Matsuda, M.
Osaka J. Math.
11 (1974), 23-36

INTEGRATION OF ORDINARY DIFFERENTIAL EQUATIONS OF THE FIRST ORDER BY QUADRATURES

Michihiko MATSUDA

(Received December 26, 1972)

Abstract. A differential equation $y'=f(x, y)$ can be solved by quadrature if an infinitesimal transformation $\xi\partial/\partial x+\eta\partial/\partial y$ leaving $y'=f$ invariant is known. This theorem is due to Lie. Here, the converse will be proved in the following form:

Suppose that a one-parameter family of equations $y'=\theta(x, y; a)$ each of which is left invariant by $\xi\partial/\partial x+\eta\partial/\partial y$ is known. Then the equation $\xi dy-\eta dx=0$ can be solved by quadrature.

Through this theorem we shall give a method different from that of Lie for integrating $y'=f(x, y)$ by quadratures.

1. Introduction. Consider a differential equation

$$y'=f(x, y)\,. \tag{1}$$

Suppose that an infinitesimal transformation

$$\xi(x, y)\frac{\partial}{\partial x}+\eta(x, y)\frac{\partial}{\partial y} \tag{2}$$

leaves (1) invariant. Then the Pfaffian form

$$(\eta-f\xi)^{-1}(dy-f\,dx)$$

is exactly integrable. This theorem is due to Lie [2, p.97].

Here, we shall consider an infinitesimal contact transformation leaving (1) invariant. Every infinitesimal contact transformation is expressed in the form

$$-\psi_z\frac{\partial}{\partial x}+(\psi-z\psi_z)\frac{\partial}{\partial y}+(\psi_x+z\psi_y)\frac{\partial}{\partial z}\,, \tag{3}$$

AMS 1970 subject classifications. Primary 34A05.
Key words and phrases. Integration by quadratures, infinitesimal contact transformation, Jacobi's multiplier.

where $z=y'$ and ψ is a function of x, y, z. Equation (1) is left invariant by (3) if and only if $z=f(x,y)$ is a solution of the partial differential equation of the first order

$$(4) \qquad -p\psi_z+(\psi-z\psi_z)q = \psi_x+z\psi_y\,,$$

where $p=\partial z/\partial x$, $q=\partial z/\partial y$. By Jacobi's method of the last multiplier we shall prove the following (Theorem 1):

Suppose that $\lambda(x,y,z)=a$ is an integral of the system of ordinary differential equations

$$(5) \qquad \frac{dx}{-\psi_z} = \frac{dy}{\psi-z\psi_z} = \frac{dz}{\psi_x+z\psi_y}\,.$$

Then the two Pfaffian forms

$$(6) \qquad \psi^{-1}(dy-zdx)\,,$$

$$(7) \qquad (\psi^2\lambda_z)^{-1}\{-\psi_z dy-(\psi-z\psi_z)dx\}$$

are exactly integrable for each value of parameter a. Here, we replace z in (6), (7) by its value $\theta(x,y;a)$ obtained from $\lambda(x,y,z)=a$.

In this theorem take $\psi=\eta-z\xi$, where ξ, η are functions of x, y. Then the infinitesimal transformation (3) is the prolonged one of (2) in the space of line elements, and we have

$$-\psi_z = \xi, \quad \psi-z\psi_z = \eta, \quad \psi_x+z\psi_y = \zeta(z)\,,$$

where

$$\zeta(z) = \eta_x+(\eta_y-\xi_x)z-\xi_y z^2\,.$$

The system (5) becomes

$$(8) \qquad \frac{dx}{\xi} = \frac{dy}{\eta} = \frac{dz}{\zeta(z)}\,,$$

and the Pfaffian form (7) takes on the form

$$(9) \qquad (\psi^2\lambda_z)^{-1}(\xi dy-\eta dx)\,.$$

Since $\lambda=a$ is an integral of (5), $z=\theta(x,y;a)$ is a solution of (4) for every a. Hence we can state the converse of Lie's theorem stated above as follows:

The equation $\xi dy-\eta dx=0$ can be solved by quadrature if a one-parameter family of equations $y'=\theta(x,y;a)$ each of which is left invariant by (2) is known.

