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CLASSIFICATION OF REAL ANALYTIC SL(n, R) ACTIONS ON n-SPHERE

Dedicated to Professor A. Komatu on his 70th birthday

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0. Introduction

C.R. Schneider [5] classified real analytic $SL(2, \mathbf{R})$ actions on closed surfaces. Except for the work, there seems to be no work on the classification problem about non-compact Lie group actions.

In this paper, we classify real analytic $SL(n, \mathbf{R})$ actions on the standard *n*-sphere for each $n \ge 3$. Here $SL(n, \mathbf{R})$ denotes the special linear group over the field of real numbers. The result can be stated roughly as follows: there is a one-to-one correspondence between real analytic $SL(n, \mathbf{R})$ actions on the *n*-sphere and real valued real analytic functions on an interval satisfying certain conditions (see Theorem 2.2 and Theorem 4.2). It is important to consider the restricted actions of $SL(n, \mathbf{R})$ to a maximal compact subgroup SO(n).

It is still open to classify C^{∞} actions of SL(n, R) on the standard *n*-sphere, by lack of C^{∞} analogue of a local theory due to Guillemin and Sternberg (see Lemma 4.3).

1. Real analytic SO(n) actions on certain *n*-manifolds

First we prepare the following two lemmas of which proof is given in the last section.

Lemma 1.1. Let G be a closed connected subgroup of O(n). Suppose that $n \ge 3$ and

$$\dim O(n) > \dim G \ge \dim O(n) - n$$
.

Suppose that G is not conjugate to SO(n-1) which is canonically imbedded in O(n). Then the pair (O(n), G) is pairwise isomorphic to one of the following:

$$(O(8), Spin(7)), (O(7), G_2), (O(6), U(3)), (O(4), U(2)),$$

 $(O(4), SU(2)), (O(4), SO(2) \times SO(2)) and (O(3), \{1\}),$

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up to inner automorphisms of O(n). In these cases the subgroups are standardly imbedded in O(n).

Lemma 1.2. Suppose $n \ge 3$. Let $h: SO(n) \rightarrow O(n)$ be a continuous homomorphism with a finite kernel. Then there is an element x of O(n) such that $h(y) = xyx^{-1}$ for each y of SO(n).

Now we shall prove the following result.

Theorem 1.3. Suppose $n \ge 3$. Let M be a closed connected n-dimensional real analytic manifold. Suppose that

$$\pi_1(M) = \pi_2(M) = \{1\}$$
.

Suppose that SO(n) acts on M real analytically and almost effectively. Then the SO(n)-manifold M is real analytically diffeomorphic to the standard n-sphere S^n as SO(n)-manifolls. Here the SO(n) action on S^n is the restriction of the standard SO(n+1) action on S^n .

Proof. (i) First we show that the SO(n)-manifold M is C^{∞} diffeomorphic to the standard sphere S^n as SO(n)-manifolds. Let G be the identity component of a principal isotropy group. Then

$$\dim SO(n) > \dim G \ge \dim SO(n) - n,$$

and SO(n) acts almost effectively on the homogeneous space SO(n)/G by the assumption that SO(n) acts almost effectively on M, and hence Lemma 1.1 is applicable. The pair (SO(n), G) is not pairwise isomorphic to (SO(4), U(2)) nor (SO(4), SU(2)), because SU(2) is a normal subgroup of SO(4). If

$$\dim \mathbf{SO}(n)/G = \dim M,$$

then the SO(n) action on M is transitive and the pair (SO(n), G) is pairwise isomorphic to one of the following by Lemma 1.1:

$$(SO(7), G_2), (SO(6), U(3)), (SO(4), SO(2) \times SO(2))$$
 and $(SO(3), \{1\})$.

But

$$\begin{aligned} \pi_1(\boldsymbol{SO}(7)/\boldsymbol{G}_2) &= \pi_1(\boldsymbol{SO}(3)/\{1\}) = \boldsymbol{Z}_2 ,\\ \pi_2(\boldsymbol{SO}(6)/\boldsymbol{U}(3)) &= \boldsymbol{Z} \quad \text{and} \quad \pi_2(\boldsymbol{SO}(4)/\boldsymbol{SO}(2)\times\boldsymbol{SO}(2)) = \boldsymbol{Z}\times\boldsymbol{Z} . \end{aligned}$$

This is a contradiction to the assumption

$$\pi_1(M) = \pi_2(M) = \{1\}$$
.

Consequently G is conjugate to SO(n-1) or the pair (SO(n), G) is pairwise isomorphic to (SO(8), Spin(7)) by Lemma 1.1 and hence the SO(n)-manifold

M has codimension one principal orbits and just two singular orbits (cf. [6], Lemma 1.2.1). Since SO(n-1) in SO(n) (resp. Spin(7) in SO(8)) is a maximal closed connected subgrou_I, the singular orbits are fixed points. It follows that the SO(n)-manifold M is C^{∞} diffeomorphic to $M' = D^n \cup D^n$ as SO(n)manifolds. Here the SO(n) action on D^n is standard by Lemma 1.2, and $f: \partial D^n \to \partial D^n$ is an SO(n) equivariant diffeomorphism. It follows that f is the identity map or the antipodal map, and hence M' is C^{∞} diffeomorphic to the standard n-sphere S^n as SO(n)-manifolds.

(ii) Here we assume that M_1 and M_2 are *n*-dimensional real analytic manifolds on which SO(n) acts real analytically. Assume that the SO(n)manifolds M_1 and M_2 are C^{∞} diffeomorphic to the standard *n*-sphere S^n as SO(n)-manifolds. According to a theorem of Grauert ([3], Theorem 3), M_i is real analytically imbedded in a euclidean space of sufficiently high dimension; hence M_i possesses a real analytic Riemannian metric. By averaging the real analytic Riemannian metric on M_i with respect to the SO(n) action, we have an SO(n) invariant real analytic Riemannian metric g_i on M_i . Denote by $\{N_i, S_i\}$ the fixed point set of the SO(n)-manifold M_i . We can assume that

$$d_1(N_1, S_1) = d_2(N_2, S_2),$$

where d_i is a distance function on M_i defined by the Riemannian metric g_i . Denote by F_i the fixed point set of the restricted SO(n-1) action on M_i . It follows that F_i is a real analytic submanifold of M_i which is NSO(n-1) invariant and C^{∞} diffeomorphic to S^1 by the assumption. Here NSO(n-1) denotes the normalizer of SO(n-1) in SO(n). Then there exsists an isometry $\varphi: F_1 \rightarrow F_2$ such that $\varphi(N_1) = N_2$ and $\varphi(S_1) = S_2$. The isometry φ is a real analytic diffeomorphism and φ is compatible with the action of NSO(n-1) on F_i . It is easy to see that the SO(n)-manifold $M_i - \{N_i, S_i\}$ is real analytically diffeomorphic to

$$SO(n) \underset{NSO(n-1)}{\times} (F_i - \{N_i, S_i\})$$

as SO(n)-manifolds; hence φ extends uniquely to an SO(n) equivariant homeomorphism $\Phi: M_1 \rightarrow M_2$. By the construction, the restriction of Φ to $M_1 - \{N_1, S_1\}$ is a real analytic diffeomorphism of $M_1 - \{N_1, S_1\}$ onto $M_2 - \{N_2, S_2\}$.

