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CLASSIFICATION OF REAL ANALYTIC
SL(n, R) ACTIONS ON n-SPHERE

Dedicated to Professor A. Komatu on his 70th birthday

FUICHI UCHIDA*)

(Received March 17, 1978)

0. Introduction

C.R. Schneider [5] classified real analytic $SL(2, R)$ actions on closed surfaces. Except for the work, there seems to be no work on the classification problem about non-compact Lie group actions.

In this paper, we classify real analytic $SL(n, R)$ actions on the standard $n$-sphere for each $n \geq 3$. Here $SL(n, R)$ denotes the special linear group over the field of real numbers. The result can be stated roughly as follows: there is a one-to-one correspondence between real analytic $SL(n, R)$ actions on the $n$-sphere and real valued real analytic functions on an interval satisfying certain conditions (see Theorem 2.2 and Theorem 4.2). It is important to consider the restricted actions of $SL(n, R)$ to a maximal compact subgroup $SO(n)$.

It is still open to classify $C^\infty$ actions of $SL(n, R)$ on the standard $n$-sphere, by lack of $C^\infty$ analogue of a local theory due to Guillemin and Sternberg (see Lemma 4.3).

1. Real analytic $SO(n)$ actions on certain $n$-manifolds

First we prepare the following two lemmas of which proof is given in the last section.

Lemma 1.1. Let $G$ be a closed connected subgroup of $O(n)$. Suppose that $n \geq 3$ and

\[
\dim O(n) > \dim G \geq \dim O(n) - n.
\]

Suppose that $G$ is not conjugate to $SO(n-1)$ which is canonically imbedded in $O(n)$. Then the pair $(O(n), G)$ is pairwise isomorphic to one of the following:

\[
(O(8), Spin(7)), (O(7), G_2), (O(6), U(3)), (O(4), U(2)),
\]

\[
(O(4), SU(2)), (O(4), SO(2) \times SO(2)) \text{ and } (O(3), \{1\}),
\]

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up to inner automorphisms of $O(n)$. In these cases the subgroups are standardly imbedded in $O(n)$.

**Lemma 1.2.** Suppose $n \geq 3$. Let $h: SO(n) \rightarrow O(n)$ be a continuous homomorphism with a finite kernel. Then there is an element $x$ of $O(n)$ such that $h(y) = x y x^{-1}$ for each $y$ of $SO(n)$.

Now we shall prove the following result.

**Theorem 1.3.** Suppose $n \geq 3$. Let $M$ be a closed connected $n$-dimensional real analytic manifold. Suppose that

\[ \pi_1(M) = \pi_2(M) = \{1\} . \]

Suppose that $SO(n)$ acts on $M$ real analytically and almost effectively. Then the $SO(n)$-manifold $M$ is real analytically diffeomorphic to the standard $n$-sphere $S^n$ as $SO(n)$-manifolds. Here the $SO(n)$ action on $S^n$ is the restriction of the standard $SO(n+1)$ action on $S^n$.

**Proof.** (i) First we show that the $SO(n)$-manifold $M$ is $C^\infty$ diffeomorphic to the standard sphere $S^n$ as $SO(n)$-manifolds. Let $G$ be the identity component of a principal isotropy group. Then

\[ \dim SO(n) > \dim G \geq \dim SO(n) - n , \]

and $SO(n)$ acts almost effectively on the homogeneous space $SO(n)/G$ by the assumption that $SO(n)$ acts almost effectively on $M$, and hence Lemma 1.1 is applicable. The pair $(SO(n), G)$ is not pairwise isomorphic to $(SO(4), U(2))$ nor $(SO(4), SU(2))$, because $SU(2)$ is a normal subgroup of $SO(4)$. If

\[ \dim SO(n)/G = \dim M , \]

then the $SO(n)$ action on $M$ is transitive and the pair $(SO(n), G)$ is pairwise isomorphic to one of the following by Lemma 1.1:

\[ (SO(7), G_2), (SO(6), U(3)), (SO(4), SO(2) \times SO(2)) \quad \text{and} \quad (SO(3), \{1\}) . \]

But

\[ \pi_1(SO(7)/G_2) = \pi_1(SO(3)/\{1\}) = \mathbb{Z}_2 , \]

\[ \pi_2(SO(6)/U(3)) = \mathbb{Z} \quad \text{and} \quad \pi_2(SO(4)/SO(2) \times SO(2)) = \mathbb{Z} \times \mathbb{Z} . \]

This is a contradiction to the assumption

\[ \pi_1(M) = \pi_2(M) = \{1\} . \]

Consequently $G$ is conjugate to $SO(n-1)$ or the pair $(SO(n), G)$ is pairwise isomorphic to $(SO(8), Spin(7))$ by Lemma 1.1 and hence the $SO(n)$-manifold
$M$ has codimension one principal orbits and just two singular orbits (cf. [6], Lemma 1.2.1). Since \( SO(n-1) \) in \( SO(n) \) (resp. \( Spin(7) \) in \( SO(8) \)) is a maximal closed connected subgroup, the singular orbits are fixed points. It follows that the \( SO(n) \)-manifold \( M \) is \( C^\infty \) diffeomorphic to \( M' = D^n \cup D^m \) as \( SO(n) \)-manifolds. Here the \( SO(n) \) action on \( D^n \) is standard by Lemma 1.2, and \( f: \partial D^n \to \partial D^m \) is an \( SO(n) \) equivariant diffeomorphism. It follows that \( f \) is the identity map or the antipodal map, and hence \( M' \) is \( C^\infty \) diffeomorphic to the standard \( n \)-sphere \( S^n \) as \( SO(n) \)-manifolds.

(ii) Here we assume that \( M_1 \) and \( M_2 \) are \( n \)-dimensional real analytic manifolds on which \( SO(n) \) acts real analytically. Assume that the \( SO(n) \)-manifolds \( M_1 \) and \( M_2 \) are \( C^\infty \) diffeomorphic to the standard \( n \)-sphere \( S^n \) as \( SO(n) \)-manifolds. According to a theorem of Grauert ([3], Theorem 3), \( M_i \) is real analytically imbedded in a euclidean spuce of sufficiently high dimension; hence \( M_i \) posesses a real analytic Riemannian metric. By averaging the real analytic Riemannian metric on \( M_i \) with respect to the \( SO(n) \) action, we have an \( SO(n) \) invariant real analytic Riemannian metric \( g_i \) on \( M_i \). Denote by \( \{N_i, S_i\} \) the fixed point set of the \( SO(n) \)-manifold \( M_i \). We can assume that

\[
d_i(N_1, S_1) = d_2(N_2, S_2),
\]

where \( d_i \) is a distance function on \( M_i \) defined by the Riemannian metric \( g_i \). Denote by \( F_i \) the fixed point set of the restricted \( SO(n-1) \) action on \( M_i \). It follows that \( F_i \) is a real analytic submanifold of \( M_i \) which is \( NSO(n-1) \) in-variant and \( C^\infty \) diffeomorphic to \( S^1 \) by the assumption. Here \( NSO(n-1) \) denotes the normalizer of \( SO(n-1) \) in \( SO(n) \). Then there exists an isometry \( \varphi : F_1 \to F_2 \) such that \( \varphi(N_1) = N_2 \) and \( \varphi(S_1) = S_2 \). The isometry \( \varphi \) is a real analytic diffeomorphism and \( \varphi \) is compatible with the action of \( NSO(n-1) \) on \( F_i \). It is easy to see that the \( SO(n) \)-manifold \( M_i - \{N_i, S_i\} \) is real analytically diffeomorphic to

\[
SO(n) \times_{NSO(n-1)} (F_i - \{N_i, S_i\})
\]

as \( SO(n) \)-manifolds; hence \( \varphi \) extends uniquely to an \( SO(n) \) equivariant homeomorphism \( \Phi : M_i \to M_2 \). By the construction, the restriction of \( \Phi \) to \( M_i - \{N_i, S_i\} \) is a real analytic diffeomorphism of \( M_i - \{N_i, S_i\} \) onto \( M_2 - \{N_2, S_2\} \).

