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ACYCLIC ALGEBRAIC SURFACES
BOUNDED BY SEIFERT SPHERES

S. Yu. OREVKO

(Received November 13, 1995)

Let \( Y \) be a complex algebraic surface. We say that it is \( \mathbb{Z} \)-acyclic (respectively \( \mathbb{Q} \)-acyclic) if its reduced homology with coefficients in \( \mathbb{Z} \) (resp. in \( \mathbb{Q} \)) vanishes. Topologically one can represent \( Y \) as a compact 4-manifold with boundary (denote the boundary by \( S \)), attached by a collar \( S \times [0, 1) \). Call \( S \) the boundary of \( Y \). If \( Y \) is an affine surface in \( \mathbb{C}^m \) then \( S \) is the intersection of \( Y \) with a sufficiently large sphere. We say that \( Y \) is \( A \)-acyclic at infinity if \( S \) is an \( A \)-homology 3-sphere. (\( A = \mathbb{Z}, \mathbb{Q} \)). If \( Y \) is \( A \)-acyclic then it is \( A \)-acyclic at infinity. If \( Y \) is \( \mathbb{Q} \)-acyclic and \( \mathbb{Z} \)-acyclic at infinity, then it is \( \mathbb{Z} \)-acyclic.

In the paper [18] Ramanujam proved that the only \( \mathbb{Z} \)-acyclic surface bounded by a homotopy 3-sphere is \( \mathbb{C}^2 \), and he also constructed there the first example of a non-trivial \( \mathbb{Z} \)-acyclic (and even contractible) surface. Later on Gurjar and Shastri [7] proved that all \( \mathbb{Z} \)-acyclic surfaces are rational. Tom Dieck and Petri [1] classified all algebraic surfaces which rise out of line configurations on \( \mathbb{P}^2 \). Fujita [5] (resp. Miyanishi, Tsunoda [11] and Gurjar, Miyanishi [6]) classified surfaces with \( \kappa = 0 \) (resp. \( \infty \) and 1), where \( \kappa \) denotes the log-Kodaira dimension. Zaidenberg [21] pointed out the connection of \( \mathbb{Z} \)-acyclic surfaces with exotic algebraic and analytic structures on \( \mathbb{C}^n \), \( n \geq 3 \). Flenner and Zaidenberg [4] studied deformations of acyclic surfaces.

A Seifert fibration (see [19], [17]) on a smooth compact 3-manifold \( M \) is a mapping onto a 2-manifold \( \pi : M \to B \), which is a locally trivial fibration with fiber \( S^1 \) over \( B - \{ p_1, \ldots, p_r \} \) and which looks near \( p_j \) like \( D^2 \times S^1 \to D^2, (z_1, z_2) \mapsto z_1^{\mu_j} z_2^{\nu_j} \), where \( D^2 = \{ \|z\| < 1 \} \subset \mathbb{C}, S^1 = \partial D^2 \) and \( \mu_j, \nu_j \) are coprime integers, \( \mu_j \geq 2 \). The \( \pi^{-1}(p_j) \) are called multiple fibers; \( M \) is called Seifert manifold if it admits a Seifert fibration. Seifert \( A \)-homology sphere (\( A \) stands for \( \mathbb{Z} \) or \( \mathbb{Q} \)) is a Seifert manifold \( M \) with \( H_*(M; A) = H_*(S^3; A) \). In this case the base \( B \) is a 2-sphere. The question, when a Seifert homology sphere bounds an acyclic 4-manifold, was studied, for instance, in [3], [15].

Our main result is:

**Theorem 1.** Let \( Y \) be a smooth algebraic \( \mathbb{Q} \)-acyclic surface of logarithmic Kodaira dimension 2, bounded by a Seifert \( \mathbb{Q} \)-homology sphere with \( r \) multiple fibers.
Then:
(a) $Y$ cannot be $Z$-acyclic.
(b) $r \leq 16$.

Let $Y$ be a $Q$-acyclic surface. Consider an algebraic compactification $X$ of $Y$ such that $Y = X - D$, where $D$ is a reduced curve with simple normal crossings (an SNC-curve). Then all irreducible components of $D$ are rational, and the dual weighted graph of $D$ (denote it by $\Gamma_D$) is a tree (see [13]). (The dual graph of a curve is the weighted graph, whose vertices correspond to the irreducible components, edges correspond to their intersection points and the weight of a vertex is the self-intersection number.) A tree is called $r$-fork if it has one vertex of valence $r$ and valences are $\leq 2$. Suppose that $D$ is minimal, i.e. it contains no $(-1)$-curve intersecting one or two others. A $Q$-acyclic surface $Y$ with $\kappa(Y) = 2$ is bounded by a Seifert sphere if and only if $\Gamma_D$ (with minimal $D$) is a fork.\(^1\) Thus, we can reduce Theorems 1 to:

**Theorem 1'**. Let $D$ be a minimal SNC-curve on a smooth projective surface $X$. Suppose that $Y = X - D$ is $Q$-acyclic, $\kappa(Y) = 2$ and the dual graph $\Gamma_D$ is an $r$-fork. Then:
(a) $Y$ cannot be $Z$-acyclic.
(b) $r \leq 16$.

**Remark 1.** As we mentioned above, acyclic surfaces with $\kappa < 2$ are classified [5], [11], [6]. Using this classification and the classification of Seifert homology spheres [17], one can see that if $Y$ is a $Z$-acyclic surface which is bounded either by a Seifert sphere or by a fork, then $Y = C^2$. If $Y$ is $Q$-acyclic and $\kappa(Y) < 2$ then all the possible values for $r$ are shown in the following table:

<table>
<thead>
<tr>
<th>$\kappa(Y)$</th>
<th>$-\infty$</th>
<th>0</th>
<th>1</th>
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<tr>
<td>$\partial Y$ is a Seifert sphere with $r$ mult. fibers</td>
<td>${0,1,2,3}$</td>
<td>${3,4,5}$</td>
<td>${4,5,\ldots}$</td>
</tr>
<tr>
<td>$\Gamma_D$ is an $r$-fork</td>
<td>${0,1,\ldots}$</td>
<td>${3}$</td>
<td>$\emptyset$</td>
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</tbody>
</table>

This fact can be easily deduced from the results in [5], [6] and [10]. Note only that the cases with $\kappa = 0, 1$ and $r \geq 4$ correspond to the surfaces $X - D$ with $\Gamma_D$ of the form $\ldots \rightarrow 0 \rightarrow \ldots \leftarrow 0 \ldots$. Such a surface is bounded by a Seifert sphere because $\Gamma_D$ becomes a fork after a 0-absorption (see [2], [14]).

\(^1\)It is so, because when $\kappa = 2$, the tree $\Gamma_D$ satisfies so called Negative Chains Condition: If the valence of a vertex is $\leq 2$ then its weight is $\leq -2$. When $\kappa < 2$, the both assertions "if" and "only if" are wrong.
Remark 2. Zaidenberg asked [22; Question 1.6] if there is only a finite list of possibilities for the topological type of the dual graph at infinity of an acyclic (resp. contractible) surface with $\bar{k} = 2$. Theorem 1' can be considered as a very first step toward the positive answer to this question.

Remark 3. The proof of the part (b) of Theorem 1' is based on the logarithmic Bogomolov-Miyaoka-Yau (log-BMY) inequality [12], strengthened by Kobayashi-Nakamura-Sakai [9], and Fujita’s computation [5] of the Zariski decomposition of $K + D$. The part (a) also can be obtained as a direct consequence of the elementary formulas from §§1-3 (most of them needed for the part (b)) using the rationality of $\mathbb{Z}$-acyclic surfaces [7] and the log-BMY inequality\(^2\). However, these two results are quite non-trivial, while, as the referee of the first version of the paper has pointed out,

"...a very elementary proof is possible. Using Lemma 4.1 in part I of [7], we can show: Write $K_X \sim a_0D_0 + \sum_{i \geq 1} a_iD_i$ where $D_0$ is the central curve. Then $a_0 \geq 0 \implies$ all $a_i \geq 0$ and $a_0 < 0 \implies$ all $a_i < 0$. But if all $a_i \leq 0$, then $p_g(X) > 0$. This is not possible. Hence all $a_i < 0$. But then $K + D$ is either trivial or a strictly negative divisor. In the latter case, $\bar{k}(Y) = -\infty$. If $K + D \sim 0$, then $(K + D) \cdot D_0 = -2 + r = 0 \implies r = 2$. Hence $\Gamma_D$ is linear. This completes the proof."

In fact, only the implication "$a_0 \leq 0 \implies$ all $a_i \leq 0$" is proven in [7, Lemma 4.1]. However, the proof can be easily completed to derive the implication "$a_0 < 0 \implies$ all $a_i < 0$" as well. Indeed, if $a_0 < 0$ then by [7, (4.1)] all $a_i \leq 0$. If some of them were $= 0$ then (due to connectedness of $D$) would exist two components $D_i$ and $D_j$ such that $a_i = 0, a_j \neq 0$ and $D_i \cdot D_j = 1$. Then, since $D_i^2 + 2 \leq 0$, one would have $0 = g(D_i) = KD_i + D_i^2 + 2 \leq KD_i = a_j + \sum_{k \neq i,j} a_kD_kD_i \leq a_j < 0$.

