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Author(s)	Lee, Sang Youl; Park, Chan-Young
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ON THE MODIFIED GOERITZ MATRICES OF 2-PERIODIC LINKS

SANG YOUL LEE and CHAN-YOUNG PARK

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1. Introduction

An oriented link $\ell = k_1 \cup \cdots \cup k_{\mu}$ of μ components in S^3 is called a 2-*periodic* link if there is a \mathbb{Z}_2 -action on the pair (S^3, ℓ) such that the fixed point set f of the action is homeomorphic to a 1-sphere in S^3 disjoint from ℓ . It is known that f is unknotted. Hence the quotient map $p: S^3 \to S^3/\mathbb{Z}_2$ is an 2-fold cyclic branched covering branched over $p(f) = f_*$ and $p(\ell) = \ell_*$ is also an oriented link in the orbit space $S^3/\mathbb{Z}_2 \cong S^3$, which is called the *factor link* of ℓ .

In this paper, we express a relationship between the modified Goeritz matrices of a 2-periodic link ℓ and those of its factor link ℓ_* and the link $\ell_* \cup \bar{f}_*$. As an application, we give an alternative proof of the Gordon and Litherland's formular([3]): $\sigma(\ell) - Lk(\ell, \bar{f}) = \sigma(\ell_*) + \sigma(\ell_* \cup \bar{f}_*)$ for the signature $\sigma(\ell)$ of a 2-periodic null homologous oriented link ℓ in a closed 3-manifold M in the case of a 2-periodic oriented link in S^3 . We also show that $n(\ell) = n(\ell_*) + n(\ell_* \cup \bar{f}_*) - 1$, where $n(\ell)$ denotes the nullity of an oriented link ℓ and \bar{f}_* denotes the knot f_* with an arbitrary orientation.

2. Preliminaries

Let ℓ be an oriented link in S^3 and let L be its link diagram in the plane $\mathbb{R}^2 \subset \mathbb{R}^3 = S^3 - \{\infty\}$. Colour the regions of $\mathbb{R}^2 - L$ alternately black and white. Denote the white regions by X_0, X_1, \dots, X_w (We always take the unbounded region to be white and denote it by X_0). Let C(L) be the set of all crossings of L. Assign an incidence number $\eta(c) = \pm 1$ to each crossing $c \in C(L)$ as in Fig. 2.1 and define a crossing $c \in C(L)$ to be of type I or type II as indicated in Fig. 2.1.

Let S(L) denote the compact surface with boundary L, more precisely, S(L) is built up out of discs and bands. Each disc lies in $S^2 = \mathbb{R}^2 \cup \{\infty\}$ and is a closed black region less a small neighbourhood of each crossing. Each crossing gives a small half-twisted band. Let $\beta_0(L)$ denote the number of the connected components of the surface S(L).

Let $G'(L) = (g_{ij})_{0 \le i,j \le w}$, where $g_{ij} = -\sum_{c \in C_L(X_i, X_j)} \eta(c)$ for $i \ne j$ and $g_{ii} = \sum_{c \in C_L(X_i)} \eta(c)$, where $C_L(X_i) = \{c \in C(L) | c \text{ is incident to } X_i\}$ and $C_L(X_i, X_j) = \{c \in C(L) | c \text{ is incident to both } X_i \text{ and } X_j\}$.

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Fig. 2.1.

The principal minor $G(L) = (g_{ij})_{1 \le i,j \le w}$ of G'(L) is called the *Goeritz matrix* of ℓ associated to the diagram L([1], [2]).

Let $C_{II}(L) = \{c_1, c_2, \dots, c_p\}$ denote the set of all crossings of type II in Land let $A(L) = diag(-\eta(c_1), -\eta(c_2), \dots, -\eta(c_p))$ be the $p \times p$ diagonal matrix. Then Traldi([5]) defined the *modified Goeritz matrix* H(L) of ℓ associated to L by $H(L) = G(L) \oplus A(L) \oplus B(L)$, where B(L) denotes the $(\beta_0(L) - 1) \times (\beta_0(L) - 1)$ zero matrix.

