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LAGRANGEAN CONTACT STRUCTURES ON PROJECTIVE COTANGENT BUNDLES

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Introduction

Let (M, D) be a contact manifold of dimension $2n-1$, $n \geq 2$, and (E, E') a pair of subbundles of D . We say that $(D; E, E')$ is a Lagrangean contact structure on M if for each point $x \in M$ the fibres E_x and E'_x are transversal Lagrangean subspaces of D_x with respect to the natural conformal symplectic structure of D_x .

An example of Lagrangean contact structure is given on the projective cotangent bundle $P(T^*M)$ of a manifold M of dimension n in the following way. Let D be the canonical contact structure on $P(T^*M)$. Suppose that a projective structure Q on M is given. For $[\lambda] \in P(T^*M)$, we define $E'_{[\lambda]}$ to be the space of vertical vectors in $T_{[\lambda]}(P(T^*M))$ for the projection $\varpi: P(T^*M) \rightarrow M$. Furthermore, choosing a local torsionfree connection η belonging to Q defined over a neighbourhood of $x = \varpi([\lambda]) \in M$, we define $E_{[\lambda]}$ to be the space of horizontal lifts to $[\lambda]$ of vectors $X \in T_x M$ with $\lambda(X) = 0$. It is determined by Q independently on the choice of η . These subspaces $E_{[\lambda]}$, $E'_{[\lambda]}$ of $T_{[\lambda]}(P(T^*M))$, $[\lambda] \in P(T^*M)$, constitute subbundles E, E' of D such that $(D; E, E')$ becomes a Lagrangean contact structure on $P(T^*M)$ (Theorem 4.2).

A typical one is the Lagrangean contact structure $(D_0; E_0, E'_0)$ on the projective cotangent bundle of n -projective space P^n associated to the flat projective structure Q_0 on P^n . A Lagrangean contact structure is said to be flat if it is locally isomorphic to $(D_0; E_0, E'_0)$. The purpose of the present note is to prove:

The Lagrangean contact structure on $P(T^*M)$ associated to a projective structure Q on M is flat if and only if Q is projectively flat.

A conformal analogue to our theorem in the following form was proved by Miyaoka [2], Sato-Yamaguchi [3]: The Lie contact structure on the tangential sphere bundle $S(TM)$ associated to a conformal structure C on a manifold M is flat if and only if C is conformally flat, provided $\dim M \geq 3$.

The proof of our theorem is based on the theory of Tanaka [5] of G -structures associated to simple graded Lie algebras as in [2], [3]. First, we show that the Lagrangean contact structures are in bijective correspondence with the \tilde{G} -structures of type \mathfrak{m} associated to $\mathfrak{sl}(n+1)$ endowed with gradation of contact type in the sense of Tanaka [5] (Theorem 5.1). Next, we construct a normal Cartan connection ω associated to the \tilde{G} -structure of type \mathfrak{m} which corresponds to our Lagrangean contact structure on $P(T^*M)$, making use of the normal Cartan connection for the projective structure Q (Theorem 6.4). It turns out that the curvature of ω vanishes if and only if the projective curvature of Q vanishes. This implies our theorem.

1. Lagrangean pairs

In this paper we work in C^∞ -category though all the arguments are valid also in complex analytic category, replacing the real number field \mathbf{R} by the complex number field \mathbf{C} .

Let (W, A) be a symplectic vector space over \mathbf{R} of dimension $2n$. A subspace E of W is said to be *Lagrangean* with respect to A (or with respect to the conformal symplectic structure determined by A) if $\dim E = n$ and $A(E, E) = \{0\}$. A pair (E, E') of subspaces of W is called a *Lagrangean pair* if E and E' are Lagrangean subspaces of (W, A) such that $E \cap E' = \{0\}$. A symplectic basis $\{e_1, \dots, e_{2n}\}$ of (W, A) with $A(e_i, e_{n+j}) = \delta_{ij}$ is said to be *adapted to (E, E')* if $E = [e_1, \dots, e_n]$ and $E' = [e_{n+1}, \dots, e_{2n}]$, where $[*]$ denotes the subspace spanned by $*$. Any Lagrangean pair admits an adapted symplectic basis. The Lagrangean pairs are conjugate to each other under the symplectic automorphisms or the conformal symplectic automorphisms of (W, A) .

Now let us recall the notion of torsionfree connection in order to give a geometric example of Lagrangean pair. Let M be a manifold of dimension n and fix a vector space V over \mathbf{R} of dimension n . Let $\pi: F(M) \rightarrow M$ be the frame bundle of M , with structure group $GL(V)$. Denote by θ the canonical form on $F(M)$, which is a V -valued 1-form on $F(M)$. A connection η in $F(M)$ is said to be *torsionfree* if

$$d\theta + [\eta, \theta] = 0.$$

It is also described in the following way (see Kobayashi [1]). Let $\pi^2: F^2(M) \rightarrow M$ be the second order frame bundle of M , with structure group $G^2(V)$. We may consider $GL(V)$ as a subgroup of $G^2(V)$ through the natural monomorphism $GL(V) \rightarrow G^2(V)$. Then the natural projection $\pi_1^2: F^2(M) \rightarrow F(M)$ is $GL(V)$ -equivariant. Denote by θ^2 the second

canonical form on $F^2(M)$, which is a $V + \mathfrak{gl}(V)$ -valued 1-form on $F^2(M)$. We decompose it to the sum

$$\theta^2 = \Theta_{-1} + \Theta_0$$

of the V -component Θ_{-1} and the $\mathfrak{gl}(V)$ -component Θ_0 . Then the torsionfree connections η are in bijective correspondence with the $GL(V)$ -equivariant sections $s: F(M) \rightarrow F^2(M)$ of $\pi_1^*: F^2(M) \rightarrow F(M)$ in such a way that $s^*\Theta_0 = \eta$. The section s corresponding to η is constructed as follows. For a given $u \in F(M)$ a local diffeomorphism $f: (V, 0) \rightarrow M$ is defined by $f(v) = \text{Exp}^u u(v)$, where Exp^u denotes the exponential map for the linear connection in the tangent bundle TM induced by η . Then the correspondence $u \mapsto j_0^2(f)$, the second jet of f at 0, provides the required section s .

Let η be a connection in $F(M)$ and ∇ the linear connection in the cotangent bundle $p: T^*M \rightarrow M$ induced by η . For given $\lambda \in T_x^*M$ and $X \in T_xM$, we denote by $X_\lambda^H \in T_\lambda(T^*M)$ the horizontal lift of X to T^*M with respect to ∇ . It may be also described as follows. Identify T^*M with the associated bundle $F(M) \times_{GL(V)} V^*$ with respect to the natural (contragredient) action $(\text{id})^*$ of $GL(V)$ on the dual space V^* of V , and denote the projection $F(M) \times V^* \rightarrow T^*M$ by $(u, \xi) \mapsto u \cdot \xi$. For a fixed $\xi \in V^*$, the differential $T(F(M)) \rightarrow T(T^*M)$ of the map $F(M) \rightarrow T^*M$ defined by $u \mapsto u \cdot \xi$ will be denoted by $X \mapsto X \cdot \xi$. Then we have

$$X_\lambda^H = X_u^* \cdot \xi \quad \text{for } \lambda = u \cdot \xi,$$

where $X_u^* \in T_u(F(M))$ is the horizontal lift of X to $F(M)$ with respect to η .

EXAMPLE 1.1. Let η be a torsionfree connection in $F(M)$. For a given $\lambda \in T_x^*M$ we define subspaces E_λ and E'_λ of $T_\lambda(T^*M)$ by

$$E_\lambda = \{X_\lambda^H; X \in T_xM\}, \quad E'_\lambda = \{\mu_\lambda^V; \mu \in T_x^*M\},$$

where $\mu \mapsto \mu_\lambda^V$ denotes the identification $T_x^*M = T_\lambda(T_x^*M)$. Further, we define a 1-form α on T^*M by

$$\alpha(X) = \lambda(p_*X) \quad \text{for } X \in T_\lambda(T^*M),$$

whose exterior differential $d\alpha$ is known to be a symplectic form on each $T_\lambda(T^*M)$. Then (E_λ, E'_λ) is a Lagrangean pair of $(T_\lambda(T^*M), d\alpha)$. More precisely, we have that

$$d\alpha(E_\lambda, E_\lambda) = d\alpha(E'_\lambda, E'_\lambda) = \{0\},$$

$$d\alpha(\mu_\lambda^V, X_\lambda^H) = \frac{1}{2}\mu(X) \quad \text{for } \mu \in T_x^*M, \quad X \in T_xM.$$

2. Lagrangean contact structures

In this section we assume that $n \geq 2$. A graded Lie algebra (abbreviated to GLA) over \mathbf{R}

$$\mathfrak{m} = \mathfrak{g}_{-2} + \mathfrak{g}_{-1}, \quad [\mathfrak{g}_p, \mathfrak{g}_q] \subset \mathfrak{g}_{p+q}$$

is called a *fundamental GLA of contact type* of degree n , if $\dim \mathfrak{g}_{-2} = 1$, $\dim \mathfrak{g}_{-1} = 2n - 2$, and $[\mathfrak{g}_{-1}, X] = \{0\}$ implies $X = 0$ for $X \in \mathfrak{g}_{-1}$. Such a GLA is unique up to GLA-isomorphism. If we take an $e_0 \in \mathfrak{g}_{-2}$ with $e_0 \neq 0$, a symplectic form A_0 on \mathfrak{g}_{-1} is defined by

$$[X, Y] = A_0(X, Y)e_0 \quad \text{for } X, Y \in \mathfrak{g}_{-1},$$

whose conformal class is determined by \mathfrak{m} independently on the choice of e_0 . We define $C(\mathfrak{m})$ to be the subgroup of $GL(\mathfrak{m})$ consisting of $a \in GL(\mathfrak{m})$ such that $a\mathfrak{g}_{-1} = \mathfrak{g}_{-1}$ and that the graded linear automorphism \bar{a} of \mathfrak{m} induced by a is a GLA-automorphism.