Remark 4. After the old proof of Theorem 1'(a) was omitted, the propositions 1.4-1.6 remained without applications. However, we decided to leave them because they are simple but maybe they are of some independent interest.

Remark 5. The estimate $r \leq 16$ in Theorem 1', requires messy calculations (see §8). However, the fact that $r$ is bounded from above, can be obtained without them. Therefore, we presented in §7 a shorter proof of Theorem 1' with a weaker estimate for $r$.

Remark 6. The estimate $r \leq 16$ still does not seem to be the best possible. However, a stronger estimate needs other techniques, because an attempt to prove it by the methods of this paper leads to so huge volume of calculations that the result does not worth them.

\(^2\)see the preliminary version of this paper in "Mathematica Gottingensis", 38 (1995).
I am grateful to M.G. Zaidenberg for introducing me to the subject and useful discussions. I am grateful to the referee for pointing out the elementary proof of Theorem 1(a) and other useful remarks. This work was done during my stay at the University Bordeaux-I (supported by the program “PAST”) and partially in Göttingen Mathematical Institute (supported by SFB-170). I thank these institutions for the hospitality and the financial support.

1. Weighted trees and their discriminants

We list in this section some well-known elementary facts about discriminants of weighted trees. A **weighted tree** is a finite tree (finite graph without cycles) with an integer weight $w(v)$ assigned to each vertex $v$. Let $\Gamma$ be a weighted tree and $v_1, \ldots, v_n$ be its vertices. The **incidence matrix** of $\Gamma$ is $A_\Gamma = (a_{ij})$, where

$$a_{ij} = \begin{cases} w(v_i) & \text{if } i = j, \\ 1 & \text{if } v_i \text{ is connected to } v_j \text{ by an edge,} \\ 0 & \text{otherwise.} \end{cases}$$

The **discriminant** of $\Gamma$ is defined as $d(\Gamma) = \det(-A_\Gamma)$. By convention, $d(\emptyset) = 1$. Clearly, this definition is independent of the order of the vertices and that the discriminant of a disjoint union is the product of the discriminants of the connected components.

The following lemma can be easily obtained, using the Cramer rule (see e.g. [2] for details).

**Lemma 1.1.** Let $\Gamma$ be a weighted tree with $d(\Gamma) \neq 0$. Let $B_\Gamma = (b_{ij}) = A_\Gamma^{-1}$ be the inverse matrix. Then

$$b_{ij} = -d(\Gamma - [v_i, v_j])/d(\Gamma),$$

where $[v_i, v_j]$ is the minimal connected subgraph of $\Gamma$, which contains $v_i$ and $v_j$.

**Lemma 1.2.** Let $\Gamma$ be a weighted tree, $v$ a vertex of $\Gamma$ and $w(v)$ the weight of $v$. Denote by $\Gamma_1, \ldots, \Gamma_r$ the connected components of $\Gamma - v$, and let $\Gamma'_j = \Gamma_j - v_j$, $j = 1, \ldots, r$, where $v_j$ is the vertex of $\Gamma_j$, connected by an edge to $v$. Then (remind that $d(\emptyset) = 1$)

$$d(\Gamma) = -w(v) \prod_{j=1}^r d(\Gamma_j) - \sum_{j=1}^r \left( d(\Gamma'_j) \prod_{k \neq j} d(\Gamma_k) \right).$$

**Proof.** Expand the determinant of $A_\Gamma$ according to the row, corresponding to $v$. \qed
The valence of a vertex $v$ of a graph is the number of edges, incident to $v$. A graph is called a linear chain if its vertices $v_1, \ldots, v_n$ can be orders so, that $v_i$ is connected to $v_j$ iff $|i - j| = 1$.

**Corollary 1.3.** Let $T$ be a linear chain with all weights $\leq -2$.

a) If $v$ is one of the ends of $T$ then $d(T) > d(T - v) > 0$.

b) Let $u$ be any vertex of $T$. Denote by $T_1$ and $T_2$ the connected components of $T - u$, and let $a = d(T), b = d(T_1), c = d(T_2)$. Then $a \geq b + c$.

Proof. a) Induction by the number of vertices, using Lemma 1.2.

b) For $i = 1, 2$ let $u_i$ be the vertex of $T_i$, nearest to $u$, and $T'_i = T_i - u_i$. Put $b' = d(T'_1), c' = d(T'_2)$ (if $T'_1 = \emptyset$, put $b' = 0$). Let $w$ be the weight of $u$. Then by Lemma 1.2 we have $a = -wbc - b'c - b'c = (-w - 2)bc + (c - c') + c(b - b') \geq b + c$, because $-w - 2 \geq 0$, and by (a), $c - c' \geq 1$, $b - b' \geq 1$. □

The following three propositions will not be used in the rest of the paper.

**Proposition 1.4.** Let $\Gamma$ be a weighted tree; $u$ and $v$ two its vertices. Let $A_0, \ldots, A_k$ be the connected components of $\Gamma - u$, and $B_0, \ldots, B_m$ be those of $\Gamma - v$, indexed in such a way that $v \in A_0, u \in B_0$. Denote: $a_i = d(A_i), b_i = d(B_i), a = a_1 \cdots a_k, b = b_1 \cdots b_m, \Delta = d(\Gamma), \delta = d(A_0 \cap B_0), c = d((A_0 \cap B_0) - [u, v])$. Suppose that $a \neq 0, b \neq 0, \delta \neq 0$. Then

\[(27) \delta \Delta = a_0b_0 - abc^2.\]

Proof. Let $M$ be the minor of $A_\Gamma$ obtained by deleting the two rows and the two columns, corresponding to $u$ and $v$. Clearly, $M = \delta ab$. On the other hand, by Jacobi formula for the minor of the inverse matrix,

\[
\frac{\Delta}{M} = \begin{vmatrix} b_{uu} & b_{uv} \\ b_{vu} & b_{vv} \end{vmatrix},
\]

where, by Lemma 1.1, $b_{uu} = aa_0/\Delta, b_{uv} = b_{vu} = abc/\Delta, b_{vv} = bb_0/\Delta$. □

**Remarks.** 1. If $\gamma$ is a linear chain and $d(\Gamma) = \pm 1$ then (1) is the formula for the "edgede determinant" due Eisenbud-Neumann.

2. In fact, (1) is still true even if any of its ingredients are zeros.

A tree $\Gamma$ is called $r$-fork, if it contains a vertex $v_0$ of valence $r$ and the valences of other vertices are $\leq 2$.

**Proposition 1.5.** Let $\Gamma$ be an $r$-fork and $v_0$ the vertex of valence $r$. Suppose
that the weights of the other vertices are \( \leq -2 \). Let \( Q_\Gamma \) be the quadratic form, defined by \( A_\Gamma \). Then:

(i) if \( d(\Gamma) > 0 \) then \( Q_\Gamma \) is negatively definite;
(ii) if \( d(\Gamma) < 0 \) then \( Q_\Gamma \) has the signature \((+, -, \ldots, -)\).

Proof. Apply the Sylvester criterium, choosing an increasing sequence \( 1 = M_0, M_1, \ldots, M_n = d(\Gamma) \) of principal minors of the matrix \(-A\). where \( M_{n-1} \) is obtained from \( M_n \) by deleting the row and the column, which correspond to \( v_0 \). It follows from Corollary 1.3, that \( M_i > 0 \) for \( i < n \).

**Proposition 1.6.** In the hypothesis of Proposition 1.5 if \( d(\Gamma) = -1 \) then all the entries \( b_{ij} \) of \( B_\Gamma = A_\Gamma^{-1} \) are non-negative.

Proof. Denote by \( T_1, \ldots, T_r \) the connected components of \( \Gamma - v_0 \), and by \( v_j \) the end vertex of \( T_j \) (the vertex of \( T_j \), whose valence in \( \Gamma \) is 1). Denote also: \( \Gamma'_j = \Gamma - v_j \), \( T'_j = T - v_j \), \( \Delta'_j = d(\Gamma'_j) \), \( a_j = d(T_j) \), \( a'_j = d(T'_j) \), \( e_j = a'_j/a_j \) \((j = 1, \ldots, r)\), and \( p = a_1 \cdots a_r \).

By Lemma 1.1 it is enough to show that the descriminant of any connected proper (i.e. \( \neq \Gamma \)) subgraph of \( \Gamma \) is non-negative. First, we prove this for the subgraphs \( \Gamma'_j \). Indeed, applying 1.4 with \( u = v_0 \) and \( v = v_j \), we obtain \( a'_j \cdot (-1) = \Delta'_j a_j - p/a_j \), or, dividing by \( a_j \), \( \Delta'_j = p/a_j^2 - e_j \). But \( p/a_j^2 > 0 \) and \( e_j < 1 \). Hence, \( \Delta'_j > -1 \), but \( \Delta'_j \in \mathbb{Z} \), so, \( \Delta'_j \geq 0 \).

