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## HEAT KERNEL AND SINGULAR VARIATION OF DOMAINS

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### 1. Introduction

We consider a bounded region  $M$  in  $\mathbf{R}^n$  ( $n=2$  or  $3$ ) whose boundary is smooth. Let  $w$  be a fixed point in  $M$ . By  $B(\varepsilon; w)$  we denote a ball of radius  $\varepsilon$  with the center  $w$ . We put  $M_\varepsilon = M \setminus \overline{B(\varepsilon; w)}$ .

Let  $U(x, y, t)$  ( $U^{(\varepsilon)}(x, y, t)$ ; respectively) be the heat kernel in  $M$  ( $M_\varepsilon$ ; respectively) with the Dirichlet condition on its boundary  $\partial M$  ( $\partial M_\varepsilon$ ; respectively). That is, it satisfies

$$(1.1) \quad \begin{cases} (\partial_t - \Delta_x)U(x, y, t) = 0 & x, y \in M, \quad t > 0 \\ U(x, y, t) = 0 & x \in \partial M, \quad y \in M, \quad t > 0 \\ \lim_{t \rightarrow 0} U(x, y, t) = \delta(x - y) & x, y \in M \end{cases}$$

$$(1.1) \quad \begin{cases} (\partial_t - \Delta_x)U^{(\varepsilon)}(x, y, t) = 0 & x, y \in M_\varepsilon, \quad t > 0 \\ U^{(\varepsilon)}(x, y, t) = 0 & x \in \partial M_\varepsilon, \quad y \in M_\varepsilon, \quad t > 0 \\ \lim_{t \rightarrow 0} U^{(\varepsilon)}(x, y, t) = \delta(x - y) & x, y \in M_\varepsilon \end{cases}$$

We put

$$(1.2) \quad (U_t f)(x) = \int_M U(x, y, t) f(y) dy, \quad f \in L^p(M)$$

and

$$(1.3) \quad (U_t^{(\varepsilon)} f)(x) = \int_{M_\varepsilon} U^{(\varepsilon)}(x, y, t) f(y) dy, \quad f \in L^p(M_\varepsilon).$$

Then,  $U_t f$  and  $U_t^{(\varepsilon)} f$  satisfy the following.

$$(\partial_t - \Delta_x)(U_t f)(x) = 0 \quad x \in M, \quad t > 0$$

$$\left\{ \begin{array}{ll} (U_t f)(x) = 0 & x \in \partial M, \quad t > 0 \\ \|U_t f - f\|_{L^p(M)} \rightarrow 0 & \text{as } t \rightarrow 0 \end{array} \right. \\ \left\{ \begin{array}{ll} (\partial_t - \Delta_x)(U_t^{(\varepsilon)} f)(x) = 0 & x \in M_\varepsilon, \quad t > 0 \\ (U_t^{(\varepsilon)} f)(x) = 0 & x \in \partial M_\varepsilon, \quad t > 0 \\ \|U_t^{(\varepsilon)} f - f\|_{L^p(M_\varepsilon)} \rightarrow 0 & \text{as } t \rightarrow 0 \end{array} \right.$$

We want to construct an approximate kernel of  $U^{(\varepsilon)}(x, y, t)$  by using  $U(x, y, t)$ . We put

$$(1.4) \quad V^{(\varepsilon)}(x, y, t) = U(x, y, t) - L_n(\varepsilon) \int_0^t U(x, w, \tau) U(w, y, t - \tau) d\tau,$$

where

$$L_n(\varepsilon) = \begin{cases} -2\pi(\log \varepsilon)^{-1} & (\text{if } n=2) \\ 4\pi\varepsilon & (\text{if } n=3). \end{cases}$$

and we put

$$(V_t^{(\varepsilon)} f)(x) = \int_M V^{(\varepsilon)}(x, y, t) f(y) dy, \quad f \in L^p(M).$$

Let  $T$  and  $T_\varepsilon$  be operators on  $M$  and  $M_\varepsilon$ , respectively. Then,  $\|T\|_p$ ,  $\|T_\varepsilon\|_{p,\varepsilon}$  denotes the operator norm on  $L^p(M)$ ,  $L^p(M_\varepsilon)$ , respectively. Let  $f$  and  $f_\varepsilon$  be functions on  $M$  and  $M_\varepsilon$ , respectively. Then,  $\|f\|_p$ ,  $\|f_\varepsilon\|_{p,\varepsilon}$  denotes the norm on  $L^p(M)$ ,  $L^p(M_\varepsilon)$ , respectively.

Let  $\chi_\varepsilon$  denote the characteristic function of  $M_\varepsilon$ . Then, we have the following Theorems 1 and 2.

**Theorem 1.** Assume that  $n=2$ . Then, there exists a constant  $C_p$ , which may depend on  $t$  but which is independent of  $\varepsilon$  such that

$$\begin{aligned} & \|U_t^{(\varepsilon)} - \chi_\varepsilon V_t^{(\varepsilon)} \chi_\varepsilon\|_{p,\varepsilon} \\ & \leq \begin{cases} C_t \varepsilon^{1/p} |\log \varepsilon|^{-1} & \text{if } p \in (2, \infty) \\ C_t \varepsilon^{(1-s)/2} |\log \varepsilon|^{-1} & \text{if } p = 2 \\ C_t \varepsilon^{1-(1/p)} |\log \varepsilon|^{-1} & \text{if } p \in (1, 2) \end{cases} \end{aligned}$$

hold. Here  $s \in (0, 1)$  is an arbitrary fixed constant.