Let M be a manifold of dimension $2n - 1$ and D a subbundle of TM of codimension 1. Denote by $\kappa: TM \rightarrow TM/D$ the projection to the quotient line bundle TM/D . For a point $x \in M$ we define a GLA $\mathfrak{m}(x)$ as follows. Let $\mathfrak{g}_{-2}(x) = (TM/D)_x$, $\mathfrak{g}_{-1}(x) = D_x$, and $\mathfrak{m}(x) = \mathfrak{g}_{-2}(x) + \mathfrak{g}_{-1}(x)$. For $X, Y \in \mathfrak{g}_{-1}(x)$ we define

$$[X, Y] = \kappa[\tilde{X}, \tilde{Y}]_x \in \mathfrak{g}_{-2}(x),$$

taking local sections \tilde{X} and \tilde{Y} of D around x which extend X and Y , respectively. Further, we set $[\mathfrak{m}(x), \mathfrak{g}_{-2}(x)] = \{0\}$. If $\mathfrak{m}(x)$ is GLA-isomorphic to \mathfrak{m} for every point $x \in M$, D is called a *contact structure* on M . Note that then D carries a natural conformal symplectic structure determined by the $\mathfrak{m}(x)$'s. A contact structure may be also defined by a *system of contact forms* $\{U_i, \gamma_i\}$, where $\{U_i\}$ is an open cover of M , and γ_i is a 1-form defined on U_i with $\gamma_i \wedge (d\gamma_i)^{n-1} \neq 0$ everywhere on U_i which satisfies $\gamma_i = f_{ij}\gamma_j$ on $U_i \cap U_j$ with a function f_{ij} on $U_i \cap U_j$. Then

$$D_x = \{X \in T_xM; \gamma_i(X) = 0\} \quad \text{if } x \in U_i$$

defines a contact structure D . And every contact structure D is obtained in this way. Note that in this case the conformal symplectic structure on D is given by $(d\gamma_i)_x|D_x \times D_x$.

EXAMPLE 2.1. Let M be a manifold and $\varpi: P(T^*M) \rightarrow M$ the projective cotangent bundle of M . We set $\mathring{T}^*M = T^*M - \{\text{zero section}\}$ and denote by $q: \mathring{T}^*M \rightarrow P(T^*M)$ the natural projection $\lambda \mapsto [\lambda]$. Let α be the 1-form on T^*M defined in Example 1.1. If we take local sections $s_i: U_i \rightarrow \mathring{T}^*M$ of q and set $\gamma_i = s_i^*\alpha$, then $\{U_i, \gamma_i\}$ becomes a system of contact forms on $P(T^*M)$. The contact structure D determined by this system is called the *canonical contact structure* on $P(T^*M)$.

For a contact structure D on a manifold M of dimension $2n-1$, a frame $u: \mathfrak{m} \rightarrow T_x M$ at $x \in M$ is called a *contact frame* of (M, D) if $u g_{-1} = D_x$ and the graded linear isomorphism $\bar{u}: \mathfrak{m} \rightarrow \mathfrak{m}(x)$ induced by u is a GLA-isomorphism. Then the subset $F_D(M)$ of $F(M)$ consisting of the contact frames of (M, D) becomes a $C(\mathfrak{m})$ -structure. Furthermore, $P = F_D(M)$ is a $C(\mathfrak{m})$ -structure of type \mathfrak{m} in the sense that

$$d\theta_{-2} + \frac{1}{2}[\theta_{-1}, \theta_{-1}] \equiv 0 \quad \text{mod } \theta_{-2},$$

where θ_{-2} and θ_{-1} denote the \mathfrak{g}_{-2} -component and the \mathfrak{g}_{-1} -component, respectively, of the restriction θ to P of the canonical form on $F(M)$. Conversely, for every $C(\mathfrak{m})$ -structure P of type \mathfrak{m} there exists uniquely a contact structure D such that $F_D(M) = P$.

Let D_i be a contact structure on M_i , $i=1,2$. A diffeomorphism $\varphi: M_1 \rightarrow M_2$ is called a *contact isomorphism* of (M_1, D_1) to (M_2, D_2) if $\varphi_* D_1 = D_2$, which is equivalent to that φ_* induces a GLA-isomorphism of $\mathfrak{m}_1(x)$ to $\mathfrak{m}_2(\varphi(x))$ for each point $x \in M_1$, or to that φ is a $C(\mathfrak{m})$ -structure isomorphism of $(M_1, F_{D_1}(M_1))$ to $(M_2, F_{D_2}(M_2))$, namely, the first prolongation $\varphi^{(1)}: F(M_1) \rightarrow F(M_2)$ of φ sends $F_{D_1}(M_1)$ onto $F_{D_2}(M_2)$.

EXAMPLE 2.2. Let D_i be the canonical contact structure on $P(T^*M_i)$, $i=1,2$. A diffeomorphism $\varphi: M_1 \rightarrow M_2$ induces a diffeomorphism $\hat{\varphi}: P(T^*M_1) \rightarrow P(T^*M_2)$ such that the diagram

$$\begin{array}{ccc} \mathring{T}^*M_1 & \xrightarrow{(\varphi^*)^{-1}} & \mathring{T}^*M_2 \\ q_1 \downarrow & & q_2 \downarrow \\ P(T^*M_1) & \xrightarrow{\hat{\varphi}} & P(T^*M_2) \end{array}$$

is commutative, where $q_i: T^*M_i \rightarrow P(T^*M_i)$, $i=1,2$, are natural projections. Then ϕ is a contact isomorphism of $(P(T^*M_1), D_1)$ to $(P(T^*M_2), D_2)$.

Let $\mathfrak{m} = \mathfrak{g}_{-2} + \mathfrak{g}_{-1}$ be a fundamental GLA of contact type of degree n . If a Lagrangean pair (e, e') of the symplectic vector space (\mathfrak{g}_{-1}, A_0) is given, the triple $(\mathfrak{m}; e, e')$ is called a *fundamental GLA of Lagrangean contact type* of degree n . Such a triple is unique up to isomorphism. Here for two such triples $(\mathfrak{m}_i; e_i, e'_i)$, $i=1,2$, a GLA-isomorphism ϕ of \mathfrak{m}_1 to \mathfrak{m}_2 such that $\phi e_1 = e_2$, $\phi e'_1 = e'_2$ is called an *isomorphism* of $(\mathfrak{m}_1; e_1, e'_1)$ to $(\mathfrak{m}_2; e_2, e'_2)$. For a fundamental GLA $(\mathfrak{m}; e, e')$ of Lagrangean contact type, a basis $\{e_0, e_1, \dots, e_{2n-2}\}$ of \mathfrak{m} is called a *Lagrangean contact basis* if $e_0 \in \mathfrak{g}_{-2}$ and $\{e_1, \dots, e_{2n-2}\}$ is a symplectic basis of (\mathfrak{g}_{-1}, A_0) adapted to (e, e') . We define the *Lagrangean contact group* $C(\mathfrak{m}; e, e')$ to be the subgroup of $C(\mathfrak{m})$ consisting of $a \in C(\mathfrak{m})$ such that $ae = e, ae' = e'$. With respect to a Lagrangean contact basis, it is represented by

$$C(\mathfrak{m}; e, e') = \left\{ \begin{pmatrix} c & 0 & 0 \\ b_1 & a & 0 \\ b_2 & 0 & c^t a^{-1} \end{pmatrix} ; c \in \mathbf{R}^*, b_1, b_2 \in \mathbf{R}^{n-1}, a \in GL(n-1) \right\}.$$

Let D be a contact structure on a manifold M of dimension $2n-1$. Suppose that two subbundles E, E' of D are given. We say that $(D; E, E')$ is a *Lagrangean contact structure* if for every $x \in M$, (E_x, E'_x) is a Lagrangean pair of D_x with respect to the natural conformal symplectic structure on D_x . A frame $u: \mathfrak{m} \rightarrow T_x M$ of M is called a *Lagrangean contact frame* of $(M, D; E, E')$ if it is a contact frame of (M, D) such that $ue = E_x, ue' = E'_x$. Then the subset $F_{(D; E, E')}(M)$ of $F(M)$ consisting of the Lagrangean contact frames of $(M, D; E, E')$ becomes a $C(\mathfrak{m}; e, e')$ -structure of type \mathfrak{m} . Conversely, for every $C(\mathfrak{m}; e, e')$ -structure \tilde{F} of type \mathfrak{m} there exists uniquely a Lagrangean contact structure $(D; E, E')$ such that $F_{(D; E, E')}(M) = \tilde{F}$. Let $(D_i; E_i, E'_i)$ be a Lagrangean contact structure on M_i , $i=1,2$. A diffeomorphism $\phi: M_1 \rightarrow M_2$ is called a *Lagrangean contact isomorphism* if it is a contact isomorphism of (M_1, D_1) to (M_2, D_2) such that $\phi_* E_1 = E_2$, $\phi_* E'_1 = E'_2$, which is equivalent to that ϕ is a $C(\mathfrak{m}; e, e')$ -structure isomorphism of $(M_1, F_{(D_1; E_1, E'_1)}(M_1))$ to $(M_2, F_{(D_2; E_2, E'_2)}(M_2))$.