Let \( \Gamma'' \) be any proper connected subgraph of \( \Gamma \). It is contained in some \( \Gamma'_j \). Chose an increasing sequence of principal minors which involves \( d(\Gamma'') \) as well as \( d(\Gamma'_j) \), and estimate the signature of \( Q_\Gamma \), by Sylvester criterium. Clearly, the inequality \( d(\Gamma'') < 0 \) contradicts Proposition 1.5.

2. Some elementary linear algebra on dual graphs

Let \( X \) be a smooth projective algebraic surface and \( D \) a reduced SNC-curve on \( X \). Denote by \( V_D \) the subspace of \( H^2(X; \mathbb{Q}) \) generated by the irreducible components \( D_1, \ldots, D_n \) of \( D \). We shall call elements of \( V_D \) by \( Q \)-divisors.

Denote by \( A_D = (D_i \cdot D_j)_{ij} \) the intersection matrix of \( D \). Let \( \Gamma_D \) be the dual weighted graph of \( D \). Clearly that \( A_D \) is the incidence matrix (see §1) of \( \Gamma_D \). Define the **discriminant** of \( D \) as \( d(D) = d(\Gamma_D) := \det(-A_D) \).

Suppose that \( d(D) \neq 0 \) (in particular \( D_i \)'s are linearly independent), and let \( B_D = A_D^{-1} \).

**Lemma 2.1.** For \( C_1, C_2 \in V_D \) one has \( C_1 \cdot C_2 = \sum_{i,j} b_{ij}(C_1 \cdot D_i) \cdot (C_2 \cdot D_j) \)

Proof. Any bilinear form defines a homomorphism to the dual space. One can interpret \( A_D \) as the matrix of that for the intersection form. Then the required
equality is just $C_1 \cdot C_2 = (A_D C_1, C_2) = (Z_1, B_D Z_2)$ for $Z_k = A_D C_k, k = 1, 2$.

Let $K_X$ be the canonical class of $V$ and let $K = K_D$ be its orthogonal projection onto $V_D$. Actually, for the main purpose of this paper we need only the case, when $V_D = \text{Pic} X \otimes \mathbb{Q}$, and hence $K_D = K_X$ (it is so if $X - D$ is $\mathbb{Q}$-acyclic). However, this assumption does not simplify the statements (nor the proofs), in this and next §§, so we do not restrict ourselves by this case here.

For an irreducible component $C$ of $D$ denote by $\nu_D(C)$ its valence in $\Gamma_D$, i.e. $\nu_D(C) = C \cdot (D - C)$, and put $\nu_i = \nu_D(D_i)$. Let $\chi_i$ be the Euler characteristic of $D_i$.

**Lemma 2.2.** $(K + D) \cdot D_i = \nu_i - \chi_i$.

**Proof.** Apply adjunction formula: $D_i \cdot (K + D) = D_i \cdot (K + D_i) + \nu_i = \nu_i - \chi_i$. 

**Corollary 2.3** (cf. [16]). $(K + D)^2 = \sum i, j b_{ij} (\nu_i - \chi_i)(\nu_j - \chi_j)$.

Following Fujita [5], define a **twig** of $D$ as a maximal linear rational branch. It means that $T$ is a twig, if $T = C_1 \cup \cdots \cup C_k$, where each $C_i$ is a rational irreducible component of $D$; $\nu_D(C_k) = 1; \nu_D(C_i) = 2$ and $C_i \cdot C_{i+1} = 1$ for $1 \leq i < k$; and if we denote by $C_0$ the component of $D - T$, which intersects $C_1$, then either $C_0$ is not rational or $\nu_D(C_0) \neq 2$. In this case $C_0$ is called the **root** of the twig $T$ (it is not contained in $T$); $C_k$ is called the **tip** of $T$. The rational number $d(T - C_k)/d(T)$ is called **inductance** of $T$ and is denoted by $e(T)$ (we use the convention: $d(\emptyset) = 1, e(\emptyset) = 0$). The twig is called **admissible** if $C_i^2 < -1$ for all $i = 1, \ldots, k$. Clearly, that if a twig $T$ is admissible then $d(T) > 0$ and $0 < e(T) < 1$ (see Corollary 1.3)

For a twig $T$ of $D$ with $d(T) \neq 0$ we define the **bark** of $T$ (see [5]) as the unique $\mathbb{Q}$-divisor $Bk(T)$ in $V_T$ (i.e. $\text{Supp}(Bk(T)) \subset T$), such that $Bk(T) \cdot \text{tip}(T) = -1, Bk(T) \cdot C = 0$ for a component $C$ of $T$, which is not the tip. The following lemma is an immediate consequence of Lemmas 1.1 and 2.1, applied to the matrix $B_T$.

**Lemma 2.4** (Fujita, [5, (6.16)]). Let $T$ be a twig of $D$, and $d(T) \neq 0$. Then

(i) $Bk(T)^2 = -e(T)$.

(ii) If $C$ is a vertex of a twig $T$ then the coefficient of $C$ in $Bk(T)$ is equal to $d(T_C)/d(T)$, where $T_C$ is the connected component of $T - C$ which is between $C$ and the root of $T$.

(iii) In particular, if $C$ is the vertex, nearest to the root, then the coefficient of $C$ is equal to $1/d(T)$.
3. Local Zariski-Fujita decomposition

Let, as in §2, D be an SNC-curve on a smooth projective algebraic surface X, K = KD be the projection of KX onto VD, and suppose that D is not a linear chain of rational components, and that all the twigs of D are admissible.

In this case we define the local Zariski-Fujita decomposition of K + D near D as K + D = H + N, where N = ND is the sum of the barks of all the twigs of D. The Q-divisors H = HD and ND are called respectively positive and negative parts of K + D near D. From Lemma 2.2 and the definition of bark we obtain immediately the following properties of the local Zariski-Fujita decomposition:

Lemma 3.1 (Fujita, [5, (6.12)]).
(i) K + D = H + N, where H, N ∈ VD;
(ii) Supp(N) is contained in the union of all twigs of D;
(iii) H is orthogonal to each irreducible component of N.

Remark. It is proved in [5] (we do not use this here), that H and N are uniquely defined by the conditions (i)–(iii) in Lemma 3.1. Fujita has also proved (see [5, (6.20–6.24)]) that under certain conditions Zariski decomposition of K + D coincides with the local one (see Theorem 5.2 below). Even if this is not the case, it is much more convenient to calculate separately H^2 and N^2 in order to calculate (K + D)^2 in terms of discriminants of subgraphs (i.e. via the inverse matrix BD = AD^{-1}).

Denote by br(D) the set of all irreducible components C of D which have either positive genus or VD(C) > 2, and put

\[ h_i = \begin{cases} 
\nu_i - \chi_i - \sum \frac{1}{d(T)} & \text{for } i \in \text{br}(D) \\
0 & \text{otherwise}
\end{cases} \]

where T runs through all twigs, rooted by Di.

Lemma 3.2. If all the twigs of D are admissible, then HD · Di = hi for any i.

Proof. By Lemma 2.2 we have (K + D) · Di = νi - χi. By Lemma 2.4(iii) and the definition of bark we have

\[ ND · Di = \begin{cases} 
\sum \frac{1}{d(T)} & \text{for } i \in \text{br}(D) \\
2 - \nu_i & \text{otherwise.}
\end{cases} \]

It remains to subtract the latter equality from the former one. \[ \Box \]
Corollary 3.3 [16]. If all the twigs of $D$ are admissible, then $H_D^2 = \sum_{i,j \in \text{br}(D)} b_{ij} h_i h_j$.

Proof. Apply Lemmas 2.1 and 3.2

4. The formulas from §§2, 3 for the case of a fork

Let $D$ be a rational $r$-fork on a smooth projective algebraic surface $X$. This means that $D$ is an SNC-curve with rational components, and the dual graph of $D$ is an $r$-fork. Introduce the following notation. Denote by $D_0, \ldots, D_n$ the irreducible components of $D$ and by $\nu_i = \nu(D_i)$ their valences. Without loss of generality we may assume that $\nu_0 = r$ (and hence, $\nu_i \leq 2$ for $i > 0$). Let $T_1, \ldots, T_r$ be the twigs of $D$, i.e the connected components of $D - D_0$, and $d_1, \ldots, d_r$ their discriminants. For $i = 1, \ldots, n$ put

$$a_i = d_j, \quad b_i = d(T^+_j, i), \quad c_i = d(T^-_j, i),$$

where $T_j$ is the twig containing $D_j$ and $T^+_j$ (resp., $T^-_j$) is the connected components of $T_j - D_0$, which does not intersect (resp., does intersect) the “central” curve $D_0$ (see Fig. 1). Extend this notation for $i = 0$, putting $a_0 = b_0 = 1, c_0 = 0$.