(iii) Finally we show that \( \Phi \) is real analytic on neighborhoods of \( N_i \) and \( S_i \). Notice that the tangent space of \( M_i \) at \( N_i \) with the induced \( SO(n) \) action is naturally isomorphic to \( R^n \) with the standard \( SO(n) \) action by the assumption. Denote by \( D_i \) an \( \varepsilon \)-neighborhood of the origin \( 0 \) in \( R^n \). Denote by \( e_i : D_i \to M_i \) the exponential map with respect to the Riemannian metric \( g_i \) such that \( e_i(0) = N_i \). Then \( e_i \) is an \( SO(n) \) equivariant real analytic diffeomorphism onto an open neighborhood of \( N_i \) for sufficiently small \( \varepsilon \). Denote by \( D'_i \) the fixed point set of the restricted \( SO(n-1) \) action on \( D_i \). Define
\[ \Phi' = e^{-1} \Phi e_1 : D_\tau \rightarrow D_\tau. \]

Then \( \Phi' \) is an \( \text{SO}(n) \) equivariant homeomorphism. Since \( \Phi \) is an extension of the isometry \( \varphi \), the restriction of \( \Phi' \) to \( D_\tau \) onto itself is the identity map or the antipodal map. It follows that \( \Phi' \) is the identity map or the antipodal map of \( D_\tau \) onto itself, because \( \Phi' \) is \( \text{SO}(n) \) equivariant. Therefore \( \Phi \) is real analytic on a neighborhood of \( N_1 \). Similarly \( \Phi \) is real analytic on a neighborhood of \( S_1 \). Consequently \( \Phi \) is a real analytic diffeomorphism of \( M_1 \) onto \( M_2 \).

This completes the proof of Theorem 1.3.

**Remark.** The real analytic diffeomorphism \( \Phi : M_1 \rightarrow M_2 \) in the proof of Theorem 1.3 is not necessary an isometry with respect to the Riemannian metrics \( g_1 \) and \( g_2 \).

2. **Construction of real analytic \( \text{SL}(n, \mathbb{R}) \) actions**

Consider the following conditions for a real valued real analytic function \( f(t) \):

(A) \( f(t) \) is defined on an open interval \((-1-\epsilon, 1+\epsilon)\) and \( f(-1) = f(1) = 0 \),

(B) \( t \cdot f(t) < 0 \) for \( 1-\epsilon < |t| < 1 \),

where \( \epsilon \) is a sufficiently small positive real number. If \( f(t) \) is a real analytic function satisfying the condition (A), then the corresponding vector field \( f(t) \frac{d}{dt} \) on \((-1, 1)\) is complete; hence the vector field induces a real analytic \( \mathbb{R} \) action

\[ \psi = \psi_f : \mathbb{R} \times (-1, 1) \rightarrow (-1, 1) \]

such that

\[ f(t) = \lim_{s \rightarrow 0} \frac{\psi(s, t) - t}{s} \quad \text{for } -1 < t < 1. \]

Denote by \( F \) the set of all real analytic functions satisfying the conditions (A) and (B). Define an equivalence relation in \( F \) as follows: we say that \( f(t) \) is equivalent to \( g(t) \) if there is a real analytic diffeomorphism \( h \) of the open interval \((-1,1)\) onto itself such that

\[ h_\ast \left( f(t) \frac{d}{dt} \right) = g(t) \frac{d}{dt}. \]

The relation means that the corresponding \( \mathbb{R} \) actions \( \psi_f \) and \( \psi_g \) are compatible under the real analytic diffeomorphism \( h \). Denote by \( F_\ast \) the set of all equivalence classes of \( F \).

**Example.** The polynomial
\[ f_{m,a}(t) = at \cdot \prod_{i=1}^{m} (kt+1)(kt-1) \]
satisfies the conditions (A), (B) for each positive integer \( m \) and each positive real number \( a \).

**Proposition 2.1.** If \((m, a) \neq (m', a')\), then the functions \( f_{m,a}(t) \) and \( f_{m',a'}(t) \) are not equivalent.

Proof. Suppose that there is a real analytic diffeomorphism \( h \) of the interval \((-1, 1)\) onto itself such that
\[
 h\left( f_{m,a}(t) \frac{d}{dt} \right) = f_{m',a'}(t) \frac{d}{dt}.
\]
Then it follows that
\[
 m = m', \quad h(0) = 0
\]
and
\[
 f_{m',a'}(t) = f_{m,a}(h^{-1}(t)) \frac{dh}{dt}(h^{-1}(t)).
\]
Therefore we have
\[
 (-1)^n a' = \frac{df_{m',a'}}{dt}(0) = \frac{df_{m,a}}{dt}(0) = (-1)^n a.
\]
It follows that \( a = a' \).

Put
\[
 L(n) = \{(a_{ij}) \in SL(n, R) : a_{11} = 1, a_{21} = a_{31} = \cdots = a_{n1} = 0\} ,
\]
\[
 N(n) = \{(a_{ij}) \in SL(n, R) : a_{11} > 0, a_{21} = a_{31} = \cdots = a_{n1} = 0\} .
\]
Then \( L(n) \) and \( N(n) \) are closed connected subgroups of \( SL(n, R) \), and \( L(n) \) is a normal subgroup of \( N(n) \). Consider the standard action of \( SL(n, R) \) on \( R^n \). Then the action is transitive on \( R^n - \{0\} \), and \( L(n) \) is the isotropy group at \( e_1 = (1, 0, \cdots, 0) \).

Let \( f(t) \) be a real analytic function satisfying the conditions (A) and (B). Here we shall construct a real analytic \( SL(n, R) \) action on a closed connected \( n \)-dimensional real analytic manifold \( M_f \) associated with the function \( f(t) \). Let \( \psi_f \) be the real analytic \( R \) action on \((-1, 1)\) corresponding to \( f(t) \). Since the factor group \( N(n)/L(n) \) is naturally isomorphic to \( R \) as Lie groups by a correspondence
\[
 (a_{ij} \cdot L(n) \to \log a_{11}, \quad \text{for} \quad (a_{ij}) \in N(n) ,
\]
we consider \( \psi_f \) as a real analytic \( N(n)/L(n) \) action on \((-1, 1)\). Define \( X_f \) the quotient manifold of the product
$SL(n, \mathbb{R})/\mathbb{Z}(n) \times (-1, 1)$

by the relation

$(xL(n), t) = (x y^{-1} L(n), \psi(y L(n), t));$

$x \in SL(n, \mathbb{R}), y \in N(n), |t| < 1.$

Then $X_f$ is an $n$-dimensional real analytic manifold with a natural $SL(n, \mathbb{R})$ action. Denote by $[xL(n), t]$ the element of $X_f$ represented by $(xL(n), t)$.