Through this theorem let us give a method different from that of Lie for integrating (1) by quadratures. An equation $y'=\theta(x,y)$ is left invariant by (2) if and only if $\theta(x,y)$ is a solution of

$$\xi\frac{\partial\theta}{\partial x}+\eta\frac{\partial\theta}{\partial y}=\zeta(\theta)\,. \tag{10}$$

We try to find such a pair of $\xi(x,y)$, $\eta(x,y)$ that $\eta/\xi=f$ and the equation (10) has a solution of the form $\theta=\theta(f;a)$. Suppose that there exists such a pair of ξ, η that $\eta/\xi=f$ and each of the coefficients of the quadratic form

$$(\xi f_x+\eta f_y)^{-1}\zeta(\theta) \tag{11}$$

is a function of f. Then f is a solution of Riccati's equation

$$\frac{d\theta}{df}=(\xi f_x+\eta f_y)^{-1}\zeta(\theta) \tag{12}$$

derived from (8), since we have the identity

$$\xi\frac{\partial}{\partial x}\left(\frac{\eta}{\xi}\right)+\eta\frac{\partial}{\partial y}\left(\frac{\eta}{\xi}\right)=\zeta\left(\frac{\eta}{\xi}\right).$$

Hence the general solution $\theta(f;a)$ of (12) can be obtained by quadratures. It is a solution of (10) for each a. Let us define the class Ω as all of equations (1) for which we can find such a pair of $\xi(x,y)$, $\eta(x,y)$ that $f=\eta/\xi$ and each of the coefficients of (11) is a function of f. Suppose that equation (1) is a member of Ω and that the pair of ξ, η is given by $\exp(\rho(x,y))$, $f\exp\rho$. Then ρ_x, ρ_y are determined from $z=f(x,y)$ by

$$\begin{cases}\Delta\rho_x=\beta\delta\gamma-\gamma\delta\beta+\beta(B\alpha-A\gamma)\,,\\ \Delta\rho_y=\gamma\delta\alpha-\alpha\delta\gamma-\alpha(B\alpha-A\gamma)\,,\end{cases} \tag{13}$$

where

$$A=\delta\log(p+zq),\quad B=(p+zq)^{-1}p^2\delta\left(\frac{q}{p}\right),\quad C=p^2\delta\left(\frac{q}{p}\right),$$

$$\alpha=p\delta\left\{\frac{q^2}{C}\frac{\partial}{\partial y}\frac{A}{q}\right\},\quad \beta=-q\delta\left\{\frac{p^2}{C}\frac{\partial}{\partial x}\frac{A}{p}\right\},$$

$$\gamma=p\left[\delta\left\{\frac{q^2}{C}\frac{\partial}{\partial y}\frac{B}{q}\right\}-\frac{q}{C}A^2\frac{\partial}{\partial y}\frac{B}{A}\right],\quad \Delta=\alpha\delta\beta-\beta\delta\alpha\,,$$

and δ is the operator $p\partial/\partial y-q\partial/\partial x$. Suppose that $\Delta\neq 0$. Then, integrating the exactly integrable Pfaffian form $\rho_x dx+\rho_y dy$, we have the ρ by quadrature. For this ρ, let $\lambda(f,z)=a$ be the integral of (8) obtained from the general solution $\theta(f;a)$ of (12). Then the Pfaffian form (9) takes on the form

$$-[\exp\{-\rho-\textstyle\int(p+zq)^{-1}(q-\rho_x-z\rho_y)dz\}](dy-fdx)\,. \tag{14}$$

Here the integrand $(p+zq)^{-1}(q-\rho_x-z\rho_y)$ is a function of z. Hence, equation (1) in Ω is solved by quadratures if $\Delta\neq 0$. For defining Ω, we shall give in

Theorem 2 such a system of partial differential equations that equation (1) is a member of Ω if and only if f is a solution of the system.

Let us define the subclass Ω_0 of Ω as all of equations for which we can find such a pair of the ξ, η that $\xi_y=0$. It is a necessary and sufficient condition that Riccati's equation (12) be linear. Equation (1) is a member of Ω_0 if and only if $z=f(x, y)$ satisfies

$$\frac{\partial}{\partial y}\left\{\left(\frac{B}{q}\right)_y \Big/ \left(\frac{A}{q}\right)_y\right\} = 0$$

and

$$\frac{B}{q} - X\frac{A}{q} - X' = 0\,, \tag{15}$$

where $X=\left(\frac{B}{q}\right)_y\Big/\left(\frac{A}{q}\right)_y$. The ρ is determined by $\rho_x=-X$, $\rho_y=0$. Suppose that X is an arbitrary function of x. Then each solution $z=f(x, y)$ of (15) gives a member of Ω_0. The equation (15) is of Monge-Ampère's type, since by the definitions

$$A = (p+zq)^{-1}\{p(s+zt)-q(r+zs)\}\,,$$
$$B = (p+zq)^{-1}(q^2r-2pqs+p^2t)\,,$$

where $r=\partial^2 z/\partial x^2$, $s=\partial^2 z/\partial x\partial y$, $t=\partial^2 z/\partial y^2$. This equation can be solved by Monge's method of integration, and the general solution will be given in a finite form in Theorem 3. In particular, Ω_0 contains the following three equations:

$$y' = X_0(x)Y_0(y); \tag{16}$$

$$y' = X_1(x)+X_2(x)y; \tag{17}$$

$$y = \phi_1(y')x+\phi_2(y') \qquad \text{(Lagrange's type)}. \tag{18}$$

Here, X_0, Y_0, X_1, X_2, ϕ_1, ϕ_2 are arbitary functions.