**Theorem 2.** Assume that  $n=3$ . Then, there exists a constant  $C_t$  independent

of  $\varepsilon$  such that

$$\begin{aligned} & \|U_t^{(\varepsilon)} - \chi_\varepsilon V_t^{(\varepsilon)} \chi_\varepsilon\|_{p,\varepsilon} \\ & \leq \begin{cases} C_t \varepsilon^{1+(2/p)} & \text{if } p \in (3, \infty) \\ C_t \varepsilon^{(5-s)/3} & \text{if } p \in [3/2, 3] \\ C_t \varepsilon^{3-(2/p)} & \text{if } p \in (1, 3/2) \end{cases} \end{aligned}$$

hold. Here  $s \in (0,1)$  is an arbitrary fixed constant.

REMARK. Thus, by Theorems 1 and 2, we know that

$$-L_n(\varepsilon) \chi_\varepsilon(x) \chi_\varepsilon(y) \int_0^t U(x, w, \tau) U(w, y, t - \tau) d\tau$$

gives a main asymptotic term of the difference between  $U^{(\varepsilon)}(x, y, t)$  and  $U(x, y, t)$ .

The Hadamard variation of the heat kernel was discussed in [2]. And we have various papers on singular variation of domain. See, for example, [3], [4], [5], [6].

We give the proof of Theorems 1 and 2 in section 2 and section 3, respectively. In Appendix we give some properties of  $U(x, y, t)$  and  $U^{(\varepsilon)}(x, y, t)$ . Following an usual custom, we use the same letter  $C$  in inequalities which are independent of  $\varepsilon$ .

## 2. Proof of Theorem 1

Throughout this section we assume that  $n=2$ . We put

$$(2.1) \quad U(x, y, t) = W(x, y, t) + S(x, y, t),$$

where

$$(2.2) \quad W(x, y, t) = (4\pi t)^{-n/2} \exp(-|x-y|^2/4t).$$

We write  $B(\varepsilon; w) = B_\varepsilon$ . Without loss of generality we may assume that  $w=0$ .

We take arbitrary  $f \in L^p(M_\varepsilon)$ . Let  $\hat{f}$  be an extension of  $f$  to  $M$  which is 0 on  $B_\varepsilon$ . At first we want to estimate  $|(V_t^{(\varepsilon)} \hat{f})(x)|_{|x \in \partial B_\varepsilon}$ . By (2.2), we have

$$\begin{aligned} (2.3) \quad & \int_0^t W(x, w, \tau) d\tau \mid x \in \partial B_\varepsilon \\ & = \int_0^t (4\pi\tau)^{-1} \exp(-\varepsilon^2/4\tau) d\tau \\ & = (4\pi)^{-1} \int_{\varepsilon^2/4t}^\infty s^{-1} e^{-s} ds \end{aligned}$$

$$\begin{aligned}
&= (4\pi)^{-1} (e^{-s} \log s) \Big|_{s=\varepsilon^2/4t}^{s=\infty} + \int_{\varepsilon^2/4t}^{\infty} e^{-s} (\log s) ds \\
&= (4\pi)^{-1} (-2 \log \varepsilon + R(\varepsilon, t)),
\end{aligned}$$

where

$$\begin{aligned}
R(\varepsilon, t) &= 2(1 - \exp(-\varepsilon^2/4t)) \log \varepsilon + (\exp(-\varepsilon^2/4t)) \log(4t) \\
&\quad + \int_{\varepsilon^2/4t}^{\infty} e^{-s} (\log s) ds.
\end{aligned}$$

Let  $\gamma$  be the Euler constant. Then,

$$\gamma = - \int_0^{\infty} e^{-s} (\log s) ds.$$

Thus, we have

$$\begin{aligned}
(2.4) \quad R(\varepsilon, t) &= 2(1 - \exp(-\varepsilon^2/4t)) \log \varepsilon \\
&\quad + (\exp(-\varepsilon^2/4t)) \log(4t) \\
&\quad - \gamma - \int_0^{\varepsilon^2/4t} e^{-s} (\log s) ds \\
&= -\gamma + \log(4t) + \int_0^{\varepsilon^2/4t} e^{-s} \log(\varepsilon^2/(4ts)) ds.
\end{aligned}$$

Since  $0 \leq \log(\varepsilon^2/(4ts)) \leq 2(\varepsilon^2/(4ts))^{1/2} = \varepsilon(ts)^{-1/2}$  hold for any  $s \in (0, \varepsilon^2/4t)$ , we have

$$\begin{aligned}
(2.5) \quad & \left| \int_0^{\varepsilon^2/4t} e^{-s} \log(\varepsilon^2/(4ts)) ds \right| \\
& \leq \varepsilon t^{-1/2} \int_0^{\varepsilon^2/4t} s^{-1/2} e^{-s} ds \\
& \leq \varepsilon t^{-1/2} \int_0^{\infty} s^{-1/2} e^{-s} ds = \pi^{1/2} \varepsilon t^{-1/2}.
\end{aligned}$$

It is easy to see that  $|\log t| \leq 2(t+t^{-1/2})$  holds for any  $t \in (0, \infty)$ . Thus, by (2.3), (2.4) and (2.5), we get

$$(2.6) \quad \int_0^t W(x, w, \tau) d\tau|_{x \in \partial B_\varepsilon} = -(2\pi)^{-1} \log \varepsilon + R(\varepsilon, t),$$

where

$$\begin{aligned}|R(\varepsilon, t)| &\leq C(|\log t| + t^{-1/2} + 1) \\ &\leq C(t + t^{-1/2} + 1)\end{aligned}$$

hold for any sufficiently small  $\varepsilon > 0$ .

