

Title	The Miyazawa polynomial of periodic virtual links
Author(s)	Kim, Joonoh; Lee, Sang Youl; Seo, Myoungsoo
Citation	Osaka Journal of Mathematics. 2009, 46(3), p. 769–781
Version Type	VoR
URL	https://doi.org/10.18910/9330
rights	
Note	

# Osaka University Knowledge Archive : OUKA

https://ir.library.osaka-u.ac.jp/

Osaka University

# THE MIYAZAWA POLYNOMIAL OF PERIODIC VIRTUAL LINKS

JOONOH KIM, SANG YOUL LEE and MYOUNGSOO SEO

(Received January 28, 2008, revised May 9, 2008)

# Abstract

In this paper, we investigate the behavior of the Miyazawa polynomial of periodic virtual links. As applications, we give some criteria to detect possible periods of a given oriented virtual link.

#### 1. Introduction

A classical link L in  $S^3$  is called a *p*-periodic link  $(p \ge 2$  an integer) if there exists an orientation preserving auto-homeomorphism h of  $S^3$  such that h(L) = L, h is of order p and the set of fixed points of h is a circle disjoint from L. In this case,  $L_* = L/\langle h \rangle$  is called the *factor link* of L. A link diagram D in  $\mathbb{R}^2 \setminus \{0\}$  is said to have period p if there exists a rotation  $\phi$  of  $\mathbb{R}^2$  about the origin **0** through  $2\pi/p$  such that  $\phi(D) = D$ . It is well known that every p-periodic link has a diagram of period p.

In 1988, Murasugi [10] found some relationships between the Jones polynomials of a periodic link and its factor link and showed that the knot  $10_{105}$  has no period. In 1990, Traczyk [13] gave a periodicity criterion for links in  $S^3$  by mapping Kauffman's bracket polynomial homomorphically into the group ring over  $Z_p$  of a cyclic group  $C_{p^n}$  of order  $p^n$  (p a prime), and proved that the knots  $10_{101}$  and  $10_{105}$  have no period seven. In addition, several people found criteria to detect possible periods for an oriented link by using polynomial invariants [1, 6, 7, 9, 11, 12, 14, 15, 16].

In 1996, Kauffman introduced the concept of a virtual link [5]. A virtual link diagram is a link diagram in  $\mathbb{R}^2$  possibly with some encircled crossings without over/under information. Such an encircled crossing is called a virtual crossing. Fig. 1 shows an example of a virtual link diagram. If two virtual link diagrams are related by a finite sequence of generalized Reidemeister moves as described in Fig. 2, they are said to be equivalent. A virtual link is defined to be an equivalence class of virtual link diagrams.

In [5], Kauffman defined a polynomial invariant  $f_L \in \mathbb{Z}[A^{\pm 2}]$  for a virtual link L which we call the *Jones-Kauffman polynomial*. For a classical link L, it is equal to the Jones polynomial  $V_L(t)$  after substituting  $\sqrt{t}$  for  $A^2$ . In 2005, Kamada and Miyazawa [4] introduced the concept of virtual magnetic graph diagrams and defined a 2-variable polynomial invariant for a virtual link derived from virtual magnetic graph diagrams. In [8], Miyazawa defined a virtual link invariant, which generalizes the Jones-Kauffman

<sup>2000</sup> Mathematics Subject Classification. 57M25.



Fig. 1. A virtual link diagram.



Classical Reidemeister moves



Virtual Reidemeister moves

Fig. 2. Generalized Reidemeister moves.

polynomial and the 2-variable polynomial invariant. In [3], Kamada gave some relations of the 2-variable polynomial invariant for a virtual skein triple.

In this paper, we investigate the behavior of the Miyazawa polynomial of periodic virtual links. As applications, we give some criteria to detect possible periods of a given oriented virtual link.

# 2. The Miyazawa polynomial

In this section, we review the Miyazawa polynomial of a virtual link [3, 4, 8].

Let G be an oriented 2-valent graph in  $S^3$ . G is called *magnetic* if the edges of G are oriented alternately as in Fig. 3. We allow G to have components consisting of closed edges without vertices. A *magnetic graph diagram* of a magnetic graph G is a projection image of G on a plane equipped with over/under information on each crossing as in Fig. 4. A *virtual magnetic graph diagram* (or shortly *VMG diagram*) is a magnetic graph diagram possibly with some virtual crossings as in Fig. 5. Two VMG



Fig. 4. A magnetic graph diagram.