Two integral matrices H_1 and H_2 are said to be *equivalent*, denoted by $H_1 \approx H_2$, if they can be transformed into each other by a finite number of the following two types of transformations and their inverses:

(I) $H \to UHU^t$, where U is a unimodular matrix of integers,

(II)
$$H \to H \oplus \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}$$
.

If L_1 and L_2 are link diagrams of ambient isotopic links, then $H(L_1)$ and $H(L_2)$ are equivalent([5]).

The signature $\sigma(\ell)$ and the nullity $n(\ell)$ of an oriented link ℓ in S^3 are given by the formulars: $\sigma(\ell) = \sigma(H(L))$, $n(\ell) = n(H(L)) + 1$, where $\sigma(H(L))$ and n(H(L))are the signature and nullity of the matrix H(L) respectively([4], [5]). The absolute value of the determinant of the modified Goeritz matrix H(L) associated to a diagram L of a link ℓ is clearly an invariant of the link type ℓ , denoted by $|det(\ell)|$.

3. The modified Goeritz matrices of 2-periodic links

Let $\ell = k_1 \cup \cdots \cup k_{\mu}$ be a 2-periodic oriented link of μ components in S^3 . Then we may assume that the homeomorphism of the pair (S^3, ℓ) induced by the periodic \mathbb{Z}_2 -action is the standard rotation ϕ of \mathbb{R}^3 through π about the z-axis and hence the fixed point set f is the z-axis $\cup \infty$. We choose the standard orientation on the z-axis and denote it by \bar{f} . Define $Lk(\ell, \bar{f}) = \sum_{i=1}^{\mu} link(k_i, \bar{f})$, where $link(k_i, \bar{f})$ denotes the linking number of k_i and \bar{f} .

Applying an isotopy deformation if necessary, we may assume that ℓ is represented by a 2-periodic oriented diagram L in an annulus in \mathbb{R}^2 , which is divided into 2 pieces L_1 and L_2 such that $\varphi(L_1) = L_2, \varphi(L_2) = L_1$, where φ is the rotation of \mathbb{R}^2 through π about the origin. Let $a_1, a_2, \dots, a_r, a_{r+1}, \dots, a_m$ denote the intersec-



Fig. 3.1.

tion points of L with the line $\theta = 0$ and let $\varphi(a_i) = b_i, i = 1, 2, \dots, m$, as shown in Fig. 3.1. Note that $Lk(\ell, \bar{f}) = 2r - m$.

Colour the regions of $\mathbb{R}^2 - L$ alternately black and white. Without the loss of generality we may assume that the surface S(L) is connected and the orientation of ℓ is as indicated in Fig. 3.1. If not, by applying ambient isotopy deformations in $\mathbb{R}^3 - f$, i.e., the Reidemeister moves in $\mathbb{R}^2 - \{0\}$ (hence $Lk(\ell, \bar{f})$ is not changed), L can be deformed to L' so that L' is also a 2-periodic link diagram of ℓ , which has the required orientation and S(L') is connected. Now let $\varphi_* : \mathbb{R}^2 \to \mathbb{R}^2/\varphi \cong \mathbb{R}^2$ be the quotient map and let $\varphi_*(L) = L_*$. Then L_* is a link diagram of the factor link ℓ_* of ℓ .

In the case of $Lk(\ell, \bar{f}) \equiv 1 \pmod{2}$ we denote the white regions as follows. We denote the unbounded white region by X_0^0 . Notice that the bounded region containing the origin is then a black region. Let $X_1^1, X_1^2, \dots, X_1^k$ denote the white regions in $L_1 \subset L$ which do not intersect the line $\theta = 0$, and let $X_1^{k+1}, X_1^{k+2}, \dots, X_1^w$ (w = k + (m-1)/2) denote the white regions in L which intersect the line $\theta = 0$. For each $j = 1, 2, \dots, w$, let $X_2^j = \varphi(X_1^j)$. Note that $\varphi(X_0^0) = X_0^0$ (see Fig. 3.1). For $p \neq q$ or $i \neq j$, let $g_{pq}^{ij} = -\sum_{c \in C_L(X_p^i, X_q^j)} \eta(c)$. For p = q and i = j, let $g_{pq}^{ij} = \sum_{c \in C_L(X_p^i)} \eta(c)$. Denote $M = (g_{11}^{ij})_{1 \leq i \leq k, k+1 \leq j \leq w}, P = (g_{11}^{ij})_{1 \leq i \leq k, k+1 \leq j \leq w}, Q = (g_{12}^{ij})_{1 \leq i \leq k, k+1 \leq j \leq w}, R = (g_{12}^{ij})_{k+1 \leq i, j \leq w}, A(L_1) = diag(-\eta(c_1), -\eta(c_2), \dots, -\eta(c_s))$, where $c_i \in C_{II}(L_1)$, and I_k the $k \times k(k \geq 1)$ identity matrix.