On the other hand, since

$$\begin{aligned}U(w, y, t - \tau) - U(w, y, t) &= \int_0^\tau \frac{\partial}{\partial s} U(w, y, t - s) ds \\ &= - \int_0^\tau \frac{\partial}{\partial t} U(w, y, t - s) ds\end{aligned}$$

hold for  $\tau \in (0, t)$  and  $y \in M_\varepsilon$ , we see that

$$\begin{aligned}&\int_0^\tau W(x, w, \tau) U(w, y, t - \tau) d\tau \\ &= \left( \int_0^\tau W(x, w, \tau) d\tau \right) U(w, y, t) \\ &\quad - \int_0^\tau W(x, w, \tau) \left( \int_0^\tau \frac{\partial}{\partial t} U(w, y, t - s) ds \right) d\tau.\end{aligned}$$

Thus, we have

$$\begin{aligned}&\int_{M_\varepsilon} \left( \int_0^\tau W(x, w, \tau) U(w, y, t - \tau) d\tau \right) f(y) dy \\ &= \left( \int_0^\tau W(x, w, \tau) d\tau \right) \int_{M_\varepsilon} U(w, y, t) f(y) dy \\ &\quad - \int_0^\tau W(x, w, \tau) \left( \int_0^\tau \int_{M_\varepsilon} \frac{\partial}{\partial t} U(w, y, t - s) f(y) dy ds \right) d\tau \\ &= \left( \int_0^\tau W(x, w, \tau) d\tau \right) (U_t \hat{f})(w) - \int_0^\tau W(x, w, \tau) \left( \int_0^\tau \frac{\partial}{\partial t} (U_{t-s} \hat{f})(w) ds \right) d\tau\end{aligned}$$

for  $x \in M$ . Combining this equality with (1.2), (1.4) and (2.1), we can easily see

$$\begin{aligned}(2.7) \quad &(V_t^{(\varepsilon)} \hat{f})(x) \\ &= \int_{M_\varepsilon} U(x, y, t) f(y) dy \\ &\quad - L_n(\varepsilon) \int_{M_\varepsilon} \left( \int_0^\tau W(x, w, \tau) U(w, y, t - \tau) d\tau \right) f(y) dy\end{aligned}$$

$$\begin{aligned}
& -L_n(\varepsilon) \int_{M_\varepsilon} \left( \int_0^t S(x, w, \tau) U(w, y, t - \tau) d\tau \right) f(y) dy \\
& = (U_t \hat{f})(x) - L_n(\varepsilon) \left( \int_0^t W(x, w, \tau) d\tau \right) (U_t \hat{f})(w) \\
& \quad + L_n(\varepsilon) \int_0^t W(x, w, \tau) \left( \int_0^\tau \frac{\partial}{\partial t} (U_{t-s} \hat{f})(w) ds \right) d\tau \\
& \quad - L_n(\varepsilon) \int_0^t S(x, w, \tau) (U_{t-\tau} \hat{f})(w) d\tau
\end{aligned}$$

for  $x \in M$ .

We recall that  $L_n(\varepsilon) = -2\pi(\log \varepsilon)^{-1}$  for  $n=2$ . Thus, by (2.6) and (2.7), we have

$$(2.8) \quad (V_t^{(e)} \hat{f})(x)|_{x \in \partial B_\varepsilon} = \sum_{i=1}^3 I_i(\varepsilon, t),$$

where

$$\begin{aligned}
I_1(\varepsilon, t) &= (U_t \hat{f})(x)|_{x \in \partial B_\varepsilon} - (U_t \hat{f})(w) \\
I_2(\varepsilon, t) &= -2\pi(\log \varepsilon)^{-1} \int_0^t W(x, w, \tau) \left( \int_0^\tau \frac{\partial}{\partial t} (U_{t-s} \hat{f})(w) ds \right) d\tau \\
I_3(\varepsilon, t) &= 2\pi(\log \varepsilon)^{-1} (R(\varepsilon, t)(U_t \hat{f})(w) + \int_0^t S(x, w, \tau) (U_{t-\tau} \hat{f})(w) d\tau)
\end{aligned}$$

for  $x \in \partial B_\varepsilon$ .

Notice that  $S(x, w, \tau)$  is uniformly bounded for  $x \in M$  and  $\tau \in [0, t]$ . Thus, by (2.6) and Lemma A.3 in Appendix,

$$\begin{aligned}
(2.9) \quad & |R(\varepsilon, t)(U_t \hat{f})(w)| \\
& \leq C |R(\varepsilon, t)| t^{-1/p} \|\hat{f}\|_p \leq C t^{-1/p} (t + t^{-1/2} + 1) \|f\|_{p, \varepsilon}
\end{aligned}$$

and

$$\begin{aligned}
(2.10) \quad & \left| \int_0^t S(x, w, \tau) (U_{t-\tau} \hat{f})(w) d\tau \right| \\
& \leq C \int_0^t |(U_{t-\tau} \hat{f})(w)| d\tau \\
& \leq C \|\hat{f}\|_p \int_0^t (t - \tau)^{-1/p} d\tau \leq C t^{1-(1/p)} \|f\|_{p, \varepsilon}
\end{aligned}$$

hold for  $p > 1$  and  $x \in M$ . Therefore, by (2.8), (2.9) and (2.10), we have

$$(2.11) \quad |I_3(\varepsilon, t)| \leq C |\log \varepsilon|^{-1} t^{-1/p} (t + t^{-1/2} + 1) \|f\|_{p, \varepsilon}$$

for  $p > 1$ . The same calculation as above yields

$$(2.12) \quad \begin{aligned} & \left| \int_0^\tau \frac{\partial}{\partial t} (U_{t-s} \hat{f})(w) ds \right| \\ & \leq C \|\hat{f}\|_p \int_0^\tau (t-s)^{-1-(1/p)} ds \\ & \leq C \|f\|_{p, \varepsilon} ((t-\tau)^{-1/p} - t^{-1/p}) \\ & \leq C \|f\|_{p, \varepsilon} t^{-1/p} \tau^{1/p} (t-\tau)^{-1/p} \end{aligned}$$