Fig. 5. A virtual magnetic graph diagram.

diagrams are said to be *equivalent* if they are related by a finite sequence of generalized Reidemeister moves. We note that virtual link diagrams are VMG diagrams without vertices. For a VMG diagram D, we denote the sum of the signs of real crossings of D by w(D). It is called the *writhe* of D. A *pure VMG diagram* is a VMG diagram whose crossings are all virtual.

Let D be a pure VMG diagram and E(D) the set of edges of D. A weight map of D is a map  $f: E(D) \rightarrow \{+1, -1\}$  such that the product of images of two adjacent edges by f is -1. We denote the set of weight maps of D by WM(D). For a weight map f of D, we denote  $D_f$  a pure VMG diagram of which each edge is labeled its weight as in Fig. 6. It is called a weighted diagram corresponding to f. If c is a virtual crossing of a weighted diagram  $D_f$ , there exist two types of virtual crossings on  $D_f$ . If the product of weights of two edges which intersect at c is +1 (resp. -1), c is called a regular crossing (resp. irregular crossing).

Let D be a pure VMG diagram and f a weight map of D. Let c be an irregular virtual crossing of  $D_f$ . Suppose that c is formed with two edges  $e_1$  and  $e_{-1}$  whose



Fig. 6. A weighted pure VMG diagram.



Fig. 7. The raised diagram of the diagram in Fig. 6.

weights are +1 and -1, respectively. Then *c* can be replaced with a real crossing  $\hat{c}$  so that the edges  $e_1$  and  $e_{-1}$  are changed into the overpath and the underpath at  $\hat{c}$ , respectively. Such a replacement is called a *raise* of an irregular crossing. The *raised diagram* of *D* with respect to *f*, which is denoted by  $\hat{D}_f$ , is defined to be the VMG diagram obtained from the weighted diagram  $D_f$  by doing raises of all irregular crossings of  $D_f$ . For example, the raised diagram derived from the weighted diagram in Fig. 6 is given in Fig. 7.

For a pure VMG diagram D, let  $F_D$  be a map from WM(D) to  $\mathbb{Z}$  defined by  $F_D(f) = w(\hat{D}_f)$  for all weight map f of D. If we put WM<sub>n</sub>(D) = { $f \in WM(D) | F_D(f) = n$ } for any integer n, then we have

**Lemma 2.1.** For a pure VMG diagram D and an integer n, there exists a oneto-one correspondence between  $WM_n(D)$  and  $WM_{-n}(D)$ .

Proof. For a weight map f of D, we define a map  $\tilde{f}$  from E(D) to  $\{+1, -1\}$  by

$$\tilde{f}(e) = -f(e)$$
, for all  $e \in E(D)$ .



Then  $\tilde{f}$  is also a weight map of D. Let c be a real crossing of the raised diagram  $\hat{D}_f$ and  $\tilde{c}$  the real crossing of the raised diagram  $\hat{D}_{\tilde{f}}$  corresponding to c. Then  $\operatorname{sign}(\tilde{c}) = -\operatorname{sign}(c)$  and hence  $w(\hat{D}_{\tilde{f}}) = -w(\hat{D}_f)$ . It follows that  $\tilde{f} \in WM_{-n}(D)$  if  $f \in WM_n(D)$ . Now we define a map  $\phi_n$  from  $WM_n(D)$  to  $WM_{-n}(D)$  by

$$\phi_n(f) = \hat{f}$$
, for all  $f \in WM_n(D)$ .

Then  $\phi_n$  is well-defined. Since  $\phi_{-n} \circ \phi_n$  and  $\phi_n \circ \phi_{-n}$  are the identity maps,  $\phi_n$  is a one-to-one correspondence between WM<sub>n</sub>(D) and WM<sub>-n</sub>(D).

