Uchida, F.
Osaka J. Math.
16 (1979), 561-579

CLASSIFICATION OF REAL ANALYTIC $SL(n, R)$ ACTIONS ON n-SPHERE

Dedicated to Professor A. Komatu on his 70th birthday

FUICHI UCHIDA*)

(Received March 17, 1978)

0. Introduction

C.R. Schneider [5] classified real analytic $SL(2, R)$ actions on closed surfaces. Except for the work, there seems to be no work on the classification problem about non-compact Lie group actions.

In this paper, we classify real analytic $SL(n, R)$ actions on the standard n-sphere for each $n \geqslant 3$. Here $SL(n, R)$ denotes the special linear group over the field of real numbers. The result can be stated roughly as follows: there is a one-to-one correspondence between real analytic $SL(n, R)$ actions on the n-sphere and real valued real analytic functions on an interval satisfying certain conditions (see Theorem 2.2 and Theorem 4.2). It is important to consider the restricted actions of $SL(n, R)$ to a maximal compact subgroup $SO(n)$.

It is still open to classify C^∞ actions of $SL(n, R)$ on the standard n-sphere, by lack of C^∞ analogue of a local theory due to Guillemin and Sternberg (see Lemma 4.3).

1. Real analytic $SO(n)$ actions on certain n-manifolds

First we prepare the following two lemmas of which proof is given in the last section.

Lemma 1.1. *Let G be a closed connected subgroup of $O(n)$. Suppose that $n \geqslant 3$ and*

$$\dim O(n) > \dim G \geqslant \dim O(n) - n .$$

Suppose that G is not conjugate to $SO(n-1)$ which is canonically imbedded in $O(n)$. Then the pair $(O(n), G)$ is pairwise isomorphic to one of the following:

$$(O(8), Spin(7)), (O(7), G_2), (O(6), U(3)), (O(4), U(2)),$$
$$(O(4), SU(2)), (O(4), SO(2)\times SO(2)) \text{ and } (O(3), \{1\}),$$

*) Supported by Grant-in-Aid for Scientific Research

up to inner automorphisms of $O(n)$. In these cases the subgroups are standardly imbedded in $O(n)$.

Lemma 1.2. *Suppose $n \geqslant 3$. Let h: $SO(n) \to O(n)$ be a continuous homomorphism with a finite kernel. Then there is an element x of $O(n)$ such that $h(y) = xyx^{-1}$ for each y of $SO(n)$.*

Now we shall prove the following result.

Theorem 1.3. *Suppose $n \geqslant 3$. Let M be a closed connected n-dimensional real analytic manifold. Suppose that*

$$\pi_1(M) = \pi_2(M) = \{1\}\ .$$

Suppose that $SO(n)$ acts on M real analytically and almost effectively. Then the $SO(n)$-manifold M is real analytically diffeomorphic to the standard n-sphere S^n as $SO(n)$-manifolds. Here the $SO(n)$ action on S^n is the restriction of the standard $SO(n+1)$ action on S^n.

Proof. (i) First we show that the $SO(n)$-manifold M is C^∞ diffeomorphic to the standard sphere S^n as $SO(n)$-manifolds. Let G be the identity component of a principal isotropy group. Then

$$\dim SO(n) > \dim G \geqslant \dim SO(n) - n\ ,$$

and $SO(n)$ acts almost effectively on the homogeneous space $SO(n)/G$ by the assumption that $SO(n)$ acts almost effectively on M, and hence Lemma 1.1 is applicable. The pair $(SO(n), G)$ is not pairwise isomorphic to $(SO(4), U(2))$ nor $(SO(4), SU(2))$, because $SU(2)$ is a normal subgroup of $SO(4)$. If

$$\dim SO(n)/G = \dim M\ ,$$

then the $SO(n)$ action on M is transitive and the pair $(SO(n), G)$ is pairwise isomorphic to one of the following by Lemma 1.1:

$$(SO(7), G_2),\ (SO(6), U(3)),\ (SO(4), SO(2)\times SO(2)) \quad\text{and}\quad (SO(3), \{1\})\ .$$

But

$$\pi_1(SO(7)/G_2) = \pi_1(SO(3)/\{1\}) = Z_2\ ,$$
$$\pi_2(SO(6)/U(3)) = Z \quad\text{and}\quad \pi_2(SO(4)/SO(2)\times SO(2)) = Z\times Z\ .$$

This is a contradiction to the assumption

$$\pi_1(M) = \pi_2(M) = \{1\}\ .$$

Consequently G is conjugate to $SO(n-1)$ or the pair $(SO(n), G)$ is pairwise isomorphic to $(SO(8), Spin(7))$ by Lemma 1.1 and hence the $SO(n)$-manifold

M has codimension one principal orbits and just two singular orbits (cf. [6], Lemma 1.2.1). Since $\boldsymbol{SO}(n-1)$ in $\boldsymbol{SO}(n)$ (resp. $\boldsymbol{Spin}(7)$ in $\boldsymbol{SO}(8)$) is a maximal closed connected subgroup, the singular orbits are fixed points. It follows that the $\boldsymbol{SO}(n)$-manifold M is C^∞ diffeomorphic to $M'=\boldsymbol{D}^n \underset{f}{\cup} \boldsymbol{D}^n$ as $\boldsymbol{SO}(n)$-manifolds. Here the $\boldsymbol{SO}(n)$ action on $\boldsymbol{D}^n$ is standard by Lemma 1.2, and f: $\partial\boldsymbol{D}^n \to \partial\boldsymbol{D}^n$ is an $\boldsymbol{SO}(n)$ equivariant diffeomorphism. It follows that f is the identity map or the antipodal map, and hence M' is C^∞ diffeomorphic to the standard n-sphere S^n as $\boldsymbol{SO}(n)$-manifolds.

(ii) Here we assume that M_1 and M_2 are n-dimensional real analytic manifolds on which $\boldsymbol{SO}(n)$ acts real analytically. Assume that the $\boldsymbol{SO}(n)$-manifolds M_1 and M_2 are C^∞ diffeomorphic to the standard n-sphere S^n as $\boldsymbol{SO}(n)$-manifolds. According to a theorem of Grauert ([3], Theorem 3), M_i is real analytically imbedded in a euclidean space of sufficiently high dimension; hence M_i posesses a real analytic Riemannian metric. By averaging the real analytic Riemannian metric on M_i with respect to the $\boldsymbol{SO}(n)$ action, we have an $\boldsymbol{SO}(n)$ invariant real analytic Riemannian metric g_i on M_i. Denote by $\{N_i, S_i\}$ the fixed point set of the $\boldsymbol{SO}(n)$-manifold M_i. We can assume that

$$d_1(N_1, S_1) = d_2(N_2, S_2),$$

where d_i is a distance function on M_i defined by the Riemannian metric g_i. Denote by F_i the fixed point set of the restricted $\boldsymbol{SO}(n-1)$ action on M_i. It follows that F_i is a real analytic submanifold of M_i which is $N\boldsymbol{SO}(n-1)$ invariant and C^∞ diffeomorphic to S^1 by the assumption. Here $N\boldsymbol{SO}(n-1)$ denotes the normalizer of $\boldsymbol{SO}(n-1)$ in $\boldsymbol{SO}(n)$. Then there exsists an isometry φ: $F_1 \to F_2$ such that $\varphi(N_1)=N_2$ and $\varphi(S_1)=S_2$. The isometry φ is a real analytic diffeomorphism and φ is compatible with the action of $N\boldsymbol{SO}(n-1)$ on F_i. It is easy to see that the $\boldsymbol{SO}(n)$-manifold $M_i-\{N_i, S_i\}$ is real analytically diffeomorphic to

$$\boldsymbol{SO}(n) \underset{N\boldsymbol{SO}(n-1)}{\times} (F_i-\{N_i, S_i\})$$

as $\boldsymbol{SO}(n)$-manifolds; hence φ extends uniquely to an $\boldsymbol{SO}(n)$ equivariant homeomorphism Φ: $M_1 \to M_2$. By the construction, the restriction of Φ to $M_1-\{N_1, S_1\}$ is a real analytic diffeomorphism of $M_1-\{N_1, S_1\}$ onto $M_2-\{N_2, S_2\}$.