In these notations we have the following Lemma 3.1 and Lemma 3.2.

Lemma 3.1. Let ℓ be an oriented 2-periodic link with $Lk(\ell, \bar{f}) \equiv 1 \pmod{2}$ and let L be the 2-periodic diagram of ℓ as shown in Fig. 3.1 and let $\varphi_*(L) = L_*$. Then

(1)

$$H(L) = \begin{pmatrix} M & P & O & Q \\ P^{t} & N & Q^{t} & R \\ O & Q & M & P \\ Q^{t} & R & P^{t} & N \end{pmatrix} \oplus A(L_{1}) \oplus A(L_{1}).$$

(2)

$$TH(L)T^{-1} = U \begin{bmatrix} H(L_*) \oplus \begin{pmatrix} M & P-Q \\ P^t - Q^t & N-R \end{pmatrix} \oplus A(L_1) \end{bmatrix} U^t,$$

where $T = I_w \otimes \begin{pmatrix} 1 & 1 \\ 1 & -1 \end{pmatrix} \oplus I_s \oplus I_s$ and U is a unimodular integral matrix.

Proof. (1) For p,q = 0, 1, 2, let $G_{pq} = (g_{pq}^{ij})_{1 \le i,j \le w}$. Then $G'(L) = (G_{pq})_{0 \le p,q \le 2}$. It is easy to see that the Goeritz matrix G(L) of ℓ associated to L is the matrix of the form: for an integral matrix X,

$$G(L) = \begin{pmatrix} G_{11} & G_{12} \\ G_{21} & G_{22} \end{pmatrix} = \begin{pmatrix} M & P & O & Q \\ P^t & N & X & R \\ O & X^t & M & P \\ Q^t & R^t & P^t & N \end{pmatrix}.$$

For $k+1 \leq i \leq w$ and $1 \leq j \leq w$, X_1^i and X_2^j are incident if and only if $\varphi(X_1^i) = X_2^i$ and $\varphi(X_2^j) = X_1^j$ are incident, and their corresponding crossing types are the same. Thus $g_{12}^{ij} = g_{21}^{ij}$ for $k+1 \leq i \leq w, 1 \leq j \leq w$. Hence $X = (g_{12}^{ij})_{k+1 \leq i \leq w, 1 \leq j \leq k} = (g_{21}^{ij})_{k+1 \leq i \leq w, 1 \leq j \leq k} = Q^t$ and $R^t = (g_{21}^{ij})_{k+1 \leq i, j \leq w} = (g_{12}^{ij})_{k+1 \leq i, j \leq w} = R$. It is obvious that $A(L) = A(L_1) \oplus A(L_1)$.