Let g be a map from  $\mathbb{Z}$  to a Laurent polynomial ring  $\mathbb{Z}[h^{\pm 1}]$ . The *double bracket* polynomial  $\langle\!\langle D \rangle\!\rangle_g$  of a pure VMG diagram D associated to g is a Laurent polynomial in  $\mathbb{Z}[2^{-1}, h^{\pm 1}]$  defined by

$$\langle\!\langle D \rangle\!\rangle_g = 2^{-\mu(D)} \sum_{f \in \mathrm{WM}(D)} (g \circ F_D)(f).$$

If c is a real crossing of D, then there are two kinds of splices at c, which are called 0-splice and  $\infty$ -splice at c as in Fig. 8. A state of D is a pure VMG diagram obtained from D by doing 0-splice or  $\infty$ -splice at each real crossing of D. We denote the set of states of D by S(D). For a state s of D, let  $C_0(D; s)$  (resp.  $C_\infty(D; s)$ ) be the set of real crossings of D where 0-splices (resp.  $\infty$ -splices) are applied to obtain s from D. We put

$$P(D; s) = \sum_{c \in C_0(D; s)} \operatorname{sign}(c) - \sum_{c \in C_\infty(D; s)} \operatorname{sign}(c),$$

where sign(c) is the crossing sign of c.

Let *D* be a virtual link diagram of a virtual link *L* and  $g: \mathbb{Z} \to \mathbb{Z}[h^{\pm 1}]$ . In [8], Miyazawa gave a Laurent polynomial  $H_{D,g}(A, h)$  (or briefly, H(D, g)) of *D* associated with *g* in  $\mathbb{Z}[2^{-1}, A^{\pm 1}, h^{\pm 1}]$  defined by

$$H_{D,g}(A, h) = \sum_{s \in \mathcal{S}(D)} A^{P(D;s)} d^{\mu(s)-1} \langle\!\langle s \rangle\!\rangle,$$

where  $d = -A^2 - A^{-2}$  and  $\mu(s)$  is the number of components of *s*. The *Miyazawa polynomial*  $R_{L,g}(A, h)$  (or briefly, R(L, g)) of *L* associated with *g* is a Laurent polynomial

in  $\mathbb{Z}[2^{-1}, A^{\pm 1}, h^{\pm 1}]$  defined by

$$R_{L,g}(A, h) = R_{D,g}(A, h) = (-A^3)^{-w(D)} H_{D,g}(A, h).$$

In [8], Miyazawa showed that  $R_{L,g}(A,h)$  is a virtual link invariant and gave some properties.

**Proposition 2.2** ([8]). (1) If  $g: \mathbb{Z} \to \mathbb{Z}[h^{\pm 1}]$  is defined by g(n) = 1, then R(L, g) is identical with the Jones-Kauffman polynomial of L. (2) If  $g: \mathbb{Z} \to \mathbb{Z}[h^{\pm 1}]$  is defined by g(n) = |n| and L is a classical link, then R(L, g)

(2) If  $g: \mathbb{Z} \to \mathbb{Z}[n^{-1}]$  is defined by g(n) = |n| and L is a classical link, then R(L, g) is equal to zero.

(3) If  $g: \mathbb{Z} \to \mathbb{Z}[h^{\pm 1}]$  is defined by  $g(n) = h^{(1-(-1)^n)/2}$ , then R(L, g) coincides with the 2-variable polynomial defined by Kamada and Miyazawa.

(4) If  $g: \mathbb{Z} \to \mathbb{Z}[h^{\pm 1}]$  is defined by  $g(n) = h^n$  and v(L) is the virtual crossing number of L, then  $v(L) \ge \max \deg_h R(L, g)$ .

REMARK 2.3. In [8], Miyazawa used an arbitrary Laurent polynomial ring  $\Gamma$  over  $\mathbb{Q}$  as the range of g. If  $\Gamma = \mathbb{Q}[h^{\pm 1}]$ , then  $\langle\!\langle D \rangle\!\rangle_g \in \mathbb{Q}[h^{\pm 1}]$  and  $R(L, g) \in \mathbb{Q}[h^{\pm 1}, A^{\pm 1}]$ . Since the ideal of  $\mathbb{Q}[h^{\pm 1}, A^{\pm 1}]$  generated by a non-zero integer is itself, our theorems in Section 3 are meaningless for  $g: \mathbb{Z} \to \mathbb{Q}[h^{\pm 1}]$ . On the other hand, the rage of g in propositions of [8] can be restricted in  $\mathbb{Z}[h^{\pm 1}]$ . Thus we can use the Laurent polynomial ring  $\mathbb{Z}[h^{\pm 1}]$  as the range of g. Since the ideals in Section 3 are proper, our theorems are meaningful.