Let $a'$ (resp. $a''$) be the largest (resp. the smallest) zero of $f(t)$ on $(-1, 1)$. Let $a_+, a_- : \mathbb{R}^n - \{0\} \rightarrow X_f$ be the equivariant $SL(n, \mathbb{R})$ maps determined by

$$a_+(e_i) = \left[ L(n), \frac{1+a'}{2} \right], \quad a_-(e_i) = \left[ L(n), \frac{a''-1}{2} \right]$$

respectively, where $e_i = (1, 0, \ldots, 0)$. Let $\mathbb{R}^*_+$ and $\mathbb{R}^*_-$ be copies of $\mathbb{R}^*$, and consider $a_+, a_-$ as the maps

$$a_+: \mathbb{R}^*_+ - \{0\} \rightarrow X_f, \quad a_- : \mathbb{R}^*_+ - \{0\} \rightarrow X_f$$

respectively. Define $M_f$ the quotient space of a disjoint union

$$\mathbb{R}^*_+ \cup X_f \cup \mathbb{R}^*_-$$

given by the attaching maps $a_+, a_-$.

Since $f(t)$ satisfies the conditions (A) and (B), the space $M_f$ posesses naturally a real analytic structure as a compact connected $n$-dimensional manifold with a natural $SL(n, \mathbb{R})$ action. Notice that $M_f$ is a two points compactification of $X_f$.

For each $k \leq n-2$, $\pi_k(M_f) = \pi_k(X_f)$ by a general position theorem. The natural projection of $X_f$ onto $SL(n, \mathbb{R})/\mathbb{Z}(n) = S^{n-1}$ is a fibre bundle with a contractible fibre. It follows that $M_f$ is $(n-2)$-connected. In particular, $\pi_k(M_f) = \pi_k(M_f) = \{1\}$ for each $n \geq 3$. Since the restricted $SO(n)$ action on $M_f$ is effective, $M_f$ is real analytically diffeomorphic to the standard $n$-sphere $S^n$ by Theorem 1.3.

Denote by $A(n)$ the set of all real analytic non-trivial $SL(n, \mathbb{R})$ actions on the standard $n$-sphere $S^n$. Two such actions $\psi$ and $\psi'$ are said to be equivalent if there is a real analytic diffeomorphism $h$ of $S^n$ onto itself such that the following diagram is commutative:

$$SL(n, \mathbb{R}) \times S^n \xrightarrow{\psi} S^n \quad \xrightarrow{1 \times h} \quad S^n \xrightarrow{h} S^n.$$
A_{f} = \{a_{f}\}$ of real analytic $SL(n, \mathbb{R})$ actions on $S^{n}$ such that the $n$-sphere $S^{n}$ with a real analytic $SL(n, \mathbb{R})$ action $a_{f}$ is real analytically diffeomorphic to $M_{f}$ as $SL(n, \mathbb{R})$-manifolds. If $f(t)$ and $g(t)$ are equivalent, then it is easy to see that $M_{f}$ and $M_{g}$ are real analytically diffeomorphic as $SL(n, \mathbb{R})$-manifolds. It follows that the correspondence $f(t) \rightarrow A_{f}$ induces a map $c_{n}: F_{k} \rightarrow A_{d}(n)$ for each $n \geq 3$.

**Theorem 2.2.** The map $c_{n}: F_{k} \rightarrow A_{d}(n)$ is injective for each $n \geq 3$.

**Proof.** Let $f(t)$, $g(t)$ be real analytic functions satisfying the conditions (A), (B). Suppose that the induced real analytic $SL(n, \mathbb{R})$-manifolds $M_{f}$ and $M_{g}$ are real analytically diffeomorphic as $SL(n, \mathbb{R})$-manifolds. Then the open manifolds $X_{f}$ and $X_{g}$ are real analytically diffeomorphic as $SL(n, \mathbb{R})$-manifolds. Compare the fixed point sets of the restricted $L(n)$ action. Then the fixed point sets $F(L(n), X_{f})$ and $F(L(n), X_{g})$ are one dimensional real analytic submanifolds of $X_{f}$ and $X_{g}$, respectively and real analytically diffeomorphic as $NL(n)$-manifolds. Here $NL(n)$ denotes the normalizer of $L(n)$ in $SL(n, \mathbb{R})$. Since $NL(n)/L(n)$ is naturally isomorphic to $\mathbb{Z}_{2} \times N(n)/L(n)$ as Lie groups, it is easy to see that $f(t)$ and $g(t)$ are equivalent. q.e.d.

3. Certain closed subgroups of $SL(n, \mathbb{R})$

Put

\[ L(n) = \{(a_{i,j}) \in SL(n, \mathbb{R}) : a_{11} = 1, a_{21} = a_{31} = \cdots = a_{n1} = 0\} \]
\[ N(n) = \{(a_{i,j}) \in SL(n, \mathbb{R}) : a_{11} > 0, a_{21} = a_{31} = \cdots = a_{n1} = 0\} \]
\[ L^{*}(n) = \{(a_{i,j}) \in SL(n, \mathbb{R}) : a_{11} = 1, a_{12} = a_{13} = \cdots = a_{1n} = 0\} \]
\[ N^{*}(n) = \{(a_{i,j}) \in SL(n, \mathbb{R}) : a_{11} > 0, a_{12} = a_{13} = \cdots = a_{1n} = 0\} . \]

Consider $SL(n-1, \mathbb{R})$ and $SO(n-1)$ as subgroups of $SL(n, \mathbb{R})$ as follows:

\[ SL(n-1, \mathbb{R}) = L(n) \cap L^{*}(n), \quad SO(n-1) = SO(n) \cap SL(n-1, \mathbb{R}) . \]

**Lemma 3.1.** Suppose $n \geq 3$. Let $G$ be a connected Lie subgroup of $SL(n, \mathbb{R})$. Suppose that $G$ contains $SO(n-1)$ and

\[ \dim SL(n, \mathbb{R}) - n \leq \dim G < \dim SL(n, \mathbb{R}) . \]

Then $G$ is one of the following : $L(n), N(n), L^{*}(n)$ and $N^{*}(n)$.