(iii) Finally we show that Φ is real analytic on neighborhoods of N_i and S_i. Notice that the tangent space of M_i at N_i with the induced $\boldsymbol{SO}(n)$ action is naturally isomorphic to $\boldsymbol{R}^n$ with the standard $\boldsymbol{SO}(n)$ action by the assumption. Denote by $\boldsymbol{D}_\varepsilon$ an ε-neighborhood of the origin 0 in $\boldsymbol{R}^n$. Denote by e_i: $\boldsymbol{D}_\varepsilon \to M_i$ the exponential map with respect to the Riemannian metric g_i such that $e_i(0)=N_i$. Then e_i is an $\boldsymbol{SO}(n)$ equivariant real analytic diffeomorphism onto an open neighborhood of N_i for sufficiently small ε. Denote by $\boldsymbol{D}'_\varepsilon$ the fixed point set of the restricted $\boldsymbol{SO}(n-1)$ action on $\boldsymbol{D}_\varepsilon$. Define

$$\Phi' = e_2^{-1}\Phi e_1 \colon \boldsymbol{D}_\varepsilon \to \boldsymbol{D}_\varepsilon .$$

Then Φ' is an $\boldsymbol{SO}(n)$ equivariant homeomorphism. Since Φ is an extension of the isometry φ, the restriction of Φ' to $\boldsymbol{D}'_\varepsilon$ onto itself is the identity map or the antipodal map. It follows that Φ' is the identity map or the antipodal map of $\boldsymbol{D}_\varepsilon$ onto itself, because Φ' is $\boldsymbol{SO}(n)$ equivariant. Therefore Φ is real analytic on a neighborhood of N_1. Similarly Φ is real analytic on a neighborhood of S_1. Consequently Φ is a real analytic diffeomorphism of M_1 onto M_2.

This completes the proof of Theorem 1.3.

REMARK. The real analytic diffeomorphism $\Phi\colon M_1 \to M_2$ in the proof of Theorem 1.3 is not necessary an isometry with respect to the Riemannian metrics g_1 and g_2.

2. Construction of real analytic $SL(n, R)$ actions

Consider the following conditions for a real valued real analytic function $f(t)$:

(A) $f(t)$ is defined on an open interval $(-1-\varepsilon, 1+\varepsilon)$ and $f(-1)=f(1)=0$,

(B) $t\cdot f(t)<0$ for $1-\varepsilon<|t|<1$,

where ε is a sufficiently small positive real number. If $f(t)$ is a real analytic function satisfying the condition (A), then the corresponding vector field $f(t)\dfrac{d}{dt}$ on $(-1, 1)$ is complete; hence the vector field induces a real analytic $\boldsymbol{R}$ action

$$\psi = \psi_f \colon \boldsymbol{R}\times(-1, 1) \to (-1, 1)$$

such that

$$f(t) = \lim_{s\to 0}\frac{\psi(s, t)-t}{s} \qquad \text{for } -1<t<1 .$$

Denote by $\boldsymbol{F}$ the set of all real analytic functions satisfying the conditions (A) and (B). Define an equivalence relation in $\boldsymbol{F}$ as follows: we say that $f(t)$ is equivalent to $g(t)$ if there is a real analytic diffeomorphism h of the open interval $(-1,1)$ onto itself such that

$$h_*\left(f(t)\frac{d}{dt}\right) = g(t)\frac{d}{dt} .$$

The relation means that the corresponding $\boldsymbol{R}$ actions ψ_f and ψ_g are compatible under the real analytic diffeomorphism h. Denote by $\boldsymbol{F}_*$ the set of all equivalence classes of $\boldsymbol{F}$.

EXAMPLE. The polynomial

$$f_{m,a}(t) = at\cdot\prod_{k=1}^{m}(kt+1)(kt-1)$$

satisfies the conditions (A), (B) for each positive integer m and each positive real number a.

Proposition 2.1. *If $(m, a)\neq(m', a')$, then the functions $f_{m,a}(t)$ and $f_{m',a'}(t)$ are not equivalent.*

Proof. Suppose that there is a real analytic diffeomorphism h of the interval $(-1, 1)$ onto itself such that

$$h_*\left(f_{m,a}(t)\frac{d}{dt}\right) = f_{m',a'}(t)\frac{d}{dt}\,.$$

Then it follows that

$$m = m', \quad h(0) = 0$$

and

$$f_{m',a'}(t) = f_{m,a}(h^{-1}(t))\frac{dh}{dt}(h^{-1}(t))\,.$$

Therefore we have

$$(-1)^{m'}a' = \frac{df_{m',a'}}{dt}(0) = \frac{df_{m,a}}{dt}(0) = (-1)^m a\,.$$

It follows that $a=a'$. q.e.d.

Put

$$L(n) = \{(a_{ij})\in \boldsymbol{SL}(n, \boldsymbol{R})\colon a_{11} = 1,\, a_{21} = a_{31} = \cdots = a_{n1} = 0\}\,,$$
$$N(n) = \{(a_{ij})\in \boldsymbol{SL}(n, \boldsymbol{R})\colon a_{11} > 0,\, a_{21} = a_{31} = \cdots = a_{n1} = 0\}\,.$$

Then $L(n)$ and $N(n)$ are closed connected subgroups of $\boldsymbol{SL}(n, \boldsymbol{R})$, and $L(n)$ is a normal subgroup of $N(n)$. Consider the standard action of $\boldsymbol{SL}(n, \boldsymbol{R})$ on $\boldsymbol{R}^n$. Then the action is transitive on $\boldsymbol{R}^n - \{0\}$, and $L(n)$ is the isotropy group at $e_1=(1, 0, \cdots, 0)$.

Let $f(t)$ be a real analytic function satisfying the conditions (A) and (B). Here we shall construct a real analytic $\boldsymbol{SL}(n, \boldsymbol{R})$ action on a closed connected n-dimensional real analytic manifold M_f associated with the function $f(t)$. Let ψ_f be the real analytic $\boldsymbol{R}$ action on $(-1, 1)$ corresponding to $f(t)$. Since the factor group $N(n)/L(n)$ is naturally isomorphic to $\boldsymbol{R}$ as Lie groups by a correspondence

$$(a_{ij})\cdot L(n) \to \log a_{11}\,, \qquad \text{for } (a_{ij})\in N(n)\,,$$

we consider ψ_f as a real analytic $N(n)/L(n)$ action on $(-1, 1)$. Define X_f the quotient manifold of the product

$$SL(n, \boldsymbol{R})/L(n)\times(-1, 1)$$

by the relation

$$(xL(n), t) = (xy^{-1}L(n), \psi_f(yL(n), t));$$
$$x\in SL(n, \boldsymbol{R}),\ y\in N(n),\ |t|<1\,.$$

Then X_f is an n-dimensional real analytic manifold with a natural $SL(n, \boldsymbol{R})$ action. Denote by $[xL(n), t]$ the element of X_f represented by $(xL(n), t)$.

Let a' (resp. a'') be the largest (resp. the smallest) zero of $f(t)$ on $(-1, 1)$. Let a_+, a_-: $\boldsymbol{R}^n-\{0\}\to X_f$ be the equivariant $SL(n, \boldsymbol{R})$ maps determined by

$$a_+(e_1) = \left[L(n), \frac{1+a'}{2}\right],\quad a_-(e_1) = \left[L(n), \frac{a''-1}{2}\right]$$

respectively, where $e_1=(1, 0, \cdots, 0)$. Let $\boldsymbol{R}^n_+$ and $\boldsymbol{R}^n_-$ be copies of $\boldsymbol{R}^n$, and consider a_+, a_- as the maps

$$a_+\colon \boldsymbol{R}^n_+-\{0\}\to X_f,\quad a_-\colon \boldsymbol{R}^n_--\{0\}\to X_f$$

respectively. Define M_f the quotient space of a disjoint union

$$\boldsymbol{R}^n_+\cup X_f\cup \boldsymbol{R}^n_-$$

given by the attaching maps a_+, a_-. Since $f(t)$ satisfies the conditions (A) and (B), the space M_f posesses naturally a real analytic structure as a compact connected n-dimensional manifold with a natural $SL(n, \boldsymbol{R})$ action. Notice that M_f is a two points compactification of X_f.