(iii) Finally we show that Φ is real analytic on neighborhoods of N_i and S_i . Notice that the tangent space of M_i at N_i with the induced SO(n) action is naturally isomorphic to \mathbb{R}^n with the standard SO(n) action by the assumption. Denote by D_e an \mathcal{E} -neighborhood of the origin 0 in \mathbb{R}^n . Denote by $e_i: D_e \to M_i$ the exponential map with respect to the Riemannian metric g_i such that $e_i(0) = N_i$. Then e_i is an SO(n) equivariant real analytic diffeomorphism onto an open neighborhood of N_i for sufficiently small \mathcal{E} . Denote by D'_e the fixed point set of the restricted SO(n-1) action on D_e .

$$\Phi' = e_2^{-1} \Phi e_1 \colon \boldsymbol{D}_t \to \boldsymbol{D}_t \,.$$

Then Φ' is an SO(n) equivariant homeomorphism. Since Φ is an extension of the isometry φ , the restriction of Φ' to D'_{ε} onto itself is the identity map or the antipodal map. It follows that Φ' is the identity map or the antipodal map of D_{ε} onto itself, because Φ' is SO(n) equivariant. Therefore Φ is real analytic on a neighborhood of N_1 . Similarly Φ is real analytic on a neighborhood of S_1 . Consequently Φ is a real analytic diffeomorphism of M_1 onto M_2 .

This completes the proof of Theorem 1.3.

REMARK. The real analytic diffeomorphism $\Phi: M_1 \rightarrow M_2$ in the proof of Theorem 1.3 is not necessary an isometry with respect to the Riemannian metrics g_1 and g_2 .

2. Construction of real analytic SL(n, R) actions

Consider the following conditions for a real valued real analytic function f(t):

(A) f(t) is defined on an open interval $(-1-\varepsilon, 1+\varepsilon)$ and f(-1)=f(1)=0,

(B) $t \cdot f(t) < 0$ for $1 - \varepsilon < |t| < 1$, where ε is a sufficiently small positive real number. If f(t) is a real analytic function satisfying the condition (A), then the corresponding vector field $f(t)\frac{d}{dt}$ on (-1, 1) is complete; hence the vector field induces a real analytic **R** action

$$\psi = \psi_f \colon \boldsymbol{R} \times (-1, 1) \to (-1, 1)$$

such that

$$f(t) = \lim_{s \to 0} \frac{\psi(s, t) - t}{s}$$
 for $-1 < t < 1$.

Denote by F the set of all real analytic functions satisfying the conditions (A) and (B). Define an equivalence relation in F as follows: we say that f(t) is equivalent to g(t) if there is a real analytic diffeomorphism h of the open interval (-1,1) onto itself such that

$$h_*\left(f(t)\frac{d}{dt}\right) = g(t)\frac{d}{dt}$$
.

The relation means that the corresponding R actions ψ_f and ψ_g are compatible under the real analytic diffeomorphism h. Denote by F_* the set of all equivalence classes of F.

EXAMPLE. The polynomial

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$$f_{m,a}(t) = at \cdot \prod_{k=1}^{m} (kt+1)(kt-1)$$

satisfies the conditions (A), (B) for each positive integer m and each positive real number a.

Proposition 2.1. If $(m, a) \neq (m', a')$, then the functions $f_{m,a}(t)$ and $f_{m',a'}(t)$ are not equivalent.

Proof. Suppose that there is a real analytic diffeomorphism h of the interval (-1, 1) onto itself such that

$$h_*\left(f_{m,a}(t)\frac{d}{dt}\right) = f_{m',a'}(t)\frac{d}{dt}.$$

Then it follows that

$$m = m', h(0) = 0$$

and

$$f_{m',a'}(t) = f_{m,a}(h^{-1}(t))\frac{dh}{dt}(h^{-1}(t))$$
.

Therefore we have

$$(-1)^{m'}a' = \frac{df_{m',a'}}{dt}(0) = \frac{df_{m,a}}{dt}(0) = (-1)^m a \,.$$

It follows that a = a'.

Put

$$L(n) = \{(a_{ij}) \in SL(n, R): a_{11} = 1, a_{21} = a_{31} = \dots = a_{n1} = 0\},\$$

$$N(n) = \{(a_{ij}) \in SL(n, R): a_{11} > 0, a_{21} = a_{31} = \dots = a_{n1} = 0\}.$$

Then L(n) and N(n) are closed connected subgroups of SL(n, R), and L(n) is a normal subgroup of N(n). Consider the standard action of SL(n, R) on \mathbb{R}^n . Then the action is transitive on $\mathbb{R}^n - \{0\}$, and L(n) is the isotropy group at $e_1 = (1, 0, \dots, 0)$.

Let f(t) be a real analytic function satisfying the conditions (A) and (B). Here we shall construct a real analytic SL(n, R) action on a closed connected *n*-dimensional real analytic manifold M_f associated with the function f(t). Let ψ_f be the real analytic R action on (-1, 1) corresponding to f(t). Since the factor group N(n)/L(n) is naturally isomorphic to R as Lie groups by a correspondence

 $(a_{ii}) \cdot L(n) \rightarrow \log a_{11}$, for $(a_{ii}) \in N(n)$,

we consider ψ_f as a real analytic N(n)/L(n) action on (-1, 1). Define X_f the quotient manifold of the product

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q.e.d.

$$SL(n, R)/L(n) \times (-1, 1)$$

by the relation

$$(xL(n), t) = (xy^{-1}L(n), \psi_f(yL(n), t));$$

 $x \in SL(n, R), y \in N(n), |t| < 1.$

Then X_f is an *n*-dimensional real analytic manifold with a natural SL(n, R) action. Denote by [xL(n), t] the element of X_f represented by (xL(n), t).