#### 3. Periodic virtual links

An oriented virtual link L is said to have period  $p \ge 2$  if it admits an oriented virtual link diagram D in  $\mathbb{R}^2 \setminus \{0\}$  that invariant under the rotation  $\zeta$  of  $\mathbb{R}^2$  about the origin **0** through  $2\pi/p$ . The virtual link  $L_*$  represented by the quotient  $D/\langle \zeta \rangle$  is called the *factor link* of L. The diagram described in Fig. 1 is a virtual link diagram of a virtual link having period 3.

**Theorem 3.1** (Fermat's little theorem, [2]). If p is a prime and a an integer relatively prime to p, then

$$a^{p-1} \equiv 1 \mod p.$$

**Theorem 3.2.** Let p be an odd prime and L a virtual link that has period  $p^r$   $(r \ge 1)$ . Let g be a map from  $\mathbb{Z}$  to  $\mathbb{Z}[h^{\pm 1}]$ . (1) If  $g: (\mathbb{Z}, +) \to (\mathbb{Z}[h^{\pm 1}], \cdot)$  is a homomorphism, then

$$R(L, g) \equiv [R(L_*, g)]^{p^r} \mod (p, (-A^2 - A^{-2})^{p-1} - 1).$$



Fig. 9.

(2) If 
$$g: \mathbb{Z} \to \mathbb{Z}[h^{\pm 1}]$$
 is defined by  $g(n) = h^{(1-(-1)^n)/2}$ , then

$$R(L, g) \equiv [R(L_*, g)]^{p^r} \mod (p, (-A^2 - A^{-2})^{p-1} - 1, h^{p-1} - 1)$$

Proof. It suffices to prove the theorem for r = 1 (the theorem for r > 1 is proved by applying the argument for r = 1 repeatedly). Let D be a virtual link diagram of Lin  $\mathbb{R}^2 \setminus \{\mathbf{0}\}$  that invariant under the rotation  $\zeta$  of  $\mathbb{R}^2$  about the origin  $\mathbf{0}$  through  $2\pi/p$ . Then D can be divided into p pieces  $D_0, D_1, \ldots, D_{p-1}$  such that  $\zeta(D_i) = D_{i+1}$  ( $i = 0, 1, \ldots, p - 1$ ) and  $D_p = D_0$ . Let  $I(0, 2\pi/p)$  be the closed domain bounded by two half lines  $\theta = 0$  and  $\theta = 2\pi/p$  in the polar coordinate system. We may assume that  $D_0 = D \cap I(0, 2\pi/p)$ . Let  $A_1, A_2, \ldots, A_l$  be the points of intersection of  $D_0$  and the line  $\theta = 0$  and let  $\zeta(A_i) = B_i$  ( $i = 1, 2, \ldots, l$ ). By joining  $A_i$  and  $B_i$  on  $\mathbb{R}^2 \setminus I(0, 2\pi/p)$ by circle  $C_i$  centered 0, we obtain a diagram  $D_*$  of the factor link  $L_*$ . For example, see Fig. 9. For simplicity, we write  $D_* = D/\zeta$ . We note that the rotation  $\zeta : \mathbb{R}^2 \to \mathbb{R}^2$ maps D onto itself preserving the sign of each crossing. If s is a state in  $\mathcal{S}(D)$ , then either  $\zeta(s) \neq s$  or  $\zeta(s) = s$ .

If  $\zeta(s) \neq s$ , then  $s, \zeta(s), \zeta^2(s), \ldots, \zeta^{p-1}(s)$  are all distinct. Since any two of these are isomorphic, we have p identical terms in H(D, g), and they vanish by reducing modulo p.