(2) Note that the colouring of the diagram L induces the colouring of the diagram L_* of ℓ_* . Let $X^0 = \varphi_*(X_0^0), X^j = \varphi_*(X_1^j)$ for each $j = 1, \dots, w$. Then $\{X^j | j = 0, 1, \dots, w\}$ is the set of all white regions of L_* . Hence $G(L_*) = (g_{ij})_{1 \leq i, j \leq w}$, where $g_{ij} = -\sum_{c \in C_{L_*}(X^i, X^j)} \eta(c) (i \neq j), g_{ii} = \sum_{c \in C_{L_*}(X^i)} \eta(c)$. If $1 \leq i \leq k$, then X_1^i intersect neither the line $\theta = 0$ nor the line $\theta = \pi$. So $(g_{ij})_{1 \leq i, j \leq k} = (g_{11}^{ij})_{1 \leq i, j \leq k} = M$. Notice that for $k + 1 \leq j \leq w$, the region X^j of L_* is $\varphi_*((X_1^j \cup X_2^j) \cap L_1)$. So $(g_{ij})_{1 \leq i \leq k, k+1 \leq j \leq w} = (g_{11}^{ij} + g_{12}^{ij})_{1 \leq i \leq k, k+1 \leq j \leq w} = P + Q$. Let $\bar{g}_{pq}^{ij} = -\sum_{c \in C_{L_1}(X_p^i, X_q^j)} \eta(c)$, and $\bar{g}_{pq}^{ij} = -\sum_{c \in C_{L_2}(X_p^i, X_q^j)} \eta(c)$. Then $(g_{ij})_{k+1 \leq i, j \leq w} = (\bar{g}_{11}^{ij} + \bar{g}_{12}^{ij})_{k+1 \leq i, j \leq w} = (\bar{g}_{11}^{ij} + \bar{g}_{12}^{ij})_{k+1 \leq i, j \leq w} + (g_{12}^{ij})_{k+1 \leq i, j \leq w}$ = N + R. Since $A(L_*) = A(L_1), H(L_*) = \begin{pmatrix} M & P + Q \\ P^t + Q^t & N + R \end{pmatrix} \oplus A(L_1)$ and so the result follows by easy calculations.



Fig. 3.2.

Now we consider the modified Goeritz matrix of the oriented link $\ell_* \cup \bar{f}_*$ in $S^3/\mathbb{Z}_2 \cong S^3$, where $\bar{f}_* = p(\bar{f})$. From Fig. 3.1 we can obtain the diagram $L_* \cup \bar{F}_*$ in Fig. 3.2 as a diagram of $\ell_* \cup \bar{f}_*$ and denote the white regions of the coloured diagram $L_* \cup \bar{F}_*$ by X^0, X^1, \dots, X^{w+m} as indicated in Fig. 3.2. Then we obtain the following

Lemma 3.2. Let $L_* \cup \overline{F}_*$ be the diagram of $\ell_* \cup \overline{f}_*$ shown in Fig. 3.2. Then

$$VH(L_* \cup \bar{F}_*)V^t = \begin{pmatrix} M & P-Q \\ P^t - Q^t & N-R \end{pmatrix} \oplus A(L_1) \oplus \begin{pmatrix} N_2 & I_{(m-1)/2} \\ I_{(m-1)/2} & O \end{pmatrix}$$
$$\oplus E \oplus (2),$$

where V is an unimodular integral matrix, N_2 is an integral matrix, $E = -I_r \oplus I_{m-r-1}$ if r is even, and $E = -I_{r+1} \oplus I_{m-r}$ if r is odd.

Proof. Let $G'(L_* \cup \bar{F}_*) = (g_{ij})_{0 \le i,j \le w+m}$, where $g_{ij} = -\sum_{c \in C_{L_* \cup \bar{F}_*}(X^i, X^j)} \eta(c)$. We may identify X^i in $L_* \cup \bar{F}_*$ with X_1^i in L for $i = 1, \dots, w$. Let $\bar{g}_{pq}^{ij} = -\sum_{c \in C_{L_1}(X_p^i, X_q^j)} \eta(c)$ for p, q = 0, 1, 2 and $L_1 \subset L$. Denote $E_1 = (g_{0j})_{1 \le j \le k} = (g_{01}^{0j})_{1 \le j \le k}, E_2 = (g_{0j})_{k+1 \le j \le w} = (\bar{g}_{01}^{0j})_{k+1 \le j \le w}, E_3 = (g_{0w+j})_{1 \le j \le (m-1)/2} = (\bar{g}_{02}^{0k+j})_{1 \le j \le (m-1)/2}$, and $E_4 = (g_{0j})_{w+(m-1)/2+1 \le j \le w+m} = (-2 \ 0 \ \cdots \ 0)$. Notice that $E_2 + E_3 = (g_{01}^{0j})_{k+1 \le j \le w}$.