![Fig. 1.](image)

Let $V_D$ be the $Q$-vector space generated by $D_0, \ldots, D_n$. Denote by $V_j, j = 1, \ldots, r$ the subspace of $V_D$ generated by the irreducible components of $T_j$, and let $V_H$ be the orthogonal complement of $\bigoplus_{j=1}^r V_j$. Denote by $pr_1, \ldots, pr_r$ and $pr_H$ the orthogonal projections onto $V_1, \ldots, V_r$ and $V_H$ respectively. Let $K + D = H + N$ be the local Fujita decomposition of $K + D$ near $D$. Since $V_H$ is one-dimensional, it is generated by $H$ unless $H = 0$. Let $N_j = Bk(T_j)$ (clearly, that $pr_j(N) = N_j$, $pr_H(N) = 0$ and $N = \sum N_j$). Denote:

$$p = \prod_{j=1}^r d_j; \quad \Delta = d(D); \quad h = r - 2 - \sum_{j=1}^r \frac{1}{d_j}; \quad \varepsilon = -ph/\Delta.$$

Lemma 4.1. Let $C$ be a $Q$-divisor in $V_D$. Put $x_i = C \cdot D_i, i = 0, \ldots, r$ and
\( C_H = \text{pr}_H(C). \) Then

(a) \( H^2 = \varepsilon h; \)
(b) \( C \cdot H = \varepsilon \sum_{i=0}^{n} \frac{b_i}{a_i}; \)
(c) \( C \cdot N = \sum_{i=0}^{n} \frac{c_i}{a_i}; \)
(d) \( C \cdot D = \sum_{i=1}^{n} x_i; \)
(e) \( C \cdot K = \sum_{i=0}^{n} x_i \left( \frac{c_i + \varepsilon b_i}{a_i} - 1 \right); \)
(f) \( C_H^2 = \frac{(C \cdot H^2)}{\varepsilon h} = \frac{\varepsilon}{h} \left( \sum_{i=0}^{n} \frac{b_i}{a_i} \right)^2. \)

Proof. (a) is an immediate consequences of Corollary 3.3. By Lemma 1.1 the entry \( b_{0i} \) of the matrix \( B_D \) is equal to \(-(b_i \cdot (p/a_i))\Delta. \) Thus, (b) follows from Lemmas 2.1 and 3.2. (c) follows from Lemma 2.4(ii); (d) is trivial; (e) follows from (b,c,d) since \( K = H + N - D; \) (f) follows from (b) and (a).

**Corollary 4.2.** If \( r \geq 4 \) and all twigs of \( D \) are admissible then there exists no smooth rational \((-1\)-curve \( C \) on \( X \) such that \( C \cdot D = 1 \) and \( C \not\subseteq D. \)

Proof. Suppose that such a curve \( C \) exists. Then \( C \cdot K = -1 \) and \( C \cdot D = 1 \) implies that for some \( i \) we have \( x_i = 1, \) \( x_k = 0 \) for \( k \neq i. \) Hence, by Lemma 4.1(e) we have \(-1 = C \cdot K = (c_i + \varepsilon b_i)/a_i - 1. \) But if \( r > 3 \) then \( \varepsilon > 0. \) Contradiction.

### 5. Zariski decomposition and refined log-BMY inequality

Let \( D \) be an SNC-curve on a smooth projective surface \( X, \) and \( Y = X - D. \) Remind the following definition (see e.g. [5], [8]). If \( \kappa(Y) \geq 0, \) then there exists the *Zariski decomposition* \( K + D = H + N, \) where \( H, N \) are \( \mathbb{Q} \)-divisors in \( X \) such that

(i) the intersection form is negatively definite on the subspace \( V_N \) generated by the irreducible components of \( N \) (in particular, \( N^2 \leq 0; \))

(ii) \( HC \geq 0 \) for any complete irreducible curve \( C \subset X; \)

(iii) \( H \) is orthogonal to \( V_N \) (and hence, \( (K + D)^2 = H^2 + N^2). \)

The main tool, used in the proof of Theorem 1', is the following refined version of the log-BMY inequality.

**Theorem 5.1** Kobayashi-Nakamura-Sakai [9]. \( \) If \( \overline{\kappa}(Y) = 2, \) then \( H^2 \leq 3e(Y), \) where \( e \) is the topological Euler characteristic.

The following theorem is a partial case of [5, (6.20)].
Theorem 5.2 (Fujita). Let $Y = X - D$ be a smooth projective surface with $\kappa(Y) \geq 0$ and $D$ a connected SNC-curve on it. Suppose that all twigs of $D$ are admissible and $D$ is neither a linear rational chain, nor a 3-fork. Then the (global) Zariski decomposition of $(K + D)$ coincides with the local Zariski-Fujita decomposition near $D$ unless there exists a smooth rational $(-1)$-curve $C$ on $X$, which is not contained in $D$ and which satisfies one of the following conditions.

(i) $D \cdot C = 0$, i.e. $D \cap C = \emptyset$.

(ii) $T \cdot C = 1$ for some twig $T$ of $D$.

Corollary 5.3. Let $Y = X - D$ be a $Q$-acyclic surface with $\kappa(Y) = 2$, and $D$ be a minimal rational $r$-fork with $r > 4$. Then Zariski decomposition of $K + D$ coincides with its local Zariski-Fujita decomposition near $D$.

Proof. Let $C$ be some smooth rational $(-1)$-curve on $X$. Since $\kappa(X) = 2$, according to [5, (6.13)], all the twigs are admissible, so, according to the Theorem 5.2 it suffices to check that $C$ does not satisfies (i), (ii) of 5.2. The condition (i) evidently contradicts to $H_2(Y) = 0$. The condition (ii) contradicts Corollary 4.2.

6. Beginning of the proof of Theorem 1'

Let $D$ be a minimal SNC-curve on smooth projective $X$, such that $\Gamma_D$ is a $r$-fork with $r \geq 4$, $Y = X - D$ is a $Q$-acyclic surface and $\kappa(Y) = 2$. Introduce the notation as in §4. Since $\kappa(Y) = 2$, it follows from [5, (6.13)], that all twigs are admissible, so, all $a_i, b_i, c_i$ are positive for $i > 0$.

Lemma 6.1. $r \leq 2h + 4$.

Proof. By (2), $h = r - 2 - 1/d_1 - \cdots - 1/d_r \geq r - 2 - 1/2 - \cdots - 1/2 = (r/2) - 2$.

Due to the refined log-BMY inequality (Theorem 5.1) and Corollary 5.3, we have (see Lemma 4.1(a))

$$\varepsilon h \leq 3.$$

Thus, by Lemma 6.1 we must estimate $h$ from above, or, equivalently, $\varepsilon$ from below.

Lemma 6.2. If $D_0^2 \leq 0$ then $h < (3 + \sqrt{33})/2 \approx 4.3722$.

Proof. Denote: $d_j = d(T_j)$, $d'_j = d(T'_j)$, $j = 1, \ldots, r$, where $T'_j$ is obtained from the twig $T_j$ by deleting the component, nearest to $D_0$. Then, by Lemma 1.2,
if \( D_0^2 \leq 0 \), we have

\[
-\Delta = p \cdot \left( D_0^2 + \sum_{j=1}^{r} \frac{d_j}{d_j} \right) \leq p \cdot \left( 0 + \sum_{j=1}^{r} \frac{d_j - 1}{d_j} \right) = p \cdot (h + 2).
\]

Thus, (3) implies \( 3 \geq h \varepsilon = -ph^2/\Delta \geq h^2/(h + 2) \), hence \( h^2 - 3h - 6 \leq 0 \).

**Corollary 6.3.** If \( r > 12 \) then \( X \) is rational.

**Proof.** If \( r > 12 \) then by 6.1 and 6.2 we have \( D_0^2 > 0 \). Hence, [20; Ch. II, §4, Theorem 2] implies that \( X \) is rational. \( \square \)

From now on we suppose that \( r > 12 \), hence by 6.3, \( X \) is rational, and there exists a smooth rational \((-1)\)-curve \( C \) on \( X \). Hence,

\( C^2 = -1; \quad C \cdot K = 1 \)

Like in Lemma 4.1, put \( x_i = C \cdot D_i, i = 0, \ldots, n \) and \( C_H = \text{pr}_H(C) \). Put also \( C_j = \text{pr}_j(C), j = 1, \ldots, r, C_N = \sum_{j=1}^{r} D_j \). By Lemma 6.2, \( C \neq D_0 \), and from minimality of \( D \) we know that \( C \neq D_i, i > 0 \). So, \( C \notin D \), hence, all \( x_i \) are \( \geq 0 \).