If  $\zeta(s) = s$ , then *s* defines a unique quotient state  $s_* (= s/\zeta)$ . Let  $\alpha$  and  $\alpha_*$  be the terms in H(D, g) and  $H(D_*, g)$  which are associated with *s* and  $s_*$ , respectively. Since  $\sum_{C_0(D;s)} \operatorname{sign}(c) = p \cdot \sum_{C_0(D_*;s_*)} \operatorname{sign}(c)$  and  $\sum_{C_\infty(D;s)} \operatorname{sign}(c) = p \cdot \sum_{C_\infty(D_*;s_*)} \operatorname{sign}(c)$ , we have

$$P(D;s) = p \cdot P(D_*;s_*).$$

Then we have that

(3.1) 
$$\alpha = A^{p \cdot P(D_*;s_*)} d^{\mu(s)-1} \langle \! \langle s \rangle \! \rangle, \quad \alpha_* = A^{P(D_*;s_*)} d^{\mu(s_*)-1} \langle \! \langle s_* \rangle \! \rangle.$$

We will compare  $\mu(s) - 1$  and  $\mu(s_*) - 1$ . Let  $G = \{id, \zeta, \dots, \zeta^{p-1}\}$  and  $\mathcal{C} = \{C \mid C \text{ is a component of } s\}$ , where *id* is the identity of  $\mathbb{R}^2$ . Then *G* acts on *C* by  $\zeta^i \cdot C = \zeta^i(C)$ . We put  $\mathcal{C}_G = \{C \in \mathcal{C} \mid gC = C, \forall g \in G\}$  and  $\mathcal{C}/G = \{G(C) \mid G(C) \text{ is the orbit of } C \in \mathcal{C}\}$ . For a set *S*, we denote by |S| the number of elements in *S*. If  $\zeta^i(C) = C$  for some *i*  $(1 \leq i \leq p - 1)$ , then  $\zeta^j(C) = C$  for all *j* because *p* is prime. Thus |G(C)| = p or 1. We note that |G(C)| = 1 if and only if  $C \in \mathcal{C}_G$ . Since  $\mu(s_*) = |\mathcal{C}/G|$ , we calculate that

(3.2) 
$$\mu(s) = |\mathcal{C}| = |\mathcal{C}_G| + p(|\mathcal{C}/G| - |\mathcal{C}_G|) = p \cdot \mu(s_*) - (p-1)|\mathcal{C}_G|.$$

Since  $\mu(s) - 1 = p(\mu(s_*) - 1) - (p - 1)(|\mathcal{C}_G| - 1)$ , we have that

(3.3) 
$$d^{\mu(s)-1} \equiv d^{p \cdot (\mu(s_*)-1)} \mod (d^{p-1}-1).$$

By Theorem 3.1 and (3.2), it follows that

(3.4) 
$$2^{-\mu(s)} \equiv 2^{p \cdot (-\mu(s_*))} \mod p$$

Let f be a wight map of s. We define a weight map  $\zeta(f)$  of s by, for each edge e of s,

$$\zeta(f)(e) = f(e')$$
 whenever  $\zeta(e') = e$ .

If  $\zeta(f) \neq f$ , then  $f, \zeta(f), \ldots, \zeta^{p-1}(f)$  are all distinct but  $\widehat{s_f}, \widehat{s_{\zeta(f)}}, \ldots, \widehat{s_{\zeta^{p-1}(f)}}$  are equivalent. Thus  $w(\widehat{s_f}) = w(\widehat{s_{\zeta(f)}}) = \cdots = w(\widehat{s_{\zeta^{p-1}(f)}})$ . If  $\zeta(f) = f$ , then f defines a unique weight map  $f_* (= f/\zeta)$  of  $s_*$ . Let WD(s) denote the set of weighted diagram of s, that is, WD(s) = { $s_f \mid f \in WM(s)$ }. Then G acts on WD(s) by

$$\zeta(s_f) = s_{\zeta(f)}.$$

We can put that WD(s) =  $\{s_{f_1}, s_{f_2}, \ldots, s_{f_m}\} \cup \{s_{f_{1,0}}, s_{f_{1,1}}, \ldots, s_{f_{1,p-1}}\} \cup \cdots \cup \{s_{f_{n,0}}, s_{f_{n,1}}, \ldots, s_{f_{n,p-1}}\}$  where  $\zeta(s_{f_i}) = s_{f_i}$  for all i  $(1 \le i \le m)$  and  $f_{j,k} = \zeta^k(f_{j,0})$  for each  $k = 1, 2, \ldots, p-1, j = 1, 2, \ldots, n$ . We set that  $w_i = w(\widehat{s_{f_i}})$  and  $w_{j,k} = w(\widehat{s_{f_{j,k}}})$  for each  $i = 1, 2, \ldots, m, k = 1, 2, \ldots, p-1, j = 1, 2, \ldots, n$  and set  $w_i^* = w(\widehat{(s_*)_{(f_i)_*}})$  for each  $i = 1, 2, \ldots, m$ . For each  $i = 1, 2, \ldots, m$ , we have