Let a' (resp. a'') be the largest (resp. the smallest) zero of f(t) on (-1, 1). Let a_+ , a_- : $\mathbb{R}^n - \{0\} \rightarrow X_f$ be the equivariant $SL(n, \mathbb{R})$ maps determined by

$$a_{+}(e_{1}) = \left[L(n), \frac{1+a'}{2}\right], \quad a_{-}(e_{1}) = \left[L(n), \frac{a''-1}{2}\right]$$

respectively, where $e_1 = (1, 0, \dots, 0)$. Let \mathbf{R}_+^n and \mathbf{R}_-^n be copies of \mathbf{R}^n , and consider a_+, a_- as the maps

$$a_+: \mathbf{R}^n_+ - \{0\} \to X_f, \quad a_-: \mathbf{R}^n_- - \{0\} \to X_f$$

respectively. Define M_f the quotient space of a disjoint union

$$\boldsymbol{R}_{+}^{n} \cup X_{f} \cup \boldsymbol{R}_{-}^{n}$$

given by the attaching maps a_+ , a_- . Since f(t) satisfies the conditions (A) and (B), the space M_f possesses naturally a real analytic structure as a compact connected *n*-dimensional manifold with a natural SL(n, R) action. Notice that M_f is a two points compactification of X_f .

For each $k \leq n-2$, $\pi_k(M_f) = \pi_k(X_f)$ by a general position theorem. The natural projection of X_f onto $SL(n, \mathbf{R})/N(n) = S^{n-1}$ is a fibre bundle with a contractible fibre. It follows that M_f is (n-2)-connected. In particular, $\pi_1(M_f) = \pi_2(M_f) = \{1\}$ for each $n \geq 3$. Since the restricted SO(n) action on M_f is effective, M_f is real analytically diffeomorphic to the standard *n*-sphere S^n by Theorem 1.3.

Denote by A(n) the set of all real analytic non-trivial SL(n, R) actions on the standard *n*-sphere S^n . Two such actions ψ and ψ' are said to be equivalent if there is a real analytic diffeomorphism h of S^n onto itself such that the following diagram is commutative:

$$\begin{aligned} \boldsymbol{SL}(n, \boldsymbol{R}) \times S^{n} & \stackrel{\boldsymbol{\psi}}{\longrightarrow} S^{n} \\ & \downarrow 1 \times h & \downarrow h \\ \boldsymbol{SL}(n, \boldsymbol{R}) \times S^{n} & \stackrel{\boldsymbol{\psi}'}{\longrightarrow} S^{n}. \end{aligned}$$

Denote by $A_*(n)$ the set of all equivalence classes of A(n). By the above construction of M_f , the real analytic function f(t) defines an equivalence class

 $A_f = \{a_f\}$ of real analytic SL(n, R) actions on S^n such that the *n*-sphere S^n with a real analytic SL(n, R) action a_f is real analytically diffeomorphic to M_f as SL(n, R)-manifolds. If f(t) and g(t) are equivalent, then it is easy to see that M_f and M_g are real analytically diffeomorphic as SL(n, R)-manifolds. It follows that the correspondence $f(t) \rightarrow A_f$ induces a map $c_n \colon F_* \rightarrow A_*(n)$ for each $n \ge 3$.

Theorem 2.2. The map $c_n: F_* \rightarrow A_*(n)$ is injective for each $n \ge 3$.

Proof. Let f(t), g(t) be real analytic functions satisfying the conditions (A), (B). Suppose that the induced real analytic SL(n, R)-manifolds M_f and M_g are real analytically diffeomorphic as SL(n, R)-manifolds. Then the open manifolds X_f and X_g are real analytically diffeomorphic as SL(n, R)-manifolds. Compare the fixed point sets of the restricted L(n) action. Then the fixed point sets $F(L(n), X_f)$ and $F(L(n), X_g)$ are one dimensional real analytic submanifolds of X_f and X_g respectively and real analytically diffeomorphic as NL(n)-manifolds. Here NL(n) denotes the normalizer of L(n) in SL(n, R). Since NL(n)/L(n) is naturally isomorphic to $Z_2 \times N(n)/L(n)$ as Lie groups, it is easy to see that f(t) and g(t) are equivalent.

3. Certain closed subgroups of SL(n, R)

Put

$$L(n) = \{(a_{ij}) \in SL(n, R): a_{11} = 1, a_{21} = a_{31} = \dots = a_{n1} = 0\},$$

$$N(n) = \{(a_{ij}) \in SL(n, R): a_{11} > 0, a_{21} = a_{31} = \dots = a_{n1} = 0\},$$

$$L^*(n) = \{(a_{ij}) \in SL(n, R): a_{11} = 1, a_{12} = a_{13} = \dots = a_{1n} = 0\},$$

$$N^*(n) = \{(a_{ij}) \in SL(n, R): a_{11} > 0, a_{12} = a_{13} = \dots = a_{1n} = 0\}.$$

Consider SL(n-1, R) and SO(n-1) as subgroups of SL(n, R) as follows:

$$SL(n-1, R) = L(n) \cap L^{*}(n), SO(n-1) = SO(n) \cap SL(n-1, R)$$

Lemma 3.1. Suppose $n \ge 3$. Let G be a connected Lie subgroup of SL(n, R). Suppose that G contains SO(n-1) and

dim $SL(n, R) - n \leq \dim G < \dim SL(n, R)$.

Then G is one of the following: L(n), N(n), $L^*(n)$ and $N^*(n)$.

Proof. Denote by $M_n(\mathbf{R})$ the set of all $n \times n$ matrices in the field of real numbers \mathbf{R} . As usual we consider $M_n(\mathbf{R})$ as the Lie algebra of the general linear group $GL(n, \mathbf{R})$. Denote by $\mathfrak{Sl}(n, \mathbf{R})$ and $\mathfrak{So}(n)$ the Lie subalgebras of $M_n(\mathbf{R})$ corresponding to the Lie subgroups $SL(n, \mathbf{R})$ and SO(n) of $GL(n, \mathbf{R})$ respectively. Then

$$\mathfrak{sl}(n, \mathbf{R}) = \{X \in M_n(\mathbf{R}): \text{trace } X = 0\},\$$

 $\mathfrak{so}(n) = \{X \in M_n(\mathbf{R}): X \text{ is skew-symmetric}\}$

Denote by $\mathfrak{Sl}(n-1, \mathbb{R})$ the Lie subalgebra of $\mathfrak{Sl}(n, \mathbb{R})$ corresponding to the Lie subgroup $SL(n-1, \mathbb{R})$ of $SL(n, \mathbb{R})$. Put

$$\mathfrak{so}(n-1) = \mathfrak{so}(n) \cap \mathfrak{sl}(n-1, \mathbf{R}),$$

$$\mathfrak{som}(n-1) = \{X \in \mathfrak{sl}(n-1, \mathbf{R}): X \text{ is symmetric} \},$$

$$\mathfrak{a} = \{(a_{ij}) \in \mathfrak{sl}(n, \mathbf{R}): a_{ij} = 0 \text{ for } i \neq 1\},$$

$$\mathfrak{a}^* = \{(a_{ij}) \in \mathfrak{sl}(n, \mathbf{R}): a_{ij} = 0 \text{ for } j \neq 1\},$$

$$\mathfrak{b} = \{(a_{ij}) \in \mathfrak{sl}(n, \mathbf{R}): a_{ij} = 0 \text{ for } i \neq j, a_{22} = a_{33} = \dots = a_{nn}\}.$$