**Proof.** Denote by $M_{n}(\mathbb{R})$ the set of all $n \times n$ matrices in the field of real numbers $\mathbb{R}$. As usual we consider $M_{n}(\mathbb{R})$ as the Lie algebra of the general linear group $GL(n, \mathbb{R})$. Denote by $\mathfrak{sl}(n, \mathbb{R})$ and $\mathfrak{so}(n)$ the Lie algebras of $M_{n}(\mathbb{R})$ corresponding to the Lie subgroups $SL(n, \mathbb{R})$ and $SO(n)$ of $GL(n, \mathbb{R})$ respectively. Then
\[ \mathfrak{sl}(n, \mathbb{R}) = \{ X \in M_n(\mathbb{R}) : \text{trace } X = 0 \} , \]
\[ \mathfrak{so}(n) = \{ X \in M_n(\mathbb{R}) : \text{X is skew-symmetric} \} . \]

Denote by \( \mathfrak{sl}(n-1, \mathbb{R}) \) the Lie subalgebra of \( \mathfrak{sl}(n, \mathbb{R}) \) corresponding to the Lie subgroup \( SL(n-1, \mathbb{R}) \) of \( SL(n, \mathbb{R}) \). Put
\[ \mathfrak{so}(n-1) = \mathfrak{so}(n) \cap \mathfrak{sl}(n-1, \mathbb{R}) , \]
\[ \mathfrak{sym}(n-1) = \{ X \in \mathfrak{sl}(n-1, \mathbb{R}) : \text{X is symmetric} \} , \]
\[ a = \{(a_{ij}) \in \mathfrak{sl}(n, \mathbb{R}) : a_{ii} = 0 \text{ for } i \neq 1 \} , \]
\[ a^* = \{(a_{ij}) \in \mathfrak{sl}(n, \mathbb{R}) : a_{ij} = 0 \text{ for } j \neq 1 \} , \]
\[ b = \{(a_{ij}) \in \mathfrak{sl}(n, \mathbb{R}) : a_{ij} = 0 \text{ for } i \neq j , a_{22} = a_{33} = \cdots = a_{nn} \} . \]

These are linear subspaces of \( \mathfrak{sl}(n, \mathbb{R}) \) and
\[ \mathfrak{sl}(n, \mathbb{R}) = \mathfrak{sl}(n-1, \mathbb{R}) \oplus a \oplus a^* \oplus b , \]
\[ \mathfrak{sl}(n-1, \mathbb{R}) = \mathfrak{so}(n-1) \oplus \mathfrak{sym}(n-1) \]
as direct sums of vector spaces. Moreover we have
\[ [a, a] = \{0\} , \quad [a^*, a^*] = \{0\} , \quad [b, b] = \{0\} , \]
\[ (1) \quad [a, b] = a , \quad [a^*, b] = a^* , \quad [a, a^*] = \mathfrak{sl}(n-1, \mathbb{R}) \oplus b , \]
\[ [a, \mathfrak{sl}(n-1, \mathbb{R})] = a , \quad [a^*, \mathfrak{sl}(n-1, \mathbb{R})] = a^* . \]

Denote by \( Ad : SL(n, \mathbb{R}) \to GL(\mathfrak{sl}(n, \mathbb{R})) \) the adjoint representation. Then the linear subspaces \( \mathfrak{sl}(n-1, \mathbb{R}) \), \( a \), \( a^* \) and \( b \) are \( Ad(SL(n-1, \mathbb{R})) \) invariant, and the linear subspaces \( \mathfrak{so}(n-1) \) and \( \mathfrak{sym}(n-1) \) are \( Ad(SO(n-1)) \) invariant. Moreover the linear subspaces \( \mathfrak{sym}(n-1) \), \( a \), \( a^* \) and \( b \) are irreducible \( Ad(SO(n-1)) \) spaces respectively for each \( n \geq 3 \). The Lie subalgebras
\[ \mathfrak{sl}(n-1, \mathbb{R}) \oplus a , \quad \mathfrak{sl}(n-1, \mathbb{R}) \oplus a^* \oplus b , \]
\[ \mathfrak{sl}(n-1, \mathbb{R}) \oplus a^* , \quad \mathfrak{sl}(n-1, \mathbb{R}) \oplus a^* \oplus b \]
of \( \mathfrak{sl}(n, \mathbb{R}) \) corresponds to the connected Lie subgroups \( L(n) \), \( N(n) \), \( L^*(n) \) and \( N^*(n) \) of \( SL(n, \mathbb{R}) \) respectively.

Let \( G \) be a connected Lie subgroup of \( SL(n, \mathbb{R}) \). Denote by \( g \) the corresponding Lie subalgebra of \( \mathfrak{sl}(n, \mathbb{R}) \). Suppose that
\[ (3) \quad G \text{ contains } SO(n-1) , \quad \text{and} \]
\[ (4) \quad \dim SL(n, \mathbb{R}) - n \leq \dim G < \dim SL(n, \mathbb{R}) . \]

By (3), \( g \) is an \( Ad(SO(n-1)) \) invariant linear subspace of \( \mathfrak{sl}(n, \mathbb{R}) \) which contains \( \mathfrak{so}(n-1) \). Hence we derive that
\[ g = \mathfrak{so}(n-1) \oplus (g \cap \mathfrak{sym}(n-1)) \oplus (g \cap (a \oplus a^*)) \oplus (g \cap b) \]
as a direct sum of $Ad(SO(n-1))$ invariant linear subspaces. The inequality (4) implies that $g$ contains $\mathfrak{sym}(n-1)$ or $\alpha \oplus \alpha^*$, because $\mathfrak{sym}(n-1)$, $\alpha$ and $\alpha^*$ are irreducible $Ad(SO(n-1))$ spaces respectively and

$$\dim \alpha = \dim \alpha^* = n-1, \quad \dim \mathfrak{sym}(n-1) > n-1$$

for any $n \geq 3$. If $\alpha \oplus \alpha^*$ is contained in $g$, then $g = \mathfrak{sl}(n, \mathbb{R})$ by (1). This is a contradiction to (4). It follows that

$$\mathfrak{sym}(n-1) \subset g, \quad \alpha \oplus \alpha^* \subset g.$$ 

In particular, $g$ contains $\mathfrak{sl}(n-1, \mathbb{R})$, and hence $G$ contains $SL(n-1, \mathbb{R})$. Then we derive that

$$\mathfrak{g} = \mathfrak{sl}(n-1, \mathbb{R}) \oplus (\mathfrak{g} \cap (\alpha \oplus \alpha^*)) \oplus (\mathfrak{g} \cap \mathfrak{b})$$

as a direct sum of $Ad(SL(n-1, \mathbb{R})$ invariant linear subspaces.

Suppose first $n \geq 4$. Then $\alpha$ and $\alpha^*$ are mutually non-equivalent irreducible $Ad(SL(n-1, \mathbb{R}))$ spaces; hence $Ad(SL(n-1, \mathbb{R}))$ invariant subspaces of $\alpha \oplus \alpha^*$ are one of the following: $\{0\}$, $\alpha$, $\alpha^*$ and $\alpha \oplus \alpha^*$. It follows that $g$ is one of the Lie algebras in (2), by (1), (4), (5) and (6).

Suppose next $n = 3$. Then $\alpha$ and $\alpha^*$ are equivalent irreducible $Ad(SL(2, \mathbb{R}))$ spaces. Put

$$h(p, q) = \begin{pmatrix} 0 & qy & -qx \\ px & 0 & 0 \\ py & 0 & 0 \end{pmatrix} : x, y \in \mathbb{R}$$

for each real numbers $p, q$. Then $h(p, q)$ is an $Ad(SL(2, \mathbb{R}))$ invariant linear subspace of $\alpha \oplus \alpha^*$ for each $p, q$. It is easy to see that any $Ad(SL(2, \mathbb{R})$ invariant proper linear subspace of $\alpha \oplus \alpha^*$ is one of $h(p, q)$ for certain $p, q$. It follows that

$$\mathfrak{g} \cap (\alpha \oplus \alpha^*) = h(p, q)$$

for certain real numbers $p, q$. Suppose $pq \neq 0$. Then we derive

$$[h(p, q), h(p, q)] = b,$$

$$[h(p, q), b] = h(-p, q),$$

$$h(p, q) + h(-p, q) = \alpha \oplus \alpha^*.$$ 

It follows that $g$ contains $\alpha \oplus \alpha^*$; this is a contradiction to (5). Hence we obtain $pq = 0$, namely

$$\mathfrak{g} \cap (\alpha \oplus \alpha^*) = \{0\}, \alpha \text{ or } \alpha^*.$$

It follows that $g$ is one of the Lie algebras in (2), by (1), (4) and (6).
Consequently the assumptions (3) and (4) implies that the Lie algebra \( g \) is one of the Lie algebras in (2) for each \( n \geq 3 \), and hence the connected Lie subgroup \( G \) is one of the following: \( L(n), N(n), L^*(n) \) and \( N^*(n) \).