$$(3.5) w_i = p \cdot w_i^*.$$

For any map  $g: \mathbb{Z} \to \mathbb{Z}[h^{\pm 1}]$ , we have that

$$\langle\!\langle s_* \rangle\!\rangle = 2^{-\mu(s_*)} [g(w_1^*) + \cdots + g(w_m^*)]$$

and

$$\langle\!\langle s \rangle\!\rangle = 2^{-\mu(s)} [g(w_1) + \dots + g(w_m) + p \cdot g(w_{1,0}) + \dots + p \cdot g(w_{m,0})].$$

By (3.4) and (3.5), it follows that

$$\langle\!\langle s \rangle\!\rangle \equiv 2^{p \cdot (-\mu(s_*))} [g(p \cdot w_1^*) + \dots + g(p \cdot w_m^*)] \mod p.$$

(1) If  $g: (Z, +) \to (\mathbb{Z}[h^{\pm 1}], \cdot)$  is a homomorphism, then

(3.6)  

$$\langle\!\langle s_* \rangle\!\rangle^p = 2^{p \cdot (-\mu(s_*))} [g(w_1^*) + \dots + g(w_m^*)]^p \\
\equiv 2^{p \cdot (-\mu(s_*))} [g(w_1^*)^p + \dots + g(w_m^*)^p] \mod p \\
\equiv 2^{p \cdot (-\mu(s_*))} [g(p \cdot w_1^*) + \dots + g(p \cdot w_m^*)] \mod p \\
\equiv \langle\!\langle s \rangle\!\rangle \mod p.$$

By (3.1), (3.3) and (3.6), it follows that

$$\alpha_*^p \equiv \alpha \mod (p, (-A^2 - A^{-2})^{p-1} - 1).$$

Hence we have

$$H(D, g) \equiv [H(D_*, g)]^p \mod (p, (-A^2 - A^{-2})^{p-1} - 1).$$

Since  $w(D) = p \cdot w(D_*)$ ,  $(-A^3)^{-w(D)} = [(-A^3)^{-w(D_*)}]^p$ . Therefore we have

$$R(L, g) \equiv [R(L_*, g)]^p \mod (p, (-A^2 - A^{-2})^{p-1} - 1).$$

(2) Suppose that  $g: \mathbb{Z} \to \mathbb{Z}[h^{\pm 1}]$  is defined by  $g(n) = h^{(1-(-1)^n)/2}$ . Since  $w_i = p \cdot w_i^*$  and p is an odd prime,  $w_i$  and  $w_i^*$  have the same parity and hence  $g(w_i) = g(w_i^*)$ . Since  $g(w_i^*)$  is either h or 1, we have

$$g(w_i^*)^p \equiv g(w_i^*) \mod (h^{p-1} - 1).$$

Then we know that

(3.7)  

$$\langle \langle s_* \rangle \rangle^p = 2^{p \cdot (-\mu(s_*))} [g(w_1^*) + \dots + g(w_m^*)]^p \\
\equiv 2^{p \cdot (-\mu(s_*))} [g(w_1^*)^p + \dots + g(w_m^*)^p] \mod p \\
\equiv 2^{p \cdot (-\mu(s_*))} [g(w_1^*) + \dots + g(w_m^*)] \mod (p, h^{p-1} - 1) \\
\equiv 2^{p \cdot (-\mu(s_*))} [g(w_1) + \dots + g(w_m)] \mod (p, h^{p-1} - 1) \\
\equiv \langle \langle s \rangle \rangle \mod (p, h^{p-1} - 1).$$

By (3.1), (3.3) and (3.7), it follows that

$$\alpha_*^p \equiv \alpha \mod (p, (-A^2 - A^{-2})^{p-1} - 1, h^{p-1} - 1).$$



Fig. 10.