3. Projective structures

Let W be a vector space over \mathbf{R} of dimension $n+1$, $n \geq 1$, and $P^n = P(W)$ the projective space associated to W . We denote by L the group of projective transformations of P^n , which is isomorphic to the

quotient group $GL(W)/C$ of $GL(W)$ by its center C . The Lie algebra $\mathfrak{l} = \text{Lie } L$ of L is identified with $\mathfrak{sl}(W)$, and L may be considered as a subgroup of the automorphism group $\text{Aut}(\mathfrak{l})$ of \mathfrak{l} through the adjoint representation. We define an L -invariant nondegenerate symmetric bilinear form B on \mathfrak{l} by

$$B(X, Y) = \text{tr}(XY) \quad \text{for } X, Y \in \mathfrak{l}.$$

In the following we fix a basis $\{w_0, w_1, \dots, w_n\}$ of W , and denote by $\{\zeta^0, \zeta^1, \dots, \zeta^n\}$ the basis of W^* dual to this. Thus we have identifications: $L = PL(n+1) = GL(n+1)/C$ where $C = \mathbf{R}^* 1_{n+1}$, $\mathfrak{l} = \mathfrak{sl}(n+1)$, and $P^n = P(\mathbf{R}^{n+1})$. We set

$$\bar{E} = \frac{1}{n+1} \begin{pmatrix} n & 0 \\ 0 & -1_n \end{pmatrix} \in \mathfrak{l},$$

$$\mathfrak{l}_p = \{X \in \mathfrak{l}; [\bar{E}, X] = pX\} \quad p = -1, 0, 1,$$

which determines a GLA-structure on \mathfrak{l} :

$$\mathfrak{l} = \mathfrak{l}_{-1} + \mathfrak{l}_0 + \mathfrak{l}_1, \quad [\mathfrak{l}_p, \mathfrak{l}_q] \subset \mathfrak{l}_{p+q}.$$

We set $V = \mathfrak{l}_{-1}$. Then, since $B|_{\mathfrak{l}_{-1} \times \mathfrak{l}_1}$ is nondegenerate, \mathfrak{l}_1 is identified with V^* through B . These subspaces \mathfrak{l}_p are explicitly given as follows.

$$\begin{aligned} \mathfrak{l}_{-1} &= \left\{ \begin{pmatrix} 0 & 0 \\ v & 0 \end{pmatrix}; v \in \mathbf{R}^n \right\}, \quad \mathfrak{l}_1 = \left\{ \begin{pmatrix} 0 & \xi \\ 0 & 0 \end{pmatrix}; \xi \in \mathbf{R}^n \right\}, \\ \mathfrak{l}_0 &= \left\{ \begin{pmatrix} \alpha & 0 \\ 0 & A \end{pmatrix}; \alpha \in \mathbf{R}, A \in \mathfrak{gl}(n), \text{tr } A = -\alpha \right\}. \end{aligned}$$

We set

$$\mathfrak{l}' = \mathfrak{l}_0 + \mathfrak{l}_1,$$

which is a subalgebra of \mathfrak{l} with $\mathfrak{l} = \mathfrak{l}_{-1} + \mathfrak{l}'$ (direct sum as vector space). Let L_0 denote the group of GLA-automorphisms of \mathfrak{l} . It is given by

$$L_0 \cong \left\{ \begin{pmatrix} a & 0 \\ 0 & b \end{pmatrix}; a \in \mathbf{R}^*, b \in GL(n) \right\} / C,$$

and thus $L_0 \subset L$ and $\text{Lie } L_0 = \mathfrak{l}_0$. Further, we have that $L = L_0 \text{Inn}(\mathfrak{l})$, where $\text{Inn}(\mathfrak{l})$ denotes the group of inner automorphisms of \mathfrak{l} . We define $L_1 = \exp \mathfrak{l}_1$ and $L' = N_L(\mathfrak{l}')$, the normalizer of \mathfrak{l}' in L , whose Lie algebras

are I_1 and I' , respectively. The subgroup L' has a semidirect decomposition $L' = L_0 L_1$ and is identical with the isotropy subgroup in L at the point $[w_0] \in P^n$. Therefore, we have an identification

$$L/L' = P^n,$$

which implies an identification $T_{[w_0]} P^n = I_{-1}$. Let $\bar{\rho}: L' \rightarrow GL(I_{-1}) = GL(V)$ be the linear isotropy representation at $[w_0]$. Then we have that $\text{Kernel } \bar{\rho} = L_1$ and $\text{Image } \bar{\rho} = GL(V)$, and hence, $\bar{\rho}$ maps L_0 isomorphically onto $GL(V)$. We shall identify L_0 with $GL(V)$ through $\bar{\rho}$, and also I_0 with $\mathfrak{gl}(V)$ through $\bar{\rho}_*$. We define

$$\bar{e}_i = E_{i+1,1}, \quad \bar{e}_i^* = E_{1,i+1} \quad \text{for } 1 \leq i \leq n,$$

where the E_{ij} 's denote the standard matrix units in $\mathfrak{gl}(n+1)$. Then $\{\bar{e}_i\}$ is an orthonormal basis of I_{-1} with respect to the inner product

$$(X, Y) = \text{tr}({}^tXY) \quad \text{for } X, Y \in I$$

on I . (In complex analytic category, one should replace $\text{tr}({}^tXY)$ by $\text{tr}({}^tX\bar{Y})$.) Furthermore, $\{\bar{e}_i^*\}$ is the basis of I_1 dual to $\{\bar{e}_i\}$ under the previous identification $I_1 = V^*$. We may identify $I_0 = \mathfrak{gl}(V)$ with $\mathfrak{gl}(n)$ through the basis $\{\bar{e}_i\}$ of V . It is easy to see the following.

Lemma 3.1. *Under the identification above, for*

$$v = ({}^t(v^1, \dots, v^n) = \sum_i v^i \bar{e}_i \in V = I_{-1},$$

$$\xi = (\xi_1, \dots, \xi_n) = \sum_i \xi_i \bar{e}_i^* \in V^* = I_1,$$

$[v, \xi] \in \mathfrak{gl}(n) = I_0$ is given by

$$[v, \xi] = v\xi + (\xi v)1_n.$$

Next, we embed $V = I_{-1}$ into P^n as an open set containing $[w_0]$ by the map $v \mapsto (\exp v)[w_0]$, and so every $a \in L$ determines a local diffeomorphism $a: (V, 0) \rightarrow P^n$. We define a map $\epsilon: L \rightarrow F^2(P^n)$ by

$$\epsilon(a) = j_0^2(a) \quad \text{for } a \in L.$$

Then ϵ is an embedding which induces a monomorphism $\epsilon: L' \rightarrow G^2(V)$ such that $\pi_1^2 \circ \epsilon = \bar{\rho}$, where $\pi_1^2: G^2(V) \rightarrow GL(V)$ is the natural projection. In the following we shall consider L' as a subgroup of $G^2(V)$ through the

monomorphism ι .

Now let M be a manifold of dimension n . An L' -subbundle $Q \subset F^2(M)$ of the second order frame bundle $\pi^2: F^2(M) \rightarrow M$ is called a *projective structure* on M . For example, let η be a torsionfree connection in $F(M)$, with the corresponding $GL(V)$ -equivariant section $s: F(M) \rightarrow F^2(M)$ of $\pi_1^2: F^2(M) \rightarrow F(M)$. Then $Q_\eta = s(F(M)) \cdot L_1 \subset F^2(M)$ is a projective structure on M , which we call the *projective structure associated to η* . Let η and η' be torsionfree connections in $F(M)$. They are said to be *projectively equivalent* if $Q_\eta = Q_{\eta'}$, which is equivalent to that there exists an $L_1 = V^*$ -valued function p on $F(M)$ of type $\text{Ad} = (\text{id})^*$ such that

$$\eta - \eta' = [\theta, p],$$

which is the case that $s' = s \cdot \exp p$ for the corresponding sections $s, s': F(M) \rightarrow F^2(M)$. Let $Q \subset F^2(M)$ be a projective structure and $U \subset M$ an open set. A torsionfree connection η in $F(M)|_U = \pi^{-1}(U)$ is called a *local torsionfree connection belonging to Q* if $Q_\eta = Q|_U$. For any projective structure $Q \subset F^2(M)$ there exists a family $\{U_i, \eta_i\}$ of local torsionfree connections belonging to Q , where $(*)$: $\{U_i\}$ is an open cover of M ; η_i is a torsionfree connection in $F(M)|_{U_i}$; η_i and η_j are projectively equivalent over $U_i \cap U_j$. Conversely, for any family $\{U_i, \eta_i\}$ with $(*)$, there exists uniquely a projective structure $Q \subset F^2(M)$ such that each η_i belongs to Q .