These are linear subspaces of $\mathfrak{Sl}(n, \mathbf{R})$ and

$$\mathfrak{Sl}(n, \mathbf{R}) = \mathfrak{Sl}(n-1, \mathbf{R}) \oplus \mathfrak{a} \oplus \mathfrak{a}^* \oplus \mathfrak{b},$$

 $\mathfrak{Sl}(n-1, \mathbf{R}) = \mathfrak{So}(n-1) \oplus \mathfrak{Sym}(n-1)$

as direct sums of vector spaces. Moreover we have

$$[a, a] = \{0\}, [a^*, a^*] = \{0\}, [b, b] = \{0\},$$

$$(1) \qquad [a, b] = a, [a^*, b] = a^*, [a, a^*] = \mathfrak{SI}(n-1, \mathbf{R}) \oplus b,$$

$$[a, \mathfrak{SI}(n-1, \mathbf{R})] = a, [a^*, \mathfrak{SI}(n-1, \mathbf{R})] = a^*.$$

Denote by $Ad: SL(n, \mathbb{R}) \rightarrow GL(\mathfrak{A}(n, \mathbb{R}))$ the adjoint representation. Then the linear subspaces $\mathfrak{A}(n-1, \mathbb{R})$, \mathfrak{a} , \mathfrak{a}^* and \mathfrak{b} are $Ad(SL(n-1, \mathbb{R}))$ invariant, and the linear subspaces $\mathfrak{So}(n-1)$ and $\mathfrak{Sym}(n-1)$ are Ad(SO(n-1)) invariant. Moreover the linear subspaces $\mathfrak{Sym}(n-1)$, \mathfrak{a} , \mathfrak{a}^* and \mathfrak{b} are irreducible Ad(SO(n-1)) spaces respectively for each $n \ge 3$. The Lie subalgebras

(2)
$$\begin{split} \widehat{\mathfrak{sl}}(n-1, \mathbf{R}) \oplus \mathfrak{a}, \, \widehat{\mathfrak{sl}}(n-1, \mathbf{R}) \oplus \mathfrak{a} \oplus \mathfrak{b}, \\ \widehat{\mathfrak{sl}}(n-1, \mathbf{R}) \oplus \mathfrak{a}^*, \, \widehat{\mathfrak{sl}}(n-1, \mathbf{R}) \oplus \mathfrak{a}^* \oplus \mathfrak{b} \end{split}$$

of $\mathfrak{Sl}(n, \mathbf{R})$ corresponds to the connected Lie subgroups L(n), N(n), $L^*(n)$ and $N^*(n)$ of $SL(n, \mathbf{R})$ respectively.

Let G be a connected Lie subgroup of SL(n, R). Denote by g the corresponding Lie subalgebra of $\mathfrak{Sl}(n, R)$. Suppose that

(3) G contains SO(n-1), and

(4)
$$\dim SL(n, R) - n \leq \dim G < \dim SL(n, R).$$

By (3), g is an Ad(SO(n-1)) invariant linear subspace of $\mathfrak{SI}(n, \mathbb{R})$ which contains $\mathfrak{so}(n-1)$. Hence we derive that

$$g = \mathfrak{so}(n-1) \oplus (g \cap \mathfrak{sym}(n-1)) \oplus (g \cap (\mathfrak{a} \oplus \mathfrak{a}^*)) \oplus (g \cap \mathfrak{b})$$

as a direct sum of Ad(SO(n-1)) invariant linear subspaces. The inequality (4) implies that g contains $\mathfrak{Sym}(n-1)$ or $\mathfrak{a}\oplus\mathfrak{a}^*$, because $\mathfrak{Sym}(n-1)$, \mathfrak{a} and \mathfrak{a}^* are irreducible Ad(SO(n-1)) spaces respectively and

dim
$$\mathfrak{a} = \dim \mathfrak{a}^* = n-1$$
, dim $\mathfrak{sym}(n-1) \ge n-1$

for any $n \ge 3$. If $\mathfrak{a} \oplus \mathfrak{a}^*$ is contained in g, then $g = \mathfrak{sl}(n, \mathbb{R})$ by (1). This is a contradiction to (4). It follows that

(5)
$$\mathfrak{sym}(n-1) \subset \mathfrak{g}, \ \mathfrak{a} \oplus \mathfrak{a}^* \subset \mathfrak{g}.$$

In particular, g contains $\mathfrak{SL}(n-1, \mathbf{R})$, and hence G contains $SL(n-1, \mathbf{R})$. Then we derive that

$$(6) \qquad \qquad g = \mathfrak{S}(n-1, \mathbf{R}) \oplus (g \cap (\mathfrak{a} \oplus \mathfrak{a}^*)) \oplus (g \cap \mathfrak{b})$$

as a direct sum of Ad(SL(n-1, R)) invariant linear subspaces.

Suppose first $n \ge 4$. Then a and a* are mutually non-equivalent irreducible $Ad(SL(n-1, \mathbf{R}))$ spaces; hence $Ad(SL(n-1, \mathbf{R}))$ invariant subspaces of $\mathfrak{a} \oplus \mathfrak{a}^*$ are one of the following : $\{0\}$, a, a* and $\mathfrak{a} \oplus \mathfrak{a}^*$. It follows that g is one of the Lie algebras in (2), by (1), (4), (5) and (6).

Suppose next n=3. Then a and a* are equivalent irreducible $Ad(SL(2, \mathbf{R}))$ spaces. Put

$$h(p, q) = \left\{ \begin{pmatrix} 0 & qy & -qx \\ px & 0 & 0 \\ py & 0 & 0 \end{pmatrix} : x, y \in \mathbf{R} \right\}$$

for each real numbers p, q. Then h(p, q) is an $Ad(SL(2, \mathbb{R}))$ invariant linear subspace of $a \oplus a^*$ for each p, q. It is easy to see that any $Ad(SL(2, \mathbb{R}))$ invariant proper linear subspace of $a \oplus a^*$ is one of h(p, q) for certain p, q. It follows that

$$\mathfrak{g} \cap (\mathfrak{a} \oplus \mathfrak{a}^*) = h(p, q)$$

for certain real numbers p, q. Suppose $pq \neq 0$. Then we derive

$$[h(p, q), h(p, q)] = b,$$

 $[h(p, q), b] = h(-p, q),$
 $h(p, q)+h(-p, q) = a \oplus a^*$

It follows that g contains $a \oplus a^*$; this is a contradiction to (5). Hence we obtain pq=0, namely

$$\mathfrak{g} \cap (\mathfrak{a} \oplus \mathfrak{a}^*) = \{0\}, \mathfrak{a} \text{ or } \mathfrak{a}^*.$$

It follows that g is one of the Lie algebras in (2), by (1), (4) and (6).