For $1 \leq i, j \leq k$, $(g_{ij}) = (g_{11}^{ij}) = M$. For $1 \leq i \leq k$ and $k+1 \leq j \leq w$, $(g_{ij}) = (g_{11}^{ij}) = P$. For $1 \leq i \leq k$ and $1 \leq j \leq (m-1)/2$, since $g_{iw+j} = g_{12}^{ik+j}$,

 $(g_{iw+j}) = (g_{12}^{ik+j}) = (g_{12}^{st})_{1 \le s \le k, k+1 \le t \le w} = Q.$

For $1 \le i \le k$ and $w + (m-1)/2 + 1 \le j \le w + m$, X^i and X^j are not incident for each pair of i and j. Thus $(g_{ij}) = O$, the $k \times (m-1)/2$ zero matrix.

If $k+1 \leq i \leq w$ and $w + (m-1)/2 + 1 \leq j \leq w + m$, then the regions X^i and X^j are incident exactly at one crossing only for j = w + (m-1)/2 + i and j = w + (m-1)/2 + i + 1 whose incidence number is -1 for j = w + (m-1)/2 + i and 1 for j = w + (m-1)/2 + i + 1. So $(g_{ij}) = J$, where J is the $(m-1)/2 \times ((m-1)/2 + 1)$ matrix of the form: for m = 1, $J = \emptyset$ and for m > 1,

$$J = \begin{pmatrix} 1 & -1 & 0 & \cdots & 0 & 0 \\ 0 & 1 & -1 & \cdots & 0 & 0 \\ \vdots & \vdots & \vdots & \ddots & \vdots & \vdots \\ 0 & 0 & 0 & \cdots & 1 & -1 \end{pmatrix}$$

Similarly for $w + 1 \le i \le w + (m - 1)/2$ and $w + (m - 1)/2 + 1 \le j \le w + m$, $(g_{ij}) = J$.

Now let $N_1 = (g_{ij})_{k+1 \le i,j \le w}, N_2 = (g_{ij})_{w+1 \le i,j \le w+(m-1)/2}, R_1 = (g_{ij})_{k+1 \le i \le w, w+1 \le j \le w+(m-1)/2}, K = (g_{ij})_{w+(m-1)/2+1 \le i,j \le w+m}$. Then

$$G'(L_* \cup \bar{F}_*) = (g_{ij})_{0 \le i,j \le w+m} = \begin{pmatrix} -a & E_1 & E_2 & E_3 & E_4 \\ E_1^t & M & P & Q & O \\ E_2^t & P^t & N_1 & R_1 & J \\ E_3^t & Q^t & R_1^t & N_2 & J \\ E_4^t & O & J^t & J^t & K \end{pmatrix},$$

where a is the sum of all entries of the row matrices E_1, E_2, E_3 , and E_4 .

For $w + (m-1)/2 + 1 \le i, j \le w + m$, $g_{ij} = 0 (i \ne j)$ and $g_{ii} = -d_i$, where d_i is the sum of all the *i*-th rows of E_4^t and $2J^t$. But $d_{w+m} = -2$ and the other d_i 's are all zero. Hence $K = \begin{pmatrix} O & O \\ O & 2 \end{pmatrix}$.

Now by deleting the first row and the first column of $G'(L_* \cup \overline{F}_*)$, we obtain the Goeritz matrix $G(L_* \cup \overline{F}_*)$ of $\ell_* \cup \overline{f}_*$ associated to $L_* \cup \overline{F}_*$.

Let E be the diagonal matrix whose diagonal entry corresponding to each type II crossing c in the diagram $L_* \cup \bar{F}_*$ generated by intersecting L_* with \bar{F}_* is $-\eta(c)$. If r is even, then the number of these type II crossings with incidence number +1 is equal to r and the number of these type II crossings with incidence number -1 is equal to m - r - 1 and hence $E = -I_r \oplus I_{m-r-1}$. Similarly for r odd, we have $E = -I_{r+1} \oplus I_{m-r}$. Thus $A(L_* \cup \bar{F}_*) = A(L_1) \oplus E$. Since $S(L_*)$ is connected, $S(L_* \cup \bar{F})$ is also connected. Hence $B(L_* \cup \bar{F})$ is the empty matrix. Therefore $H(L_* \cup \bar{F}_*) = G(L_* \cup \bar{F}_*) \oplus A(L_1) \oplus E$.