EXAMPLE 3.2. Set $Q_1 = L$ and regard it as a submanifold of $F^2(P^n)$ through the embedding ι . Then $Q_1 \subset F^2(P^n)$ is a projective structure on P^n , which we call the *flat projective structure on P^n* .

Let $Q_i \subset F^2(M_i)$ be a projective structure on M_i , $i=1,2$. A diffeomorphism $\varphi: M_1 \rightarrow M_2$ is called a *projective isomorphism* of (M_1, Q_1) to (M_2, Q_2) if the second prolongation $\varphi^{(2)}: F^2(M_1) \rightarrow F^2(M_2)$ of φ sends Q_1 onto Q_2 . A projective structure $Q \subset F^2(M)$ is said to be *projectively flat* if (M, Q) is locally projectively isomorphic to (P^n, Q_1) , that is, for each point $x \in M$ there exist an open neighbourhood U of x and an open set U_0 of P^n such that $(U, Q|_U)$ is projectively isomorphic to $(U_0, Q_1|_{U_0})$.

Now we recall the theory of Cartan connections for projective structures following the formulation by Tanaka [4]. Let Q be a projective structure on a manifold M of dimension n . An L -valued 1-form $\bar{\omega}$ on Q is called a *Cartan connection* in Q of type L/L' if

- (1) for each $z \in Q$, $\bar{\omega}: T_z Q \rightarrow L$ is a linear isomorphism;
- (2) $R_a^* \bar{\omega} = \text{Ad} a^{-1} \bar{\omega}$ for $a \in L'$; and
- (3) $\bar{\omega}(A^*) = A$ for $A \in L'$,

where R_a denotes the action of $a \in L'$ on Q , and A^* the fundamental vector field on Q generated by $A \in \mathfrak{l}'$. Let

$$\bar{\omega} = \bar{\omega}_{-1} + \bar{\omega}_0 + \bar{\omega}_1$$

be the decomposition of $\bar{\omega}$ into the sum of \mathfrak{l}_p -components $\bar{\omega}_p$. We call

$$\bar{\Omega} = d\bar{\omega} + \frac{1}{2}[\bar{\omega}, \bar{\omega}]$$

the *curvature* of $\bar{\omega}$, which is semibasic in the sense that $\bar{\Omega}(X, Y) = 0$ if X or $Y \in T_z Q$ is tangent to the fibre of π^2 . Thus there exists an $\mathfrak{l} \otimes \Lambda^2 \mathfrak{l}_{-1}^*$ -valued function \bar{K} on Q , called the *curvature function* of $\bar{\omega}$, such that

$$\bar{\Omega} = \frac{1}{2} \bar{K}(\bar{\omega}_{-1} \wedge \bar{\omega}_{-1}).$$

Let

$$\bar{K} = \bar{K}_{-1} + \bar{K}_0 + \bar{K}_1$$

be the decomposition of \bar{K} into the sum of \mathfrak{l}_p -components \bar{K}_p . Recall that the second canonical form θ^2 on $F^2(M)$ is a $V + \mathfrak{gl}(V) = \mathfrak{l}_{-1} + \mathfrak{l}_0$ -valued 1-form with decomposition into the sum

$$\theta^2 = \Theta_{-1} + \Theta_0$$

of \mathfrak{l}_p -components Θ_p . A Cartan connection $\bar{\omega}$ is said to be *normal* if it satisfies the following two conditions.

(1) The restrictions of Θ_{-1} and Θ_0 to Q are identical with $\bar{\omega}_{-1}$ and $\bar{\omega}_0$, respectively. (In this case $\bar{K}_{-1} = 0$.)

(2) If $\{\bar{e}_1, \dots, \bar{e}_n\}$ is a basis of \mathfrak{l}_{-1} with $(\bar{e}_i, \bar{e}_j) = \delta_{ij}$, and $\{\bar{e}_1^*, \dots, \bar{e}_n^*\}$ the basis of \mathfrak{l}_1 dual to $\{\bar{e}_i\}$ with respect to B , then

$$(\bar{\partial}^* \bar{K})(X) = \sum_i [\bar{e}_i^*, \bar{K}(\bar{e}_i, X)] = 0 \quad \text{for } X \in \mathfrak{l}_{-1}.$$

EXAMPLE 3.3. The Maurer-Cartan form $\bar{\omega}$ of $L = Q_1$ is a normal Cartan connection in $Q_1 \subset F^2(P^n)$ of type L/L' with the curvature $\bar{\Omega} = 0$.

Theorem 3.4. (Tanaka [4]) *For any projective structure Q on a manifold M of dimension $n \geq 2$, there exists uniquely a normal Cartan connection $\bar{\omega}$ in Q of type L/L' .*

The following follows from Example 3.3 and Frobenius' theorem.

Corollary 3.5. *Q is projectively flat if and only if the curvature $\bar{\Omega}$ of $\bar{\omega}$ vanishes on Q , provided $n \geq 2$.*

4. Lagrangean contact structures on projective cotangent bundles

Let M be a manifold of dimension $n \geq 2$ and $\varpi: P(T^*M) \rightarrow M$ the projective cotangent bundle of M . We identify $P(T^*M)$ with the associated bundle $F(M) \times_{GL(V)} P(V^*)$ with respect to the natural projective action of $GL(V)$ on $P(V^*)$. In the same way as in Section 1, the projection $F(M) \times P(V^*) \rightarrow P(T^*M)$ will be denoted by $(u, [\xi]) \mapsto u \cdot [\xi]$, and for a fixed $[\xi] \in P(V^*)$ the differential of the map $F(M) \rightarrow P(T^*M)$ defined by $u \mapsto u \cdot [\xi]$ will be denoted by $X \mapsto X \cdot [\xi]$. Then we have

$$X \cdot [\xi] = q_*(X \cdot \xi) \quad \text{for } \xi \in \mathring{V}^* = V^* - \{0\}, \quad X \in T(F(M)),$$

for the natural projection $q: \mathring{T}^*M \rightarrow P(T^*M)$. Let η be a connection in $F(M)$. For given $[\lambda] \in P(T^*M)$ and $X \in T_x M$ with $\varpi([\lambda]) = x$, the *horizontal lift* $X_{[\lambda]}^H \in T_{[\lambda]}(P(T^*M))$ of X to $P(T^*M)$ with respect to η is defined by

$$X_{[\lambda]}^H = X_u^* \cdot [\xi] \quad \text{for } [\lambda] = u \cdot [\xi],$$

where $X_u^* \in T_u(F(M))$ is the horizontal lift of X to $F(M)$ with respect to η . It is also described as follows. Choose an element $\lambda \in \mathring{T}_x^*M$ so that $q(\lambda) = [\lambda]$, and let $X_\lambda^H \in T_\lambda(T^*M)$ be the horizontal lift of X to T^*M with respect to the linear connection in T^*M induced by η . Then $q_*X_\lambda^H$ is independent on the choice of λ , and is equal to $X_{[\lambda]}^H$.

Lemma 4.1. *Suppose that η and η' are torsionfree connections in $F(M)$ which are projectively equivalent. Let $[\lambda] \in P(T^*M)$ with $\varpi([\lambda]) = x$. Then for every X in*

$$[\lambda]_x^\perp = \{X \in T_x M; \lambda(X) = 0\},$$

the corresponding horizontal lifts are identical:

$$X_{[\lambda]}^H = X_{[\lambda]}^{H'}.$$

Proof. It follows from the assumption that there exists an $l_1 = V^*$ -valued function p on $F(M)$ such that $\eta - \eta' = [\theta, p]$. Take an element $\lambda = u \cdot \xi \in \mathring{T}_x^*M$ with $q(\lambda) = [\lambda]$. We shall show that

$$X_\lambda^H - X_\lambda^{H'} = p_u(\theta(X_u^*))\lambda_\lambda^V \in T_\lambda(T^*M).$$

Then, applying q_* to this we obtain the assertion. Indeed, since $\pi_* X_u^* = \pi_* X_u^{*'} = X$, there is an $A \in \mathfrak{gl}(V)$ such that $X_u^* - X_u^{*'} = A_u^*$. Then

$$\begin{aligned} (\eta - \eta')(X_u^*) &= -\eta'(X_u^*) = -\eta'(X_u^* - X_u^{*'}) \\ &= -\eta'(A_u^*) = -A. \end{aligned}$$

On the other hand, the lefthand side is equal to $[\theta, p](X_u^*) = [v, p_u]$ where $v = \theta(X_u^*)$, and hence $A = -[v, p_u]$. Therefore, we have

$$\begin{aligned} X_\lambda^H - X_\lambda^{H'} &= (X_u^* - X_u^{*'}) \cdot \xi = A_u^* \cdot \xi \\ &= u \cdot (A \cdot \xi) = -u \cdot ([v, p_u] \cdot \xi), \end{aligned}$$

under the identification $T_\xi V^* = V^*$, where $A \cdot \xi$ denotes the natural action of $\mathfrak{gl}(V)$ on V^* . Here for $v = {}^t(v^1, \dots, v^n) \in V = \mathbb{I}_{-1}$, $p_u = (p_1, \dots, p_n) \in V^* = \mathbb{I}_1$ and $\xi = (\xi_1, \dots, \xi_n) \in \dot{V}^*$, by Lemma 3.1 we have

$$\begin{aligned} [v, p_u] \cdot \xi &= (vp_u + (p_u v)1_n) \cdot \xi = -\xi(vp_u + (p_u v)1_n) \\ &= -\xi(v)p_u - p_u(v)\xi. \end{aligned}$$

Thus we obtain

$$X_\lambda^H - X_\lambda^{H'} = \xi(v)u \cdot p_u + p_u(v)u \cdot \xi = p_u(v)\lambda,$$

since $\xi(v) = (u \cdot \xi)(X) = \lambda(X) = 0$. This implies the required equality.