**Lemma 6.4.** \(-C_N^2 \geq CN\).

**Proof.** Let \( I_j = \{ i \mid D_i \subset T_j \} \). Then by Lemma 2.1 and lemma 4.1(c)

\[
-C_j^2 = \sum_{i \in I_j} x_i^2 \frac{c_i b_i}{a_i} + 2 \sum_{i, k \in I_j, i < k} x_i x_k \frac{c_i c_k b_i}{a_i} \geq \sum_{i \in I_j} x_i^2 \frac{c_i b_i}{a_i} \geq \sum_{i \in I_j} x_i \frac{c_i}{a_i} = CN_j.
\]

\( \square \)

**Lemma 6.5.** If \( C \cdot D > 2 \) then \( h \leq (9 + \sqrt{21})/2 \approx 6.7912 \ldots \)

**Proof.** By Corollary 1.3(b) we have \( b_i/a_i + c_i/a_i \leq 1 \), hence, by Lemma 4.1(b,c,d), \((CH)/\varepsilon + CN \leq CD\). Therefore, by (4),

\[
1 = -CK = -CH - CN + CD \geq -CH + \frac{CH}{\varepsilon} = CH \frac{1 - \varepsilon}{\varepsilon}.
\]

Thus, \( CH \leq \varepsilon/(1 - \varepsilon) \), hence, by Lemma 4.1(f), \( C_H^2 \leq \varepsilon/((1 - \varepsilon)^2 h) \), and by (4) and Lemma 6.4, \( 2 = -C^2 - CK = (C_N^2 - CN) - (C_H^2 + CH) + CD \geq CD - \varepsilon_1 \), where

\[
\varepsilon_1 = \frac{\varepsilon}{1 - \varepsilon} \left( 1 + \frac{1}{(1 - \varepsilon)h} \right).
\]
Since $CD$ is integer, $CD > 2$ implies $\varepsilon > 1$, hence $2\varepsilon^2 - (3 + 1/h)\varepsilon + 1 \geq 0$, hence $\varepsilon \geq 1/4h (3h + 1 - \sqrt{h^2 + 6h + 1})$, and by (3) it implies $h^2 - 9h + 15 \leq 0$. 

7. **Proof of Theorem 1’ with a weaker estimate**

Let all the notation be like in §§4, 6, but in this section we shall suppose, that $CD = 2$. Let $i$ and $k$ be such indices that $CD_i + CD_k = CD = 2$. Thus, if $i = k$ then $x_i = 2$, $x_l = 0$ for $l \neq i$, and if $i \neq k$ then $x_i = x_k = 1$, $x_l = 0$ for $l \neq i, k$. In any case we rewrite the last two formulas of Lemma 4.1 as

\[
(CK) = \left(\frac{c_i}{a_i} + \frac{c_k}{a_k}\right) + \varepsilon \left(\frac{b_i}{a_i} + \frac{b_k}{a_k}\right) - 2; \quad (C_H^2) = \frac{\varepsilon}{h} \left(\frac{b_i}{a_i} + \frac{b_k}{a_k}\right)^2.
\]

Denote by $Q_{ik}$ “the predicate of belonging $D_i$ and $D_k$ to the same twig”, i.e. $Q_{ik} = 1$ if $D_i \cup D_k \subset T_j$ for some $j$, and $Q_{ik} = 0$ otherwise. When $Q_{ik} = 1$, without loss of generality we can assume that $D_i$ is between $D_0$ and $D_k$. In this notation we have

\[
-C_N^2 = \frac{b_ic_i}{a_i} + \frac{b_kc_k}{a_k} + 2Q_{ik} \frac{c_ib_k}{a_i}.
\]

Using (5), (6) and the fact that $C^2 = C_H^2 + C_N^2$, we rewrite (4) as

\[
\left(\frac{c_i}{a_i} + \frac{c_k}{a_k}\right) + \varepsilon \left(\frac{b_i}{a_i} + \frac{b_k}{a_k}\right) = 1,
\]

\[
\left(\frac{b_ic_i}{a_i} + \frac{b_kc_k}{a_k}\right) + 2Q_{ik} \frac{c_ib_k}{a_i} - \varepsilon \frac{b_i}{a_i} \frac{b_k}{a_k} = 1.
\]

**Lemma 7.1.** Suppose that one of the following conditions holds:

(i) $x_0 > 0$;  
(ii) $x_0 = 0$ (i.e. $i \neq 0$ and $k \neq 0$) and $b_i \geq 2, b_k \geq 2$. Then there exists a constant $A_1$ such that $h > A_1$.

**Proof.** In the case (i) without loss of generality we suppose that $k = 0$, and, putting $a_k = b_k = 1, c_k = Q_{ik} = 0$, into (8), and using $c_i/a_i < 1$, we see that $b_i > 1$, hence, $b_i \geq 2$. Thus, in the both cases (i) and (ii) we have $(c_i/a_i) \cdot (b_i - 2) \geq 0$ for $\nu = i, k$. Hence, subtracting (7) multiplied by 2 from (8), we obtain

\[
\frac{\varepsilon}{h} u^2 + 2\varepsilon u - 1 = \sum_{\nu=i,k} \frac{c_i}{a_i} \cdot (b_i - 2) + 2Q_{ik} \frac{c_ib_k}{a_i} < 0., \quad \text{where} \quad u = \frac{b_i}{a_i} + \frac{b_k}{a_k}.
\]

Since $u < 2$ and $\varepsilon \leq 3/h$, we see that $h$ can not be arbitrary big. 

\[\square\]
Lemma 7.2. If \( x_0 = 0 \) (i.e. \( i \neq 0 \) and \( k \neq 0 \)), \( b_k = 1 \) and \( Q_{ik} = 1 \) then \( h < (3 + \sqrt{21})/2 \approx 3.791 \ldots \)

Proof. Putting \( b_k = Q_{ik} = 1 \), \( a_i = a_k = a \) into (7) and (8), subtracting (7) from (8) and multiplying the result by \( a/(b_i+1) \), we see that \( c_i - \varepsilon - (\varepsilon/h) \cdot (1+b_i)/a = 0 \). Hence, using the estimates \( c_i \geq 1 \) and \((b_i+1)/a \leq 1 \), we get \( 1 - \varepsilon - (\varepsilon/h) \leq 0 \), and applying (3), we obtain \( h^2 - 3h - 3 < 0 \).

Lemma 7.3. Let \( Q_{ik} = 0 \) and \( b_k = 1 \). Then \( b_i \geq 2 \).

Proof. If \( b_i = 1 \), then subtracting (7) from (8) we would obtain \( \varepsilon = 0 \).

Lemma 7.4. If \( x_0 = 0 \) (i.e. \( k \neq 0 \) and \( i \neq 0 \)), \( b_k = 1 \) and \( Q_{ik} = 0 \) then there exists a constant \( A_2 \) such that \( h < A_2 \).

Proof. Putting \( b_k = 1 \), \( Q_{ik} = 0 \) into (7) and (8), subtracting (7) from (8) and multiplying the result by \( a_i \), we see that

\[
b_i c_i - c_i = \left( b_i + \frac{a_i}{a_k} \right) \varepsilon_1, \quad \text{where} \quad \varepsilon_1 = \varepsilon \cdot \left( 1 + \frac{1}{h} \left( \frac{b_i}{a_i} + 1 \right) \right) = O(\varepsilon)
\]

or, equivalently,

\[
\frac{a_i}{a_k} = \frac{b_i c_i - c_i}{\varepsilon_1} - b_i.
\]

On the other hand, applying the estimate \( c_k \leq a_k - 1 \) (see 1.3(a)) to (7), putting \( b_i = 1 \) and multiplying the obtained inequality by \( a_i \), we see that

\[
c_i + \varepsilon b_i \geq \frac{a_i}{a_k} (1 - \varepsilon).
\]

Substituting (9) into (10), we obtain \((1 - \varepsilon) b_i c_i \leq \varepsilon_1 b_i + (1 + \varepsilon_1 - \varepsilon) c_i \). Replacing \( b_i \) with \( b' + 1 \), this inequality can be transformed into \((b' - \varepsilon_2)(c_i - \varepsilon_3) \leq \varepsilon_4 \) where \( \varepsilon_2, \varepsilon_3 \) and \( \varepsilon_4 \) are \( O(\varepsilon) \). Since \( b' \geq 1 \) (by 7.3) and \( c_i \geq 1 \), we see that \( \varepsilon \) can not be arbitrary small.

Proposition 7.5. Under the hypothesis of Theorem 1' one has \( r \leq 30 \).

Proof. Lemmas 6.2-7.4 imply \( h < \max(A_1, A_2) \). Easy to see that these constants can be chosen to be less than 13 1/2. Hence, by 6.1 we have \( r \leq 2h + 4 < 31 \).
8. More precise estimates for the case $C \cdot D = 2$

In this and the next section we are going to prove Theorem 1' in full volume (with the estimate $r \leq 16$). To this end we strengthen here the estimates for $h$ given in §7. Thus, let $C$ be a smooth rational $(-1)$-curve on $X$, where $X - D$ is a $Q$-acyclic surface with $\overline{\kappa} = 2$, and $CD = 2$. Let the notation be like in §§4, 6, 7. Denote also $h + (1/a_i) + (1/a_k)$ by $h^+$. We shall need the following evident identity:

\[
 b(x - y)^2 = (x^2 + y^2)b + xy ((b - 1)^2 - b^2 - 1) = (y - bx)(by - x) + xy(b - 1)^2.
\]

Lemma 8.1. Let $k \neq 0$, $Q_{ik} = 0$, $b_k = 1$ and $h^+ \geq 7 1/2$. Then $h^+ = 8$, $b_i = 5$, $c_i = 1$, $c_i = a_k - 1$, $a_i = 5a_k - 1$ and $a_k = 2, 3$ or 4.

