $R = k[a, b]$. Let α be a root of an equation $aX^2 + bX + a = 0$ and put $A = R[\alpha]$. Then $J_{[\alpha]} = (a, b)R$ and $\text{grade}((a, b)R) = 2$ so that α is a super-primitive element by Theorem 1.12. In this case, $J_{[\alpha]} = B_{[\alpha]} = (a, b)R$. Thus $\text{Spec}(A) \rightarrow \text{Spec}(R)$ is surjective, but not flat. Hence the implication (iii) \Rightarrow (i) in Proposition 3.4 does not necessarily hold.

Theorem 3.10. *Assume that α is an anti-integral element over R and let $p \in \text{Spec}(R)$. If $R[\alpha]$ is not a blowing-up at q , then $\text{depth } R[\alpha]_q = \text{depth } R_q$ for $Q \in \text{Spec}(R[\alpha])$ with $Q \cap R = q$.*

Proof. Since α is an anti-integral element over R and q is not a blowing-up point, $R[\alpha]_q$ is flat over R_q by Theorem 1.8. Since $R[\alpha]_q$ is obtained from $R[\alpha]_q$ by localizing at $QR[\alpha]_q$, $R[\alpha]_q$ is flat over Rq . So we have $\text{depth } R_q \leq \text{depth } R[\alpha]_q$. As q is not a blowing-up point, there exists $a \in I_{[\alpha]}$ such that $a\varphi_\alpha(X)R_q[X] = (A_{[\alpha]})_q$. Put $f(X) := a\varphi_\alpha(X)$. Since $Q \in \text{Spec}(R[\alpha])$, there exists $P \in \text{Spec}(R[X])$ such that $P \supset A_{[\alpha]}$ and $Q = P/A_{[\alpha]}$. Then $Q_q = P_q/(A_{[\alpha]})_q = P_q/f(X)R_q[X]$. So $QR[\alpha]_q = PR[X]_p/f(X)R[X]_p$ implies that $\text{depth } R[\alpha]_q = \text{depth } R[X]_p - 1$. Now since $P \cap R = q$, we have $P \supset pR[X]$. Suppose that $P = qR[X]$. Then $qR[X] = P \supset A_{[\alpha]}$, which asserts that q is a blowing-up point. So we have $P \neq qR[X]$. Since $PR_q[X]/qR_q[X] (\subset k(P)[X]) \neq 0$, we have $PR_q[X] = qR_q[X] + g(X)R_q[X]$ for some $g(X) \in R[X]_q R[X]$. Hence $\text{depth } R[X]_p \leq \text{depth } R[X]_q R[X] + 1$. We obtain that $\text{depth } R[\alpha]_q \leq \text{depth } R_q$ because $\text{depth } R[X]_q R[X] = \text{depth } R_q$. Thus $\text{depth } R_q = \text{depth } R[\alpha]_q$. Q.E.D.

4. Unramifiedness and Etaleness of Super-Primitive Extensions

The following result can be proved by using [1, VI (6.8)] but we give a direct proof. If α is super-primitive and integral over R , $R[\alpha]$ is finite, flat over

R (cf. Proposition 1.11).

Proposition 4.1. *Assume that α is an anti-integral element which is integral over R . Then $R[\alpha]$ is unramified over R if and only if $R[\alpha]_{\mathfrak{p}}$ is unramified over $R_{\mathfrak{p}}$ for any $\mathfrak{p} \in D\mathfrak{p}_1(R)$.*