Consequently the assumptions (3) and (4) implies that the Lie algebra g is one of the Lie algebras in (2) for each $n \ge 3$, and hence the connected Lie subgroup G is one of the following: L(n), N(n), $L^*(n)$ and $N^*(n)$.

This completes the proof of Lemma 3.1.

4. Real analytic SL(n, R) actions on the *n*-sphere

Let $\psi: SL(n, \mathbb{R}) \times S^n \to S^n$ be a real analytic non-trivial action of $SL(n, \mathbb{R})$ on the standard *n*-sphere S^n . For each subgroup *H* of $SL(n, \mathbb{R})$, we put

$$F(H) = \{x \in S^n : \psi(h, x) = x \text{ for all } h \in H\},\$$

namely, F(H) is the fixed point set of the restricted action of ψ to H. Then F(H) is a closed subset of S^n , but it is not necessary a submanifold of S^n .

Lemma 4.1. Suppose $n \ge 3$. Then

$$F(SO(n)) = F(SL(n, R)) = F(L(n)) \cap F(L^*(n)),$$

$$F(SO(n-1)) = F(L(n)) \text{ or } F(L^*(n))$$

for any real analytic non-trivial SL(n, R) action on the n-sphere.

Proof. From Lemma 3.1, we derive

$$F(SO(n)) = F(SL(n, R)) = F(L(n)) \cap F(L^*(n)),$$

$$F(SO(n-1)) = F(L(n)) \cup F(L^*(n)).$$

According to Theorem 1.3, we see that the set F(SO(n-1))-F(SO(n)) has just two connected components. Each connected component is contained in F(L(n)) or $F(L^*(n))$. Put

$$g = \begin{pmatrix} -1 & & \\ & -1 & \\ & & 1 \\ & & \ddots \\ & & & 1 \end{pmatrix}.$$

Then it follows easily from Theorem 1.3 that x and gx belong distinct connected components respectively for each element x of F(SO(n-1))-F(SO(n)). Then we conclude that

$$F(SO(n-1)) = F(L(n))$$
 or $F(L^*(n))$. q.e.d.

Denote by $\sigma(g)$ the transpose of g^{-1} for each $g \in SL(n, \mathbb{R})$. Then the correspondence $g \rightarrow \sigma(g)$ defines an automorphism σ of $SL(n, \mathbb{R})$. The automorphism σ is an involution and

$$\sigma(L(n)) = L^*(n)$$
.

Let ψ be a real analytic non-trivial SL(n, R) action on S^n . Define a new action $\sigma_* \psi$ of SL(n, R) on S^n as follows:

$$(\sigma_*\psi)(g, x) = \psi(\sigma(g), x)$$
 for $g \in SL(n, R), x \in S^n$.

Then it is seen that if F(SO(n-1)) = F(L(n)) (resp. $F(L^*(n))$) for the action ψ , then $F(SO(n-1)) = F(L^*(n))$ (resp. F(L(n))) for the action $\sigma_{\sharp}\psi$.

As in the section 2, let A(n) denote the set of all real analytic non-trivial SL(n, R) actions on S^n , and let $A_*(n)$ denote the set of all equivalence classes of A(n). Then the mapping $\sigma_* : A(n) \rightarrow A(n)$ is an involution, and σ_* induces naturally an involution $\sigma_* : A_*(n) \rightarrow A_*(n)$.

Denote by $A^+(n)$ (resp. $A^-(n)$) the set of all real analytic non-trivial SL(n, R) actions on S^n such that

$$F(SO(n-1)) = F(L(n)) (resp. F(L^*(n))).$$

Denote by $A_*^+(n)$ (resp. $A_*^-(n)$) the set of all equivalence classes represented by an element of $A^+(n)$ (resp. $A^-(n)$). Then we derive

$$\sigma_{\sharp}A^{+}(n) = A^{-}(n), \quad \sigma_{\sharp}A^{-}(n) = A^{+}(n),$$

$$\sigma_{\ast}A^{+}_{\ast}(n) = A^{-}_{\ast}(n), \quad \sigma_{\ast}A^{-}_{\ast}(n) = A^{+}_{\ast}(n).$$

Moreover $A_*(n)$ is a disjoint union of $A_*^+(n)$ and $A_*^-(n)$ by Lemma 4.1. Let $c_n: F_* \to A_*(n)$ be the mapping defined in the section 2. Then it is seen that the image $c_n(F_*)$ is contained in $A_*^+(n)$.

We shall show the following result.

Theorem 4.2. $c_n(F_*) = A_*^+(n)$ for each $n \ge 3$.

In order to prove this theorem, we require the following result due to Guillemin and Sternberg [4]:

Lemma 4.3. Let g be a real semi-simple Lie algebra and let $\rho: g \rightarrow L(M)$ be a homomorphism of g into the Lie algebra of real analytic vector fields on a real analytic n-manifold M. Let p be a point at which the vector fields in the image $\rho(g)$ have a common zero. Then there exists an analytic system of coordinates $(U; x_1, \dots, x_n)$, with origin at p, in which all of the vector fields in $\rho(g)$ are linear. Namely, there exists

$$a_{ij} \in \mathfrak{g}^* = \operatorname{Hom}_{R}(\mathfrak{g}, R)$$

such that

$$\rho(X)_q = \sum_{i,j} a_{ij}(X) x_i(q) \frac{\partial}{\partial x_j} \quad \text{for } X \in \mathfrak{g}, \ q \in U.$$

REMARK. The correspondence $X \rightarrow (a_{ij}(X))$ defines a Lie algebra homomorphism of g into $\mathfrak{gl}(n, \mathbf{R})$.

Lemma 4.4. Suppose $n \ge 3$. Let ψ be a real analytic non-trivial SL(n, R) action on S^n such that F(SO(n-1)) = F(L(n)). Let $p \in S^n$ be a fixed point of the SL(n, R) action ψ . Then there is an equivariant real analytic diffeomorphism h of R^n onto an invariant open set of S^n such that h(0) = p. Here SL(n, R) acts standardly on R^n .