2. Infinitesimal contact transformation. To prove the first theorem stated in §1, let us recall here Jacobi's method of the last multiplier ([1, p.356]). Consider a system of ordinary differential equations

$$\frac{dx}{P} = \frac{dy}{Q} = \frac{dz}{R}\,, \tag{19}$$

where P, Q, R are functions of x, y, z. Then a function M of x, y, z is called the last multiplier of (19) if it satisfies

$$PM_x+QM_y+RM_z+M(P_x+Q_y+R_z) = 0.$$

Suppose that M is the last multiplier of (19) and $g(x, y, z)=a$ is an integral of

(19) satisfying $g_z \neq 0$. Then the Pfaffian form $g_z^{-1}M(Pdy-Qdx)$ is exactly integrable for each a, where we replace z by its value obtained from $g(x, y, z)=a$. Effecting the quadrature, we have

$$H(x, y; a) = \int g_z^{-1}M(Pdy-Qdx)\,.$$

Suppose that $P\neq 0$ or $Q\neq 0$, and $M\neq 0$. Then the second integral of (19) is given by $H(x, y; g)=b$, and any integral of (19) is expressed in the form

$$\Phi(g(x, y, z),\ H(x, y; g)) = c\,.$$

Theorem 1. *Suppose that* $\psi\neq 0$ *and* $\lambda(x, y, z)=a$ *is an integral of* (5) *satisfying* $\lambda_z\neq 0$. *Then the two Pfaffian forms* (6), (7) *are exactly integrable for each* a.

Proof. Since $\lambda=a$ is an integral of (5),

$$-\psi_z\lambda_x+(\psi-z\psi_z)\lambda_y+(\psi_x+z\psi_y)\lambda_z = 0\,. \tag{20}$$

Consider a system

$$\frac{dx}{\lambda_z} = \frac{dy}{z\lambda_z} = \frac{dz}{-(\lambda_x+z\lambda_y)}\,. \tag{21}$$

Then ψ^{-1} is the last multiplier of (21) by (20). Hence the Pfaffian form (6) is exactly integrable, because $\lambda=a$ is an integral of (21). The function ψ^{-2} is the last multiplier of (5). Hence the Pfaffian form (7) is exactly integrable.

Effecting the quadratures, we have

$$\Sigma(x, y; a) = \int \psi^{-1}(dy-zdx)\,,$$
$$\Pi(x, y; a) = -\int (\lambda_z\psi^2)^{-1}\{\psi_z dy+(\psi-z\psi_z)dx\}\,.$$

Suppose tnat $\psi_z\neq 0$ or $\psi-z\psi_z\neq 0$. Then $\Sigma(x, y; \lambda)=b$ and $\Pi(x, y; \lambda)=c$ give the second integral integral of (21) and (5) respectively.

Proposition 1. *The transformation* $x_1=\Sigma(x, y; \lambda)$, $y_1=\lambda(x, y, z)$, $z_1=\Pi^{-1}(x, y; \lambda)$ *is a contact one, and the infinitesimal transformation* (3) *is written in the form* $\partial/\partial x_1$ *by the coordinate system* (x_1, y_1, z_1).

Proof. By (20) we have

$$-\psi_z\frac{\partial x_1}{\partial x}+(\psi-z\psi_z)\frac{\partial x_1}{\partial y}+(\psi_x+z\psi_y)\frac{\partial x_1}{\partial z} = 1.$$

Hence x_1, y_1, z_1 are functionally independent, and the infinitesimal transformation (3) is written in the form $\partial/\partial x_1$. Since $\Sigma_a=\Pi$, we have

$$\frac{\partial y_1}{\partial z}-z_1\frac{\partial x_1}{\partial z}=0,$$

$$\frac{\partial y_1}{\partial x}+z\frac{\partial y_1}{\partial y}-z_1\left(\frac{\partial x_1}{\partial x}+z\frac{\partial x_1}{\partial y}\right)=0.$$

Hence our transformation is a contact one.