For each $k\leqslant n-2$, $\pi_k(M_f)=\pi_k(X_f)$ by a general position theorem. The natural projection of X_f onto $SL(n, \boldsymbol{R})/N(n)=S^{n-1}$ is a fibre bundle with a contractible fibre. It follows that M_f is $(n-2)$-connected. In particular, $\pi_1(M_f)=\pi_2(M_f)=\{1\}$ for each $n\geqslant 3$. Since the restricted $SO(n)$ action on M_f is effective, M_f is real analytically diffeomorphic to the standard n-sphere S^n by Theorem 1.3.

Denote by $A(n)$ the set of all real analytic non-trivial $SL(n, \boldsymbol{R})$ actions on the standard n-sphere S^n. Two such actions ψ and ψ' are said to be equivalent if there is a real analytic diffeomorphism h of S^n onto itself such that the following diagram is commutative:

$$\begin{array}{ccc} SL(n, \boldsymbol{R})\times S^n & \xrightarrow{\psi} & S^n \\ \big\downarrow 1\times h & & \big\downarrow h \\ SL(n, \boldsymbol{R})\times S^n & \xrightarrow{\psi'} & S^n\,. \end{array}$$

Denote by $A_*(n)$ the set of all equivalence classes of $A(n)$. By the above construction of M_f, the real analytic function $f(t)$ defines an equivalence class

$A_f=\{a_f\}$ of real analytic $\boldsymbol{SL}(n, \boldsymbol{R})$ actions on S^n such that the n-sphere S^n with a real analytic $\boldsymbol{SL}(n, \boldsymbol{R})$ action a_f is real analytically diffeomorphic to M_f as $\boldsymbol{SL}(n, \boldsymbol{R})$-manifolds. If $f(t)$ and $g(t)$ are equivalent, then it is easy to see that M_f and M_g are real analytically diffeomorphic as $\boldsymbol{SL}(n, \boldsymbol{R})$-manifolds. It follows that the correspondence $f(t)\to A_f$ induces a map c_n: $\boldsymbol{F}_*\to \boldsymbol{A}_*(n)$ for each $n\geqslant 3$.

Theorem 2.2. *The map c_n: $\boldsymbol{F}_*\to \boldsymbol{A}_*(n)$ is injective for each $n\geqslant 3$.*

Proof. Let $f(t)$, $g(t)$ be real analytic functions satisfying the conditions (A), (B). Suppose that the induced real analytic $\boldsymbol{SL}(n, \boldsymbol{R})$-manifolds M_f and M_g are real analytically diffeomorphic as $\boldsymbol{SL}(n, \boldsymbol{R})$-manifolds. Then the open manifolds X_f and X_g are real analytically diffeomorphic as $\boldsymbol{SL}(n, \boldsymbol{R})$-manifolds. Compare the fixed point sets of the restricted $L(n)$ action. Then the fixed point sets $F(L(n), X_f)$ and $F(L(n), X_g)$ are one dimensional real analytic submanifolds of X_f and X_g respectively and real analytically diffeomorphic as $NL(n)$-manifolds. Here $NL(n)$ denotes the normalizer of $L(n)$ in $\boldsymbol{SL}(n, \boldsymbol{R})$. Since $NL(n)/L(n)$ is naturally isomorphic to $\boldsymbol{Z}_2\times N(n)/L(n)$ as Lie groups, it is easy to see that $f(t)$ and $g(t)$ are equivalent. q.e.d.

3. Certain closed subgroups of $SL(n, R)$

Put

$$\begin{aligned}
L(n) &= \{(a_{ij})\in \boldsymbol{SL}(n, \boldsymbol{R}):\ a_{11} = 1,\ a_{21} = a_{31} = \cdots = a_{n1} = 0\}\,,\\
N(n) &= \{(a_{ij})\in \boldsymbol{SL}(n, \boldsymbol{R}):\ a_{11} > 0,\ a_{21} = a_{31} = \cdots = a_{n1} = 0\}\,,\\
L^*(n) &= \{(a_{ij})\in \boldsymbol{SL}(n, \boldsymbol{R}):\ a_{11}=1,\ a_{12} = a_{13} = \cdots = a_{1n} = 0\}\,,\\
N^*(n) &= \{(a_{ij})\in \boldsymbol{SL}(n, \boldsymbol{R}):\ a_{11}>0,\ a_{12} = a_{13} = \cdots = a_{1n} = 0\}\,.
\end{aligned}$$

Consider $\boldsymbol{SL}(n-1, \boldsymbol{R})$ and $\boldsymbol{SO}(n-1)$ as subgroups of $\boldsymbol{SL}(n, \boldsymbol{R})$ as follows:

$$\boldsymbol{SL}(n-1, \boldsymbol{R}) = L(n)\cap L^*(n),\quad \boldsymbol{SO}(n-1) = \boldsymbol{SO}(n)\cap \boldsymbol{SL}(n-1, \boldsymbol{R})\,.$$

Lemma 3.1. *Suppose $n\geqslant 3$. Let G be a connected Lie subgroup of $\boldsymbol{SL}(n, \boldsymbol{R})$. Suppose that G contains $\boldsymbol{SO}(n-1)$ and*

$$\dim \boldsymbol{SL}(n, \boldsymbol{R})-n\leqslant \dim G<\dim \boldsymbol{SL}(n, \boldsymbol{R})\,.$$

Then G is one of the following: $L(n)$, $N(n)$, $L^*(n)$ and $N^*(n)$.

Proof. Denote by $M_n(\boldsymbol{R})$ the set of all $n\times n$ matrices in the field of real numbers $\boldsymbol{R}$. As usual we consider $M_n(\boldsymbol{R})$ as the Lie algebra of the general linear group $\boldsymbol{GL}(n, \boldsymbol{R})$. Denote by $\mathfrak{sl}(n, \boldsymbol{R})$ and $\mathfrak{so}(n)$ the Lie subalgebras of $M_n(\boldsymbol{R})$ corresponding to the Lie subgroups $\boldsymbol{SL}(n, \boldsymbol{R})$ and $\boldsymbol{SO}(n)$ of $\boldsymbol{GL}(n, \boldsymbol{R})$ respectively. Then

$$\mathfrak{sl}(n, \boldsymbol{R}) = \{X \in M_n(\boldsymbol{R})\colon \text{trace } X = 0\}\ ,$$
$$\mathfrak{so}(n) = \{X \in M_n(\boldsymbol{R})\colon X \text{ is skew-symmetric}\}\ .$$

Denote by $\mathfrak{sl}(n-1, \boldsymbol{R})$ the Lie subalgebra of $\mathfrak{sl}(n, \boldsymbol{R})$ corresponding to the Lie subgroup $\boldsymbol{SL}(n-1, \boldsymbol{R})$ of $\boldsymbol{SL}(n, \boldsymbol{R})$. Put

$$\begin{aligned}
&\mathfrak{so}(n-1) = \mathfrak{so}(n) \cap \mathfrak{sl}(n-1, \boldsymbol{R})\,,\\
&\mathfrak{sym}(n-1) = \{X \in \mathfrak{sl}(n-1, \boldsymbol{R})\colon X \text{ is symmetric}\}\ ,\\
&\mathfrak{a} = \{(a_{ij}) \in \mathfrak{sl}(n, \boldsymbol{R})\colon a_{ij} = 0 \text{ for } i \neq 1\}\ ,\\
&\mathfrak{a}^* = \{(a_{ij}) \in \mathfrak{sl}(n, \boldsymbol{R})\colon a_{ij} = 0 \text{ for } j \neq 1\}\ ,\\
&\mathfrak{b} = \{(a_{ij}) \in \mathfrak{sl}(n, \boldsymbol{R})\colon a_{ij} = 0 \text{ for } i \neq j,\ a_{22} = a_{33} = \cdots = a_{nn}\}\ .
\end{aligned}$$