This completes the proof of Lemma 3.1.

4. Real analytic \( SL(n, R) \) actions on the \( n \)-sphere

Let \( \varphi : SL(n, R) \times S^n \rightarrow S^n \) be a real analytic non-trivial action of \( SL(n, R) \) on the standard \( n \)-sphere \( S^n \). For each subgroup \( H \) of \( SL(n, R) \), we put

\[
F(H) = \{ x \in S^n : \varphi(h, x) = x \text{ for all } h \in H \},
\]

namely, \( F(H) \) is the fixed point set of the restricted action of \( \varphi \) to \( H \). Then \( F(H) \) is a closed subset of \( S^n \), but it is not necessary a submanifold of \( S^n \).

**Lemma 4.1.** Suppose \( n \geq 3 \). Then

\[
F(SO(n)) = F(SL(n, R)) = F(L(n)) \cap F(L^*(n)),
\]

\[
F(SO(n-1)) = F(L(n)) \text{ or } F(L^*(n))
\]

for any real analytic non-trivial \( SL(n, R) \) action on the \( n \)-sphere.

**Proof.** From Lemma 3.1, we derive

\[
F(SO(n)) = F(SL(n, R)) = F(L(n)) \cap F(L^*(n)),
\]

\[
F(SO(n-1)) = F(L(n)) \cup F(L^*(n)).
\]

According to Theorem 1.3, we see that the set \( F(SO(n-1)) - F(SO(n)) \) has just two connected components. Each connected component is contained in \( F(L(n)) \) or \( F(L^*(n)) \). Put

\[
g = \begin{pmatrix}
-1 \\
-1 \\
1 \\
\vdots \\
1
\end{pmatrix}.
\]

Then it follows easily from Theorem 1.3 that \( x \) and \( gx \) belong distinct connected components respectively for each element \( x \) of \( F(SO(n-1)) - F(SO(n)) \). Then we conclude that

\[
F(SO(n-1)) = F(L(n)) \text{ or } F(L^*(n)). \quad \text{q.e.d.}
\]

Denote by \( \sigma(g) \) the transpose of \( g^{-1} \) for each \( g \in SL(n, R) \). Then the correspondence \( g \rightarrow \sigma(g) \) defines an automorphism \( \sigma \) of \( SL(n, R) \). The automorphism \( \sigma \) is an involution and
Let $\psi$ be a real analytic non-trivial $SL(n, \mathbb{R})$ action on $S^n$. Define a new action $\sigma_\psi \psi$ of $SL(n, \mathbb{R})$ on $S^n$ as follows:

$$(\sigma_\psi \psi)(g, x) = \psi(\sigma(g), x) \quad \text{for } g \in SL(n, \mathbb{R}), \ x \in S^n.$$ 

Then it is seen that if $F(SO(n-1)) = F(L(n))$ (resp. $F(L^*(n))$) for the action $\psi$, then $F(SO(n-1)) = F(L^*(n))$ (resp. $F(L(n))$) for the action $\sigma_\psi \psi$.

As in the section 2, let $A(n)$ denote the set of all real analytic non-trivial $SL(n, \mathbb{R})$ actions on $S^n$, and let $A_\#(n)$ denote the set of all equivalence classes of $A(n)$. Then the mapping $\sigma_\#: A(n) \to A(n)$ is an involution, and $\sigma_\#$ induces naturally an involution $\sigma_\#: A_\#(n) \to A_\#(n)$.

Denote by $A^+(n)$ (resp. $A^-(n)$) the set of all real analytic non-trivial $SL(n, \mathbb{R})$ actions on $S^n$ such that $F(SO(n-1)) = F(L(n))$ (resp. $F(L^*(n))$).

Denote by $A_\#^+(n)$ (resp. $A_\#^-(n)$) the set of all equivalence classes represented by an element of $A^+(n)$ (resp. $A^-(n)$). Then we derive

$$\sigma_\#A^+(n) = A^-(n), \quad \sigma_\#A^-(n) = A^+(n),$$

$$\sigma_\#A_\#^+(n) = A_\#^-(n), \quad \sigma_\#A_\#^-(n) = A_\#^+(n).$$

Moreover $A_\#(n)$ is a disjoint union of $A_\#^+(n)$ and $A_\#^-(n)$ by Lemma 4.1. Let $c_\#: F_\# \to A_\#(n)$ be the mapping defined in the section 2. Then it is seen that the image $c_\#(F_\#)$ is contained in $A_\#^+(n)$.

We shall show the following result.

**Theorem 4.2.** $c_\#(F_\#) = A_\#^+(n)$ for each $n \geq 3$.

In order to prove this theorem, we require the following result due to Guillem and Sternberg [4]:

**Lemma 4.3.** Let $g$ be a real semi-simple Lie algebra and let $\rho: g \to L(M)$ be a homomorphism of $g$ into the Lie algebra of real analytic vector fields on a real analytic $n$-manifold $M$. Let $p$ be a point at which the vector fields in the image $\rho(g)$ have a common zero. Then there exists an analytic system of coordinates $(U; x_1, \ldots, x_n)$, with origin at $p$, in which all of the vector fields in $\rho(g)$ are linear. Namely, there exists

$$a_{ij} \in \mathfrak{g}^* = \text{Hom}_R(\mathfrak{g}, \mathbb{R})$$

such that

$$\rho(X)_q = \sum_{i,j} a_{ij}(X) x_i(q) \frac{\partial}{\partial x_j} \quad \text{for } X \in \mathfrak{g}, \ q \in U.$$
Remark. The correspondence \( X \mapsto (a_{ij}(X)) \) defines a Lie algebra homomorphism of \( g \) into \( \mathfrak{sl}(n, \mathbb{R}) \).

Lemma 4.4. Suppose \( n \geq 3 \). Let \( \psi \) be a real analytic non-trivial \( SL(n, \mathbb{R}) \) action on \( S^* \) such that \( F(SO(n-1)) = F(L(n)) \). Let \( p \in S^* \) be a fixed point of the \( SL(n, \mathbb{R}) \) action \( \psi \). Then there is an equivariant real analytic diffeomorphism \( h \) of \( \mathbb{R}^* \) onto an invariant open set of \( S^* \) such that \( h(0) = p \). Here \( SL(n, \mathbb{R}) \) acts standardly on \( \mathbb{R}^* \).