3. Integration of $\xi dy-\eta dx=0$. Suppose that ξ, η are functions of x, y and that $\lambda(x, y, z)=a$ is an integral of (8). Then by Theorem 1 an integrating factor of the Pfaffian equation

$$\xi dy-\eta dx=0 \tag{22}$$

is given by $(\psi^2\lambda_z)^{-1}$. Let us see how it depends on a. Suppose that $\omega(x, y)=b$ is an integral of (22) and $\sigma(x, y)$ is a solution of $\xi\sigma_x+\eta\sigma_y=1$. Then $\omega=b$ is an integral of (8). The second integral of (8) is obtained as follows. Consider Riccati's equation

$$\frac{dz}{d\sigma}=\zeta(z) \tag{23}$$

under the condition that $\omega=b$. Then there exists such a pair of $u(\sigma; b)$, $v(\sigma; b)$ that the general solution of (23) is

$$z=(u+c\xi)^{-1}(v+c\eta).$$

When the quantity b in u, v is replaced by $\omega(x,y)$, the second integral of (8) is given by

$$(\eta-z\xi)^{-1}(uz-v)=c.$$

Let μ denote the left-hand member. Then the integral $\lambda=a$ of (8) is expressed in the form $\Lambda(\omega,\mu)=a$, and the integrating factor $(\psi^2\lambda_z)^{-1}$ of (22) takes on

$$-\left\{\frac{\partial\Lambda}{\partial\mu}(\xi v-\eta u)\right\}^{-1},$$

where we replace μ by its value obtained from $\Lambda(\omega, \mu)=a$.

In the case where $\lambda(x, y, \eta/\xi)$ is not constant, we can obtain the integral of (22) without integrating (9).

Proposition 2. *Suppose that $\lambda(x, y, z)=a$ is an integral of* (8), *and $\lambda(x, y, \eta/\xi)$ is not constant. Then the integral of* (22) *is given by $\lambda(x, y, \eta/\xi)=a$.*

Proof. Let $\omega(x, y)$ denote $\lambda(x, y, \eta/\xi)$. Then,

$$\xi\omega_x+\eta\omega_y=\xi\left\{\lambda_x+\lambda_z\frac{\partial}{\partial x}\frac{\eta}{\xi}\right\}+\eta\left\{\lambda_y+\lambda_z\frac{\partial}{\partial y}\frac{\eta}{\xi}\right\}$$

$$= \xi\lambda_x+\eta\lambda_y+\zeta\left(\frac{\eta}{\xi}\right)\lambda_z = 0\,.$$

Suppose that there exists such a function $w(x, y)$ that each of coefficients of $(\xi w_x+\eta w_y)^{-1}\zeta(z)$ is a function of w. Then we can make an integral $\lambda=a$ of (8) from the generzal solution $z=\theta(w; a)$ of Riccati's equation

$$\frac{dz}{dw} = (\xi w_x+\eta w_y)^{-1}\zeta(z)\,, \tag{24}$$

solving $z=\theta(w; a)$ with respect to a. In this case $\lambda(x, y, \eta/\xi)$ is constant if and only if w is a function of η/ξ.

EXAMPLE 1. Suppose that $\xi=y-x(\log x-1)$, $\eta=-(\log x-1)y$. Then we can take $w=y/x$, and it is functionally independent on η/ξ. Riccati's equation (24) is

$$\frac{dz}{dw} = \frac{1}{w}-\frac{z}{w^2}+\frac{z^2}{w^2}\,,$$

and its general solution is

$$z = w+w^2\{a-\textstyle\int\exp(w^{-1})dw\}^{-1}\exp(w^{-1})\,.$$

Hence,

$$\lambda = \textstyle\int\exp(w^{-1})dw+w^2(z-w)^{-1}\exp(w^{-1})\,,$$

and the integral of

$$ydy-(\log x-1)\,(xdy-ydx) = 0$$

is given by

$$\textstyle\int\exp(x/y)\,d(y/x)+(\log x-1-y/x)\exp\,(x/y) = a\,.$$

4. Integration of equation in Ω. We shall prove the statements on Ω given in §1.

Proposition 3. *Equation* (1) *is a member of Ω if and only if the system of two Monge-Ampère's equations*

$$\begin{cases} q\rho_{xx}-p\rho_{xy}+A\rho_x+B = 0\,, \\ q\rho_{xy}-p\rho_{yy}+A\rho_y = 0 \end{cases} \tag{25}$$

has a solution $\rho(x, y)$.