Proof. Since $A := R[\alpha]$ is integral over R , $\varphi_{\omega}(X) \in R[X]$ by Theorem 2.2. For a polynomial f , we denote the derivative of f by f' . Then $\varphi'_{\omega}(\alpha) = d\alpha^{d-1} + (d-1)\eta_1\alpha^{d-2} + \dots + \eta_{d-1}$ and let $\mathfrak{p} \in \text{Spec}(R)$. Then $\varphi'_{\omega}(\alpha)A \not\subset P$ for any $P \in \text{Spec}(A)$ with $P \cap R = \mathfrak{p}$ if and only if $A_{\mathfrak{p}}$ is unramified over $R_{\mathfrak{p}}$ (cf. [1, VI (6.12)]). Suppose that $\varphi'_{\omega}(\alpha)A \neq A$. Then there exists $P \in \text{Ht}_1(A)$ such that $\varphi'_{\omega}(\alpha) \in P$. Put $\mathfrak{p} = P \cap R$. Then $\text{depth } A_{\mathfrak{p}} = 1$ implies $\text{depth } R_{\mathfrak{p}} = 1$ because $A_{\mathfrak{p}}$ is flat over $R_{\mathfrak{p}}$. Thus $A_{\mathfrak{p}}$ is unramified over $R_{\mathfrak{p}}$ by the assumption. Hence $A_{\mathfrak{p}}$ is unramified over $R_{\mathfrak{p}}$, which is a contradiction. So $\varphi'_{\omega}(\alpha)A = A$, which means that A is unramified over R . Q.E.D.

REMARK 4.2. Let the notation be the same as in Proposition 4.1 and its proof. Let $B = A[1/\alpha]$. Then for $P \in \text{Spec}(B)$, $B_{\mathfrak{p}}$ is unramified over $R_{R \cap B}$ if and only if $P \not\supset \varphi'_{\omega}(\alpha)B$. Indeed, let $P \subset B$ be a prime ideal and put $Q = P \cap A$ and $\mathfrak{p} = P \cap R$. When $B_{\mathfrak{p}}/R_{\mathfrak{p}}$ is ramified, $A_{\mathfrak{q}}/R_{\mathfrak{p}}$ is ramified. So $\varphi'_{\omega}(\alpha) \in Q \subset P$. Conversely, if $\varphi'_{\omega}(\alpha) \in P$, then $Q = P \cap A \ni \varphi'_{\omega}(\alpha)$. So $B_{\mathfrak{p}} = A_{\mathfrak{q}}$ is ramified over $R_{\mathfrak{p}}$.

It is known that the purity of branch locus holds for a finite flat extension [1]. The following is a result similar to this fact.

Proposition 4.3. *Assume that α is a super-primitive element which is flat over R and that R contains an infinite field k . Then $R[\alpha]$ is unramified over R if and only if $R[\alpha]_{\mathfrak{p}}$ is unramified over $R_{\mathfrak{p}}$ for any $\mathfrak{p} \in D\mathfrak{p}_1(R)$.*

Proof. We have only to consider the case that R is a local ring. So we may assume that (R, \mathfrak{m}) is a local ring. If $A := R[\alpha]$ is integral over R , we have shown this in Proposition 4.1. Assume that A is not integral over R . Since $\int_{[\omega]} = R$ by Theorem 2.2, replacing α by $\alpha - \lambda$ for some $\lambda \in k$, we may assume by Proposition 1.14, that α satisfies that

- (a) $1/\alpha \in R[\alpha]$,
- (b) $1/\alpha$ is a super-primitive element of degree d over R ,
- (c) $1/\alpha$ is integral over R .

Hence we have

$$R \subset R[1/\alpha] \subset R[\alpha, 1/\alpha] = R[\alpha] = A.$$

Apply Remark 4.2 to $B = R[1/\alpha][(1/\alpha)^{-1}] = A$. We conclude that for $P \in \text{Spec}(A)$, $A_{\mathfrak{p}}$ is unramified over $R_{P \cap R}$ if and only if $P \not\supset \varphi'_{[\omega]}(1/\alpha)A$. In the

same way as in the proof of Proposition 4.1, the assumption that A_p is unramified over R_p for any $p \in Dp_1(R)$ yields that $R[\alpha]$ is unramified over R . Q.E.D.

As a consequence of Propositions 4.1 and 4.3, we obtain the following theorem.

Theorem 4.4. *Assume that α is a super-primitive element over R and that R contains an infinite field k . Then there exist $p_1, \dots, p_t \in Dp_1(R)$ (t may be 0) such that the non-etale locus of $R[\alpha]$ is given by $V(J_{[\alpha]}) \cup \bigcup_{i=1}^t V(p_i)$.*

EXAMPLE 4.5. Let k be a field, a, b indeterminates and $R = k[a, b]$. Let α be a root of an equation $aX^2 + bX + a = 0$ and put $A = R[\alpha]$. Then $J_{[\alpha]} = (a, b)R$. Assume that $p \in \text{Spec}(R)$ and $p \not\supset J_{[\alpha]}$. When $a \notin p$, $(2\alpha + b/c)A_p$ is the ramification locus. When $a \in p$ and $b \notin p$, $(\alpha + 1)A_p$ is the ramification locus.