These are linear subspaces of $\mathfrak{sl}(n, \boldsymbol{R})$ and

$$\begin{aligned}
\mathfrak{sl}(n, \boldsymbol{R}) &= \mathfrak{sl}(n-1, \boldsymbol{R}) \oplus \mathfrak{a} \oplus \mathfrak{a}^* \oplus \mathfrak{b}\,,\\
\mathfrak{sl}(n-1, \boldsymbol{R}) &= \mathfrak{so}(n-1) \oplus \mathfrak{sym}(n-1)
\end{aligned}$$

as direct sums of vector spaces. Moreover we have

$$\begin{aligned}
&[\mathfrak{a}, \mathfrak{a}] = \{0\},\ [\mathfrak{a}^*, \mathfrak{a}^*] = \{0\},\ [\mathfrak{b}, \mathfrak{b}] = \{0\}\ ,\\
(1)\qquad &[\mathfrak{a}, \mathfrak{b}] = \mathfrak{a},\ [\mathfrak{a}^*, \mathfrak{b}] = \mathfrak{a}^*,\ [\mathfrak{a}, \mathfrak{a}^*] = \mathfrak{sl}(n-1, \boldsymbol{R}) \oplus \mathfrak{b}\,,\\
&[\mathfrak{a}, \mathfrak{sl}(n-1, \boldsymbol{R})] = \mathfrak{a},\ [\mathfrak{a}^*, \mathfrak{sl}(n-1, \boldsymbol{R})] = \mathfrak{a}^*\ .
\end{aligned}$$

Denote by $Ad\colon \boldsymbol{SL}(n, \boldsymbol{R}) \to \boldsymbol{GL}(\mathfrak{sl}(n, \boldsymbol{R}))$ the adjoint representation. Then the linear subspaces $\mathfrak{sl}(n-1, \boldsymbol{R})$, $\mathfrak{a}$, $\mathfrak{a}^*$ and $\mathfrak{b}$ are $Ad(\boldsymbol{SL}(n-1, \boldsymbol{R}))$ invariant, and the linear subspaces $\mathfrak{so}(n-1)$ and $\mathfrak{sym}(n-1)$ are $Ad(\boldsymbol{SO}(n-1))$ invariant. Moreover the linear subspaces $\mathfrak{sym}(n-1)$, $\mathfrak{a}$, $\mathfrak{a}^*$ and $\mathfrak{b}$ are irreducible $Ad(\boldsymbol{SO}(n-1))$ spaces respectively for each $n \geqslant 3$. The Lie subalgebras

$$\begin{aligned}
(2)\qquad &\mathfrak{sl}(n-1, \boldsymbol{R}) \oplus \mathfrak{a},\ \mathfrak{sl}(n-1, \boldsymbol{R}) \oplus \mathfrak{a} \oplus \mathfrak{b}\,,\\
&\mathfrak{sl}(n-1, \boldsymbol{R}) \oplus \mathfrak{a}^*,\ \mathfrak{sl}(n-1, \boldsymbol{R}) \oplus \mathfrak{a}^* \oplus \mathfrak{b}
\end{aligned}$$

of $\mathfrak{sl}(n, \boldsymbol{R})$ corresponds to the connected Lie subgroups $L(n)$, $N(n)$, $L^*(n)$ and $N^*(n)$ of $\boldsymbol{SL}(n, \boldsymbol{R})$ respectively.

Let G be a connected Lie subgroup of $\boldsymbol{SL}(n, \boldsymbol{R})$. Denote by $\mathfrak{g}$ the corresponding Lie subalgebra of $\mathfrak{sl}(n, \boldsymbol{R})$. Suppose that

(3) G contains $\boldsymbol{SO}(n-1)$, and

(4) $\dim \boldsymbol{SL}(n, \boldsymbol{R}) - n \leqslant \dim G < \dim \boldsymbol{SL}(n, \boldsymbol{R})$.

By (3), $\mathfrak{g}$ is an $Ad(\boldsymbol{SO}(n-1))$ invariant linear subspace of $\mathfrak{sl}(n, \boldsymbol{R})$ which contains $\mathfrak{so}(n-1)$. Hence we derive that

$$\mathfrak{g} = \mathfrak{so}(n-1) \oplus (\mathfrak{g} \cap \mathfrak{sym}(n-1)) \oplus (\mathfrak{g} \cap (\mathfrak{a} \oplus \mathfrak{a}^*)) \oplus (\mathfrak{g} \cap \mathfrak{b})$$

as a direct sum of $Ad(\boldsymbol{SO}(n-1))$ invariant linear subspaces. The inequality (4) implies that $\mathfrak{g}$ contains $\mathfrak{Sym}(n-1)$ or $\mathfrak{a}\oplus\mathfrak{a}^*$, because $\mathfrak{Sym}(n-1)$, $\mathfrak{a}$ and $\mathfrak{a}^*$ are irreducible $Ad(\boldsymbol{SO}(n-1))$ spaces respectively and

$$\dim \mathfrak{a} = \dim \mathfrak{a}^* = n-1, \ \dim \mathfrak{Sym}(n-1) \geqslant n-1$$

for any $n\geqslant 3$. If $\mathfrak{a}\oplus\mathfrak{a}^*$ is contained in $\mathfrak{g}$, then $\mathfrak{g}=\mathfrak{sl}(n, \boldsymbol{R})$ by (1). This is a contradiction to (4). It follows that

$$(5) \qquad \mathfrak{Sym}(n-1)\subset\mathfrak{g}, \ \mathfrak{a}\oplus\mathfrak{a}^*\not\subset\mathfrak{g}\,.$$

In particular, $\mathfrak{g}$ contains $\mathfrak{sl}(n-1, \boldsymbol{R})$, and hence G contains $\boldsymbol{SL}(n-1, \boldsymbol{R})$. Then we derive that

$$(6) \qquad \mathfrak{g} = \mathfrak{sl}(n-1, \boldsymbol{R})\oplus(\mathfrak{g}\cap(\mathfrak{a}\oplus\mathfrak{a}^*))\oplus(\mathfrak{g}\cap\mathfrak{b})$$

as a direct sum of $Ad(\boldsymbol{SL}(n-1, \boldsymbol{R}))$ invariant linear subspaces.

Suppose first $n\geqslant 4$. Then $\mathfrak{a}$ and $\mathfrak{a}^*$ are mutually non-equivalent irreducible $Ad(\boldsymbol{SL}(n-1, \boldsymbol{R}))$ spaces; hence $Ad(\boldsymbol{SL}(n-1, \boldsymbol{R}))$ invariant subspaces of $\mathfrak{a}\oplus\mathfrak{a}^*$ are one of the following : $\{0\}$, $\mathfrak{a}$, $\mathfrak{a}^*$ and $\mathfrak{a}\oplus\mathfrak{a}^*$. It follows that $\mathfrak{g}$ is one of the Lie algebras in (2), by (1), (4), (5) and (6).