Proof. Notice that, for each $n \ge 3$, any non-trivial endomorphism of $\mathfrak{Sl}(n, \mathbf{R})$ is of the form Ad(g) or $Ad(g) \cdot d\sigma$, where $g \in \mathbf{GL}(n, \mathbf{R})$ and $d\sigma$ is the differential of the automorphism σ . Define a Lie algebra homomorphism

$$\rho: \mathfrak{sl}(n, \mathbf{R}) \to L(S^n)$$

as follows:

(1)
$$\rho(X)_{q}(f) = \lim_{t \to 0} \frac{f(\psi(\exp(-tX), q)) - f(q)}{t}$$

for $X \in \mathfrak{Sl}(n, \mathbf{R})$, $q \in S^n$. Here f is a real valued real analytic function on S^n . Then $\rho(X)_p = 0$ for each $X \in \mathfrak{Sl}(n, \mathbf{R})$. According to Lemma 4.3, there exists an analytic system of coordinates $(U; x_1, \dots, x_n)$, with origin at p, and there exists $a_{ij} \in \mathfrak{Sl}(n, \mathbf{R})^*$ such that

(2)
$$\rho(X)_q = \sum_{i,j} a_{ij}(X) x_i(q) \frac{\partial}{\partial x_j}$$
 for $X \in \mathfrak{Sl}(n, \mathbf{R}), q \in U$.

By the above notice, it can be assumed that

(3)
$$X = (a_{ij}(X))$$
 for each $X \in \mathfrak{Sl}(n, \mathbb{R})$, or
(3') $d\sigma(X) = (a_{ij}(X))$ for each $X \in \mathfrak{Sl}(n, \mathbb{R})$.

From the assumption F(SO(n-1)) = F(L(n)), it follows that the case (3) does not happen.

Let $k: U \to \mathbb{R}^n$ be a real analytic diffeomorphism of U onto an open set of \mathbb{R}^n defined by $k(q) = (x_1(q), \dots, x_n(q))$ for $q \in U$. Then k(p) = 0. There is a positive real number \mathcal{E} such that the \mathcal{E} -neighborhood $D_{\mathcal{E}}$ of the origin is contained in k(U). Put

$$x=\left(\frac{\varepsilon}{2},0,\cdots,0
ight).$$

Then the group L(n) is the isotropy group at x. Moreover L(n) agrees with the identity component of the isotropy group at $k^{-1}(x)$ by (1), (2) and (3'). Define a map $h: \mathbb{R}^n \to S^n$ as follows:

$$h(0) = p; h(gx) = \psi(g, k^{-1}(x)) \quad \text{for } g \in SL(n, \mathbf{R}).$$

The map h is a well-defined equivariant SL(n, R) map. It follows that

$k \cdot h = identity \ on \ on \ D_{e}$

by the uniqueness of the solution of an ordinary differential equation defined by (1), (2) and (3'). Hence the map $h: \mathbb{R}^n \to S^n$ is a real analytic submersion of \mathbb{R}^n onto an invariant open set of S^n . Since h is injective on D_{ε} , it can be seen that the isotropy group at $h(x) = k^{-1}(x)$ agrees with L(n). Then the map $h: \mathbb{R}^n \to S^n$ is injective.

This completes the proof of Lemma 4.4.

Proof of Theorem 4.2. Let ψ be an element of $A^+(n)$. According to Theorem 1.3 and Lemma 4.1, F(L(n)) is a real analytic submanifold of S^n on which N(n) acts naturally, and F(L(n)) is real analytically diffeomorphic to S^1 . Moreover $F = F(SL(n, \mathbf{R}))$ consists of two points N, S. Let $h: (-1-\varepsilon, 1+\varepsilon)$ $\rightarrow F(L(n))$ be a real analytic imbedding such that h(1) = N and h(-1) = S, where ε is a sufficiently small positive real number. Since $N(n)/L(n) \cong \mathbf{R}$ acts real analytically on F(L(n)), the action defines a real analytic vector field v on F(L(n)) naturally. Then there exists a real analytic function f(t) on the interval $(-1-\varepsilon, 1+\varepsilon)$ such that $v = h_*(f(t)\frac{d}{dt})$ on the image of h. We shall first show that the function f(t) extinction f(t) on the

first show that the function f(t) satisfies the conditions (A), (B) stated in the section 2. The condition (A) follows from $F = \{N, S\}$. Considering the standard action of SL(n, R) on R^n , we can see that the condition (B) follows from Lemma 4.4.

We shall next show that the *n*-sphere S^n with the SL(n, R) action ψ is equivariantly real analytically diffeomorphic to M_f , where M_f is a real analytic SL(n, R)-manifold constructed from f(t) as before. For this purpose, we consider the following commutative diagram:

$$\begin{array}{ccc} SO(n) & \underset{NSO(n-1)}{\times} (F(SO(n-1)) - F) \xrightarrow{\alpha} S^{n} - F \\ & \downarrow \beta \\ SL(n, R) & \underset{NL(n)}{\times} (F(L(n)) - F) & \xrightarrow{\gamma} S^{n} - F \end{array}$$

Here NSO(n-1) and NL(n) are the normalizers of SO(n-1) and L(n) respectively. According to Theorem 1.3, Lemma 3.1 and Lemma 4.1, we can show that α , β and γ are real analytic one-to-one onto mappings. Moreover α is a diffeomorphism by the differentiable slice theorem; hence β and γ are also real analytic diffeomorphisms. It follows that $S^n - F$ is equivariantly real analytically diffeomorphic to a real analytic SL(n, R)-manifold X_f constructed from f(t) as before. Consequently the *n*-sphere S^n with the action ψ is equivariantly real analytically diffeomorphic to M_f , by making use of Lemma 4.4. Hence

we conclude that $c_n(F_*) = A_*^+(n)$.

This completes the proof of Theorem 4.2.

5. Certain closed subgroups of O(n)

In this section, we shall prove Lemma 1.1 and Lemma 1.2. Put

$$D(\theta) = \begin{pmatrix} \cos \theta & \sin \theta \\ -\sin \theta & \cos \theta \end{pmatrix}, \qquad \theta \in \mathbf{R}.$$

Denote by $D(a_1, \dots, a_r)$ the one-dimensional closed subgroup of O(n) consists of the following matrices:

$$\begin{pmatrix} D(a_1 heta) & 0\\ & \ddots & \\ 0 & D(a_r heta) \end{pmatrix}, \quad heta \in \mathbf{R}$$

for n=2r, and

$$\begin{pmatrix} D(a_1\theta) & 0 \\ \ddots & \\ D(a_r\theta) \\ 0 & 1 \end{pmatrix}, \quad \theta \in \mathbf{R}$$

for n=2r+1, respectively. Here a_1, \dots, a_r are integers. Consider U(k) as the centralizer of

$$egin{pmatrix} D(\pi/2) & 0 \ 0 & \ddots & 0 \ 0 & D(\pi/2) \end{pmatrix}$$

in O(2k). Then we can derive easily the following result.