Now suppose that a projective structure $Q \subset F^2(M)$ is given. Let D be the canonical contact structure on $P(T^*M)$. For a given $[\lambda] \in P(T^*M)$ with $\varpi([\lambda]) = x$, take a local torsionfree connection η belonging to Q defined over a neighbourhood of x , and set

$$E_{[\lambda]} = \{X_{[\lambda]}^H; X \in [\lambda]_x^\perp\},$$

$X_{[\lambda]}^H$ being the horizontal lift of X to $P(T^*M)$ with respect to η . By Lemma 4.1 it is determined by Q independently on the choice of η . Further we set

$$E'_{[\lambda]} = \text{Kernel } \varpi_*: T_{[\lambda]}(P(T^*M)) \rightarrow T_x M.$$

These determine subbundles E and E' of $T(P(T^*M))$.

Theorem 4.2. *The triple $(D; E, E')$ above is a Lagrangean contact*

structure on $P(T^*M)$. This will be called associated to Q .

Proof. Recall that D is given by

$$D_{[\lambda]} = \text{Kernel}(s^*\alpha)_{[\lambda]},$$

taking a local section s of $q: \mathring{T}^*M \rightarrow P(T^*M)$ around $[\lambda]$. We set $\lambda = s([\lambda])$. First note that then we have

$$s_*X_{[\lambda]}^H - X_\lambda^H \subset \text{Kernel}(q_*)_\lambda \in \text{Kernel}(p_*)_\lambda,$$

because $q_*s_*X_{[\lambda]}^H - q_*X_\lambda^H = X_{[\lambda]}^H - X_\lambda^H = 0$. Now for each $X \in [\lambda]_x^\perp$ we have that $\alpha(s_*X_{[\lambda]}^H - X_\lambda^H) = 0$ by the remark above, and hence

$$(s^*\alpha)(X_{[\lambda]}^H) = \alpha(X_\lambda^H) = \lambda(p_*X_\lambda^H) = \lambda(X) = 0.$$

Thus we get $E_{[\lambda]} \subset D_{[\lambda]}$. Furthermore, for each $\bar{X} \in E'_{[\lambda]}$ we have

$$p_*s_*\bar{X} = \varpi_*q_*s_*\bar{X} = \varpi_*\bar{X} = 0,$$

and so

$$(s^*\alpha)(\bar{X}) = \alpha(s_*\bar{X}) = \lambda(p_*s_*\bar{X}) = 0.$$

Therefore, we have also $E'_{[\lambda]} \subset D_{[\lambda]}$.

Next, we shall show that $d(s^*\alpha)(X_{[\lambda]}^H, Y_{[\lambda]}^H) = 0$ holds for every $X, Y \in [\lambda]_x^\perp$. Indeed, by the remark above we can write

$$s_*X_{[\lambda]}^H = X_\lambda^H + a\lambda_\lambda^V, \quad s_*Y_{[\lambda]}^H = Y_\lambda^H + b\lambda_\lambda^V, \quad a, b \in \mathbf{R},$$

and so by Example 1.1 we have

$$\begin{aligned} d(s^*\alpha)(X_{[\lambda]}^H, Y_{[\lambda]}^H) &= d\alpha(X_\lambda^H + a\lambda_\lambda^V, Y_\lambda^H + b\lambda_\lambda^V) \\ &= \frac{1}{2}(a\lambda(Y) + b\lambda(X)) = 0. \end{aligned}$$

It remains to show that $d(s^*\alpha)(E'_{[\lambda]}, E'_{[\lambda]}) = \{0\}$. But this is clear since E' is an integrable subbundle of $T(P(T^*M))$.

5. Cartan connections associated to Lagrangean contact structures

In this section we assume that $n \geq 2$ and retain the notation in Section 3. Let $G = L = PL(n+1)$ and $\mathfrak{g} = \text{Lie } G = \mathfrak{l} = \mathfrak{sl}(n+1)$ so that $G \subset \text{Aut}(\mathfrak{g})$. We set

$$E = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & -1 \end{pmatrix} \in \mathfrak{g},$$

$$\mathfrak{g}_p = \{X \in \mathfrak{g}; [E, X] = pX\} \quad p = -2, -1, 0, 1, 2,$$

which determines a GLA-structure on \mathfrak{g} :

$$\mathfrak{g} = \mathfrak{g}_{-2} + \mathfrak{g}_{-1} + \mathfrak{g}_0 + \mathfrak{g}_1 + \mathfrak{g}_2, \quad [\mathfrak{g}_p, \mathfrak{g}_q] \subset \mathfrak{g}_{p+q}.$$

We set

$$\mathfrak{m} = \mathfrak{g}_{-2} + \mathfrak{g}_{-1}, \quad \mathfrak{m}^* = \mathfrak{g}_1 + \mathfrak{g}_2,$$

$$\mathfrak{g}' = \mathfrak{g}_0 + \mathfrak{g}_1 + \mathfrak{g}_2,$$

so that $\mathfrak{g} = \mathfrak{m} + \mathfrak{g}'$ (direct sum as vector space). Then \mathfrak{m} becomes a fundamental GLA of contact type of degree n . Since $B|_{\mathfrak{m} \times \mathfrak{m}^*}$ is nondegenerate, \mathfrak{m}^* is identified with the dual space of \mathfrak{m} through B . These subspaces \mathfrak{g}_p are explicitly given as follows.

$$\mathfrak{g}_{-2} = \left\{ \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ \alpha & 0 & 0 \end{pmatrix}; \alpha \in \mathbf{R} \right\}, \quad \mathfrak{g}_2 = \left\{ \begin{pmatrix} 0 & 0 & \alpha \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}; \alpha \in \mathbf{R} \right\},$$

$$\mathfrak{g}_{-1} = \left\{ \begin{pmatrix} 0 & 0 & 0 \\ b_1 & 0 & 0 \\ 0 & {}^t b_2 & 0 \end{pmatrix}; b_1, b_2 \in \mathbf{R}^{n-1} \right\}, \quad \mathfrak{g}_1 = \left\{ \begin{pmatrix} 0 & {}^t b_1 & 0 \\ 0 & 0 & b_2 \\ 0 & 0 & 0 \end{pmatrix}; b_1, b_2 \in \mathbf{R}^{n-1} \right\},$$

$$\mathfrak{g}_0 = \left\{ \begin{pmatrix} \alpha & 0 & 0 \\ 0 & A & 0 \\ 0 & 0 & \beta \end{pmatrix}; \alpha, \beta \in \mathbf{R}, A \in \mathfrak{gl}(n-1), \operatorname{tr} A = -(\alpha + \beta) \right\}.$$

Let G_0 denote the group of GLA-automorphisms of \mathfrak{g} . It is given by

$$G_0 \cong \left\{ \begin{pmatrix} a & 0 & 0 \\ 0 & b & 0 \\ 0 & 0 & c \end{pmatrix}; a, c \in \mathbf{R}^*, b \in GL(n-1) \right\} / C,$$

and thus $G_0 \subset G$ and $\text{Lie } G_0 = \mathfrak{g}_0$. It holds that $G = G_0 \text{Inn}(\mathfrak{g})$. We define $G_2 = \exp \mathfrak{g}_2$ and $G' = N_G(\mathfrak{g}')$, whose Lie algebras are \mathfrak{g}_2 and \mathfrak{g}' , respectively. Note that $G' \subset L'$. Furthermore, \mathfrak{g}_0 may be considered as an algebra of GLA-derivations of \mathfrak{m} through the adjoint action. It is known (Yamaguchi [6]) that then the GLA \mathfrak{g} is the prolongation of $(\mathfrak{m}, \mathfrak{g}_0)$ in the sense of Tanaka[5].