Suppose next $n=3$. Then $\mathfrak{a}$ and $\mathfrak{a}^*$ are equivalent irreducible $Ad(\boldsymbol{SL}(2, \boldsymbol{R}))$ spaces. Put

$$h(p, q) = \left\{\begin{pmatrix} 0 & qy & -qx \\ px & 0 & 0 \\ py & 0 & 0 \end{pmatrix} : x, y\in\boldsymbol{R}\right\}$$

for each real numbers p, q. Then $h(p, q)$ is an $Ad(\boldsymbol{SL}(2, \boldsymbol{R}))$ invariant linear subspace of $\mathfrak{a}\oplus\mathfrak{a}^*$ for each p, q. It is easy to see that any $Ad(\boldsymbol{SL}(2, \boldsymbol{R}))$ invariant proper linear subspace of $\mathfrak{a}\oplus\mathfrak{a}^*$ is one of $h(p, q)$ for certain p, q. It follows that

$$\mathfrak{g}\cap(\mathfrak{a}\oplus\mathfrak{a}^*) = h(p, q)$$

for certain real numbers p, q. Suppose $pq\neq 0$. Then we derive

$$\begin{aligned} &[h(p, q), h(p, q)] = \mathfrak{b}\,, \\ &[h(p, q), \mathfrak{b}] = h(-p, q)\,, \\ &h(p, q)+h(-p, q) = \mathfrak{a}\oplus\mathfrak{a}^*\,. \end{aligned}$$

It follows that $\mathfrak{g}$ contains $\mathfrak{a}\oplus\mathfrak{a}^*$; this is a contradiction to (5). Hence we obtain $pq=0$, namely

$$\mathfrak{g}\cap(\mathfrak{a}\oplus\mathfrak{a}^*) = \{0\}, \ \mathfrak{a} \text{ or } \mathfrak{a}^*\,.$$

It follows that $\mathfrak{g}$ is one of the Lie algebras in (2), by (1), (4) and (6).

Consequently the assumptions (3) and (4) implies that the Lie algebra $\mathfrak{g}$ is one of the Lie algebras in (2) for each $n \geqslant 3$, and hence the connected Lie subgroup G is one of the following: $L(n)$, $N(n)$, $L^*(n)$ and $N^*(n)$.

This completes the proof of Lemma 3.1.

4. Real analytic $SL(n, \boldsymbol{R})$ actions on the n-sphere

Let ψ: $\boldsymbol{SL}(n, \boldsymbol{R}) \times S^n \rightarrow S^n$ be a real analytic non-trivial action of $\boldsymbol{SL}(n, \boldsymbol{R})$ on the standard n-sphere S^n. For each subgroup H of $\boldsymbol{SL}(n, \boldsymbol{R})$, we put

$$F(H) = \{x \in S^n \colon \psi(h, x) = x \quad \text{for all } h \in H\} ,$$

namely, $F(H)$ is the fixed point set of the restricted action of ψ to H. Then $F(H)$ is a closed subset of S^n, but it is not necessary a submanifold of S^n.

Lemma 4.1. *Suppose $n \geqslant 3$. Then*

$$F(\boldsymbol{SO}(n)) = F(\boldsymbol{SL}(n, \boldsymbol{R})) = F(L(n)) \cap F(L^*(n)) ,$$
$$F(\boldsymbol{SO}(n-1)) = F(L(n)) \text{ or } F(L^*(n))$$

for any real analytic non-trivial $\boldsymbol{SL}(n, \boldsymbol{R})$ action on the n-sphere.

Proof. From Lemma 3.1, we derive

$$F(\boldsymbol{SO}(n)) = F(\boldsymbol{SL}(n, \boldsymbol{R})) = F(L(n)) \cap F(L^*(n)) ,$$
$$F(\boldsymbol{SO}(n-1)) = F(L(n)) \cup F(L^*(n)) .$$

According to Theorem 1.3, we see that the set $F(\boldsymbol{SO}(n-1)) - F(\boldsymbol{SO}(n))$ has just two connected components. Each connected component is contained in $F(L(n))$ or $F(L^*(n))$. Put

$$g = \begin{pmatrix} -1 & & & & \\ & -1 & & & \\ & & 1 & & \\ & & & \ddots & \\ & & & & 1 \end{pmatrix} .$$

Then it follows easily from Theorem 1.3 that x and gx belong distinct connected components respectively for each element x of $F(\boldsymbol{SO}(n-1)) - F(\boldsymbol{SO}(n))$. Then we conclude that

$$F(\boldsymbol{SO}(n-1)) = F(L(n)) \quad \text{or} \quad F(L^*(n)) . \qquad \text{q.e.d.}$$

Denote by $\sigma(g)$ the transpose of g^{-1} for each $g \in \boldsymbol{SL}(n, \boldsymbol{R})$. Then the correspondence $g \rightarrow \sigma(g)$ defines an automorphism σ of $\boldsymbol{SL}(n, \boldsymbol{R})$. The automorphism σ is an involution and

$$\sigma(L(n)) = L^*(n)\,.$$

Let ψ be a real analytic non-trivial $\boldsymbol{SL}(n, \boldsymbol{R})$ action on S^n. Define a new action $\sigma_\sharp\psi$ of $\boldsymbol{SL}(n, \boldsymbol{R})$ on S^n as follows:

$$(\sigma_\sharp\psi)(g, x) = \psi(\sigma(g), x) \qquad \text{for } g \in \boldsymbol{SL}(n, \boldsymbol{R}),\ x \in S^n\,.$$

Then it is seen that if $F(\boldsymbol{SO}(n-1)) = F(L(n))$ (resp. $F(L^*(n))$) for the action ψ, then $F(\boldsymbol{SO}(n-1)) = F(L^*(n))$ (resp. $F(L(n))$) for the action $\sigma_\sharp\psi$.

As in the section 2, let $\boldsymbol{A}(n)$ denote the set of all real analytic non-trivial $\boldsymbol{SL}(n, \boldsymbol{R})$ actions on S^n, and let $\boldsymbol{A}_*(n)$ denote the set of all equivalence classes of $\boldsymbol{A}(n)$. Then the mapping $\sigma_\sharp$: $\boldsymbol{A}(n) \to \boldsymbol{A}(n)$ is an involution, and $\sigma_\sharp$ induces naturally an involution σ_*: $\boldsymbol{A}_*(n) \to \boldsymbol{A}_*(n)$.

Denote by $\boldsymbol{A}^+(n)$ (resp. $\boldsymbol{A}^-(n)$) the set of all real analytic non-trivial $\boldsymbol{SL}(n, \boldsymbol{R})$ actions on S^n such that

$$F(\boldsymbol{SO}(n-1)) = F(L(n)) \text{ (resp. } F(L^*(n)))\,.$$

Denote by $\boldsymbol{A}^+_*(n)$ (resp. $\boldsymbol{A}^-_*(n)$) the set of all equivalence classes represented by an element of $\boldsymbol{A}^+(n)$ (resp. $\boldsymbol{A}^-(n)$). Then we derive

$$\sigma_\sharp\boldsymbol{A}^+(n) = \boldsymbol{A}^-(n)\,, \quad \sigma_\sharp\boldsymbol{A}^-(n) = \boldsymbol{A}^+(n)\,,$$
$$\sigma_*\boldsymbol{A}^+_*(n) = \boldsymbol{A}^-_*(n)\,, \quad \sigma_*\boldsymbol{A}^-_*(n) = \boldsymbol{A}^+_*(n)\,.$$

Moreover $\boldsymbol{A}_*(n)$ is a disjoint union of $\boldsymbol{A}^+_*(n)$ and $\boldsymbol{A}^-_*(n)$ by Lemma 4.1. Let c_n: $\boldsymbol{F}_* \to \boldsymbol{A}_*(n)$ be the mapping defined in the section 2. Then it is seen that the image $c_n(\boldsymbol{F}_*)$ is contained in $\boldsymbol{A}^+_*(n)$.

We shall show the following result.

Theorem 4.2. *$c_n(\boldsymbol{F}_*) = \boldsymbol{A}^+_*(n)$ for each $n \geqslant 3$.*

In order to prove this theorem, we require the following result due to Guillemin and Sternberg [4]:

Lemma 4.3. *Let $\mathfrak{g}$ be a real semi-simple Lie algebra and let ρ: $\mathfrak{g} \to L(M)$ be a homomorphism of $\mathfrak{g}$ into the Lie algebra of real analytic vector fields on a real analytic n-manifold M. Let p be a point at which the vector fields in the image $\rho(\mathfrak{g})$ have a common zero. Then there exists an analytic system of coordinates $(U; x_1, \cdots, x_n)$, with origin at p, in which all of the vector fields in $\rho(\mathfrak{g})$ are linear. Namely, there exists*

$$a_{ij} \in \mathfrak{g}^* = \mathrm{Hom}_{\boldsymbol{R}}(\mathfrak{g}, \boldsymbol{R})$$

such that

$$\rho(X)_q = \sum_{i,j} a_{ij}(X) x_i(q) \frac{\partial}{\partial x_j} \qquad \textit{for } X \in \mathfrak{g},\ q \in U\,.$$

REMARK. The correspondence $X \rightarrow (a_{ij}(X))$ defines a Lie algebra homomorphism of $\mathfrak{g}$ into $\mathfrak{sl}(n, \boldsymbol{R})$.