Lemma 5.1. Suppose that $b_1 > b_2 > \cdots > b_s > 0$ and

$$(a_1, \cdots, a_r) = (\underbrace{b_1, \cdots, b_1}_{n_1}, \cdots, \underbrace{b_s, \cdots, b_s}_{n_s}, 0, \cdots, 0).$$

Then the centralizer of $D(a_1, \dots, a_r)$ in O(n) agrees with

$$U(n_1) \times \cdots \times U(n_s) \times O(m)$$
,

where $m = n - 2(n_1 + \cdots + n_s)$.

Here we shall prove Lemma 1.2. Let $h: SO(n) \rightarrow O(n)$ be a continuous homomorphism with a finite kernel. Suppose $n \ge 3$. Then it is easy to see that h is an isomorphism onto SO(n). Denote by T a maximal torus of SO(n) defined by the direct product of the subgroups

$$T_k = D(\underbrace{0, \dots, 0, 1}_{k}, 0, \dots, 0)$$

for $0 < k \le n/2$. Then there is an element x_1 of SO(n) such that $h(T) = x_1 T x_1^{-1}$. Then the subgroup $x_1^{-1}h(T_k)x_1$ is of the form $D(a_{k1}, \dots, a_{kr})$ for each k. Compare the centralizer of T_k and that of $x_1^{-1}h(T_k)x_1$ in O(n). We can derive

$$x_1^{-1}h(T_k)x_1 = T_i$$

for some j, by Lemma 5.1. Hence there is an element x_2 of O(n) such that

$$h(t) = x_1 x_2 t x_2^{-1} x_1^{-1}, \quad \text{for } t \in T.$$

It follows that the representations $y \rightarrow y$ and $y \rightarrow x_2^{-1}x_1^{-1}h(y)x_1x_2$ of SO(n) are equivalent. Since the representation $y \rightarrow y$ is absolutely irreducible, there is an element x_3 of O(n) such that

$$x_3 y x_3^{-1} = x_2^{-1} x_1^{-1} h(y) x_1 x_2$$

for each $y \in SO(n)$ (cf. [6], Lemma 5.5.1). Put $x = x_1x_2x_3$. Then we derive that $x \in O(n)$ and $h(y) = xyx^{-1}$ for each $y \in SO(n)$.

This completes the proof of Lemma 1.2.

We shall next prove Lemma 1.1. Let G be a connected closed subgroup of O(n). Suppose that $n \ge 3$ and

(1)
$$\dim O(n) > \dim G \ge \dim O(n) - n.$$

The inclusion map $i: G \rightarrow O(n)$ gives an orthogonal faithful representation of G. Suppose first that the representation i is reducible. Then, by an inner automorphism of O(n), G is isomorphic to a closed subgroup G' of $O(k) \times O(n-k)$ for some k such that $0 < k \le n/2$. By (1), we derive that k=1, or k=2 and G'= $SO(2) \times SO(2)$. The codimension of $O(1) \times O(n-1)$ in O(n) is n-1. If $n \ge 4$, then SO(n-1) is semi-simple; hence there is no closed subgroup of codimension one in SO(n-1). We can conclude that

$$G' = SO(1) \times SO(n-1) \cong SO(n-1),$$

$$G' = SO(2) \times SO(2) \quad \text{for } n = 4, \text{ or}$$

$$G' = \{1\} \quad \text{for } n = 3.$$

Suppose next that the representation i is irreducible and G has a onedimensional central subgroup. By Lemma 5.1, it can be seen that n is even and G is isomorphic to a closed subgroup G' of U(n/2) by an inner automorphism of O(n). It follows from (1) that

$$G' = U(3)$$
 for $n = 6$, or
 $G' = U(2)$ for $n = 4$.

It remains to consider the case that G is semi-simple and the representa-

tion *i* is irreducible. In the following, we assume that *G* is semi-simple and the representation *i* is irreducible. Suppose that the complexification i^c of *i* is reducible. Then the representation *i* possesses a complex structure and *n* is even. Hence *G* is isomorphic to a closed subgroup of U(n/2). We can derive that n=4 by (1). Moreover, by an inner automorphism of O(4), *G* is isomorphic to SU(2) which is standardly imbedded in O(4).

Suppose that the complexification i^c of i is irreducible. Then i^c is a complex irreducible representation of G of degree n.

(i) Moreover suppose first that G is not simple. Let G^* be the universal covering group of G, and let $p: G^* \rightarrow G$ be the covering projection. Since G is not simple, there are closed semi-simple normal subgroups H_1 and H_2 of G^* such that

$$G^* = H_1 \times H_2$$
.

Consider the representation $i^c p: G^* \rightarrow U(n)$. Then there are irreducible complex representations r_1 and r_2 of H_1 and H_2 respectively, such that the tensor product $r_1 \otimes r_2$ is equivalent to $i^c p$. Since $i^c p$ has a real form ip, the representations r_1 and r_2 are self-conjugate; hence r_1 (resp. r_2) has a real form or a quaternionic structure, but not both (cf.[1], Proposition 3.56). Moreover, if r_1 has a real form (resp. quaternionic structure), then r_2 has also a real form (resp. quaternionic structure). Put $n_s = \deg r_s$ for s = 1, 2. Then

(2)
$$\dim O(n) - n = \frac{n(n-3)}{2} = \frac{n_1 n_2 (n_1 n_2 - 3)}{2}.$$

Suppose first that r_1 has a quaternionic structure. Then it follows that n_1 and n_2 are even, and

$$\dim H_s \leqslant \dim \mathbf{Sp}\left(\frac{n_s}{2}\right) \quad \text{for } s = 1, 2.$$

Hence

dim
$$G = \dim H_1 + \dim H_2 \leq \frac{n_1(n_1+1)}{2} + \frac{n_2(n_2+1)}{2}$$

Compare the above inequality with (2). We can derive easily that

$$\dim G < \dim O(n) - n$$

except the case $n_1 = n_2 = 2$. If $n_1 = n_2 = 2$, then n = 4 and dim $G = \dim O(n)$. We can conclude from (1) that r_1 has no quaternionic structure. Suppose next that r_1 has a real form. Then, since H_s is semi-simple, it follows that

$$n_s \ge 3$$
 for $s = 1, 2$.