Proof. First suppose that $\rho_y \neq 0$. Then we have the identities

$$\xi_y^{-1}(\xi p+\eta q) = \rho_y^{-1}(p+zq)\,,$$

$$\xi_y^{-1}(\eta_y-\xi_x)=\rho_y^{-1}(q-\rho_x)+z\,,$$
$$\xi_y^{-1}\eta_x=\rho_y^{-1}(p+zq)-z\rho_y^{-1}(q-\rho_x)\,.$$

All of them are functions of z if and only if ρ is a solution of (25), since

$$\delta\{\rho_y^{-1}(p+zq)\}=\rho_y^{-2}(p+zq)T\,,$$
$$\delta\{\rho_y^{-1}(q-\rho_x)\}=-\rho_y^{-1}\{S+\rho_y^{-1}(q-\rho_x)T\}\,,$$

where S and T are the left-hand member of the first and second equations of (25) respectively. Secondly suppose that $\rho_y=0$. Then we have the identities

$$(\eta_y-\xi_x)^{-1}\eta_x=(q-\rho_x)^{-1}(p+z\rho_x)\,,$$
$$(\eta_y-\xi_x)^{-1}(\xi p+\eta q)=(q-\rho_x)^{-1}(p+z\rho_x)+z\,.$$

They are functions of z if and only if ρ is a solution of

$$q\rho_{xx}+A\rho_x+B=0\,, \tag{26}$$

since

$$\delta\{(q-\rho_x)^{-1}(p+z\rho_x)\}$$
$$=-(q-\rho_x)^{-2}(p+zq)\,(q\rho_{xx}+A\rho_x+B)\,.$$

Proposition 4. *Suppose that equation* (1) *is a member of* Ω *satisfying* $C\neq 0$. *Then* ρ_x, ρ_y *satisfy* (13).

Proof. By the compatibility condition that $\partial S/\partial y-\partial T/\partial x=0$, we have

$$t\rho_{xx}-2s\rho_{xy}+r\rho_{yy}+A_y\rho_x-A_x\rho_y+B_y=0\,. \tag{27}$$

From the definition, $C=q^2r-2pqs+p^2t$. Since $C\neq 0$ by the assumption, we can solve (25), (27) with respect to ρ_{xx}, ρ_{xy}, ρ_{yy}:

$$\begin{cases}\rho_{xx}=-\left\{\dfrac{A}{q}+\dfrac{p^2q}{C}\dfrac{\partial}{\partial y}\dfrac{A}{q}\right\}\rho_x+\left\{\dfrac{p^3}{C}\dfrac{\partial}{\partial x}\dfrac{A}{p}\right\}\rho_y \\ \qquad\quad -\left\{\dfrac{B}{q}+\dfrac{p^2q}{C}\dfrac{\partial}{\partial y}\dfrac{B}{q}\right\}, \\ \rho_{xy}=-\left\{\dfrac{pq^2}{C}\dfrac{\partial}{\partial y}\dfrac{A}{q}\right\}\rho_x+\left\{\dfrac{p^2q}{C}\dfrac{\partial}{\partial x}\dfrac{A}{p}\right\}\rho_y-\dfrac{pq^2}{C}\dfrac{\partial}{\partial y}\dfrac{B}{q}\,, \\ \rho_{yy}=-\left\{\dfrac{q^3}{C}\dfrac{\partial}{\partial y}\dfrac{A}{q}\right\}\rho_x+\left\{\dfrac{A}{p}+\dfrac{pq^2}{C}\dfrac{\partial}{\partial x}\dfrac{A}{p}\right\}\rho_y-\dfrac{q^3}{C}\dfrac{\partial}{\partial y}\dfrac{B}{q}\,.\end{cases} \tag{28}$$

Let E, F, G denote the right-hand member of the first, second and third equations of (28) respectively, and D_x, D_y be the operator defined by

$$D_x = \frac{\partial}{\partial x} + E\frac{\partial}{\partial \rho_x} + F\frac{\partial}{\partial \rho_y},$$

$$D_y = \frac{\partial}{\partial y} + F\frac{\partial}{\partial \rho_x} + G\frac{\partial}{\partial \rho_y}.$$

Then we have the identity

$$[D_x, D_y] = U\left(q^{-1}\frac{\partial}{\partial \rho_x} + p^{-1}\frac{\partial}{\partial \rho_y}\right), \tag{29}$$

where U is a function defined by

$$U = \alpha\rho_x + \beta\rho_y + \gamma. \tag{30}$$

Let H be the operator $pD_y - qD_x$. Then it is written in the form

$$H = -q\frac{\partial}{\partial x} + p\frac{\partial}{\partial y} + (A\rho_x + B)\frac{\partial}{\partial \rho_x} + A\rho_y\frac{\partial}{\partial \rho_y}$$

by the identities

$$qE - pF + A\rho_x + B = 0, \quad qF - pG + A\rho_y = 0. \tag{31}$$

Operating H on U, we have the identity

$$HU - AU = (\delta\alpha)\rho_x + (\delta\beta)\rho_y + \delta\gamma + B\alpha - A\gamma. \tag{32}$$

The two equations $U = HU = 0$ imply (13) by (30), (32).