for  $p > 1$ ,  $\tau \in (0, t)$  and

$$(2.13) \quad \begin{aligned} |I_1(\varepsilon, t)| &= \varepsilon |\nabla_x (U_t \hat{f})(w + \theta(x-w))|_{x \in \partial B_\varepsilon} \\ &\leq C \varepsilon t^{-(1/p)-(1/2)} \|f\|_{p, \varepsilon} \end{aligned}$$

for  $p > 1$ , where  $\theta \in (0, 1)$  denotes some constant. Furthermore, by (2.2), (2.8) and (2.12),

$$(2.14) \quad \begin{aligned} & |I_2(\varepsilon, t)| \\ & \leq C |\log \varepsilon|^{-1} t^{-1/p} \|f\|_{p, \varepsilon} \int_0^\tau \tau^{(1/p)-1} (t-\tau)^{-1/p} \exp(-\varepsilon^2/4\tau) d\tau \\ & \leq C |\log \varepsilon|^{-1} t^{-1/p} \|f\|_{p, \varepsilon} \int_0^\tau \tau^{(1/p)-1} (t-\tau)^{-1/p} d\tau \\ & = C |\log \varepsilon|^{-1} t^{-1/p} \|f\|_{p, \varepsilon} \int_0^1 s^{(1/p)-1} (1-s)^{-1/p} ds \\ & \leq C |\log \varepsilon|^{-1} t^{-1/p} \|f\|_{p, \varepsilon} \end{aligned}$$

hold for  $p > 1$ .

Summing up (2.8), (2.11), (2.13) and (2.14), we get the following.

**Proposition 2.1.** *Fix  $p > 1$  and  $t > 0$ . Then, there exists a constant  $C$  independent of  $\varepsilon$ ,  $t$  such that*

$$|(V_t^{(\varepsilon)} \hat{f})(x)|_{x \in \partial B_\varepsilon} \leq C t^{-1/p} (t + t^{-1/2} + 1) |\log \varepsilon|^{-1} \|f\|_{p, \varepsilon}$$

holds for any  $f \in L^p(M_\varepsilon)$ .



Now we are in a position to prove Theorem 1. We put  $v(x,t) = (U_t^{(\varepsilon)} f)(x) - (V_t^{(\varepsilon)} \hat{f})(x)$ . Then  $v(x,t)$  satisfies the following.

$$(2.15) \quad \begin{cases} (\partial_t - \Delta_x)v(x,t) = 0 & x \in M_\varepsilon, \quad t > 0 \\ v(x,t) = 0 & x \in \partial M, \quad t > 0 \\ v(x,t) = -(V_t^{(\varepsilon)} \hat{f})(x) & x \in \partial B_\varepsilon, \quad t > 0 \\ \lim_{t \rightarrow 0} v(x,t) = 0 & a.a. \quad x \in M_\varepsilon. \end{cases}$$

By the maximum principle we have

$$\sup_{x \in M_\varepsilon} |v(x,t)| \leq \sup_{x \in \partial M_\varepsilon} |v(x,t)| \leq \sup_{x \in \partial B_\varepsilon} |(V_t^{(\varepsilon)} \hat{f})(x)|.$$

Thus, by Proposition 2.1,

$$(2.16) \quad \begin{aligned} & \|U_t^{(\varepsilon)} f - \chi_\varepsilon V_t^{(\varepsilon)} \hat{f}\|_{\infty, \varepsilon} \\ &= \|v(\cdot, t)\|_{\infty, \varepsilon} \leq C t^{-1/p} (t + t^{-1/2} + 1) |\log \varepsilon|^{-1} \|f\|_{p, \varepsilon} \end{aligned}$$

hold for  $p > 1$ .

On the other hand, by (1.1) <sub>$\varepsilon$</sub>  and (2.15),  $v(x,t)$  is explicitly represented as follows.

$$(2.17) \quad v(x,t) = \int_0^t \left( \int_{\partial B_\varepsilon} (V_\tau^{(\varepsilon)} \hat{f})(z) \frac{\partial U^{(\varepsilon)}}{\partial \nu_z}(x, z, t - \tau) d\sigma_z \right) d\tau$$

Here  $\partial/\partial \nu_z$  denotes the derivative along the exterior normal direction with respect to  $M_\varepsilon$ . Thus, by (2.17), Proposition 2.1 and Lemma A.5 in Appendix, we have

$$(2.18) \quad \begin{aligned} & \|U_t^{(\varepsilon)} f - \chi_\varepsilon V_t^{(\varepsilon)} \hat{f}\|_{1, \varepsilon} \\ &= \|v(\cdot, t)\|_{1, \varepsilon} \\ &\leq \int_0^t \left( \sup_{x \in \partial B_\varepsilon} |(V_\tau^{(\varepsilon)} \hat{f})(z)| \right) \left( \int_{\partial B_\varepsilon} \left( \int_{M_\varepsilon} \left| \frac{\partial U^{(\varepsilon)}}{\partial \nu_z}(x, z, t - \tau) \right| dx \right) d\sigma_z \right) d\tau \\ &\leq C \varepsilon |\log \varepsilon|^{-1} \|f\|_{p, \varepsilon} \int_0^t \tau^{-1/p} (\tau + \tau^{-1/2} + 1) (t - \tau)^{-1/2} d\tau \\ &\leq C \varepsilon |\log \varepsilon|^{-1} \|f\|_{p, \varepsilon} t^{(1/2) - (1/p)} (t + t^{-1/2} + 1) \end{aligned}$$

for  $p > 2$ .