Proof. Notice that, for each \( n \geq 3 \), any non-trivial endomorphism of \( \mathfrak{sl}(n, \mathbb{R}) \) is of the form \( Ad(g) \) or \( Ad(g) \cdot d\sigma \), where \( g \in GL(n, \mathbb{R}) \) and \( d\sigma \) is the differential of the automorphism \( \sigma \). Define a Lie algebra homomorphism

\[
\rho: \mathfrak{sl}(n, \mathbb{R}) \to L(S^*)
\]

as follows:

\[
(1) \quad \rho(X)_q(f) = \lim_{t \to 0} t^{-1}(\psi(\exp(-tX), q)) - f(q)
\]

for \( X \in \mathfrak{sl}(n, \mathbb{R}) \), \( q \in S^* \). Here \( f \) is a real valued real analytic function on \( S^* \). Then \( \rho(X)_q = 0 \) for each \( X \in \mathfrak{sl}(n, \mathbb{R}) \). According to Lemma 4.3, there exists an analytic system of coordinates \( (U; x_1, \ldots, x_n) \), with origin at \( p \), and there exists \( a_{ij} \in \mathfrak{sl}(n, \mathbb{R})^* \) such that

\[
(2) \quad \rho(X)_q = \sum_{i,j} a_{ij}(X)x_i(q) \frac{\partial}{\partial x_j} \quad \text{for } X \in \mathfrak{sl}(n, \mathbb{R}), q \in U
\]

By the above notice, it can be assumed that

\[
(3) \quad X = (a_{ij}(X)) \quad \text{for each } X \in \mathfrak{sl}(n, \mathbb{R}), \text{ or}
\]

\[
(3') \quad d\sigma(X) = (a_{ij}(X)) \quad \text{for each } X \in \mathfrak{sl}(n, \mathbb{R})
\]

From the assumption \( F(SO(n-1)) = F(L(n)) \), it follows that the case (3) does not happen.

Let \( k: U \to \mathbb{R}^* \) be a real analytic diffeomorphism of \( U \) onto an open set of \( \mathbb{R}^* \) defined by \( k(q) = (x_1(q), \ldots, x_n(q)) \) for \( q \in U \). Then \( k(p) = 0 \). There is a positive real number \( \varepsilon \) such that the \( \varepsilon \)-neighborhood \( D_\varepsilon \) of the origin is contained in \( k(U) \). Put

\[
x = \left( \frac{\varepsilon}{2}, 0, \ldots, 0 \right)
\]

Then the group \( L(n) \) is the isotropy group at \( x \). Moreover \( L(n) \) agrees with the identity component of the isotropy group at \( k^{-1}(x) \) by (1), (2) and (3'). Define a map \( h: \mathbb{R}^* \to S^* \) as follows:

\[
h(0) = p; \quad h(gx) = \psi(g, k^{-1}(x)) \quad \text{for } g \in SL(n, \mathbb{R})
\]
The map $h$ is a well-defined equivariant $SL(n, R)$ map. It follows that

$$k \cdot h = \text{identity on } D_e$$

by the uniqueness of the solution of an ordinary differential equation defined by (1), (2) and (3'). Hence the map $h: R^n \rightarrow S^n$ is a real analytic submersion of $R^n$ onto an invariant open set of $S^n$. Since $h$ is injective on $D_e$, it can be seen that the isotropy group at $h(x) = k^{-1}(x)$ agrees with $L(n)$. Then the map $h: R^n \rightarrow S^n$ is injective.

This completes the proof of Lemma 4.4.

Proof of Theorem 4.2. Let $\psi$ be an element of $A^+(n)$. According to Theorem 1.3 and Lemma 4.1, $F(L(n))$ is a real analytic submanifold of $S^n$ on which $N(n)$ acts naturally, and $F(L(n))$ is real analytically diffeomorphic to $S^1$. Moreover $F = F(SL(n, R))$ consists of two points $N$, $S$. Let $h: (-1-\varepsilon, 1+\varepsilon) \rightarrow F(L(n))$ be a real analytic imbedding such that $h(1) = N$ and $h(-1) = S$, where $\varepsilon$ is a sufficiently small positive real number. Since $N(n)/L(n) \approx R$ acts real analytically on $F(L(n))$, the action defines a real analytic vector field $v$ on $F(L(n))$ naturally. Then there exists a real analytic function $f(t)$ on the interval $(-1-\varepsilon, 1+\varepsilon)$ such that $v = h_*(f(t)\frac{d}{dt})$ on the image of $h$. We shall first show that the function $f(t)$ satisfies the conditions (A), (B) stated in the section 2. The condition (A) follows from $F = \{N, S\}$. Considering the standard action of $SL(n, R)$ on $R^n$, we can see that the condition (B) follows from Lemma 4.4.

We shall next show that the $n$-sphere $S^n$ with the $SL(n, R)$ action $\psi$ is equivariantly real analytically diffeomorphic to $M_f$, where $M_f$ is a real analytic $SL(n, R)$-manifold constructed from $f(t)$ as before. For this purpose, we consider the following commutative diagram:

$$\begin{array}{ccc}
SO(n) & \times_{NSO(n-1)} & (F(SO(n-1))-F) \\
\downarrow \beta & & \downarrow \gamma \\
SL(n, R) & \times_{NL(n)} & (F(L(n))-F)
\end{array}$$

$$\begin{array}{c}
\xrightarrow{\alpha} S^n-F \\
\xrightarrow{\gamma} S^n-F.
\end{array}$$

Here $NSO(n-1)$ and $NL(n)$ are the normalizers of $SO(n-1)$ and $L(n)$ respectively. According to Theorem 1.3, Lemma 3.1 and Lemma 4.1, we can show that $\alpha$, $\beta$ and $\gamma$ are real analytic one-to-one onto mappings. Moreover $\alpha$ is a diffeomorphism by the differentiable slice theorem; hence $\beta$ and $\gamma$ are also real analytic diffeomorphisms. It follows that $S^n-F$ is equivariantly real analytically diffeomorphic to a real analytic $SL(n, R)$-manifold $X_f$ constructed from $f(t)$ as before. Consequently the $n$-sphere $S^n$ with the action $\psi$ is equivariantly real analytically diffeomorphic to $M_f$, by making use of Lemma 4.4. Hence
we conclude that \( c_n(F^*) = A_{\frac{n}{2}}(n) \). This completes the proof of Theorem 4.2.

5. Certain closed subgroups of \( O(n) \)

In this section, we shall prove Lemma 1.1 and Lemma 1.2. Put

\[
D(\theta) = \begin{pmatrix} \cos \theta & \sin \theta \\ -\sin \theta & \cos \theta \end{pmatrix}, \quad \theta \in \mathbb{R}.
\]

Denote by \( D(a_1, \ldots, a_r) \) the one-dimensional closed subgroup of \( O(n) \) consists of the following matrices:

\[
\begin{pmatrix} D(a_1 \theta) & \cdots & 0 \\ 0 & \cdots & D(a_r \theta) \end{pmatrix}, \quad \theta \in \mathbb{R}
\]

for \( n = 2r \), and

\[
\begin{pmatrix} D(a_1 \theta) & \cdots & 0 \\ 0 & \cdots & D(a_r \theta) \\ 0 & 1 \end{pmatrix}, \quad \theta \in \mathbb{R}
\]

for \( n = 2r + 1 \), respectively. Here \( a_1, \ldots, a_r \) are integers. Consider \( U(k) \) as the centralizer of

\[
\begin{pmatrix} D(\pi/2) & \cdots & 0 \\ 0 & \cdots & D(\pi/2) \end{pmatrix}
\]

in \( O(2k) \). Then we can derive easily the following result.