DEFINITION 4.6. Let A be an extension of R with $[K(A):K] = d$. Define

$$\Delta(A) := \{q \in \text{Spec}(R) \mid \text{rank}_{k(q)} A \otimes_R k(q) = d\}.$$

It is easy to see that when α is a super-primitive element of degree d over R , we have:

$$\begin{aligned} \Delta(R[\alpha]) &\supset Dp_1(R) \\ &\Leftrightarrow R[\alpha] \text{ is integral over } R \\ &\Rightarrow R[\alpha] \text{ is flat over } R. \end{aligned}$$

When A is a finitely generated extension of R , define:

$$Ur(A) := \{p \in \text{Spec}(R) \mid A_p \text{ is unramified over } R_p\},$$

which is an open set of $\text{Spec}(R)$.

Under these preparations, we finally obtain the following.

Theorem 4.7. *Assume that $[L:k] = d$, and that $\alpha_1, \dots, \alpha_n \in L$ are super-primitive elements of degree d , and let $A = R[\alpha_1, \dots, \alpha_n]$. If $\Delta(R[\alpha_i]) \supset Dp_1(R)$ ($1 \leq i \leq n$) and $Ur(R[\alpha_j]) \supset Dp_1(R)$ for some j , then A is integral over R , and A_p is etale over R_p for any $p \in \Delta(A)$. If $\Delta(A) = \text{Spec}(R)$ in addition to the preceding assumptions, then A is integral and etale over R .*

Proof. The assumption $Dp_1(R) \subset \Delta(R[\alpha_i])$ implies that α_i is integral over R and $\Delta(R[\alpha_i]) = \text{Spec}(R)$ by Theorem 2.2, and hence A is integral over R . Take $p \in \Delta(A)$. Then $p \in \Delta(R[\alpha_j])$ and $R[\alpha_j]$ is finite, flat over R as was shown in Theorem 1.8. Thus $R[\alpha_j]_q$ is an R_p -free module of rank d . Since $Ur(R[\alpha_j]) \supset Dp_1(R)$, $R[\alpha_j]$ is unramified over R by Proposition 4.1. Hence $pR[\alpha_j]_p$ is a radical ideal. Noting that A is integral over $R[\alpha_j]$, we have $pA_p \cap R[\alpha_j]_p =$

$pR[\alpha_j]_p$. Thus $R[\alpha_j]_p/pR[\alpha_j]_p \subset A_p/pA_p$. As both of those sides have the same dimension d as vector spaces over $k(p)$, we have $R[\alpha_j]_p/pR[\alpha_j]_p = A_p/pA_p$, which means that $A_p = R[\alpha_j]_p + pA_p$. By Nakayama's lemma, we get $A_p = R[\alpha_j]_p$. Therefore A_p is unramified and flat (i.e., etale) over R_p for any $p \in \Delta(A)$.

Q.E.D.

References

- [1] A. Altman and S. Kleiman: Introduction to Grothendieck duality theory, Lecture Notes in Math. 146, Springer-Verlag, 1977.
- [2] R. Fossum: The Divisor Class Group of a Krull Domain, Springer-Verlag, Berlin, 1973.
- [3] H. Matsumura: Commutative Algebra, Benjamin, New York, 1970.
- [4] H. Matsumura: Commutative Ring Theory, Cambridge Univ. Press, Cambridge, 1986.
- [5] S. Oda and K. Yoshida: *Anti-integral extensions of Noetherian domains*, Kobe J. Math. 5 (1988), 43–56.
- [6] P. Sharma: *A note on ideals in polynomial rings*, Arch. Math. 37 (1981), 325–329.

SUSUMU ODA
 Uji-Yamada High School,
 Uruguchi, Ise, Mie 516 JAPAN

JUNRO SATO and KEN-ICHI YOSHIDA
 Department of Applied Mathematics
 Okayama University of Science
 Ridai-cho, Okayama 700 JAPAN