We fix  $p > 2$  and  $t > 0$ . Then, by (2.16), (2.18) and the interpolation inequality, we see

$$(2.19) \quad \|U_t^{(\varepsilon)} f - \chi_\varepsilon V_t^{(\varepsilon)} \hat{f}\|_{p, \varepsilon}$$

$$\begin{aligned} &\leq \|U_t^{(\varepsilon)} f - \chi_\varepsilon V_t^{(\varepsilon)} \hat{f}\|_{1,\varepsilon}^{1/p} \|U_t^{(\varepsilon)} f - \chi_\varepsilon V_t^{(\varepsilon)} \hat{f}\|_{\infty,\varepsilon}^{1-(1/p)} \\ &\leq C_t \varepsilon^{1/p} |\log \varepsilon|^{-1} \|f\|_{p,\varepsilon}. \end{aligned}$$

Therefore we get the following.

**Proposition 2.2.** *Fix  $p > 2$  and  $t > 0$ . Then, there exists a constant  $C_t$  independent of  $\varepsilon$  such that*

$$\|U_t^{(\varepsilon)} - \chi_\varepsilon V_t^{(\varepsilon)} \chi_\varepsilon\|_{p,\varepsilon} \leq C_t \varepsilon^{1/p} |\log \varepsilon|^{-1}$$

holds.

From (1.4) and Lemma A.2 in Appendix, we can see that  $(V_t^{(\varepsilon)})^* = V_t^{(\varepsilon)}$  and  $(U_t^{(\varepsilon)})^* = U_t^{(\varepsilon)}$ . Thus, by the duality argument,

$$(2.20) \quad \|U_t^{(\varepsilon)} - \chi_\varepsilon V_t^{(\varepsilon)} \chi_\varepsilon\|_{p',\varepsilon} = \|U_t^{(\varepsilon)} - \chi_\varepsilon V_t^{(\varepsilon)} \chi_\varepsilon\|_{p,\varepsilon}$$

holds for any  $p \in (2, \infty)$ , where  $p' = (1 - 1/p)^{-1}$ . Furthermore, by Proposition 2.2, (2.20) and the Riesz-Thorin interpolation theorem,

$$(2.21) \quad \|U_t^{(\varepsilon)} - \chi_\varepsilon V_t^{(\varepsilon)} \chi_\varepsilon\|_{2,\varepsilon} \leq \|U_t^{(\varepsilon)} - \chi_\varepsilon V_t^{(\varepsilon)} \chi_\varepsilon\|_{p,\varepsilon}$$

holds for any  $p \in (2, \infty)$ .

From Proposition 2.2, (2.20) and (2.21), we can easily get Theorem 1.

### 3. Proof of Theorem 2

Throughout this section we assume that  $n=3$ . We recall (2.2). Then

$$\begin{aligned} &\int_0^t W(x, w, \tau) d\tau|_{x \in \partial B_\varepsilon} \\ &= \int_0^t (4\pi\tau)^{-3/2} \exp(-\varepsilon^2/4\tau) d\tau \\ &= 4^{-1} \pi^{-3/2} \varepsilon^{-1} \int_{\varepsilon^2/4t}^\infty s^{-1/2} e^{-s} ds. \end{aligned}$$

Since

$$\int_0^\infty s^{-1/2} e^{-s} ds = \pi^{1/2}$$

and

$$\left| \int_0^{\varepsilon^2/4t} s^{-1/2} e^{-s} ds \right| \leq \int_0^{\varepsilon^2/4t} s^{-1/2} ds = \varepsilon t^{-1/2}$$

hold, we have

$$(3.1) \quad \int_0^t W(x, w, \tau) d\tau|_{x \in \partial B_\varepsilon} = (4\pi\varepsilon)^{-1} + R_1(\varepsilon, t),$$

where

$$|R_1(\varepsilon, t)| \leq 4^{-1} \pi^{-3/2} t^{-1/2}.$$

We fix an arbitrary  $f \in L^p(M_\varepsilon)$ . We recall (2.7) and  $L_n(\varepsilon) = 4\pi\varepsilon$  for  $n=3$ . Thus, by (2.7) and (3.1), we have

$$(3.2) \quad (V_t^{(e)} \hat{f})(x)|_{x \in \partial B_\varepsilon} = \sum_{i=4}^6 I_i(\varepsilon, t),$$

where

$$\begin{aligned} I_4(\varepsilon, t) &= (U_t \hat{f})(x)|_{x \in \partial B_\varepsilon} - (U_t \hat{f})(w) \\ I_5(\varepsilon, t) &= 4\pi\varepsilon \int_0^t W(x, w, \tau) \left( \int_0^\tau \frac{\partial}{\partial t} (U_{t-s} \hat{f})(w) ds \right) d\tau \\ I_6(\varepsilon, t) &= -4\pi\varepsilon (R_1(\varepsilon, t) (U_t \hat{f})(w) + \int_0^t S(x, w, \tau) (U_{t-\tau} \hat{f})(w) d\tau) \end{aligned}$$

for  $x \in \partial B_\varepsilon$ . By (3.1) and Lemma A.3 in Appendix,

$$\begin{aligned} (3.3) \quad |I_6(\varepsilon, t)| &\leq C\varepsilon(t^{-1/2} |(U_t \hat{f})(w)| + \int_0^t |(U_{t-\tau} \hat{f})(w)| d\tau) \\ &\leq C\varepsilon(t^{-(1/2)-(3/2p)} \|\hat{f}\|_p + \|\hat{f}\|_p \int_0^t (t-\tau)^{-3/2p} d\tau) \\ &\leq C\varepsilon t^{-3/2p} (t + t^{-1/2}) \|f\|_{p,\varepsilon} \quad (p > 3/2), \\ (3.4) \quad |I_4(\varepsilon, t)| &= \varepsilon |\nabla_x (U_t \hat{f})(w + \theta(x-w))|_{x \in \partial B_\varepsilon} \\ &\leq C\varepsilon t^{-(3/2p)-(1/2)} \|f\|_{p,\varepsilon} \quad (p > 1) \end{aligned}$$