**Lemma 5.1.** Suppose that \( b_1 > b_2 > \cdots > b_r > 0 \) and

\[
(a_1, \ldots, a_r) = (b_1, \ldots, b_1, \ldots, b_3, \ldots, b_3, 0, \ldots, 0).
\]

Then the centralizer of \( D(a_1, \ldots, a_r) \) in \( O(n) \) agrees with

\[
U(n_1) \times \cdots \times U(n_s) \times O(m),
\]

where \( m = n - 2(n_1 + \cdots + n_s) \).

Here we shall prove Lemma 1.2. Let \( h : SO(n) \to O(n) \) be a continuous homomorphism with a finite kernel. Suppose \( n \geq 3 \). Then it is easy to see that \( h \) is an isomorphism onto \( SO(n) \). Denote by \( T \) a maximal torus of \( SO(n) \) defined by the direct product of the subgroups

\[
T_k = D(0, \ldots, 0, 1, 0, \ldots, 0)
\]
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for 0 < k ≤ n/2. Then there is an element \( x_1 \) of \( SO(n) \) such that \( h(T) = x_1 T x_1^{-1} \). Then the subgroup \( x_1^{-1} h(T_k) x_1 \) is of the form \( D(a_{11}, \ldots, a_{kk}) \) for each \( k \). Compare the centralizer of \( T_k \) and that of \( x_1^{-1} h(T_k) x_1 \) in \( O(n) \). We can derive

\[ x_1^{-1} h(T_k) x_1 = T_j \]

for some \( j \), by Lemma 5.1. Hence there is an element \( x_2 \) of \( O(n) \) such that

\[ h(t) = x_1 x_2 t x_2^{-1} x_1^{-1}, \quad \text{for } t \in T. \]

It follows that the representations \( y \to y \) and \( y \to x_2^{-1} x_1^{-1} h(y) x_1 x_2 \) of \( SO(n) \) are equivalent. Since the representation \( y \to y \) is absolutely irreducible, there is an element \( x_3 \) of \( O(n) \) such that

\[ x_3 y x_3^{-1} = x_2^{-1} x_1^{-1} h(y) x_1 x_2 \]

for each \( y \in SO(n) \) (cf. [6], Lemma 5.5.1). Put \( x_1 x_2 x_3 \). Then we derive that \( x \in O(n) \) and \( h(y) = x y x^{-1} \) for each \( y \in SO(n) \).

This completes the proof of Lemma 1.2.

We shall next prove Lemma 1.1. Let \( G \) be a connected closed subgroup of \( O(n) \). Suppose that \( n \geq 3 \) and

\[ \dim O(n) > \dim G \geq \dim O(n) - n. \]

The inclusion map \( i: G \to O(n) \) gives an orthogonal faithful representation of \( G \). Suppose first that the representation \( i \) is reducible. Then, by an inner automorphism of \( O(n) \), \( G \) is isomorphic to a closed subgroup \( G' \) of \( O(k) \times O(n-k) \) for some \( k \) such that \( 0 < k \leq n/2 \). By (1), we derive that \( k = 1 \), or \( k = 2 \) and \( G' = SO(2) \times SO(2) \). The codimension of \( O(1) \times O(n-1) \) in \( O(n) \) is \( n-1 \). If \( n \geq 4 \), then \( SO(n-1) \) is semi-simple; hence there is no closed subgroup of codimension one in \( SO(n-1) \). We can conclude that

\[ G' = SO(1) \times SO(n-1) = SO(n-1), \]
\[ G' = SO(2) \times SO(2) \quad \text{for } n = 4, \text{ or} \]
\[ G' = \{ 1 \} \quad \text{for } n = 3. \]

Suppose next that the representation \( i \) is irreducible and \( G \) has a one-dimensional central subgroup. By Lemma 5.1, it can be seen that \( n \) is even and \( G \) is isomorphic to a closed subgroup \( G' \) of \( U(n/2) \) by an inner automorphism of \( O(n) \). It follows from (1) that

\[ G' = U(3) \quad \text{for } n = 6, \text{ or} \]
\[ G' = U(2) \quad \text{for } n = 4. \]

It remains to consider the case that \( G \) is semi-simple and the representa-
tion \( i \) is irreducible. In the following, we assume that \( G \) is semi-simple and the representation \( i \) is irreducible. Suppose that the complexification \( i^c \) of \( i \) is reducible. Then the representation \( i \) possesses a complex structure and \( n \) is even. Hence \( G \) is isomorphic to a closed subgroup of \( U(n/2) \). We can derive that \( n=4 \) by (1). Moreover, by an inner automorphism of \( O(4) \), \( G \) is isomorphic to \( SU(2) \) which is standardly imbedded in \( O(4) \).

Suppose that the complexification \( i^c \) of \( i \) is irreducible. Then \( i^c \) is a complex irreducible representation of \( G \) of degree \( n \).

(i) Moreover suppose first that \( G \) is not simple. Let \( G^* \) be the universal covering group of \( G \), and let \( p: G^* \to G \) be the covering projection. Since \( G \) is not simple, there are closed semi-simple normal subgroups \( H_1 \) and \( H_2 \) of \( G^* \) such that

\[
G^* = H_1 \times H_2.
\]

Consider the representation \( i^c p: G^* \to U(n) \). Then there are irreducible complex representations \( r_1 \) and \( r_2 \) of \( H_1 \) and \( H_2 \) respectively, such that the tensor product \( r_1 \otimes r_2 \) is equivalent to \( i^c p \). Since \( i^c p \) has a real form \( ip \), the representations \( r_1 \) and \( r_2 \) are self-conjugate; hence \( r_1 \) (resp. \( r_2 \)) has a real form or a quaternionic structure, but not both (cf.\([1], \) Proposition 3.56). Moreover, if \( r_1 \) has a real form (resp. quaternionic structure), then \( r_2 \) has also a real form (resp. quaternionic structure). Put \( n_s = \deg r_s \) for \( s = 1, 2 \). Then

\[
(2) \quad \dim O(n) - n = \frac{n(n-3)}{2} = \frac{n_1 n_2 (n_1 n_2 - 3)}{2}.
\]

Suppose first that \( r_1 \) has a quaternionic structure. Then it follows that \( n_1 \) and \( n_2 \) are even, and

\[
\dim H_s \leq \dim Sp \left( \frac{n_s}{2} \right) \quad \text{for} \quad s = 1, 2.
\]

Hence

\[
\dim G = \dim H_1 + \dim H_2 \leq \frac{n_1 (n_1 + 1)}{2} + \frac{n_2 (n_2 + 1)}{2}.
\]

Compare the above inequality with (2). We can derive easily that

\[
\dim G < \dim O(n) - n
\]

except the case \( n_1 = n_2 = 2 \). If \( n_1 = n_2 = 2 \), then \( n = 4 \) and \( \dim G = \dim O(n) \). We can conclude from (1) that \( r_1 \) has no quaternionic structure. Suppose next that \( r_1 \) has a real form. Then, since \( H_s \) is semi-simple, it follows that

\[
n_s \geq 3 \quad \text{for} \quad s = 1, 2.
\]

Moreover it follows that
\[ \dim H_s \leq \dim O(n) \quad \text{for } s = 1, 2. \]

Hence
\[ \dim G = \dim H_1 + \dim H_2 \leq \frac{n_1(n_1-1)}{2} + \frac{n_2(n_2-1)}{2}. \]

Compare the above inequality with (2). We can derive that
\[ \dim G < \dim O(n) - n. \]

This is a contradiction to (1), and hence we can conclude that \( r_1 \) has no real form. Consequently we can conclude that \( G \) must be simple.