Let $\mathcal{F}(W)$ be the manifold of flags of W of type $(1, n)$, that is, the manifold of all pairs $([w], \sigma) \in P_n \times G_n(W)$ with $[w] \subset \sigma$, where $G_n(W)$ denotes the Grassmann manifold of n -subspaces of W . Let $\tilde{\omega}_0: \mathcal{F}(W) \rightarrow P^n$ denote the projection $([w], \sigma) \mapsto [w]$ to the first factor. For $[w] \in P^n$, since $T_{[w]}P^n$ is linearly isomorphic to $W/[w]$, $T_{[w]}^*(P^n)$ is linearly isomorphic to

$$[w]^\perp = \{ \zeta \in W^*; \zeta(w) = 0 \},$$

and hence we can identify $P(T_{[w]}^*P^n)$ with $P([w]^\perp)$ in a canonical way. For $[\zeta] \in P([w]^\perp) = P(T_{[w]}^*P^n)$ we define an element $\sigma \in G_n(W)$ with $[w] \subset \sigma$ by $\sigma = \{ v \in W; \zeta(v) = 0 \}$. Then the correspondence $[\zeta] \mapsto ([w], \sigma)$ gives a diffeomorphism of $P(T^*P^n)$ onto $\mathcal{F}(W)$ by which the projection $\varpi_0: P(T^*P^n) \rightarrow P^n$ corresponds to our projection ω_0 . On the other hand, G acts transitively on $\mathcal{F}(W)$, where the isotropy subgroup in G at the standard flag $([w_0], [w_0, \dots, w_{n-1}])$ is identical with G' . Therefore, we have an identification

$$G/G' = P(T^*P^n),$$

by which the origin G' of G/G' corresponds to the point $[\zeta^n] \in P(T_{[w_0]}^*P^n) \subset P(T^*P^n)$. Note that under our identification the action of G on G/G' corresponds to the natural action $\varphi \mapsto \hat{\varphi}$ of $G = PL(n+1)$ on $P(T^*P^n)$. This induces an identification $T_{[\zeta^n]}P(T^*P^n) = \mathfrak{m}$. Let $\rho: G' \rightarrow GL(\mathfrak{m})$ denote the linear isotropy representation at $[\zeta^n]$, and set

$$\tilde{G} = \rho(G') \subset GL(\mathfrak{m}).$$

Then we have that $\text{Kernel } \rho = G_2$ and $\tilde{G} = \rho(G_0)\rho(\exp \mathfrak{g}_1)$.

Next, we set

$$\mathfrak{e} = \mathfrak{g}_{-1} \cap \mathfrak{l}_{-1}, \quad \mathfrak{e}' = \mathfrak{g}_{-1} \cap \mathfrak{l}_0.$$

Then $(\mathfrak{e}, \mathfrak{e}')$ is a Lagrangean pair of (\mathfrak{g}_{-1}, A_0) , and hence, $(\mathfrak{m}; \mathfrak{e}, \mathfrak{e}')$ is a

fundamental GLA of Lagrangean contact type. If we set

$$e_0 = -E_{n+1,1},$$

$$e_i = E_{i+1,1}, \quad e_{n-1+i} = E_{n+1,i+1} \quad \text{for } 1 \leq i \leq n-1,$$

we have $(e_i, e_j) = \delta_{ij}$, and $\{e_0, e_1, \dots, e_{2n-2}\}$ is a Lagrangean contact basis of $(\mathfrak{m}; \mathfrak{e}, \mathfrak{e}')$. With respect to this basis, ρ is given by

$$G_0 \ni \begin{pmatrix} a & 0 & 0 \\ 0 & b & 0 \\ 0 & 0 & c \end{pmatrix} \bmod C \mapsto \begin{pmatrix} a^{-1}c & 0 & 0 \\ 0 & a^{-1}b & 0 \\ 0 & 0 & c'b^{-1} \end{pmatrix},$$

$$\exp \mathfrak{g}_1 \ni \exp \begin{pmatrix} 0 & {}^t b_1 & 0 \\ 0 & 0 & b_2 \\ 0 & 0 & 0 \end{pmatrix} \mapsto \begin{pmatrix} 1 & 0 & 0 \\ b_2 & 1_{n-1} & 0 \\ -b_1 & 0 & 1_{n-1} \end{pmatrix},$$

and thus $\tilde{G} = \rho(G_0)\rho(\exp \mathfrak{g}_1)$ is represented by

$$\tilde{G} = \left\{ \begin{pmatrix} c & 0 & 0 \\ b_1 & a & 0 \\ b_2 & 0 & c'a^{-1} \end{pmatrix} ; c \in \mathbf{R}^*, b_1, b_2 \in \mathbf{R}^{n-1}, a \in GL(n-1) \right\}.$$

Therefore, \tilde{G} is identical with the Lagrangean contact group $C(\mathfrak{m}; \mathfrak{e}, \mathfrak{e}')$. Thus we have proved

Theorem 5.1. *The \tilde{G} -structures of type \mathfrak{m} are in bijective correspondence with the Lagrangean contact structures.*

EXAMPLE 5.2. Let $Q_1 \subset F^2(P^n)$ be the flat projective structure on P^n and $(D_0; E_0, E'_0)$ the Lagrangean contact structure on $P(T^*P^n)$ associated to Q_1 . Then the \tilde{G} -structure of type \mathfrak{m} on $P(T^*P^n)$ corresponding to $(D_0; E_0, E'_0)$ is given as follows. We embed \mathfrak{m} into $P(T^*P^n)$ as an open set containing $[\zeta^n]$ by the map $X \mapsto (\exp X)[\zeta^n]$, and so each $a \in G$ determines a local diffeomorphism $a: (\mathfrak{m}, 0) \rightarrow P(T^*P^n)$. We define a map $\hat{\rho}_0: G \rightarrow F(P(T^*P^n))$ by

$$\hat{\rho}_0(a) = j_0^1(a) \quad \text{for } a \in G.$$

Then we have that $\hat{\rho}_0(z \cdot a) = \hat{\rho}_0(z) \cdot \rho(a)$ for $z \in G$ and $a \in G'$, and the image

$\tilde{P}_g = \hat{\rho}_0(G) \subset F(P(T^*P^n))$ is a \tilde{G} -structure such that $\tilde{P}_g = F_{(D_0; E_0, E'_0)}(P(T^*P^n))$. We call \tilde{P}_g the flat \tilde{G} -structure of type \mathfrak{m} on $P(T^*P^n)$.

A \tilde{G} -structure $\tilde{P} \subset F(M)$ of type \mathfrak{m} or the corresponding Lagrangean contact structure is said to be *flat* if (M, \tilde{P}) is locally isomorphic to $(P(T^*P^n), \tilde{P}_g)$ as \tilde{G} -structure.

Now we recall the result of Tanaka on Cartan connections associated to \tilde{G} -structures of type \mathfrak{m} . Let P be a principal G' -bundle over a manifold M of dimension $2n-1$ and ω a Cartan connection in P of type G/G' in the same sense as in Section 3. Let

$$\omega = \omega_{-2} + \omega_{-1} + \omega_0 + \omega_1 + \omega_2$$

be the decomposition of ω into the sum of \mathfrak{g}_p -components ω_p . Then the curvature $\Omega = d\omega + \frac{1}{2}[\omega, \omega]$ of ω can be written

$$\Omega = \frac{1}{2}K((\omega_{-2} + \omega_{-1}) \wedge (\omega_{-2} + \omega_{-1}))$$

by a $\mathfrak{g} \otimes \Lambda^2 \mathfrak{m}^*$ -valued function K on P . We say that ω is *normal* if it satisfies the following two conditions.

- (1) The $\mathfrak{g}_{-2} \otimes \Lambda^2 \mathfrak{g}_{-1}^*$ -component of K vanishes on P .
- (2) If $\{e_0, e_1, \dots, e_{2n-2}\}$ is a basis of \mathfrak{m} with $(e_i, e_j) = \delta_{ij}$, and $\{e_0^*, e_1^*, \dots, e_{2n-2}^*\}$ the basis of \mathfrak{m}^* dual to $\{e_i\}$ with respect to B , then

$$(\partial^* K)(X) = \sum_i [e_i^*, K(e_i, X)] + \frac{1}{2} \sum_i K([e_i^*, X]_{\mathfrak{m}}, e_i) = 0 \quad \text{for } X \in \mathfrak{m},$$

where in general $X_{\mathfrak{m}}$ denotes the \mathfrak{m} -component of $X \in \mathfrak{g}$ with respect to the decomposition $\mathfrak{g} = \mathfrak{m} + \mathfrak{g}'$.

Let P be a principal G' -bundle over M endowed with a Cartan connection ω of type G/G' and $\tilde{P} \subset F(M)$ a \tilde{G} -structure of type \mathfrak{m} with the restriction θ to \tilde{P} of the canonical form on $F(M)$. We say that (P, ω) is *associated to* \tilde{P} , if there exists a group reduction $\hat{\rho}: P \rightarrow \tilde{P}$ relative to ρ , namely, a bundle map $\hat{\rho}$ inducing the identity on M and satisfying $\hat{\rho}(z \cdot a) = \hat{\rho}(z) \cdot \rho(a)$ for $z \in P$ and $a \in G'$, such that

$$\hat{\rho}^* \theta = \omega_{-2} + \omega_{-1}.$$

EXAMPLE 5.3. Let \tilde{P}_g be the flat \tilde{G} -structure of type \mathfrak{m} on $P(T^*P^n)$. Set $P_g = G$, which is a principal G' -bundle over $P(T^*P^n)$. Then the Maurer-Cartan form ω of G is a normal Cartan connection in P_g of

type G/G' with the curvature $\Omega=0$ such that (P_g, ω) is associated to \tilde{P}_g .