and

$$(3.5) \quad \left| \int_0^\tau \frac{\partial}{\partial t} (U_{t-s} \hat{f})(w) ds \right| \leq C \|f\|_{p,\varepsilon} \int_0^\tau (t-s)^{-1-(3/2p)} ds \quad (p > 1)$$

hold for  $\tau \in (0, t)$ , where  $\theta \in (0, 1)$  denotes some constant.

Next we want to estimate  $I_5(\varepsilon, t)$ . By (2.2), (3.2) and (3.5), we see

$$(3.6) \quad |I_5(\varepsilon, t)| \leq C\varepsilon \|f\|_{p, \varepsilon} I_7(\varepsilon, t),$$

where

$$I_7(\varepsilon, t) = \int_0^t \tau^{-3/2} \exp(-\varepsilon^2/4\tau) \left( \int_0^\tau (t-s)^{-1-(3/2)p} ds \right) d\tau.$$

Since

$$\begin{aligned} I_7(\varepsilon, t) &= \iint_{0 \leq s \leq \tau \leq t} \tau^{-3/2} \exp(-\varepsilon^2/4\tau) (t-s)^{-1-(3/2)p} ds d\tau \\ &= \int_0^t (t-s)^{-1-(3/2)p} \left( \int_s^t \tau^{-3/2} \exp(-\varepsilon^2/4\tau) d\tau \right) ds \end{aligned}$$

and

$$\begin{aligned} &\int_s^t \tau^{-3/2} \exp(-\varepsilon^2/4\tau) d\tau \\ &= 2\varepsilon^{-1} \int_{\varepsilon^2/4t}^{\varepsilon^2/4s} r^{-1/2} e^{-r} dr \\ &\leq 2\varepsilon^{-1} \int_{\varepsilon^2/4t}^{\varepsilon^2/4s} r^{-1/2} dr \\ &= 2(st)^{-1/2} (t^{1/2} - s^{1/2}) \leq 2(st)^{-1/2} (t-s)^{1/2} \end{aligned}$$

hold for  $s \in (0, t)$ , we have

$$\begin{aligned} I_7(\varepsilon, t) &\leq 2t^{-1/2} \int_0^t s^{-1/2} (t-s)^{-(1/2)-(3/2)p} ds \\ &= 2t^{-(1/2)-(3/2)p} \int_0^1 \tau^{-1/2} (1-\tau)^{-(1/2)-(3/2)p} d\tau \\ &\leq Ct^{-(1/2)-(3/2)p} \end{aligned}$$

for  $p > 3$ . Combining this inequality with (3.6), we get

$$(3.7) \quad |I_5(\varepsilon, t)| \leq C\varepsilon t^{-(1/2)-(3/2)p} \|f\|_{p, \varepsilon}$$

for  $p > 3$ .

Summing up (3.2), (3.3), (3.4) and (3.7), we can get the following.

**Proposition 3.1.** *Fix  $p > 3$  and  $t > 0$ . Then there exists a constant  $C$  independent of  $\varepsilon$ ,  $t$  such that*

$$\|(V_t^{(\varepsilon)} \hat{f})(x)\|_{|x \in \partial B_\varepsilon|} \leq C \varepsilon t^{-3/2p} (t + t^{-1/2}) \|f\|_{p,\varepsilon}$$

*holds for any  $f \in L^p(M_\varepsilon)$ .*

By Proposition 3.1, Lemma A.5 in Appendix and the same argument as in section 2, we have

$$(3.8) \quad \begin{aligned} \|U_t^{(\varepsilon)} f - \chi_\varepsilon V_t^{(\varepsilon)} \hat{f}\|_{\infty,\varepsilon} &\leq \sup_{x \in \partial \tilde{B}_\varepsilon} |(V_t^{(\varepsilon)} \hat{f})(x)| \\ &\leq C \varepsilon t^{-3/2p} (t + t^{-1/2}) \|f\|_{p,\varepsilon} \end{aligned}$$

and

$$(3.9) \quad \begin{aligned} &\|U_t^{(\varepsilon)} f - \chi_\varepsilon V_t^{(\varepsilon)} \hat{f}\|_{1,\varepsilon} \\ &\leq \int_0^t \sup_{x \in \partial \tilde{B}_\varepsilon} |(V_\tau^{(\varepsilon)} \hat{f})(z)| \left( \int_{\partial B_\varepsilon} \left( \int_{M_\varepsilon} \left| \frac{\partial U^{(\varepsilon)}}{\partial v_z}(x, z, t - \tau) \right| dx \right) d\sigma_z \right) d\tau \\ &\leq C \varepsilon^3 \|f\|_{p,\varepsilon} \int_0^t \tau^{-3/2p} (\tau + \tau^{-1/2}) (t - \tau)^{-1/2} d\tau \\ &= C \varepsilon^3 \|f\|_{p,\varepsilon} t^{(1/2) - (3/2)p} \int_0^1 s^{-3/2p} (ts + (ts)^{-1/2}) (1 - s)^{-1/2} ds \\ &\leq C \varepsilon^3 t^{(1/2) - (3/2)p} (t + t^{-1/2}) \|f\|_{p,\varepsilon} \end{aligned}$$

for  $p > 3$ .