Lemma 4.4. *Suppose $n \geqslant 3$. Let ψ be a real analytic non-trivial $\boldsymbol{SL}(n, \boldsymbol{R})$ action on S^n such that $F(\boldsymbol{SO}(n-1)) = F(L(n))$. Let $p \in S^n$ be a fixed point of the $\boldsymbol{SL}(n, \boldsymbol{R})$ action ψ. Then there is an equivariant real analytic diffeomorphism h of $\boldsymbol{R}^n$ onto an invariant open set of S^n such that $h(0) = p$. Here $\boldsymbol{SL}(n, \boldsymbol{R})$ acts standardly on $\boldsymbol{R}^n$.*

Proof. Notice that, for each $n \geqslant 3$, any non-trivial endomorphism of $\mathfrak{sl}(n, \boldsymbol{R})$ is of the form $Ad(g)$ or $Ad(g) \cdot d\sigma$, where $g \in \boldsymbol{GL}(n, \boldsymbol{R})$ and $d\sigma$ is the differential of the automorphism σ. Define a Lie algebra homomorphism

$$\rho \colon \mathfrak{sl}(n, \boldsymbol{R}) \rightarrow L(S^n)$$

as follows:

$$(1) \qquad \rho(X)_q(f) = \lim_{t \to 0} \frac{f(\psi(\exp(-tX), q)) - f(q)}{t}$$

for $X \in \mathfrak{sl}(n, \boldsymbol{R})$, $q \in S^n$. Here f is a real valued real analytic function on S^n. Then $\rho(X)_p = 0$ for each $X \in \mathfrak{sl}(n, \boldsymbol{R})$. According to Lemma 4.3, there exists an analytic system of coordinates $(U; x_1, \cdots, x_n)$, with origin at p, and there exists $a_{ij} \in \mathfrak{sl}(n, \boldsymbol{R})^*$ such that

$$(2) \qquad \rho(X)_q = \sum_{i,j} a_{ij}(X) x_i(q) \frac{\partial}{\partial x_j} \qquad \text{for } X \in \mathfrak{sl}(n, \boldsymbol{R}),\ q \in U\,.$$

By the above notice, it can be assumed that

$$(3) \qquad X = (a_{ij}(X)) \qquad \text{for each } X \in \mathfrak{sl}(n, \boldsymbol{R}), \text{ or}$$

$$(3') \qquad d\sigma(X) = (a_{ij}(X)) \qquad \text{for each } X \in \mathfrak{sl}(n, \boldsymbol{R})\,.$$

From the assumption $F(\boldsymbol{SO}(n-1)) = F(L(n))$, it follows that the case (3) does not happen.

Let $k \colon U \rightarrow \boldsymbol{R}^n$ be a real analytic diffeomorphism of U onto an open set of $\boldsymbol{R}^n$ defined by $k(q) = (x_1(q), \cdots, x_n(q))$ for $q \in U$. Then $k(p) = 0$. There is a positive real number ε such that the ε-neighborhood $\boldsymbol{D}_\varepsilon$ of the origin is contained in $k(U)$. Put

$$x = \left(\frac{\varepsilon}{2}, 0, \cdots, 0\right).$$

Then the group $L(n)$ is the isotropy group at x. Moreover $L(n)$ agrees with the identity component of the isotropy group at $k^{-1}(x)$ by (1), (2) and (3′). Define a map $h \colon \boldsymbol{R}^n \rightarrow S^n$ as follows:

$$h(0) = p;\ h(gx) = \psi(g, k^{-1}(x)) \qquad \text{for } g \in \boldsymbol{SL}(n, \boldsymbol{R})\,.$$

The map h is a well-defined equivariant $\boldsymbol{SL}(n, \boldsymbol{R})$ map. It follows that

$$k \cdot h = \textit{identity on} \text{ on } \boldsymbol{D}_\varepsilon$$

by the uniqueness of the solution of an ordinary differential equation defined by (1), (2) and (3′). Hence the map h: $\boldsymbol{R}^n \to S^n$ is a real analytic submersion of $\boldsymbol{R}^n$ onto an invariant open set of S^n. Since h is injective on $\boldsymbol{D}_\varepsilon$, it can be seen that the isotropy group at $h(x) = k^{-1}(x)$ agrees with $L(n)$. Then the map h: $\boldsymbol{R}^n \to S^n$ is injective.

This completes the proof of Lemma 4.4.

Proof of Theorem 4.2. Let ψ be an element of $\boldsymbol{A}^+(n)$. According to Theorem 1.3 and Lemma 4.1, $F(L(n))$ is a real analytic submanifold of S^n on which $N(n)$ acts naturally, and $F(L(n))$ is real analytically diffeomorphic to S^1. Moreover $F = F(\boldsymbol{SL}(n, \boldsymbol{R}))$ consists of two points N, S. Let h: $(-1-\varepsilon, 1+\varepsilon) \to F(L(n))$ be a real analytic imbedding such that $h(1) = N$ and $h(-1) = S$, where ε is a sufficiently small positive real number. Since $N(n)/L(n) \cong \boldsymbol{R}$ acts real analytically on $F(L(n))$, the action defines a real analytic vector field v on $F(L(n))$ naturally. Then there exists a real analytic function $f(t)$ on the interval $(-1-\varepsilon, 1+\varepsilon)$ such that $v = h_*\left(f(t)\dfrac{d}{dt}\right)$ on the image of h. We shall first show that the function $f(t)$ satisfies the conditions (A), (B) stated in the section 2. The condition (A) follows from $F = \{N, S\}$. Considering the standard action of $\boldsymbol{SL}(n, \boldsymbol{R})$ on $\boldsymbol{R}^n$, we can see that the condition (B) follows from Lemma 4.4.

We shall next show that the n-sphere S^n with the $\boldsymbol{SL}(n, \boldsymbol{R})$ action ψ is equivariantly real analytically diffeomorphic to M_f, where M_f is a real analytic $\boldsymbol{SL}(n, \boldsymbol{R})$-manifold constructed from $f(t)$ as before. For this purpose, we consider the following commutative diagram:

$$\begin{array}{ccc}
\boldsymbol{SO}(n) \underset{NSO(n-1)}{\times} (F(\boldsymbol{SO}(n-1)) - F) & \xrightarrow{\alpha} & S^n - F \\
\downarrow \beta & & \| \\
\boldsymbol{SL}(n, \boldsymbol{R}) \underset{NL(n)}{\times} (F(L(n)) - F) & \xrightarrow{\gamma} & S^n - F.
\end{array}$$

Here $NSO(n-1)$ and $NL(n)$ are the normalizers of $\boldsymbol{SO}(n-1)$ and $L(n)$ respectively. According to Theorem 1.3, Lemma 3.1 and Lemma 4.1, we can show that α, β and γ are real analytic one-to-one onto mappings. Moreover α is a diffeomorphism by the differentiable slice theorem; hence β and γ are also real analytic diffeomorphisms. It follows that $S^n - F$ is equivariantly real analytically diffeomorphic to a real analytic $\boldsymbol{SL}(n, \boldsymbol{R})$-manifold X_f constructed from $f(t)$ as before. Consequently the n-sphere S^n with the action ψ is equivariantly real analytically diffeomorphic to M_f, by making use of Lemma 4.4. Hence we conclude that $c_n(\boldsymbol{F}_*)=\boldsymbol{A}_*^+(n)$.

This completes the proof of Theorem 4.2.