Theorem 5.4. (Tanaka [5]) *For any \tilde{G} -structure \tilde{P} of type m on a manifold M of dimension $2n-1$, there exist a principal G' -bundle P over M and a normal Cartan connection ω in P of type G/G' such that (P, ω) is associated to \tilde{P} . Our (P, ω) is unique in the following sense. Let (P, ω) and (P', ω') be associated to \tilde{P} and \tilde{P}' with canonical forms θ and θ' by group reductions $\hat{\rho}$ and $\hat{\rho}'$, respectively. Then*

(a) *for any G' -bundle isomorphism $\varphi: P \rightarrow P'$ with $\varphi^*\omega' = \omega$ there exists a G -bundle isomorphism $\tilde{\varphi}: \tilde{P} \rightarrow \tilde{P}'$ with $\tilde{\varphi}^*\theta' = \theta$ which is induced by φ in the sense that $\hat{\rho}' \circ \varphi = \tilde{\varphi} \circ \hat{\rho}$; and*

(b) *conversely, for any \tilde{G} -bundle isomorphism $\tilde{\varphi}: \tilde{P} \rightarrow \tilde{P}'$ with $\tilde{\varphi}^*\theta' = \theta$ there exists uniquely a G' -bundle isomorphism $\varphi: P \rightarrow P'$ with $\varphi^*\omega' = \omega$ which induces $\tilde{\varphi}$.*

In the same way as in Corollary 3.5 we get the following.

Corollary 5.5. *\tilde{P} is flat if and only if the curvature Ω of ω vanishes on P .*

6. Cartan connections associated to projective cotangent bundles

Let M be a manifold of dimension ≥ 2 and $\tilde{\omega}: P(T^*M) \rightarrow M$ the projective cotangent bundle of M . Fix a projective structure $\tilde{\pi}^2: Q \rightarrow M$ on M . Let $(D; E, E')$ be the Lagrangean contact structure on $P(T^*M)$ associated to Q , and $\pi: \tilde{P} \rightarrow P(T^*M)$ the \tilde{G} -structure of the Lagrangean contact frames of $(P(T^*M), D; E, E')$. We define maps $\hat{\rho}: Q \rightarrow F(P(T^*M))$ and $\hat{\pi}: Q \rightarrow P(T^*M)$ as follows. Let $z = j_0^2(f) \in Q$ where $f: (V, 0) \rightarrow M$ is a local diffeomorphism. We embed V into P^n as an open set containing $[w_0]$ as in Section 3. Then f induces a local diffeomorphism $\hat{f}: (P(T^*P^n), [\zeta^n]) \rightarrow P(T^*M)$, and the differential

$$(\hat{f}_*)_{[\zeta^n]}: m = T_{[\lambda^n]}P(T^*P^n) \rightarrow T_{[\lambda]}P(T^*M),$$

where $[\lambda] = \hat{f}([\zeta^n])$, is a linear isomorphism. We define

$$\hat{\rho}(z) = (\hat{f}_*)_{[\zeta^n]} \in F(P(T^*M)),$$

$$\hat{\pi}(z) = \hat{f}([\zeta^n]) \in P(T^*M).$$

Then we have that $\pi \circ \hat{\rho} = \hat{\pi}$, $\tilde{\omega} \circ \hat{\pi} = \tilde{\pi}^2$, and

$$(*) \quad \hat{\rho}(z \cdot a) = \hat{\rho}(z) \cdot \rho(a) \quad \text{for } z \in Q, \quad a \in G'.$$

Note that $\hat{\pi}$ is a surjective map.

Lemma 6.1. $\hat{\rho}(Q) = \tilde{P}$.

Proof. We may assume that f above is given by

$$f(v) = \text{Exp}^\eta u(v) \quad \text{for } v \in V,$$

where η is a local torsionfree connection belonging to Q defined over a neighbourhood of $x = \bar{\pi}^2(z)$ and $u = \bar{\pi}_1^2(z) \in F(M)$, $\bar{\pi}_1^2$ being the projection $F^2(M) \rightarrow F(M)$. We make use of the basis $\{\bar{e}_i\}$ of l_{-1} in Section 3, its dual basis $\{\xi^i\}$, and the basis $\{e_i\}$ of \mathfrak{m} in Section 5. Let $q: \hat{T}^*M \rightarrow P(T^*M)$ and $q_0: \hat{T}^*P^n \rightarrow P(T^*P^n)$ be the natural projections. Then we have the following commutative diagram.

$$\begin{array}{ccc} T_{\xi^n}(\hat{T}^*P^n) = V + V^* & \xrightarrow{(f^*)_*^{-1}} & T_{\lambda}(T^*M) \\ q_{0*} \downarrow & & q_* \downarrow \\ T_{[\xi^n]}(P(T^*P^n)) = \mathfrak{m} & \xrightarrow{\hat{\rho}(z)} & T_{[\lambda]}(P(T^*M)), \end{array}$$

where $V = l_{-1} = T_{[w_0]}P^n$ and $\lambda = u \cdot \xi^n$. Here the decomposition $T_{\xi^n}(\hat{T}^*P^n) = V + V^*$ is the one induced from the trivialization of T^*P^n around $[w_0]$ through the embedding $V \subset P^n$. For $1 \leq i \leq n$ we have $q_{0*}(\bar{e}_i) = e_i$ and $(f^*)_*^{-1}(\bar{e}_i) = (X_i)_\lambda^H$, where $X_i = u(\bar{e}_i) \in [\lambda]_x^\perp \subset T_xM$, and hence $\hat{\rho}(z)e_i = (X_i)_{[\lambda]}^H$. Furthermore, we have $q_{0*}(\xi^i) = e_{n-1+i}$ and $(f^*)_*^{-1}(\xi^i) = (\lambda^i)_\lambda^V$, where $\lambda^i = u \cdot \xi^i \in T_xM$, and hence $\hat{\rho}(z)e_{n-1+i} = q_*(\lambda^i)_\lambda^V$. Thus $\hat{\rho}(z)$ maps e and e' to $E_{[\lambda]}$ and $E'_{[\lambda]}$, respectively. Together with Example 2.2, we know that $\hat{\rho}(z)$ is a Lagrangean contact frame of $(P(T^*M), D; E, E')$, that is, $\hat{\rho}(z) \in \tilde{P}$. Furthermore, it follows from (*) that $\hat{\rho}(Q)$ is invariant under \tilde{G} . Thus we obtain the lemma.

Lemma 6.2. For $z, z' \in Q$, we have $\hat{\pi}(z) = \hat{\pi}(z')$ if and only if there exists an element $a \in G'$ such that $z' = z \cdot a$. Therefore $\hat{\pi}: Q \rightarrow P(T^*M)$ is a principal G' -bundle over $P(T^*M)$.

Proof. Let $z = j_0^2(f)$ and $z' = j_0^2(f')$. Suppose that $\hat{\pi}(z) = \hat{\pi}(z')$, that is, $\hat{f}([\xi^n]) = \hat{f}'([\xi'^n])$. Since then $\bar{\pi}^2(z) = \bar{\pi}^2(z')$, there exists an element $a \in L'$ such that $j_0^2(f') = j_0^2(f \circ a)$. This implies that $\hat{f}'([\xi'^n]) = \hat{f}(\hat{a}[\xi'^n])$. Therefore, from the assumption we obtain $\hat{a}[\xi'^n] = [\xi'^n]$, which means that $a \in G'$. Thus we get $z' = z \cdot a$ where $a \in G'$. The converse is clear from (*) and $\pi \circ \hat{\rho} = \hat{\pi}$.

Let $\bar{\omega}$ be the normal Cartan connection in the L' -bundle $\pi^2: Q \rightarrow M$ of type L/L' (see Theorem 3.4). Since $G' \subset L' \subset L = G$, we may regard $\bar{\omega}$ as a Cartan connection in the G' -bundle $\hat{\pi}: Q \rightarrow P(T^*M)$ of type G/G' . Let

$$\bar{\omega} = \omega_{-2} + \omega_{-1} + \omega_0 + \omega_1 + \omega_2$$

be the decomposition of $\bar{\omega}$ into the sum of \mathfrak{g} -components ω_p .