From (3.8), (3.9) and the interpolation inequality (see (2.19)), we get the following.

**Proposition 3.2.** *Fix  $p > 3$  and  $t > 0$ . Then there exists a constant  $C_t$  independent of  $\varepsilon$  such that*

$$\|U_t^{(\varepsilon)} - \chi_\varepsilon V_t^{(\varepsilon)} \chi_\varepsilon\|_{p,\varepsilon} \leq C_t \varepsilon^{1 + (2/p)}$$

*holds.*

Furthermore, by the duality argument and the Riesz-Thorin interpolation theorem, we have (2.20) for any  $p \in (3, \infty)$  and

$$(3.10) \quad \|U_t^{(\varepsilon)} - \chi_\varepsilon V_t^{(\varepsilon)} \chi_\varepsilon\|_{r,\varepsilon} \leq \|U_t^{(\varepsilon)} - \chi_\varepsilon V_t^{(\varepsilon)} \chi_\varepsilon\|_{p,\varepsilon}$$

for any  $p \in (3, \infty)$  and  $r \in [3/2, 3]$ .

From Proposition 3.2, (2.20) and (3.10), we can easily get Theorem 2.

#### 4. Appendix

Let  $M$ ,  $M_e$ ,  $U(x,y,t)$ ,  $U^{(e)}(x,y,t)$  be as in Introduction. See Friedman [1] for the fundamental properties of the heat kernel. We have the following.

**Lemma 1.1.** *There exists a constant  $C$  independent of  $x,y,t$  such that*

$$(A.1) \quad 0 \leq U(x,y,t) \leq Ct^{-n/2} \exp(-|x-y|^2 / Ct)$$

$$(A.2) \quad \left| \frac{\partial U}{\partial x_i}(x,y,t) \right| \leq Ct^{-(n+1)/2} \exp(-|x-y|^2 / Ct) \quad (1 \leq i \leq n)$$

$$(A.3) \quad \left| \frac{\partial U}{\partial t}(x,y,t) \right| \leq Ct^{-(n+2)/2} \exp(-|x-y|^2 / Ct)$$

hold for  $x, y \in \bar{M}$ ,  $t > 0$ .

**Lemma A.2.** *We have*

$$U(x,y,t) = U(y,x,t) \quad x, y \in \bar{M}, t > 0$$

and

$$U^{(e)}(x,y,t) = U^{(e)}(y,x,t) \quad x, y \in \bar{M}_e, t > 0.$$

Let  $U_t$  be as in (1.2). Then we have the following.

**Lemma A.3.** *Fix  $p \in (1, \infty)$ . Then there exists a constant  $C$  independent of  $t$  such that*

$$(A.4) \quad \sup_{x \in \bar{M}} |(U_t f)(x)| \leq Ct^{-n/2p} \|f\|_p$$

$$(A.5) \quad \sup_{x \in \bar{M}} \left| \frac{\partial}{\partial x_i} (U_t f)(x) \right| \leq Ct^{-(n/2p) - (1/2)} \|f\|_p \quad (1 \leq i \leq n)$$

$$(A.6) \quad \sup_{x \in \bar{M}} \left| \frac{\partial}{\partial t} (U_t f)(x) \right| \leq Ct^{-(n/2p) - 1} \|f\|_p$$

hold for  $f \in L^p(M)$  and  $t > 0$ .

**Proof.** We take an arbitrary  $x \in \bar{M}$ . Then, by (1.2), (A.1) and using the transformation of co-ordinates :  $y = x + (Ct)^{1/2}z$ , we have

$$\begin{aligned}
|(U_t f)(x)| &\leq C t^{-n/2} \int_M \exp(-|x-y|^2/Ct) |f(y)| dy \\
&\leq C t^{-n/2} \|f\|_p \left( \int_M \exp(-|x-y|^2/Ct) dy \right)^{1/p'} \\
&\leq C t^{-(n/2)+(n/2p')} \|f\|_p \left( \int_{\mathbb{R}^n} \exp(-|z|^2) dz \right)^{1/p'} \\
&\leq C t^{-n/2p} \|f\|_p,
\end{aligned}$$

where  $(1/p) + (1/p') = 1$ .

Therefore we get (A.4). By the same argument as above, we get (A.5) and (A.6) from (A.2) and (A.3), respectively.

q.e.d

By  $B(r; w)$  we denote a ball of radius  $r > 0$  with the center  $w$ . And we write  $B_r = B(r; w)$  as before.

**Lemma A.4.** *There exists a constant  $C$  independent of  $\varepsilon$ ,  $x$ ,  $t$  such that*

$$(A.7) \quad 0 \leq -\frac{\partial U^{(\varepsilon)}}{\partial v_z}(x, z, t) \leq C t^{-(n+1)/2} \exp(-|x-z|^2/Ct)$$

hold for  $z \in \partial B_\varepsilon$ ,  $x \in M_\varepsilon$  and  $t > 0$ .

Here  $\partial/\partial v_z$  denotes the derivative along the exterior normal direction with respect to  $M_\varepsilon$ .

*Proof.* Let  $F^{(r)}(x, y, t)$  be the fundamental solution of the heat equation in  $\mathbb{R}^n \setminus \bar{B}_r$  under the Dirichlet condition on  $\partial B_r$ . Then we have the following identity.

$$F^{(\varepsilon)}(x, y, t) = F^{(1)}(\varepsilon^{-1}x, \varepsilon^{-1}y, \varepsilon^{-2}t) \varepsilon^{-n} \quad x, y \in \mathbb{R}^n \setminus B_\varepsilon, \quad t > 0$$

Thus,

$$\frac{\partial F^{(\varepsilon)}}{\partial y_i}(x, y, t) = \left( \frac{\partial F^{(1)}}{\partial y_i} \right) (\varepsilon^{-1}x, \varepsilon^{-1}y, \varepsilon^{-2}t) \varepsilon^{-(n+1)} \quad (1 \leq i \leq n)$$

holds for  $x, y \in \mathbb{R}^n \setminus B_\varepsilon$ ,  $t > 0$ .