Moreover it follows that

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 $\dim H_s \leqslant \dim O(n_s) \qquad \text{for } s = 1, 2.$

Hence

dim
$$G = \dim H_1 + \dim H_2 \leq \frac{n_1(n_1-1)}{2} + \frac{n_2(n_2-1)}{2}$$
.

Compare the above inequality with (2). We can derive that

dim $G < \dim O(n) - n$.

This is a contradiction to (1), and hence we can conclude that i_1 has no real form. Consequently we can conclude that G must be simple.

(ii) Suppose next that G is simple. Moreover suppose first that G is an exceptional Lie group. Then we can derive the following result from a table of the degrees of the basic representations (cf. [2], p. 378, Table 30): the possibility remains only in the case that n=7 and G is locally isomorphic to the exceptional Lie group G_2 . Consider G_2 as a closed subgroup of O(7) as usual. Then we can conclude that G is isomorphic to G_2 by an inner automorphism of O(7). It remains to consider the case that G is locally isomorphic to SU(k), Sp(k) or SO(k). Put r = rank G. Denote by G^* the universal covering group of G. Denote by L_1, \dots, L_r the fundamental weights of G^* . Then there is a one-to-one correspondence between complex irreducible representations of G^* and sequences (a_1, \dots, a_r) of non-negative integers such that $a_1L_1 + \dots + a_rL_r$ is the highest weight of a corresponding complex irreducible representation (cf. [2], Theorem 0.8, Theorem 0.9). Denote by

$$d(a_1L_1+\cdots+a_rL_r)$$

the degree of the complex irreducible representation of G^* with the highest weight $a_1L_1 + \cdots + a_rL_r$. The degree can be computed by the Weyl's formula (cf. [2], Theorem 0.24; (0.148), (0.149), (0.150)). Notice that if

 $a_1 \geqslant a'_1, \cdots, a_r \geqslant a'_r$

then

$$d(a_1L_1 + \dots + a_rL_r) \ge d(a_1'L_1 + \dots + a_r'L_r)$$

and the equality holds only if $a_1 = a'_1, \dots, a_r = a'_r$.

(a) Suppose first that G^* is isomorphic to SU(r+1) for $r \ge 1$. Since rank $G \le \operatorname{rank} SO(n)$, it follows that

 $(3) 2r \leqslant n.$

If $r \ge 6$, then we derive from (3) that

$$\dim G = \dim SU(r+1) = r(r+2) < \frac{n(n-3)}{2} = \dim O(n) - n.$$

This is a contradiction to (1). If the pair (n, r) satisfies the conditions (1) and (3), then (n, r) is one of the following:

Notice that

$$d(L_i) = {}_{r+1}C_i$$
, $d(2L_1) = d(2L_r) = \frac{(r+1)\cdot(r+2)}{2}$.

Thus there is no complex irreducible representation of SU(r+1) of degree 2r for r=4,5. Hence (n, r) is not (10,5) nor (8,4). Since

$$\begin{aligned} d(2L_1) &= d(2L_2) = 6, \quad d(L_1 + L_2) = 8 \quad \text{for } r = 2; \\ d(2L_1) &= d(2L_3) = 10, \quad d(2L_2) = d(L_1 + L_2) = d(L_2 + L_3) = 20, \\ &\text{and } d(L_1 + L_3) = 15 \quad \text{for } r = 3, \end{aligned}$$

it follows that there is no complex irreducible representation of SU(r+1) of degree 2r+1 for r=2,3. Hence (n, r) is not (7,3) nor (5,2). It remains only the case (n, r)=(4,1). But it is seen that the complex irreducible representation of SU(2) of degree 4 has no real form. Therefore we can derive that G is not locally isomorphic to SU(r+1).

(b) Suppose next that G^* is isomorphic to Sp(r) for $r \ge 2$. Since rank $G \le \operatorname{rank} SO(n)$, it follows that

$$(4) 2r \leqslant n.$$

On the other hand, since dim Sp(r) = r(2r+1), the inequality (1) implies that

(5)
$$n(n-3) \leq 2r(2r+1) < n(n-1)$$
.

It follows from (4), (5) that

$$l \leq \frac{n}{2r} \leq \frac{2r+1}{n-3}$$
.

Therefore, if the pair (n, r) satisfies the conditions (4), (5), then we derive n=2r+2. Notice that

$$d(L_i) = {}_{2r+1}C_i - {}_{2r+1}C_{i-1}, \ d(2L_1) = r(2r+1).$$

If $r \ge 3$, then we can derive that

$$d(L_i) \ge 2r+3$$
 for $i = 2, 3, ..., r$;
 $d(2L_1) \ge 2r+3$.

If r=2, then

$$d(L_1) = 4$$
, $d(L_2) = 5$, $d(2L_1) = 10$,
 $d(2L_2) = 14$ and $d(L_1+L_2) = 16$.

It follows that there is no complex irreducible representation of Sp(r) of degree 2r+2, for $r \ge 2$. Therefore we can derive that G is not locally isomorphic to Sp(r).

(c) Suppose finally that G^* is isomorphic to Spin(k) for $k \ge 5$. It follows from (1) that

$$n(n-3) \leq k(k-1) < n(n-1)$$
.

Hence we have n=k+1. Suppose k=2r. Then

$$\begin{aligned} d(L_i) &= {}_{2r}C_i \text{ for } 1 \leq i \leq r-2, \quad d(L_{r-1}) = d(L_r) = 2^{r-1}, \\ d(2L_1) &= (r+1) \cdot (2r-1), \quad d(2L_{r-1}) = d(2L_r) = {}_{2r-1}C_r, \\ d(L_1+L_{r-1}) &= d(L_1+L_r) = (2r-1)2^{r-1}, \text{ and} \\ d(L_{r-1}+L_r) &= {}_{2r}C_{r-1}. \end{aligned}$$

It follows that there is no complex irreducible representation of Spin(2r) of degree 2r+1. Suppose k=2r+1. Then

$$\begin{aligned} &d(L_i) = {}_{2r+1}C_i \text{ for } 1 \leq i \leq r-1, \quad d(L_r) = 2^r, \\ &d(2L_1) = r(2r+3), \quad d(L_1+L_r) = r \cdot 2^{r+1}, \text{ and} \\ &d(2L_r) = 2^{2r}. \end{aligned}$$

It follows that there is no complex irreducible representation of Spin(2r+1) of degree 2r+2 for $r \neq 3$, and there is a unique complex irreducible representation of Spin(7) of degree 8. It is seen that the representation of Spin(7) has a real form. Therefore we can derive that n=8 and G is isomorphic to Spin(7). Here Spin(7) is considered as a closed subgroup of O(8) by the real spin representation. Then the isomorphism of G onto Spin(7) is realized by an inner automorphism of O(8).

This completes the proof of Lemma 1.1.

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