It is not difficult to see that $R_1 + R_1^t = R$, $N_1 + N_2 = N$ and there is a unimodular integral matrix V such that $VH(L_* \cup \overline{F}_*)V^t =$

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$$\begin{pmatrix} M & P-Q \\ P^t-Q^t & N-R \end{pmatrix} \oplus A(L_1) \oplus \begin{pmatrix} N_2 & I_{(m-1)/2} \\ I_{(m-1)/2} & O \end{pmatrix} \oplus E \oplus (2). \qquad \Box$$

Theorem 3.3. Let ℓ be an oriented 2-periodic link in S^3 with the fixed point set \overline{f} and let ℓ_* be the factor link of ℓ . Then there exist 2-periodic diagrams L and $L_* \cup \overline{F}_*$ of ℓ and $\ell_* \cup \overline{f}_*$ satisfying the following: (1) $Lk(\ell, \overline{f}) \equiv 1 \pmod{2}$.

$$S\left[H(L)\oplus \begin{pmatrix} I_a & O\\ O & -I_b \end{pmatrix}\oplus (2)\right]S^{-1}\approx \begin{pmatrix} H(L_*) & O\\ O & H(L_*\cup\bar{F}_*) \end{pmatrix}.$$

(2) $Lk(\ell, \bar{f}) \equiv 0 \pmod{2}$.

Let $l \circ u$ denote the splittable 2-periodic link consisting of l and the unknot u and let h^- denote the left handed Hopf link. Then

$$S\left[H(L\cup U)\oplus \begin{pmatrix} I_a & O\\ O & -I_{b+1} \end{pmatrix}\oplus (2)\right]S^{-1}\approx \begin{pmatrix} H(L_*) & O\\ O & H((L_*\cup\bar{F}_*)\ \sharp\ D^-) \end{pmatrix}\oplus (0),$$

where S is an invertible rational matrix, $L_* = \varphi_*(L)$, $L \cup U$ and D^- are diagrams of $\ell \circ u$ and h^- respectively, and $a - b + 1 = -Lk(\ell, \overline{f})$.

Proof. (1) It follows from Lemma 3.1 and 3.2 that

$$TH(L)T^{-1} \oplus \begin{pmatrix} N_2 & I_{(m-1)/2} \\ I_{(m-1)/2} & O \end{pmatrix} \oplus E \oplus (2) = X[H(L_*) \oplus H(L_* \cup \bar{F}_*)]X^t,$$

where X is a unimodular integral matrix. Note that

$$\begin{pmatrix} N_2 & I_{(m-1)/2} \\ I_{(m-1)/2} & O \end{pmatrix} \oplus E \oplus (2) = Y \begin{bmatrix} \begin{pmatrix} I_a & O \\ O & -I_b \end{pmatrix} \oplus (2) \end{bmatrix} Y^{-1},$$

 $a-b=m-2r-1=-Lk(\ell,\bar{f})-1$, where Y is an invertible rational matrix. This leads to the result.

(2) Let $L \cup U$ be the 2-periodic diagram of $\ell \circ u$ as shown in Fig. 3.3, where U denotes the diagram of the unknot u. Note that $Lk(\ell \cup u, \overline{f}) = Lk(\ell, \overline{f}) - 1 \equiv 1 \pmod{2}$. By (1), we obtain

$$S\begin{bmatrix} H(L\cup U) \oplus \begin{pmatrix} I_a & O\\ O & -I_b \end{pmatrix} \oplus (2) \end{bmatrix} S^{-1} = \begin{pmatrix} H(L_*\cup U_*) & O\\ O & H((L_*\cup U_*)\cup \bar{F}_*) \end{pmatrix}$$

and $a - (b+1) + 1 = Lk(\ell \circ u, \overline{f}) = -Lk(\ell, \overline{f}) - 1$. Since $L_* \cup U_*$ and $(L_* \cup U_*) \cup \overline{F}_*$ are ambient isotopic to a splittable link diagram $L_* \circ U_*$ and the connected sum $(L_* \cup U_*)$



Fig. 3.3.