Proof. Denote $a_k - c_k$ by $c'_k$. Putting $Q_{ik} = 0$, $b_k = 1$, $c_k = a_k - c'_k$ into (7), (8) and resolving the obtained simultaneous equations with respect to $\varepsilon$ and $h$, we see that

\[
 \varepsilon = \frac{c'_ka_i - c_ia_k}{a_i + b_ia_k}, \quad h = \frac{(c'_ka_i - c_ia_k)(a_ka_k + a_i)}{a_ia_ku},
\]

where $u = c_ib_ia_k - c'_ka_i > 0$. Hence,

\[
 h^+ = (b_i - 1)(c_i + c'_k)/u;
\]

\[
 3 \geq \varepsilon h = \frac{(c'_ka_i - c_ia_k)^2}{a_ia_ku} = \frac{b_i(c'_ka_i - c_ia_k)^2}{b_ia_ia_ka_ku} \quad \text{by (3), (12)}
\]

\[
 = \frac{(c_ia_k - c'_kb_ia_i)u + c_ia'_ka_ia_k(b_i - 1)^2}{b_ia_ia_ku} \quad \text{by (11)}
\]

\[
 = \frac{c_i}{b_ia_i} - \frac{c'_k}{a_k} + \frac{c_ia'_ka_k(b_i - 1)^2}{b_iu} \quad \text{omitted $\frac{c_i}{b_ia_i}$}
\]

\[
 > - \frac{c'_k}{c'_k + 1} + \frac{c_ia'_ka_k(b_i - 1)^2}{b_iu} \quad \text{use $a_k \geq c'_k + 1$}
\]

\[
 u > \frac{c_ia'_k(c'_k + 1)(b_i - 1)^2}{(4c'_k + 3)b_i} \quad \text{by (16)}
\]

\[
 h^+ < \frac{(c_i + c'_k)(4c'_k + 3)b_i}{c_i'a_k(c'_k + 1)(b_i - 1)} \quad \text{by (13), (17), 7.3}
\]
Denote the right hand side of (18) by \( \eta^+(b_i) = \eta^+_{c_i,c'_k}(b_i) \). Easy to check that \( \eta^+ \) is decreasing with respect to each variable when \( b_i \geq 2, c_i \geq 1, c'_k \geq 1 \).

In the Table 1 we show the values of \( c_i, c'_k, b_i \), for which \( \eta^+(b_i) \leq 7^{1/2} \) and hence, the inequality \( h^+ < 7^{1/2} \) follows from (18).

<table>
<thead>
<tr>
<th>Table 1.</th>
<th>Table 2.</th>
</tr>
</thead>
<tbody>
<tr>
<td>( c_i = 1 \quad c_i = 2 \quad c_i \geq 3 \quad c_i \geq 14 )</td>
<td>( c_i = 1 \quad c_i = 2 \quad c_i \leq 6 )</td>
</tr>
<tr>
<td>( c'_k = 1 \quad b_i \geq 15 ) ( b_i \geq 4 ) ( b_i \geq 3 ) ( b_i \geq 2 )</td>
<td>( c'_k = 1 \quad b_i \leq 4 ) ( b_i \leq 3 ) ( b_i = 2 )</td>
</tr>
<tr>
<td>( c'_k = 2 \quad b_i \geq 4 ) ( b_i \geq 2 ) ( b_i \geq 2 ) ( b_i \geq 2 )</td>
<td>( c'_k = 2 \quad b_i \leq 3 )</td>
</tr>
<tr>
<td>( c'_k \geq 3 \quad b_i \geq 3 ) ( b_i \geq 2 ) ( b_i \geq 2 ) ( b_i \geq 2 )</td>
<td>( c'_k \leq 6 \quad b_i = 2 )</td>
</tr>
</tbody>
</table>

To see this, it is enough to verify that

\[
\eta^+_{1,1}(15) = 7^{1/2}, \quad \eta^+_{2,1}(4) = 7, \quad \eta^+_{3,1}(3) = 7, \quad \eta^+_{4,1}(2) = 7^{1/2},
\eta^+_{1,2}(4) = 7^{1/2}, \quad \eta^+_{2,2}(2) = 7^{1/2},
\eta^+_{3,2}(3) = 7^{1/2}.
\]

In the Table 2 we show the values of \( c_i, c'_k \), for which the inequality \( h^+ < 7^{1/2} \) follows from (13), using the evident estimate \( u \geq 1 \).

Comparing the two table (note that \( b_i \geq 2 \) by 7.3) shows that the only cases which are not covered by them, are:

\[
7 \leq c_i \leq 13, c'_k = 1, b_i = 2; \quad c_i = 1, c'_k \geq 7, b_i = 2; \quad c_i = c'_k = 1, 5 \leq b_i \leq 14.
\]

Consider these three cases separately:

Case 1 \((7 \leq c_i \leq 13, c'_k = 1 \text{ and } b_i = 2)\). It follows from (17) that \( u > c_i/7 \geq 1 \). Hence, \( u \geq 2 \) and (13) implies \( h^+ \leq (c_i + 1)/u \leq (13 + 1)/2 = 7 \).

Case 2 \((c_i = 1, c'_k \geq 7 \text{ and } b_i = 2)\).

Subcase 2.1 \((c'_k = 7)\). Suppose that \( u = 1 \). Then by definition of \( u \) we have

\[
(45) \quad 2a_k - 7a_i = 1.
\]

We know that \( a_i \geq b_i + 1 = 3 \). If \( a_i \) were equal to \( 3 \), then by (19) one would have \( a_k = 11 \), and hence, (14) would imply \( 3 \geq \varepsilon h = 100/33 \). Therefore, \( a_i > 3 \), but \( a_i \) is odd by (19), hence, \( a_i \geq 5 \). Thus, by (19) we have \( a_k = (7a_i + 1)/2 \geq 18 \). Hence, (14) implies \( 3 \geq 1/2a_i - 7/a_k + 7/2 > -7/a_k + 7/2 \geq -7/18 + 7/2 > 3 \).

The obtained contradiction shows that \( u \geq 2 \). Hence, (13) implies \( h^+ < 8/u \leq 4 \).

Subcase 2.2 \((c'_k \geq 8)\). It follows from (15) that \( 3 > -(c'_k/a_k) + (c'_k/2u) > -1 + (c'_k/2u) \). Hence, \( u > c'_k/8 \geq 1 \). Subtracting (14) multiplied by 2 from (13), we see that \( h^+ - 6 \leq 1/u - 1/a_i + 2c'_ka_k \). But \( 0 < u = 2a_k - c'_ka_i \) implies \( 2c'_k/a_k < 4/a_i \), hence, \( h^+ - 6 < 1/u + 3/a_i \leq 1/2 + 3/3 \).
Case 3 \((c_i = 1, c'_k = 1 \text{ and } 5 \leq b_i \leq 14)\). By (17) we have \(u \geq 2 / 7 (b_i - 1)^2 / b_i > 2 / 7 (b_i - 2)\). Hence, \(b_i < (7 u + 4) / 2\) and this implies

\[
(46) \quad b_i \leq \begin{cases} 
(7 u + 2) / 2 & \text{if } u \text{ is even} \\
(7 u + 3) / 2 & \text{if } u \text{ is odd.}
\end{cases}
\]

Thus, for \(u > 1\) by (13) we have \(h^+ = 2 (b_i - 1) / u \leq 7.1 / 3\).

Suppose that \(u = 1\). Then (20) implies \(b_i = 5\). By (15) we obtain \(3 > -1(1 / a_k) + 16 / 5\). Since \(a_k \geq 2\), we have only three solutions: \(a_k = 2, 3, 4\). For them \(a_i = b_i a_k - u = 5 a_k - 1\), and by (13) we have \(h^+ = 2 (b_i - 1) / u = 8\). This is the only case when \(h^+ \geq 7.1 / 2\).

**Lemma 8.2.** Let \(k = 0\) and \(h^+ \geq 8\). Then \(h^+ = 8\) and \((a_i, b_i, c_i) = (13, 2, 7)\).

Proof. The proof is similar to that of Lemma 8.1. The beginning of the proof of 8.1 including the formulas (12), (13), (14) and (15) is valid in the case \(k = 0\) without changes. However, the implication (15) \(\Rightarrow\) (16) does not work in this case.

Since we have \(a_k = b_k = c'_k = 1\), let us denote \(a_i, b_i\) and \(c_i\) simply by \(a, b\) and \(c\) till the end of the proof. Then \(u = bc - a\).