Lemma 6.3. *For the restriction θ to \tilde{P} of the canonical form on $F(P(T^*M))$, we have*

$$\hat{\rho}^*\theta = \omega_{-2} + \omega_{-1}.$$

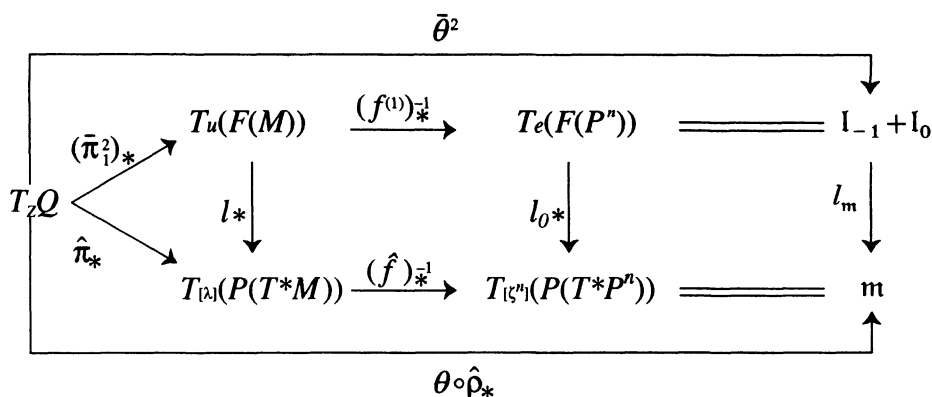
Proof. Since $l_{-1} = \mathfrak{e} + \mathfrak{g}_{-2}$, $l_0 = \mathfrak{e}' + (l_0 \cap \mathfrak{g}')$, and $\mathfrak{m} = \mathfrak{g}_{-2} + \mathfrak{e} + \mathfrak{e}'$, we get a decomposition

$$l_{-1} + l_0 = \mathfrak{m} + (l_0 \cap \mathfrak{g}') \quad (\text{direct sum as vector space}).$$

Denote by $l_m: l_{-1} + l_0 \rightarrow \mathfrak{m}$ the projection with respect to the decomposition above. Let $\bar{\theta}^2$ be the restriction to Q of the second canonical form on $F^2(M)$, which is an $l_{-1} + l_0 = V + \mathfrak{gl}(V)$ -valued 1-form on Q . First, we shall show that

$$\hat{\rho}^*\theta = l_m \circ \bar{\theta}^2.$$

For that purpose we define a map $l: F(M) \rightarrow P(T^*M)$ by $u \mapsto u \cdot [\xi^n]$, where $\{\xi^i\}$ is the one in Lemma 6.1. The corresponding map for P^n will be denoted by $l_0: F(P^n) \rightarrow P(T^*P^n)$. Note that at the point $e = \text{id}_V \in F(P^n)$ its differential $l_{0*}: T_e(F(P^n)) \rightarrow T_{[e]}(P(T^*P^n))$ corresponds to the projection l_m under the identification $T_e(F(P^n)) = l_{-1} + l_0$ induced from the local trivialization $F(P^n)|_V = V \times GL(V)$ through the embedding $V \subset P^n$. Now let $z = j_0^2(f) \in Q$ and set $u = \pi_1^2(z)$, $[\lambda] = \hat{\pi}(z) = l(u)$. Then it follows from definitions that the following diagram is commutative.


$$l_m \circ \bar{\theta}^2 = \omega_{-2} + \omega_{-1}.$$
$$\bar{\omega} = \bar{\omega}_{-1} + \bar{\omega}_0 + \bar{\omega}_1$$
$$\bar{\omega}(X) \equiv \bar{\theta}^2(X) \pmod{\mathfrak{g}'}$$
$$\bar{\omega}(X) \equiv \omega_{-2}(X) + \omega_{-1}(X) \pmod{\mathfrak{g}'}$$

Theorem 6.4. *Let Q be a projective structure on a manifold M of dimension $n \geq 2$ and $\bar{\omega}$ the normal Cartan connection in Q of type L/L' . Let \tilde{P} be the \tilde{G} -structure of type \mathfrak{m} on $P(T^*M)$ corresponding to the Lagrangean contact structure on $P(T^*M)$ associated to Q . Then $\bar{\omega}$ is a normal Cartan connection of type G/G' in the principal G' -bundle $\hat{\pi}: Q \rightarrow P(T^*M)$ such that $(Q, \bar{\omega})$ is associated to \tilde{P} .*

Proof. It follows from Lemmas 6.1 and 6.3 that $(Q, \bar{\omega})$ is associated to \tilde{P} . The curvature $\bar{\Omega} = d\bar{\omega} + \frac{1}{2}[\bar{\omega}, \bar{\omega}]$ of $\bar{\omega}$ is written in two ways:

$$\bar{\Omega} = \frac{1}{2} \bar{K}(\bar{\omega}_{-1} \wedge \bar{\omega}_{-1}) = \frac{1}{2} K((\omega_{-2} + \omega_{-1}) \wedge (\omega_{-2} + \omega_{-1})),$$

where \bar{K} is the curvature function for the L' -bundle $\bar{\pi}^2: Q \rightarrow M$, and K the one for the G' -bundle $\pi: Q \rightarrow P(T^*M)$. They are related as

$$K(X, Y) = \bar{K}(r(X), r(Y)) \quad \text{for } X, Y \in \mathfrak{m},$$

where $r: \mathfrak{m} \rightarrow \mathfrak{l}_{-1}$ denotes the projection with respect to the decomposition $\mathfrak{m} = \mathfrak{l}_{-1} + \mathfrak{e}'$. For any $X, Y \in \mathfrak{g}_{-1}$, from $\bar{K}_{-1} = 0$ we have

$$K(X, Y) = \bar{K}(r(X), r(Y)) = \bar{K}_0(r(X), r(Y)) + \bar{K}_1(r(X), r(Y))$$

$$\in \mathfrak{l}_0 + \mathfrak{l}_1 \subset \mathfrak{g}_{-1} + \mathfrak{g}',$$

that is, the \mathfrak{g}_{-2} -component of $K(X, Y)$ is 0, and hence the $\mathfrak{g}_{-2} \otimes \Lambda^2 \mathfrak{g}_{-1}^*$ -component of K vanishes. Thus it remains to show that $\partial^* K = 0$.

Let $\{\bar{e}_1, \dots, \bar{e}_n\}$ be the basis of \mathfrak{l}_{-1} with $(\bar{e}_i, \bar{e}_j) = \delta_{ij}$ and $\{\bar{e}_1^*, \dots, \bar{e}_n^*\}$ the basis of \mathfrak{l}_1 with $B(\bar{e}_i, \bar{e}_j^*) = \delta_{ij}$, defined in Section 3. Let $\{e_0, e_1, \dots, e_{2n-2}\}$ be the basis of \mathfrak{m} with $(e_i, e_j) = \delta_{ij}$ defined in Section 5, and further define a basis $\{e_0^*, e_1^*, \dots, e_{2n-2}^*\}$ of \mathfrak{m}^* with $B(e_i, e_j^*) = \delta_{ij}$ by

$$e_0^* = -E_{1, n+1},$$

$$e_i^* = E_{1, i+1}, \quad e_{n-1+i}^* = E_{i+1, n+1} \quad \text{for } 1 \leq i \leq n-1.$$

Note that $e_0 = -\bar{e}_n, e_0^* = -\bar{e}_n^*$, and $e_i = \bar{e}_i, e_i^* = \bar{e}_i^*$ for $1 \leq i \leq n-1$, and that $r(e_0) = -\bar{e}_n$ and $r(e_i) = \bar{e}_i, r(e_{n-1+i}) = 0$ for $1 \leq i \leq n-1$. Recall that for $X \in \mathfrak{m}$ we have

$$(\partial^* K)(X) = \sum_{i=0}^{2n-2} [e_i^*, K(e_i, X)] + \frac{1}{2} \sum_{i=0}^{2n-2} K([e_i^*, X]_{\mathfrak{m}}, e_i).$$

Now we have

$$\begin{aligned} [e_i^*, K(e_i, X)] &= [e_i^*, \bar{K}(r(e_i), r(X))] \\ &= \begin{cases} [\bar{e}_n^*, \bar{K}(\bar{e}_n, r(X))] & i=0, \\ [\bar{e}_i^*, \bar{K}(\bar{e}_i, r(X))] & 1 \leq i \leq n-1, \\ 0 & n \leq i \leq 2n-2, \end{cases} \end{aligned}$$

and hence

$$\sum_{i=0}^{2n-2} [e_i^*, K(e_i, X)] = \sum_{i=1}^n [\bar{e}_i^*, \bar{K}(\bar{e}_i, r(X))] = (\bar{\partial}^* \bar{K})(r(X)).$$

Furthermore, $[e_0^*, X] \in [\mathfrak{g}_2, \mathfrak{g}_{-2} + \mathfrak{g}_{-1}] \subset \mathfrak{g}_0 + \mathfrak{g}_1 \subset \mathfrak{g}'$, and so

$$K([e_0^*, X]_{\mathfrak{m}}, e_0) = 0.$$

For $1 \leq i \leq n-1$, since $[e_i^*, \mathfrak{m}] \subset \mathfrak{l}_0$ we have $[e_i^*, \mathfrak{m}]_{\mathfrak{m}} \subset \mathfrak{e}'$, and hence $r([e_i^*, X]_{\mathfrak{m}}) = 0$. Therefore

$$K([e_i^*, X]_{\mathfrak{m}}, e_i) = \bar{K}(r([e_i^*, X]_{\mathfrak{m}}), r(e_i)) = 0.$$

For $n \leq i \leq 2n-2$, since $r(e_i) = 0$ we have

$$K([e_i^*, X]_{\mathfrak{m}}, e_i) = \bar{K}(r([e_i^*, X]_{\mathfrak{m}}), r(e_i)) = 0.$$

Consequently we get

$$(\partial^* K)(X) = (\bar{\partial}^* \bar{K})(r(X)) \quad \text{for } X \in \mathfrak{m}.$$

Since $\bar{\partial}^* \bar{K} = 0$ by normality for \bar{K} , we obtain $\partial^* K = 0$.

Now Corollaries 3.5 and 5.5 imply the following.

Corollary 6.5. *The Lagrangean contact structure on $P(T^*M)$ associated to a projective structure Q on M is flat if and only if Q is projectively flat, provided $\dim M \geq 2$.*

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