Proposition 5. *Suppose that* $\lambda(f, z) = a$ *is the integral of* (8) *obtained from the general solution* $\theta(f; a)$ *of* (12). *Then the Pfaffian form* (9) *takes on* (14).

Proof. Riccati's equation (12) is

$$\frac{d\theta}{df} = (p + zq)^{-1}\{p + z\rho_x + (q + z\rho_y - \rho_x)\theta - \rho_y\theta^2\}.$$

Since $f = z$ is a solution, we take $\theta = z + \tau^{-1}$. Then the equation is changed to the linear one

$$\frac{d\tau}{df} = -(p + zq)^{-1}\{(q - z\rho_y - \rho_x)\tau - \rho_y\}.$$

Its general solution is

$$\tau = \exp(-\textstyle\int \nu dz)\{a + \int \rho_y(p + zq)^{-1}\exp(\int \nu dz)dz\},$$

where

$$\nu = (p + zq)^{-1}(q - z\rho_y - \rho_x).$$

Since

$$\psi = e^{\rho}(f-\theta) = -\tau^{-1}e^{\rho}, \quad \lambda_z = \left(\frac{\partial\theta}{\partial a}\right)^{-1} = -\tau^2\left(\frac{\partial\tau}{\partial a}\right)^{-1},$$

the Pfaffian form (9) takes on the form (14).

REMARK 1. (i) Suppose that $p+zq=0$. Then equation (1) is Clairaut's one $y=xy'+\phi(y')$. (ii) Suppose that C$=0$. Then equation (1) is of Lagrange's type (18). It is a member of Ω for which we can take $\rho=0$. (iii) Suppose that $A=0$. Then equation (1) takes on the form

$$y-\int\phi(z)zdz = \phi_0(x-\int\phi dz),$$

where ϕ, ϕ_0 are arbitrary functions of z and $x-\int\phi dz$ respectively. Its integral is obtained by eliminating z from

$$y-\int\phi(z)zdz = \phi_0(b), \quad x-\int\phi dz = b.$$

(iv) Suppose that $(A/p)_x=0$. Then $\delta\{Y^{-1}(p+zq)\}=0$, where $Y=\exp(\int(A/p)dy)$. Hence, $p+zq=\phi(z)Y$, where ϕ is an arbitrary function of z. Its general solution is obtained by elimintaing c from $x-\int z^{-1}dy=\phi_0(c)$, $\int Ydy-\int\phi^{-1}zdz=c$, where ϕ_0 is an arbitrary function of c and we replace z in the first equation by its value obtained from the second equation. This equation $y'=f(x,y)$ is changed to $y_1'=\phi(y_1)Y(x_1)$ by the transformation $x_1=y, y_1=f(x,y)$, since $y_1'=p+zq$.

REMARK 2. The Pfaffian form (14) is exactly integrable if the integrand ν is a function of z. Suppose that $\Delta\neq 0$ and ρ_x, ρ_y are defined by (13). Then, under the condition that $z\alpha-\beta\neq 0$, we have $(\rho_x)_y-(\rho_y)_x=\delta\nu=0$ if and only if ρ_x, ρ_y satisfy (25).

5. Definiing equation of Ω. Let us give a system of partial differential equations for defining Ω.

Theorem 2. *Suppose that $\Delta\neq 0$. Then equation* (1) *is a member of Ω if and only if $z=f(x,y)$ is a solution of the system of two partial differential equations*

$$\begin{aligned}(33)\qquad &\alpha(\beta_x\gamma_y-\beta_y\gamma_x)+\beta(\gamma_x\alpha_y-\gamma_y\alpha_x)+\gamma(\alpha_x\beta_y-\alpha_y\beta_x)\\ &+C^{-1}(p\alpha+q\beta)\Big[q\beta\Big\{\beta\delta\Big(\frac{\gamma}{\beta}\Big)+\varepsilon\Big\}\frac{\partial}{\partial y}\frac{A}{q}\\ &+p\alpha\Big\{\alpha\delta\Big(\frac{\gamma}{\alpha}\Big)+\varepsilon\Big\}\frac{\partial}{\partial x}\frac{A}{p}+q\alpha^2\Big\{\delta\Big(\frac{\beta}{\alpha}\Big)\Big\}\frac{\partial}{\partial y}\frac{B}{q}\Big]\\ &+\alpha(\beta\gamma_y-\beta_y\gamma)\frac{A}{q}+\beta(\gamma\alpha_x-\gamma_x\alpha)\frac{A}{p}+\alpha(\alpha\beta_y-\alpha_y\beta)\frac{B}{q}\end{aligned}$$