First, note that \(c > 1\) because otherwise \(u\) would be negative. Eliminating \(u\) from (13) and (15), we see that

\[
(47) \quad h^+ < \eta^+(b, c), \quad \text{where } \eta^+ = \frac{4 (c + 1) b}{c (b - 1)}.
\]

**Corollary 8.3.** (a) Under the hypothesis of Lemma 8.1 the graph \(\Gamma_D\) has one of the following forms:

- \(a_i = 9, a_k = 2: \quad \begin{array}{c}
\text{case 1}
\end{array}\)
- \(a_i = 14, a_k = 3: \quad \begin{array}{c}
\text{case 2}
\end{array}\)
- \(a_i = 19, a_k = 4: \quad \begin{array}{c}
\text{case 3}
\end{array}\)

(b) Under the hypothesis of Lemma 8.2 the graph \(\Gamma_D\) has the form:
Lemma 8.4. Let \( b_i \geq b_k \geq 2 \). Then

\[
(48) \quad k < \left( \frac{b_k}{a_k} + \frac{b_i}{a_i} \right) \cdot \left( \frac{2}{b_k} + \frac{3b_k}{q} \right), \quad \text{where} \quad q = (b_k - 1) + (b_i - b_k) \frac{c_k}{a_i}.
\]

Proof. Denote \((b_i/a_i) + (b_k/a_k)\) by \(u\). Multiplying (7) by \(b_k\), subtracting the result from (8) and using the estimate \(Q_{i,k,c_i}b_k/a_i \geq 0\), we obtain the inequality \((\varepsilon/h)u^2 + b_k \varepsilon u - q \geq 0\), where \(q\) denotes the same as in (22). Therefore, we have

\[
u \geq \frac{h b_k}{2} \left( -1 + \sqrt{1 + \frac{4q}{\varepsilon h b_k^2}} \right) \geq \frac{h b_k}{2} \left( -1 + \sqrt{1 + \frac{4q}{3b_k^2}} \right) = \frac{h b_k}{2} \left( -1 + \sqrt{1 + v} \right)
\]

where \(v = 4q/(3b_k^2)\). It remains to apply the evident estimate

\[-1 + \sqrt{1 + v} = -1 + (1 + v)/\sqrt{1 + v} > -1 + (1 + v)/(1 + (v/2)) = v/(2 + v) \square.
\]

Lemma 8.5. Let \( b_i \geq b_k \geq 10 \). Then \(h < 6.41/55 \approx 6.745 \cdots\).

Proof. Applying the estimates \((b_k/a_k) + (b_i/a_i) < (b_k/(b_k+1)) + 1\) and \(q \geq b_k - 1\) to the inequality (22), we see that \(h < f(b_k)\) where

\[
f(b) = \left( 1 + \frac{b}{b+1} \right) \cdot \left( \frac{2}{b} + \frac{3b}{b-1} \right) = \frac{6}{b} + \frac{3}{b-1} + \frac{2}{b+1} + \frac{3}{b^2 - 1}.
\]

\(f\) decreases when \(b > 1\). Hence, \(h < f(b_k) \leq f(10) = 6.41/55 \square\).

Lemma 8.6. Let \( b_i \geq b_k \geq 2 \). Suppose also that \( b_k \leq 9 \) and \( a_k \geq 20 \). Then \(h \leq 5.113/120\).

Proof. Case 1 \((3 \leq b_k \leq 9)\). Apply to (22) the estimates \(b_i/a_i < 1\), \(a_k \geq 20\) and \(q \geq b_k - 1\). We obtain the inequality

\[
h < f(b_k), \quad \text{where} \quad f(b) = \left( 1 + \frac{b}{2b-2} \right) \left( \frac{2}{b} + \frac{3b}{b-1} \right).
\]

Direct calculation shows that \(f(b) \leq 5.113/120\) for \(b = 3, 4, \ldots, 9\).

Case 2 \((b_k = 2)\). Substituting \(b_k = 2\) into (22) and applying the estimates \(a_k \geq 20\), \(c_i \geq 1\), we obtain \(h < f(a_o, b_i)\) where

\[
f(a, b) = \left( \frac{1}{10} + \frac{b}{a} \right) \left( 1 + \frac{6a}{a + b - 2} \right) \quad \text{and} \quad \frac{\partial f}{\partial b} = \frac{5b^2 + \gamma_1 b + \gamma_2}{5a(a + b - 2)^2}, \quad \gamma_1 = 10a - 20, \quad \gamma_2 = 32a^2 - 8a + 20.
\]
If \( a \geq 3 \) then \( \gamma_1, \gamma_2 > 0 \), hence \( f'_{\beta} > 0 \). Therefore, since \( b_i \leq a_i - 1 \), we have \( h < f(a_i, b_i) \leq f(a_i, a_i - 1) = g(a_i) \), where \( g(a) = f(a, a - 1) \). Easy to calculate that \( g'(a) < 0 \) when \( a > 1 \). Recall that \( a_i \geq b_i + 1 \geq b_k + 1 = 3 \). Hence, \( h < g(a_i) \leq g(3) = 511/30 \).

**Lemma 8.7.** Let \( b_i \geq b_k \geq 2 \). Suppose also that \( b_k \leq 9 \) and \( a_i \geq 40 \). Then \( h < 6.8 \).

**Proof.** From (22) and the estimates \( a_k \geq b_k + 1 \) and \( c_i \geq 0 \), we obtain the inequality

\[
h < f_{b_k}(a_i, b_i), \quad \text{where} \quad f_m(a, b) = \left( \frac{m}{1 + m} + \frac{b}{a} \right) \cdot \left( \frac{2}{m} + \frac{3ma}{(m-1)a + b - m} \right).
\]

If \( a \geq 6, b \geq 2, m \geq 2 \) then \( f_m \) is monotonically increasing with respect to \( b \). Indeed, one can check that

\[
\frac{\partial f_m}{\partial b} = \frac{2}{ma} + \frac{3m}{m+1} \cdot \frac{\gamma_1 a - \gamma_2}{((m-1)a + b - m)^2}, \quad \gamma_1 = m^2 - m - 1, \quad \gamma_2 = m^2 + m.
\]

\( m \geq 2 \) implies \( \gamma_1 > 0 \), hence, for \( a > 6 \) we have \( \gamma_1 a - \gamma_2 > 6 \gamma_1 - \gamma_2 = 5m^2 - 7m - 6 \geq 0 \), thus, \( \partial f_m/\partial b > 0 \). Obviously, for \( b \geq 2 \) the denominator is non-zero.

We know that \( b_i \leq a_i - 1 \) and \( a_i \geq 40 \). Hence, \( h < f_{b_k}(a_i, a_i - 1) < g_{b_k}(a_i) \), where

\[
g_m(a) := f_m(a, a - 1) + \frac{2}{ma} = 6 + \frac{m + 2}{m^2 + m} + \frac{3(m + 1)}{ma - m - 1}.
\]

Clearly, \( g_m \) is monotonically decreasing with respect to \( a \) when \( a \geq 2 \). Thus, it suffices to check that \( g_m(40) < 6.8 \) for \( m = 2, \ldots, 9 \).

**Lemma 8.8.** Suppose that \( b_i \leq b_k \leq 2 \) and \( a_k < 20 \), \( a_i < 40 \). Then \( h \leq 6.023810 \ldots \).

**Proof.** Since \( b_\nu < a_\nu \) and \( c_\nu < a_\nu \), it suffices to check only finitely many possibilities for the values of \( Q_{ik}, a_\nu, b_\nu \) and \( c_\nu \) (where \( \nu = i, k \)). In each case we can find \( \varepsilon \) and \( h \) from the equations (7), (8) and search the maximum of \( h \) under the restrictions \( \varepsilon > 0, h > 0, \varepsilon h \leq 3 \). These calculations were performed with a compute. The corresponding C-program is shown on the Fig. 2.

**Corollary 8.9.** Let \( b_i \geq b_k \geq 2 \). Then \( h < 6.8 \).

**Proof.** For \( b_k \geq 10 \) see 8.5; for \( b_k \leq 9 \) see 8.6 \(- 8.8 \).
#include <stdio.h>
main(){ int ak,bk,ck, ai,bi,ck, Q; double B,C,BC,h, hmax=0;
    for( Q=0; Q<=1; Q++ ){
        for( bk=2; bk<=9; bk++ ){
            for( ak=bk+1; ak<=21; ak++ ){
                for( bi=bk; bi<=40; bi++ ){
                    for( ai=bi+1; ai<=41; ai++ ){
                        for( ck=1; ck<=ak-bk; ck++ ){
                            for( ci=1; ci<=ai-bi; ci++ ){
                                B=(double)bi/ai + (double)bk/ak;
                                C=(double)ci/ai + (double)ck/ak;
                                BC=(double)(bi*ci)/ai + (double)(bk*ck)/ak;
                                if( ai==ak ) BC=BC+(double)(2*Q*ci*bk)/ai;
                                if( 1-C <= 0 )continue; /* eps>0 */
                                if( BC-1 <= 0 )continue; /* h>0 */
                                if( (1-C)*(1-C) > 3*(BC-1) )continue; /* BMY */
                                if( (h=(1-C)*B/(BC-1-d)) > hmax ) hmax=h;
                            }}}}}}}
    printf( "hmax=%lf", hmax );
}

Fig. 2.

9. Proof of Theorem 1'

Let things be like in §6.

**Lemma 9.1.** Suppose that \( r \geq 17 \). Then:

(a) \( h \geq 6.5 \).