(ii) Suppose next that \( G \) is simple. Moreover suppose first that \( G \) is an exceptional Lie group. Then we can derive the following result from a table of the degrees of the basic representations (cf. \cite{2}, p. 378, Table 30): the possibility remains only in the case that \( n = 7 \) and \( G \) is locally isomorphic to the exceptional Lie group \( G_2 \). Consider \( G_2 \) as a closed subgroup of \( O(7) \) as usual. Then we can conclude that \( G \) is isomorphic to \( G_2 \) by an inner automorphism of \( O(7) \). It remains to consider the case that \( G \) is locally isomorphic to \( SU(k) \), \( Sp(k) \) or \( SO(k) \). Put \( r = \text{rank } G \). Denote by \( G^\ast \) the universal covering group of \( G \). Denote by \( L_1, \ldots, L_r \) the fundamental weights of \( G^\ast \). Then there is a one-to-one correspondence between complex irreducible representations of \( G^\ast \) and sequences \( (a_1, \ldots, a_r) \) of non-negative integers such that \( a_1L_1 + \cdots + a_rL_r \) is the highest weight of a corresponding complex irreducible representation (cf. \cite{2}, Theorem 0.8, Theorem 0.9). Denote by
\[ d(a_1L_1 + \cdots + a_rL_r) \]
the degree of the complex irreducible representation of \( G^\ast \) with the highest weight \( a_1L_1 + \cdots + a_rL_r \). The degree can be computed by the Weyl’s formula (cf. \cite{2}, Theorem 0.24; (0.148), (0.149), (0.150)). Notice that if
\[ a_1 \geq a'_1, \ldots, a_r \geq a'_r, \]
then
\[ d(a_1L_1 + \cdots + a_rL_r) \geq d(a'_1L_1 + \cdots + a'_rL_r) \]
and the equality holds only if \( a_1 = a'_1, \ldots, a_r = a'_r \).

(a) Suppose first that \( G^\ast \) is isomorphic to \( SU(r+1) \) for \( r \geq 1 \). Since \( \text{rank } G \leq \text{rank } SO(n) \), it follows that
\[ 2r \leq n. \]

If \( r \geq 6 \), then we derive from (3) that
\[ \dim G = \dim SU(r+1) = r(r+2) < \frac{n(n-3)}{2} = \dim O(n) - n. \]
This is a contradiction to (1). If the pair \((n, r)\) satisfies the conditions (1) and (3), then \((n, r)\) is one of the following:

\[(10,5), (8,4), (7,3), (5,2) \text{ and } (4,1)\].

Notice that

\[d(L_i) = r+1, \quad d(2L_i) = \frac{(r+1)(r+2)}{2} .\]

Thus there is no complex irreducible representation of \(SU(r+1)\) of degree \(2r\) for \(r=4,5\). Hence \((n, r)\) is not \((10,5)\) nor \((8,4)\). Since

\[d(2L_1) = d(2L_2) = 6, \quad d(L_1 + L_2) = 8 \quad \text{for } r = 2 ;
\]
\[d(2L_1) = d(2L_3) = 10, \quad d(2L_2) = d(L_1 + L_2) = d(L_2 + L_3) = 20, \]
\[\text{and } d(L_1 + L_3) = 15 \quad \text{for } r = 3 ,\]

it follows that there is no complex irreducible representation of \(SU(r+1)\) of degree \(2r+1\) for \(r=2,3\). Hence \((n, r)\) is not \((7,3)\) nor \((5,2)\). It remains only the case \((n, r) = (4,1)\). But it is seen that the complex irreducible representation of \(SU(2)\) of degree 4 has no real form. Therefore we can derive that \(G\) is not locally isomorphic to \(SU(r+1)\).

(b) Suppose next that \(G^*\) is isomorphic to \(Sp(r)\) for \(r \geq 2\). Since rank \(G \leq \text{rank } SO(n)\), it follows that

\[2r \leq n .\]

On the other hand, since \(\dim Sp(r) = r(2r+1)\), the inequality (1) implies that

\[n(n-3) < 2r(2r+1) < n(n-1) .\]

It follows from (4), (5) that

\[1 \leq \frac{n}{2r} \leq \frac{2r+1}{n-3} .\]

Therefore, if the pair \((n, r)\) satisfies the conditions (4), (5), then we derive \(n=2r+2\). Notice that

\[d(L_i) = z_{r+1} C_{i-1}, \quad d(2L_i) = r(2r+1) .\]

If \(r \geq 3\), then we can derive that

\[d(L_i) \geq 2r+3 \quad \text{for } i = 2, 3, \ldots, r ;
\]
\[d(2L_i) \geq 2r+3 .\]

If \(r = 2\), then

\[d(L_1) = 4, \quad d(L_2) = 5, \quad d(2L_1) = 10 ,
\]
\[d(2L_2) = 14 \quad \text{and } d(L_1 + L_2) = 16 .\]
It follows that there is no complex irreducible representation of $Sp(r)$ of degree $2r+2$, for $r \geq 2$. Therefore we can derive that $G$ is not locally isomorphic to $Sp(r)$.

(c) Suppose finally that $G^*$ is isomorphic to $Spin(k)$ for $k \geq 5$. It follows from (1) that

$$n(n-3) \leq k(k-1) < n(n-1).$$

Hence we have $n=k+1$. Suppose $k=2r$. Then

$$d(L_i) = 2^r c_i \text{ for } 1 \leq i \leq r-2, \quad d(L_{r-1}) = d(L_r) = 2^{r-1},$$
$$d(2L_i) = (r+1) \cdot (2r-1), \quad d(2L_{r-1}) = d(2L_r) = 2^{r-1} c_r,$n
$$d(L_i + L_{r-1}) = d(L_i + L_r) = (2r-1) 2^{r-1}, \text{ and}$$
$$d(L_{r-1} + L_r) = 2^r c_{r-1}.$$ It follows that there is no complex irreducible representation of $Spin(2r)$ of degree $2r+1$. Suppose $k=2r+1$. Then

$$d(L_i) = 2^{r+1} c_i \text{ for } 1 \leq i \leq r-1, \quad d(L_r) = 2^r,$$
$$d(2L_i) = r(2r+3), \quad d(L_i + L_r) = r \cdot 2^{r+1}, \text{ and}$$
$$d(2L_r) = 2^{2r}.$$ It follows that there is no complex irreducible representation of $Spin(2r+1)$ of degree $2r+2$ for $r \neq 3$, and there is a unique complex irreducible representation of $Spin(7)$ of degree 8. It is seen that the representation of $Spin(7)$ has a real form. Therefore we can derive that $n=8$ and $G$ is isomorphic to $Spin(7)$. Here $Spin(7)$ is considered as a closed subgroup of $O(8)$ by the real spin representation. Then the isomorphism of $G$ onto $Spin(7)$ is realized by an inner automorphism of $O(8)$.

This completes the proof of Lemma 1.1.

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References