Thus we have

$$H(D, g) \equiv [H(D_*, g)]^p \mod (p, (-A^2 - A^{-2})^{p-1} - 1, h^{p-1} - 1).$$

Hence we get

$$R(L, g) \equiv [R(L_*, g)]^p \mod (p, (-A^2 - A^{-2})^{p-1} - 1, h^{p-1} - 1).$$

This completes the proof.

EXAMPLE 3.3. Let  $L_1$  be a virtual knot as in Fig. 10. Then the Jones-Kauffman polynomial of  $L_1$  is equal to 1 [8]. Let  $g: \mathbb{Z} \to \mathbb{Z}[h^{\pm 1}]$  be a map given by  $g(n) = h^n$ . It is known [8] that

$$R_{L_1}^g = \frac{1}{2}(1+A^{-8}) + \frac{1}{4}(1-A^{-8})(h^2+h^{-2}).$$

Suppose that  $L_1$  has period 3. From Theorem 3.2, it follows that

$$\frac{1}{4}(1 - A^{-8}) \equiv 0 \mod (3, (-A^2 - A^{-2})^2 - 1).$$

Since  $(-A^2 - A^{-2})^2 - 1 = A^4 + 1 + A^{-4} = A^{-4}(A^8 + A^4 + 1)$  and  $1 - A^{-8} = (A^{-4} - A^{-8})(A^8 + A^4 + 1) + (1 - A^4)$ ,

$$\frac{1}{4}(1 - A^{-8}) \equiv 1 + 2A^4 \mod (3, A^8 + A^4 + 1).$$

Let  $\mathcal{I}$  be the ideal of  $\mathbb{Z}[2^{-1}, A^{\pm 1}]$  generated by 3. We note that the quotient ring of  $\mathbb{Z}[2^{-1}, A^{\pm 1}]$  by  $\mathcal{I}$  is isomorphic to the ring  $\mathbb{Z}_3[A^{\pm 1}]$ . So it is not true that  $1 + 2A^4 \equiv 0 \mod (3, A^8 + A^4 + 1)$ . Hence  $L_1$  does not have period 3.

**Theorem 3.4.** Let p be a prime and L a virtual link that has period  $p^r$   $(r \ge 1)$ . Let g be a map from  $\mathbb{Z}$  to  $\mathbb{Z}[h^{\pm 1}]$ . Then

$$R_{L,g}(A, h) \equiv R_{L,g}(A^{-1}, h) \mod (p, A^{p'} - 1).$$

Proof. Let *D* be a virtual link diagram of *L* in  $\mathbb{R}^2 \setminus \{0\}$  that invariant under the rotation  $\zeta$  of  $\mathbb{R}^2$  about the origin **0** through  $2\pi/p^r$  and  $D_* = D/\zeta$ . Let *s* be a state of *D*.

If  $\zeta(s) \neq s$ , then there exist  $p^n$  distinct but equivalent states  $s, \zeta(s), \ldots, \zeta^{p^n-1}(s)$  for some  $n \ (1 \leq n \leq r)$ . Contribution of these states to the polynomial vanishes by reducing modulo p.

If  $\zeta(s) = s$ , then s defines a unique quotient states  $s_*$   $(= s/\zeta)$ . Since  $P(D; s) = p^r \cdot P(D_*; s_*)$ , we get

$$A^{P(D;s)} = A^{p^r \cdot P(D_*;s_*)} \equiv 1 \mod (A^{p^r} - 1).$$

Since  $d = -A^2 - A^{-2}$  is symmetric and  $\langle \langle s \rangle \rangle \in \mathbb{Z}[2^{-1}, h^{\pm 1}]$ , we obtain

$$H_{D,g}(A, h) \equiv H_{D,g}(A^{-1}, h) \mod (p, A^{p^r} - 1).$$

Since  $w(D) = p^r \cdot w(D_*)$ ,

$$(-A^3)^{-w(D)} \equiv (-A^{-3})^{-w(D)} \mod (A^{p^r} - 1).$$

Hence we have

$$R_{L,g}(A, h) \equiv R_{L,g}(A^{-1}, h) \mod (p, A^{p^r} - 1).$$

This completes the proof.