5. Certain closed subgroups of $O(n)$

In this section, we shall prove Lemma 1.1 and Lemma 1.2. Put

$$D(\theta) = \begin{pmatrix} \cos\theta & \sin\theta \\ -\sin\theta & \cos\theta \end{pmatrix}, \qquad \theta \in \boldsymbol{R}\,.$$

Denote by $D(a_1, \cdots, a_r)$ the one-dimensional closed subgroup of $\boldsymbol{O}(n)$ consists of the following matrices:

$$\begin{pmatrix} D(a_1\theta) & & 0 \\ & \ddots & \\ 0 & & D(a_r\theta) \end{pmatrix}, \qquad \theta \in \boldsymbol{R}$$

for $n=2r$, and

$$\begin{pmatrix} D(a_1\theta) & & & 0 \\ & \ddots & & \\ & & D(a_r\theta) & \\ 0 & & & 1 \end{pmatrix}, \qquad \theta \in \boldsymbol{R}$$

for $n=2r+1$, respectively. Here $a_1, \cdots, a_r$ are integers. Consider $U(k)$ as the centralizer of

$$\begin{pmatrix} D(\pi/2) & & 0 \\ & \ddots & \\ 0 & & D(\pi/2) \end{pmatrix}$$

in $\boldsymbol{O}(2k)$. Then we can derive easily the following result.

Lemma 5.1. *Suppose that $b_1 > b_2 > \cdots > b_s > 0$ and*

$$(a_1, \cdots, a_r) = (\underbrace{b_1, \cdots, b_1}_{n_1}, \cdots, \underbrace{b_s, \cdots, b_s}_{n_s}, 0, \cdots, 0)\,.$$

Then the centralizer of $D(a_1, \cdots, a_r)$ in $\boldsymbol{O}(n)$ agrees with

$$\boldsymbol{U}(n_1) \times \cdots \times \boldsymbol{U}(n_s) \times \boldsymbol{O}(m)\,,$$

where $m=n-2(n_1+\cdots+n_s)$.

Here we shall prove Lemma 1.2. Let h: $\boldsymbol{SO}(n) \to \boldsymbol{O}(n)$ be a continuous homomorphism with a finite kernel. Suppose $n \geqslant 3$. Then it is easy to see that h is an isomorphism onto $\boldsymbol{SO}(n)$. Denote by T a maximal torus of $\boldsymbol{SO}(n)$ defined by the direct product of the subgroups

$$T_k = D(\underbrace{0, \cdots, 0, 1}_{k}, 0, \cdots, 0)$$

for $0<k\leqslant n/2$. Then there is an element x_1 of $SO(n)$ such that $h(T)=x_1Tx_1^{-1}$. Then the subgroup $x_1^{-1}h(T_k)x_1$ is of the form $D(a_{k1}, \cdots, a_{kr})$ for each k. Compare the centralizer of T_k and that of $x_1^{-1}h(T_k)x_1$ in $O(n)$. We can derive

$$x_1^{-1}h(T_k)x_1 = T_j$$

for some j, by Lemma 5.1. Hence there is an element x_2 of $O(n)$ such that

$$h(t) = x_1x_2tx_2^{-1}x_1^{-1}, \qquad \text{for } t\in T.$$

It follows that the representations $y\to y$ and $y\to x_2^{-1}x_1^{-1}h(y)x_1x_2$ of $SO(n)$ are equivalent. Since the representation $y\to y$ is absolutely irreducible, there is an element x_3 of $O(n)$ such that

$$x_3yx_3^{-1} = x_2^{-1}x_1^{-1}h(y)x_1x_2$$

for each $y\in SO(n)$ (cf. [6],Lemma 5.5.1). Put $x=x_1x_2x_3$. Then we derive that $x\in O(n)$ and $h(y)=xyx^{-1}$ for each $y\in SO(n)$.

This completes the proof of Lemma 1.2.

We shall next prove Lemma 1.1. Let G be a connected closed subgroup of $O(n)$. Suppose that $n\geqslant 3$ and

$$(1) \qquad \dim O(n)>\dim G\geqslant \dim O(n)-n\,.$$

The inclusion map i: $G\to O(n)$ gives an orthogonal faithful representation of G. Suppose first that the representation i is reducible. Then, by an inner automorphism of $O(n)$, G is isomorphic to a closed subgroup G' of $O(k)\times O(n-k)$ for some k such that $0<k\leqslant n/2$. By (1), we derive that $k=1$, or $k=2$ and $G'=SO(2)\times SO(2)$. The codimension of $O(1)\times O(n-1)$ in $O(n)$ is $n-1$. If $n\geqslant 4$, then $SO(n-1)$ is semi-simple; hence there is no closed subgroup of codimension one in $SO(n-1)$. We can conclude that

$$\begin{aligned} G' &= SO(1)\times SO(n-1)\cong SO(n-1)\,, \\ G' &= SO(2)\times SO(2) \qquad \text{for } n = 4, \text{ or} \\ G' &= \{1\} \qquad \text{for } n = 3\,. \end{aligned}$$

Suppose next that the representation i is irreducible and G has a one-dimensional central subgroup. By Lemma 5.1, it can be seen that n is even and G is isomorphic to a closed subgroup G' of $U(n/2)$ by an inner automorphism of $O(n)$. It follows from (1) that

$$\begin{aligned} G' &= U(3) \qquad \text{for } n = 6, \text{ or} \\ G' &= U(2) \qquad \text{for } n = 4\,. \end{aligned}$$

It remains to consider the case that G is semi-simple and the representation i is irreducible. In the following, we assume that G is semi-simple and the representation i is irreducible. Suppose that the complexification i^C of i is reducible. Then the representation i posesses a complex structure and n is even. Hence G is isomorphic to a closed subgroup of $\boldsymbol{U}(n/2)$. We can derive that $n=4$ by (1). Moreover, by an inner automorphism of $\boldsymbol{O}(4)$, G is isomorphic to $\boldsymbol{SU}(2)$ which is standardly imbedded in $\boldsymbol{O}(4)$.

Suppose that the complexification i^C of i is irreducible. Then i^C is a complex irreducible representation of G of degree n.

(i) Moreover suppose first that G is not simple. Let G^* be the universal covering group of G, and let $p: G^* \to G$ be the covering projection. Since G is not simple, there are closed semi-simple normal subgroups H_1 and H_2 of G^* such that

$$G^* = H_1 \times H_2 .$$

Consider the representation $i^C p: G^* \to \boldsymbol{U}(n)$. Then there are irreducible complex representations r_1 and r_2 of H_1 and H_2 respectively, such that the tensor product $r_1 \otimes r_2$ is equivalent to $i^C p$. Since $i^C p$ has a real form ip, the representations r_1 and r_2 are self-conjugate; hence r_1 (resp. r_2) has a real form or a quaternionic structure, but not both (cf.[1], Proposition 3.56). Moreover, if r_1 has a real form (resp. quaternionic structure), then r_2 has also a real form (resp. quaternionic structure). Put $n_s = \deg r_s$ for $s=1, 2$. Then

$$(2) \qquad \dim \boldsymbol{O}(n) - n = \frac{n(n-3)}{2} = \frac{n_1 n_2 (n_1 n_2 - 3)}{2} .$$

Suppose first that r_1 has a quaternionic structure. Then it follows that n_1 and n_2 are even, and

$$\dim H_s \leqslant \dim \boldsymbol{Sp}\left(\frac{n_s}{2}\right) \qquad \text{for } s = 1, 2 .$$

Hence

$$\dim G = \dim H_1 + \dim H_2 \leqslant \frac{n_1(n_1+1)}{2} + \frac{n_2(n_2+1)}{2} .$$

Compare the above inequality with (2). We can derive easily that

$$\dim G < \dim \boldsymbol{O}(n) - n$$

except the case $n_1 = n_2 = 2$. If $n_1 = n_2 = 2$, then $n=4$ and $\dim G = \dim \boldsymbol{O}(n)$. We can conclude from (1) that r_1 has no quaternionic structure. Suppose next that r_1 has a real form. Then, since H_s is semi-simple, it follows that

$$n_s \geqslant 3 \qquad \text{for } s = 1, 2 .$$

Moreover it follows that

$$\dim H_s \leqslant \dim \boldsymbol{O}(n_s) \qquad \text{for } s = 1, 2 .$$

Hence

$$\dim G = \dim H_1 + \dim H_2 \leqslant \frac{n_1(n_1-1)}{2} + \frac{n_2(n_2-1)}{2} .$$

Compare the above inequality with (2). We can derive that

$$\dim G < \dim \boldsymbol{O}(n) - n .$$

This is a contradiction to (1), and hence we can conclude that ι_1 has no real form. Consequently we can conclude that G must be simple.