It is well known that there exists a constant  $C$  such that

$$\begin{aligned}
|F^{(1)}(x, y, t)| &\leq C t^{-n/2} \exp(-|x-y|^2/Ct) \\
\left| \frac{\partial F^{(1)}}{\partial y_i}(x, y, t) \right| &\leq C t^{-(n+1)/2} \exp(-|x-y|^2/Ct) \quad (1 \leq i \leq n)
\end{aligned}$$

hold for  $x, y \in R^n \setminus B_1$ ,  $t > 0$ . Thus, we get

$$(A.8) \quad \left| \frac{\partial F^{(e)}}{\partial y_i}(x, y, t) \right| \leq C t^{-(n+1)/2} \exp(-|x-y|^2 / Ct) \quad (1 \leq i \leq n)$$

for  $x, y \in R^n \setminus B_\varepsilon$ ,  $t > 0$ .

It should be remarked that the constant  $C$  in (A.8) does not depend on  $\varepsilon$ .

Let  $H^{(e)}(x, y, t) = F^{(e)}(x, y, t) - U^{(e)}(x, y, t)$ . Then,  $H^{(e)}(x, y, t)$  satisfies the following.

$$\left\{ \begin{array}{ll} (\partial_t - \Delta_x)H^{(e)}(x, y, t) = 0 & x, y \in M_\varepsilon, t > 0 \\ H^{(e)}(x, y, t) = F^{(e)}(x, y, t) \geq 0 & x \in \partial M, y \in M_\varepsilon, t > 0 \\ H^{(e)}(x, y, t) = 0 & x \in \partial B_\varepsilon, y \in M_\varepsilon, t > 0 \\ \lim_{t \rightarrow 0} H^{(e)}(x, y, t) = 0 & x, y \in M_\varepsilon \end{array} \right.$$

By the maximum principle,  $H^{(e)}(x, y, t) \geq 0$  holds for  $x, y \in \bar{M}_\varepsilon$ ,  $t > 0$ . Therefore,

$$(A.9) \quad 0 \leq U^{(e)}(x, y, t) \leq F^{(e)}(x, y, t)$$

hold for  $x, y \in \bar{M}$ ,  $t > 0$ .

We fix an arbitrary  $z \in \partial B_\varepsilon$ . Then,  $v_z = -(z - w) / |z - w|$  denotes the exterior normal unit vector at  $z \in \partial B_\varepsilon$  with respect to  $M_\varepsilon$ . We recall that

$$(A.10) \quad U^{(e)}(x, z, t) = F^{(e)}(x, z, t) = 0$$

for  $x \in M_\varepsilon$ ,  $t > 0$ . Therefore, by (A.9) and (A.10),

$$(A.11) \quad \begin{aligned} 0 &\leq (U^{(e)}(x, z - h v_z, t) - U^{(e)}(x, z, t)) / h \\ &\leq (F^{(e)}(x, z - h v_z, t) - F^{(e)}(x, z, t)) / h \end{aligned}$$

hold for  $x \in M_\varepsilon$ ,  $t > 0$  and any sufficiently small  $h > 0$ . Letting  $h \downarrow 0$  in (A.11), we have

$$0 \leq -\frac{\partial U^{(e)}}{\partial v_z}(x, z, t) \leq -\frac{\partial F^{(e)}}{\partial v_z}(x, z, t)$$

for  $x \in M_\varepsilon$ ,  $t > 0$ .

Combining this inequality with (A.8),

$$\begin{aligned} 0 &\leq -\frac{\partial U^{(e)}}{\partial v_z}(x, z, t) \leq |(\nabla_z F^{(e)})(x, z, t)| \\ &\leq C t^{-(n+1)/2} \exp(-|x-z|^2 / Ct) \end{aligned}$$



hold for  $x \in M_\varepsilon$ ,  $t > 0$ .

Therefore we get (A.7).

q.e.d.

Now we have the following.

**Lemma A.5.** *There exists a constant  $C$  independent of  $\varepsilon$ ,  $t$  such that*

$$(A.12) \quad \int_{\partial B_\varepsilon} \left( \int_{M_\varepsilon} \left| \frac{\partial U^{(\varepsilon)}}{\partial v_z}(x, z, t) \right| dx \right) d\sigma_z \leq C \varepsilon^{n-1} t^{-1/2}$$

holds for  $t > 0$ .

**Proof.** We fix an arbitrary  $z \in \partial B_\varepsilon$ . Then, by (A.7) and using the transformation of co-ordinates :  $x = z + (Ct)^{1/2}y$ ,

$$\begin{aligned} \int_{M_\varepsilon} \left| \frac{\partial U^{(\varepsilon)}}{\partial v_z}(x, z, t) \right| dx &\leq C t^{-(n+1)/2} \int_{M_\varepsilon} \exp(-|x-z|^2 / Ct) dx \\ &\leq C t^{-1/2} \int_{\mathbb{R}^n} \exp(-|y|^2) dy \\ &\leq C t^{-1/2} \end{aligned}$$

hold for  $t > 0$ . Here  $C$  denotes some different positive constants independent of  $\varepsilon$ ,  $t$ . Integrating this inequality on  $\partial B_\varepsilon$ , we can immediately get (A.12).

q.e.d.

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