**Corollary 3.5.** Let p be a prime and L a virtual link that has period  $p^r$ . Let  $g: (\mathbb{Z}, +) \to (\mathbb{Z}[h^{\pm 1}], \cdot)$  be a homomorphism. Then

$$R_{L,g}(A, h) \equiv R_{L,g}(A^{-1}, h^{-1}) \mod (p, A^{p'} - 1).$$

Proof. Let D a virtual link diagram of L in  $\mathbb{R}^2 \setminus \{0\}$  that invariant under the rotation  $\zeta$  of  $\mathbb{R}^2$  about the origin **0** through  $2\pi/p^r$  and  $D_* = D/\zeta$ .

Let s be a state of D. Since  $g: (\mathbb{Z}, +) \to (\mathbb{Z}[h^{\pm 1}], \cdot)$  is a homomorphism, we have  $\langle \langle s \rangle \rangle (h) = \langle \langle s \rangle \rangle (h^{-1})$  by Lemma 2.1. By the similar argument to Theorem 3.4, we have

$$R_{L,g}(A, h) \equiv R_{L,g}(A^{-1}, h^{-1}) \mod (p, A^{p^r} - 1).$$

This completes the proof.

EXAMPLE 3.6. Let  $L_1$  be a virtual knot as in Fig. 10. Then

$$R^{g}_{L_{1}}(A, h) - R^{g}_{L_{1}}(A^{-1}, h) = \frac{1}{2}(A^{-8} - A^{8}) + \frac{1}{4}(A^{8} - A^{-8})(h^{2} + h^{-2}).$$

We observe that

$$\frac{1}{2}(A^{-8} - A^8) \equiv 2A + A^2 \neq 0 \mod (3, A^3 - 1).$$

Hence this is an another proof to show that  $L_1$  does not have period 3.

ACKNOWLEDGEMENTS. The authors would like to thank the referee for many valuable comments. This work was supported for two years by Pusan National University Research Grant.

# References

- [1] J.F. Davis and C. Livingston: Alexander polynomials of periodic knots, Topology **30** (1991), 551–564.
- [2] U. Dudley: Elementary Number Theory, second edition, Freeman, San Francisco, Calif., 1978.
  [3] N. Kamada: *Some relations on Miyazawa's virtual knot invariant*, Topology Appl. **154** (2007), 1417–1429.
- [4] N. Kamada and Y. Miyazawa: A 2-variable polynomial invariant for a virtual link derived from magnetic graphs, Hiroshima Math. J. 35 (2005), 309–326.
- [5] L.H. Kauffman: Virtual knot theory, European J. Combin. 20 (1999), 663–690.
- [6] S.Y. Lee:  $\mathbb{Z}_n$ -equivariant Goeritz matrices for periodic links, Osaka J. Math. 40 (2003), 393–408.
- [7] Y. Miyazawa: Conway polynomials of periodic links, Osaka J. Math. 31 (1994), 147–163.
- Y. Miyazawa: *Magnetic graphs and an invariant for virtual links*, J. Knot Theory Ramifications 15 (2006), 1319–1334.
- [9] K. Murasugi: On periodic knots, Comment. Math. Helv. 46 (1971), 162–174.
- [10] K. Murasugi: Jones polynomials of periodic links, Pacific J. Math. 131 (1988), 319–329.
- J.H. Przytycki: On Murasugi's and Traczyk's criteria for periodic links, Math. Ann. 283 (1989), 465–478.
- [12] M. Sakuma: On the polynomials of periodic links, Math. Ann. 257 (1981), 487-494.
- [13] P. Traczyk: 10<sub>101</sub> has no period 7: a criterion for periodic links, Proc. Amer. Math. Soc. 108 (1990), 845–846.
- [14] Y. Yokota: The skein polynomial of periodic knots, Math. Ann. 291 (1991), 281-291.
- [15] Y. Yokota: The Kauffman polynomial of periodic knots, Topology 32 (1993), 309-324.
- [16] Y. Yokota: Polynomial invariants of periodic knots, J. Knot Theory Ramifications 5 (1996), 553–567.

780

Joonoh Kim Department of Mathematics Graduate School of Natural Sciences Pusan National University Pusan 609–735 Korea

Sang Youl Lee Department of Mathematics Pusan National University Pusan 609–735 Korea e-mail: sangyoul@pusan.ac.kr

Myoungsoo Seo Department of Mathematics Kyungpook National University Daegu 702–701 Korea e-mail: myseo@knu.ac.kr