(b) If \( h < 6.8 \) then \( r = 17 \), and up to a permutation, \((d_1,\ldots,d_{17})\) is either \((4,2,\ldots,2)\) or \((3,2,\ldots,2)\) or \((2,2,\ldots,2)\).

**Proof.** (a) See Lemma 6.1.

(b) If \( h < 6.8 \) then \( r = 17 \) by Lemma 6.1. Without loss of generality we may assume that \( d_1 \geq d_2 \geq \cdots \geq d_{17} \). If \( d_2 \geq 3 \), we would have \( h = 17 - 2 - 1/d_1 - \cdots - 1/d_{17} \geq 15 - 1/3 - 1/3 - 1/2 - \cdots - 1/2 = 6 5/6 > 6.8 \). Thus \( d_2 = \cdots = d_7 = 2 \) and \( 1/d_1 = 17 - 2 - 1/2 - \cdots - 1/2 = 7 - h > 1/5 \).

**Lemma 9.2.** Suppose that \( r \geq 17 \) and \( h \geq 6.8 \). Then (up to a permutation) one of the following possibilities holds:

(1) \((T_1,T_2)\) is one of the three pairs listed in 8.3(a) and either
    (1.1) \( r = 18 \) and \( d_3 = \cdots = d_{18} = 2 \), or
(1.2) \( r = 17 \) and \((d_3, \ldots, d_{17})\) is one of \((6, 3, 2, \ldots, 2), (4, 4, 2, \ldots, 2), (3, 3, 3, 2, \ldots, 2)\).

(2) \( r = 17, d_2 = \cdots = d_{17} = 2 \) and \( T_i \) is the twig depicted in 8.3(b).

Proof. By 6.3, \( X \) is rational. Hence, there exists a smooth rational \((-1)\)-curve \( C \). It does not coincide with one of \( D_1, \ldots, D_n \) by the minimality, and \( C \neq D_0 \) by Lemma 6.2. Thus, it follows from 6.5 and 4.2 that \( CD = 2 \).

Introduce the notation like in §7, §8. If the both \( b_i \) and \( b_k \) were \( \geq 2 \), then by Corollary 8.9 we would have \( h < 6.8 \). Thus, one of them, say, \( b_k \) is equal to 1 and by Lemma 7.2 we have \( Q_{ik} = 0 \).

Case 1 (Like in 8.1). \( b_k = 1, k \neq 0 \).

Since \( D_i \) and \( D_k \) do not belong to the same twig, without loss of generality we may assume that \( D_i \subset T_1, D_k \subset T_2 \) (i.e. \( d_i = a_i, d_2 = a_k \)) and that \( d_3 \leq d_4 \leq \cdots \).

Then

\[
(49) \quad h^+ = r - 2 - 1/d_3 - 1/d_4 - \cdots - 1/d_r \geq r - 2 - 1/2 - \cdots - 1/2 = (r - 2)/2.
\]

Since \( r \geq 17 \), it follows that \( h^+ \geq 71/2 \). Hence, by 8.1 we have \( h^+ = 8 \).

Subcase 1.1 \( r \geq 18 \). Then (23) turns out into \( 8 = \cdots \geq (r - 2)/2 \geq 8 \). Hence, all the \( \geq \) can be replaced with \( = \), and we have \( r = 18 \) and \( d_3 = \cdots = d_{18} = 2 \).

Subcase 1.2 \( r = 17 \). If \( d_6 = \geq 3 \), then like in (23) we would have \( 8 \geq 15 - (1/3 - 1/3 - 1/3 - 1/3) - 1/2 - \cdots = 8/6 \). Thus, \( d_6 = \cdots = d_{17} = 2 \) and \( 1/d_3 + 1/d_4 + 1/d_5 = 15 - h^+ - 1/2 - \cdots = 1 \).

Case 2 (Like in 8.2). \( k = 0 \).

Subcase 2.1 Without loss of generality assume that \( D_i \subset T_1 \), i.e. \( d_1 = a_i \). Then \( r \geq 17 \) implies like in (23) that \( h^+ = h + 1 + 1/d_1 \geq r - 1 - 1/2 - \cdots - 1/2 = (r - 1)/2 \geq 8 \), and by 8.2 we have \( h^+ = 8 \). Hence, all the \( \geq \) can be replaced with \( = \) and we obtain \( r = 17 \) and \( d_2 = \cdots = d_{17} = 2 \).

Lemma 9.3. Let \( X \) be a smooth rational projective surface. Then \( K^2 + b = 10 \) where \( K = K_X \) is the canonical class and \( b = b_2(X) \) is the second Betti number.

Proof. Since \( X \) is rational, it is obtained from \( P^2 \) by successive blow-ups and -downs. Clearly that \( K^2 + b = 10 \) for \( P^2 \) and that \( K^2 + b \) is invariant under blow-ups.

Corollary 9.4 (See e.g. [4; 1.3]). Let notation be like in 9.3. Suppose that \( D \) is an SNC-curve such that \( X - D \) is \( Q \)-acyclic. Then

\[
(50) \quad (K + D)^2 = 8 - s - 3b
\]

where \( s \) denotes the sum of all the weights of \( \Gamma_D \).
Proof. Let $D_1, \ldots, D_b$ be the irreducible components of $D$. Write $(K+D)^2 = K^2 + 2KD + D^2$ and compute each summand in the right hand side:

- $K^2 = 10 - 6$ by Lemma 9.3;
- $KD = \sum D_i (K + D_i) - \sum D_i^2 = -2b - sb$ by adjunction formula;
- $D^2 = \sum D_i^2 + \sum_{i \neq k} D_i D_k = \sum D_i^2 + 2($number of edges of $\Gamma_D) = s + 2(b - 1)$.

Now let $(X, D)$ be again as in §6. Introduce the following notation. For a twig $T$ denote $s(T) = \sum (w_v + 3)$, where $w_v$ are the weights and the summation is over all the vertices. Recall that $e(T)$ denotes the inductance of a twig $T$ (cf. §2). Let $e'(T) = e(T')$ where $T'$ is the twig obtained from a twig $T$ by reversing the order of the vertices. Denote $e(T) + e'(T) - s(T)$ by $\varphi(T)$, and put: $e_j = e(T_j), e'_j = e'(T_j)$, $s_j = s(T_j)$ and $\varphi_j = \varphi(T_j)$.

Lemma 9.5. $\sum \varphi_j > 2h - 5$.

Proof. By Lemma 1.2 and (2) we have $-\Delta = p \cdot (D_0^2 + \sum e'_j)$. Hence, $D_0^2 = -\Delta/p - \sum e'_j = h/\varepsilon - \sum e'_j$. Further, by 4.1(a) and 2.4(i) we have $(K + D)^2 = H^2 + N^2 = \varepsilon h - \sum e_j$. Putting these expressions for $D_0^2$ and $(K + D)^2$ into (24) (where, in out notation, $s + 3b = D_0^2 + 3 + \sum s_j$), we obtain $5 + \sum \varphi_j = h(\varepsilon + 1/\varepsilon) \geq 2h$.

Now let us complete the proof of Theorem 1'. Suppose that $r > 17$. Then by 9.1(a) we have $h \geq 6.5$, hence, 9.5 implies $\sum \varphi_i \geq 13 - 5 = 8$. However, each $\varphi_j$ depends only on the twig, and by 9.1 and 9.2 only few types of twigs can appear. The values of $\varphi(T)$ for these twigs are as follows:

Table 3.

<table>
<thead>
<tr>
<th>$d(T)$</th>
<th>$T$</th>
<th>$\varphi(T)$</th>
</tr>
</thead>
<tbody>
<tr>
<td>2</td>
<td>[2]</td>
<td>0</td>
</tr>
<tr>
<td>3</td>
<td>[3]</td>
<td>2/3</td>
</tr>
<tr>
<td></td>
<td>[2,2]</td>
<td>-2/3</td>
</tr>
<tr>
<td>4</td>
<td>[4]</td>
<td>1.5</td>
</tr>
<tr>
<td></td>
<td>[2,2,2]</td>
<td>-1.5</td>
</tr>
<tr>
<td>5</td>
<td>[5]</td>
<td>2.4</td>
</tr>
<tr>
<td>6</td>
<td>[6]</td>
<td>3 1/3</td>
</tr>
<tr>
<td></td>
<td>[2,2,2,2,2]</td>
<td>-3 1/3</td>
</tr>
</tbody>
</table>

Here the twig with the weights $w_1, w_2, \ldots$ is denoted by $[-w_1, -w_2, \ldots]$. In Table 3 we listed all the twigs with discriminants $\leq 6$. The values $\varphi(T)$ for those
twigs which appear in 8.3, are

\[ \varphi([2, 5]) = \frac{7}{9}, \quad \varphi([3, 5]) = \frac{2}{4}, \quad \varphi([4, 5]) = \frac{9}{19}, \quad \varphi([4, 2, 2, 2]) = -\frac{12}{13}. \]

It is easy to check that in all the cases allowed by 9.1 and 9.2 we can not have \( \sum \varphi_j \geq 8. \) Theorem 1' is proven.

References


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