 \overline{F}_*) $\#D^-$, $H(L_* \cup U_*) \approx H(L_*) \oplus (0)$ and $H((L_* \cup U_*) \cup \overline{F}_*) \approx H((L_* \cup \overline{F}_*) \# D^-)$. This implies the result.

Corollary 3.4. Let ℓ be a 2-periodic oriented link in S^3 and let ℓ_* be its factor link. Then

- (1) $\sigma(\ell) Lk(\ell, \bar{f}) = \sigma(\ell_*) + \sigma(\ell_* \cup \bar{f}_*).$
- (2) $n(\ell) = n(\ell_*) + n(\ell_* \cup \bar{f}_*) 1$, where \bar{f}_* denotes the knot f_* with an arbitrary orientation.

Proof. (1) CASE I. $Lk(\ell, \bar{f}) \equiv 1 \pmod{2}$. The relation of (1) in Theorem 3.3 gives that $\sigma(\ell) + a - b + 1 = \sigma(\ell_*) + \sigma(\ell_* \cup \bar{f}_*)$. Since $a - b + 1 = -Lk(\ell, \bar{f})$, the result follows.

CASE II. $Lk(\ell, \bar{f}) \equiv 0 \pmod{2}$. The relation of (2) in Theorem 3.3 gives that $\sigma(\ell \circ u) + a - b = \sigma(\ell_*) + \sigma((\ell_* \cup \bar{f}_*) \sharp h^-)$. Note that $\sigma(\ell \circ u) = \sigma(\ell), \sigma((\ell_* \cup \bar{f}_*) \sharp h^-) = \sigma(\ell_* \cup \bar{f}_*) + \sigma(h^-) = \sigma(\ell_* \cup \bar{f}_*) - 1$ (see [4, Lemma 7.2, 7.4]). Since $a - b = -Lk(\ell, \bar{f}) - 1$, the result follows.

(2) Since $n(H(L)) = n(\ell) + 1$, $n(H(L \cup U)) = n(H(L \circ U)) = n(H(L)) + 1$, $n(H(L_*) \oplus (0)) = n(\ell_*) + 1$, and $n(H((L_* \cup \overline{F}_*) \sharp D^-)) = n(\ell_* \cup \overline{f}_*) + n(h^-) - 1 = n(\ell_* \cup \overline{f}_*) - 1$ (see [4, Lemma 6.3, 6.4]), Theorem 3.3 implies the result.

Now reversing the orientation of the fixed point set f only changes the sign of some diagonal entries of the diagonal matrix $A(L_* \cup \overline{F}_*)$ in $H(L_* \cup \overline{F}_*)$. This implies that the equations do not depend on the choice of the orientation of f.

REMARK 3.5. Let $k_* \cup f_*$ be an oriented link in S^3 , where f_* is the unknot. Then the inverse image of k_* in the 2-fold cyclic cover $M_2(f_*)$ branched over f_* gives a 2periodic oriented link k in S^3 . Clearly any 2-periodic link in S^3 arises in this way. If k_* is a knot, then by Corollary 3.4(2) $n(k) = n(k_* \cup f_*)$.

It is well known that $|det(G(K))| = order(H_1(M_2(k);\mathbb{Z}))$ for any diagram K of a knot k, where $order(H_1(M_2(k);\mathbb{Z}))$ denotes the order of the first homology group $H_1(M_2(k);\mathbb{Z})$ of the 2-fold cyclic cover $M_2(k)$ branched over k with integer coefficients. Now if k_* is a knot and $Lk(k_*, f_*)$ is an odd integer, then k is also a knot and $n(k) = n(k_* \cup f_*) = 1$. Furthermore, |det(H(K))| = |det(G(K))| for any diagram K of the knot k. So by Theorem 3.3 (1) we obtain that

$$order(H_1(M_2(k);\mathbb{Z})) = \frac{1}{2}order(H_1(M_2(k_*);\mathbb{Z}))|det(k_* \cup f_*)|.$$

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S.Y. Lee Department of Mathematics College of Natural Sciences Kyungpook National University Taegu 702–701, Korea

C.-Y. Park Department of Mathematics College of Natural Sciences Kyungpook National University Taegu 702–701, Korea