$$+\frac{\alpha\beta}{pq}\varepsilon A = 0\,,$$

$$(34)\qquad (\beta\delta\gamma-\gamma\delta\beta+\varepsilon\beta)\delta^2\alpha+(\gamma\delta\alpha-\alpha\delta\gamma-\varepsilon\alpha)\delta^2\beta \\ +(\delta^2\gamma+\delta\varepsilon+B\delta\alpha-A\delta\gamma-A\varepsilon)\Delta = 0\,,$$

where $\varepsilon=B\alpha-A\gamma$.

Proof. We have the identities

$$(35)\qquad D_xU = \left\{\alpha_x-\alpha\frac{A}{q}-\frac{pq}{C}(p\alpha+q\beta)\frac{\partial}{\partial y}\frac{A}{q}\right\}\rho_x \\ +\left\{\beta_x+\frac{p^2}{C}(p\alpha+q\beta)\frac{\partial}{\partial x}\frac{A}{p}\right\}\rho_y \\ +\gamma_x-\alpha\frac{B}{q}-\frac{pq}{C}(p\alpha+q\beta)\frac{\partial}{\partial y}\frac{B}{q}\,,$$

$$(36)\qquad D_yU = \left\{\alpha_y-\frac{q^2}{C}(p\alpha+q\beta)\frac{\partial}{\partial y}\frac{A}{q}\right\}\rho_x \\ +\left\{\beta_y+\beta\frac{A}{p}+\frac{pq}{C}(p\alpha+q\beta)\frac{\partial}{\partial x}\frac{A}{p}\right\}\rho_y \\ +\gamma_y-\frac{q^2}{C}(p\alpha+q\beta)\frac{\partial}{\partial y}\frac{B}{q}\,,$$

and

$$(37)\qquad (H-A)^2U = (\delta^2\alpha)\rho_x+(\delta^2\beta)\rho_y \\ +\delta(\delta\gamma+\varepsilon)+B\delta\alpha-A(\delta\gamma+\varepsilon)$$

by (32). Suppose that equation (1) is a member of Ω. Then, $D_xU=D_yU=U=0$ imply (33) by (35), (36), and $H^2U=HU=U=0$ imply (34) by (32), (37). Suppose conversely that the two identities (33), (34) are satisfied by $z=f(x, y)$. Then D_xU, D_yU and H^2U are linearly dependent on U and HU, since we assumed that $\Delta\neq 0$. Let us replace ρ_x, ρ_y by their values defined by (13):

$$(38)\qquad \rho_x = \Delta^{-1}(\beta\delta\gamma-\gamma\delta\beta+\varepsilon\beta),\quad \rho_y = \Delta^{-1}(\gamma\delta\alpha-\alpha\delta\gamma-\varepsilon\alpha)\,.$$

Then, $U=HU=0$, and $D_xU=D_yU=H^2U=0$. Hence,

$$(39)\qquad \alpha\{(\rho_x)_x-E\}+\beta\{(\rho_y)_x-F\} = U_x-D_xU = 0\,,$$

$$(40)\qquad \alpha\{(\rho_x)_y-F\}+\beta\{(\rho_y)_y-G\} = U_y-D_yU = 0\,.$$

By (29) and the identity

$$[D_x, H] = -sD_x+rD_y+p[D_x, D_y]\,,$$

we have

$$D_xHU=[D_x, H]U+HD_xU=0\,,$$

since $HD_xU=0$. Hence,

$$D_yHU=p^{-1}(H+qD_x)HU=0\,,$$

and

$$\text{(41)}\qquad \begin{aligned}&(\delta\alpha)\{(\rho_x)_x-E\}+(\delta\beta)\{(\rho_y)_x-F\}\\ &=\{(H-A)U\}_x-D_x(H-A)U=0\,,\end{aligned}$$

$$\text{(42)}\qquad \begin{aligned}&(\delta\alpha)\{(\rho_x)_y-F\}+(\delta\beta)\{(\rho_y)_y-G\}\\ &=\{(H-A)U\}_y-D_y(H-A)U=0\,.\end{aligned}$$

By (39), (41) we have

$$(\rho_x)_x=E,\quad (\rho_y)_x=F\,,$$

and by (40), (42),

$$(\rho_x)_y=F,\quad (\rho_y)_y=G\,.$$

Hence, we can integrate (38), and the ρ thus obtained satisfies (28). By (31), ρ is a solution of (25). Therefore, by Proposition 3, equation (1) is a member of Ω.