(ii) Suppose next that G is simple. Moreover suppose first that G is an exceptional Lie group. Then we can derive the following result from a table of the degrees of the basic representations (cf. [2], p. 378, Table 30): the possibility remains only in the case that $n=7$ and G is locally isomorphic to the exceptional Lie group $\boldsymbol{G}_2$. Consider $\boldsymbol{G}_2$ as a closed subgroup of $\boldsymbol{O}(7)$ as usual. Then we can conclude that G is isomorphic to $\boldsymbol{G}_2$ by an inner automorphism of $\boldsymbol{O}(7)$. It remains to consider the case that G is locally isomorphic to $\boldsymbol{SU}(k)$, $\boldsymbol{Sp}(k)$ or $\boldsymbol{SO}(k)$. Put $r = \operatorname{rank} G$. Denote by G^* the universal covering group of G. Denote by $L_1, \cdots, L_r$ the fundamental weights of G^*. Then there is a one-to-one correspondence between complex irreducible representations of G^* and sequences $(a_1, \cdots, a_r)$ of non-negative integers such that $a_1L_1 + \cdots + a_rL_r$ is the highest weight of a corresponding complex irreducible representation (cf. [2], Theorem 0.8, Theorem 0.9). Denote by

$$d(a_1L_1 + \cdots + a_rL_r)$$

the degree of the complex irreducible representation of G^* with the highest weight $a_1L_1 + \cdots + a_rL_r$. The degree can be computed by the Weyl's formula (cf. [2], Theorem 0.24; (0.148), (0.149), (0.150)). Notice that if

$$a_1 \geqslant a_1', \cdots, a_r \geqslant a_r' ,$$

then

$$d(a_1L_1 + \cdots + a_rL_r) \geqslant d(a_1'L_1 + \cdots + a_r'L_r)$$

and the equality holds only if $a_1 = a_1', \cdots, a_r = a_r'$.

(a) Suppose first that G^* is isomorphic to $\boldsymbol{SU}(r+1)$ for $r \geqslant 1$. Since $\operatorname{rank} G \leqslant \operatorname{rank} \boldsymbol{SO}(n)$, it follows that

$$(3) \qquad 2r \leqslant n .$$

If $r \geqslant 6$, then we derive from (3) that

$$\dim G = \dim \boldsymbol{SU}(r+1) = r(r+2) < \frac{n(n-3)}{2} = \dim \boldsymbol{O}(n) - n .$$

This is a contradiction to (1). If the pair (n, r) satisfies the conditions (1) and (3), then (n, r) is one of the following:

$$(10,5),\ (8,4),\ (7,3),\ (5,2) \text{ and } (4,1).$$

Notice that

$$d(L_i) = {}_{r+1}C_i\,, \quad d(2L_1) = d(2L_r) = \frac{(r+1)\cdot(r+2)}{2}.$$

Thus there is no complex irreducible representation of $SU(r+1)$ of degree $2r$ for $r=4,5$. Hence (n, r) is not (10,5) nor (8,4). Since

$$\begin{aligned}
&d(2L_1) = d(2L_2) = 6, \quad d(L_1+L_2) = 8 \qquad \text{for } r = 2;\\
&d(2L_1) = d(2L_3) = 10, \quad d(2L_2) = d(L_1+L_2) = d(L_2+L_3) = 20\,,\\
&\qquad \text{and } d(L_1+L_3) = 15 \qquad \text{for } r = 3\,,
\end{aligned}$$

it follows that there is no complex irreducible representation of $SU(r+1)$ of degree $2r+1$ for $r=2,3$. Hence (n, r) is not (7,3) nor (5,2). It remains only the case $(n, r)=(4,1)$. But it is seen that the complex irreducible representation of $SU(2)$ of degree 4 has no real form. Therefore we can derive that G is not locally isomorphic to $SU(r+1)$.

(b) Suppose next that G^* is isomorphic to $Sp(r)$ for $r\geqslant 2$. Since rank $G\leqslant$ rank $SO(n)$, it follows that

$$(4) \qquad 2r\leqslant n\,.$$

On the other hand, since dim $Sp(r)=r(2r+1)$, the inequality (1) implies that

$$(5) \qquad n(n-3)\leqslant 2r(2r+1)<n(n-1)\,.$$

It follows from (4), (5) that

$$1\leqslant\frac{n}{2r}\leqslant\frac{2r+1}{n-3}\,.$$

Therefore, if the pair (n, r) satisfies the conditions (4), (5), then we derive $n=2r+2$. Notice that

$$d(L_i) = {}_{2r+1}C_i-{}_{2r+1}C_{i-1},\ d(2L_1) = r(2r+1)\,.$$

If $r\geqslant 3$, then we can derive that

$$\begin{aligned}
&d(L_i)\geqslant 2r+3 \qquad \text{for } i = 2, 3, \cdots, r;\\
&d(2L_1)\geqslant 2r+3\,.
\end{aligned}$$

If $r=2$, then

$$\begin{aligned}
&\mathrm{d}(L_1) = 4,\ d(L_2) = 5,\ d(2L_1) = 10\,,\\
&d(2L_2) = 14 \quad \text{and} \quad d(L_1+L_2) = 16\,.
\end{aligned}$$

It follows that there is no complex irreducible representation of $\boldsymbol{Sp}(r)$ of degree $2r+2$, for $r \geqslant 2$. Therefore we can derive that G is not locally isomorphic to $\boldsymbol{Sp}(r)$.

(c) Suppose finally that G^* is isomorphic to $\boldsymbol{Spin}(k)$ for $k \geqslant 5$. It follows from (1) that

$$n(n-3) \leqslant k(k-1) < n(n-1)\,.$$

Hence we have $n=k+1$. Suppose $k=2r$. Then

$$\begin{aligned}
&d(L_i) = {}_{2r}C_i \text{ for } 1 \leqslant i \leqslant r-2, \quad d(L_{r-1}) = d(L_r) = 2^{r-1}\,,\\
&d(2L_1) = (r+1)\cdot(2r-1), \quad d(2L_{r-1}) = d(2L_r) = {}_{2r-1}C_r\,,\\
&d(L_1+L_{r-1}) = d(L_1+L_r) = (2r-1)2^{r-1}, \text{ and}\\
&d(L_{r-1}+L_r) = {}_{2r}C_{r-1}\,.
\end{aligned}$$

It follows that there is no complex irreducible representation of $\boldsymbol{Spin}(2r)$ of degree $2r+1$. Suppose $k=2r+1$. Then

$$\begin{aligned}
&d(L_i) = {}_{2r+1}C_i \text{ for } 1 \leqslant i \leqslant r-1, \quad d(L_r) = 2^r\,,\\
&d(2L_1) = r(2r+3), \quad d(L_1+L_r) = r\cdot 2^{r+1}, \text{ and}\\
&d(2L_r) = 2^{2r}.
\end{aligned}$$

It follows that there is no complex irreducible representation of $\boldsymbol{Spin}(2r+1)$ of degree $2r+2$ for $r \neq 3$, and there is a unique complex irreducible representation of $\boldsymbol{Spin}(7)$ of degree 8. It is seen that the representation of $\boldsymbol{Spin}(7)$ has a real form. Therefore we can derive that $n=8$ and G is isomorphic to $\boldsymbol{Spin}(7)$. Here $\boldsymbol{Spin}(7)$ is considered as a closed subgroup of $\boldsymbol{O}(8)$ by the real spin representation. Then the isomorphism of G onto $\boldsymbol{Spin}(7)$ is realized by an inner automorphism of $\boldsymbol{O}(8)$.

This completes the proof of Lemma 1.1.

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