REMARK 3. Suppose that $\Delta=0$. Then equation (1) is a member of Ω if and only if $z=f(x, y)$ is a solution of the two equations

$$\beta(\delta\gamma+\varepsilon)-\gamma\delta\beta=\gamma\delta\alpha-\alpha(\delta\gamma+\varepsilon)=0\,.$$

REMARK 4. Let Z_1, Z_2 denote

$$(p+zq)^{-1}\rho_y,\quad (p+zq)^{-1}(q-\rho_x)$$

respectively. Then equation (1) is a member of Ω if and only if Z_1, Z_2 are functions of z. We have $\rho_x=q-(p+zq)Z_2$, $\rho_y=(p+zq)Z_1$. Hence, Ω is defined by Monge-Ampère's equation

$$\begin{aligned}&Z_1r+(Z_1z+Z_2)s+(Z_2z-1)t\\ &\quad+(p+zq)(Z_1'p+Z_2'q)+q(Z_1p+Z_2q)=0\end{aligned}$$

involving two arbitrary functions Z_1, Z_2 of z as parameters, which is the compatibility condition that $(\rho_y)_x=(\rho_x)_y$. This equation is the intermediate integral of the second order of the system of partial differential equations (33), (34) of the fifth and sixth order.

6. General solution of defining equation of Ω_0. We shall determine the form of equation (1) contained in Ω_0, solving its defining equation. By Proposition 3, equation (1) is a member of Ω_0 if and only if the equation (26) has a solution ρ depending only on x. Suppose that equation (1) is a member of Ω_0. Then ρ_x is determined by

$$-\rho_x = \left(\frac{B}{q}\right)_y \Big/ \left(\frac{A}{q}\right)_y.$$

Let X be the right-hand member. Then we have (15). Conversely suppose that X is an arbitrary function of x. Then each solution of (15) gives a member of Ω_0, for which $\rho_x = -X$, $\rho_y = 0$.

Theorem 3. *The general solution of Monge-Ampère's equation* (15) *is obtained by eliminating* c *from*

$$y - \int X^{-1}\phi(z)dz = \psi(c) \tag{43}$$

and

$$c = \int X dx - \int z^{-1}(\phi - 1)dz, \tag{44}$$

where ϕ *and* ψ *are arbitrary functions of* z *and* c *respectively. Here, we replace* x *in the integrand* $X^{-1}\phi$ *in* (43) *by its value obtained from* (44).

Proof. The equation (15) takes on the form

$$\begin{aligned} q(X-q)r + \{(zq-p)X + 2pq\}s - p(zX+p)t \\ -q(p+zq)X' = 0. \end{aligned} \tag{45}$$

To this equation, Monge's method of integration can be applied with success as follows. One of the two characteristics of (45) is

$$pdx - qdy = dz = (X-q)dp + (p+zX)dq - (p+zq)X'dx = 0.$$

The last equation is written in the form

$$z^{-1}\{(p+zX)d(p+zq) - (p+zq)d(p+zX)\} = 0$$

by $dz=0$. Hence, the two functionally independent intermediate integrals of the first order are given by $(p+zq)^{-1}(p+zX)$ and z. Therefore, the integration of (45) is reduced to that of the partial differential equation of first order

$$p + zX - (p+zq)\phi(z) = 0 \tag{46}$$

involving an arbitrary function ϕ of z. The characteristic of (46) is

$$\frac{dx}{\phi - 1} = \frac{dy}{z\phi} = \frac{dz}{zX}.$$

Hence, the general solution of (45) is expressed in the form stated in our theorem.

EXAMPLE 2. In the intermediate integral (46) let us replace ϕ or X by special values. (i) Take $\phi=0$. Then $p+zX=0$. Its general solution is $z=\exp(-\int Xdx+Y(y))$, and equation (1) is of type (16). (ii) Take $\phi=1$. Then $X=q$. Its general solution is $z=Xy+X_1(x)$, and equation (1) is of type (17). (iii) Take $X=0$. Then $pq^{-1}=-z(\phi-1)^{-1}\phi$. Its general solution is $y-\phi_1(z)x=\phi_2(z)$, where $\phi_1=z(\phi-1)^{-1}\phi$. Equation (1) is of Lagrange's type (18).

OSAKA UNIVERSITY

Bibliography

[1] A. R. Forsyth: A treatise on differential equations, 6th ed., Macmillan and Co., London, 1933.

[2] S. Lie and G. Scheffers: Vorlesungen über Differentialgleichungen, mit bekannten infinitesimalen Transformationen, Teubner, Leipzig, 1891; Chelsea, New York, 1967.