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THE MIYAZAWA POLYNOMIAL OF PERIODIC VIRTUAL LINKS

JOONOH KIM, SANG YOUL LEE and MYOUNGSOO SEO

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Abstract

In this paper, we investigate the behavior of the Miyazawa polynomial of periodic virtual links. As applications, we give some criteria to detect possible periods of a given oriented virtual link.

1. Introduction

A classical link \( L \) in \( S^3 \) is called a \( p \)-periodic link \((p \geq 2 \text{ an integer})\) if there exists an orientation preserving auto-homeomorphism \( h \) of \( S^3 \) such that \( h(L) = L \), \( h \) is of order \( p \) and the set of fixed points of \( h \) is a circle disjoint from \( L \). In this case, \( L_0 = L/h \) is called the factor link of \( L \). A link diagram \( D \) in \( \mathbb{R}^2 \) is said to have period \( p \) if there exists a rotation \( \phi \) of \( \mathbb{R}^2 \) about the origin \( 0 \) through \( 2\pi/p \) such that \( \phi(D) = D \). It is well known that every \( p \)-periodic link has a diagram of period \( p \).

In 1988, Murasugi [10] found some relationships between the Jones polynomials of a periodic link and its factor link and showed that the knot 10\(_{105}\) has no period. In 1990, Traczyk [13] gave a periodicity criterion for links in \( S^3 \) by mapping Kauffman’s bracket polynomial homomorphically into the group ring over \( \mathbb{Z}_p \) of a cyclic group \( C_{p^n} \) of order \( p^n \) (\( p \) a prime), and proved that the knots 10\(_{101}\) and 10\(_{105}\) have no period seven. In addition, several people found criteria to detect possible periods for an oriented link by using polynomial invariants [1, 6, 7, 9, 11, 12, 14, 15, 16].

In 1996, Kauffman introduced the concept of a virtual link [5]. A virtual link diagram is a link diagram in \( \mathbb{R}^2 \) possibly with some encircled crossings without over/under information. Such an encircled crossing is called a virtual crossing. Fig. 1 shows an example of a virtual link diagram. If two virtual link diagrams are related by a finite sequence of generalized Reidemeister moves as described in Fig. 2, they are said to be equivalent. A virtual link is defined to be an equivalence class of virtual link diagrams.

In [5], Kauffman defined a polynomial invariant \( f_L \in \mathbb{Z}[A^{\pm 2}] \) for a virtual link \( L \) which we call the Jones-Kauffman polynomial. For a classical link \( L \), it is equal to the Jones polynomial \( V_L(t) \) after substituting \( \sqrt{t} \) for \( A^{\pm 2} \). In 2005, Kamada and Miyazawa [4] introduced the concept of virtual magnetic graph diagrams and defined a 2-variable polynomial invariant for a virtual link derived from virtual magnetic graph diagrams. In [8], Miyazawa defined a virtual link invariant, which generalizes the Jones-Kauffman polynomial.
polynomial and the 2-variable polynomial invariant. In [3], Kamada gave some relations of the 2-variable polynomial invariant for a virtual skein triple.

In this paper, we investigate the behavior of the Miyazawa polynomial of periodic virtual links. As applications, we give some criteria to detect possible periods of a given oriented virtual link.

2. The Miyazawa polynomial

In this section, we review the Miyazawa polynomial of a virtual link [3, 4, 8].

Let $G$ be an oriented 2-valent graph in $S^3$. $G$ is called magnetic if the edges of $G$ are oriented alternately as in Fig. 3. We allow $G$ to have components consisting of closed edges without vertices. A magnetic graph diagram of a magnetic graph $G$ is a projection image of $G$ on a plane equipped with over/under information on each crossing as in Fig. 4. A virtual magnetic graph diagram (or shortly VMG diagram) is a magnetic graph diagram possibly with some virtual crossings as in Fig. 5. Two VMG
diagrams are said to be *equivalent* if they are related by a finite sequence of generalized Reidemeister moves. We note that virtual link diagrams are VMG diagrams without vertices. For a VMG diagram $D$, we denote the sum of the signs of real crossings of $D$ by $w(D)$. It is called the *writhe* of $D$. A *pure VMG diagram* is a VMG diagram whose crossings are all virtual.

Let $D$ be a pure VMG diagram and $E(D)$ the set of edges of $D$. A *weight map* of $D$ is a map $f: E(D) \to \{+1, -1\}$ such that the product of images of two adjacent edges by $f$ is $-1$. We denote the set of weight maps of $D$ by $\text{WM}(D)$. For a weight map $f$ of $D$, we denote $D_f$ a pure VMG diagram of which each edge is labeled its weight as in Fig. 6. It is called a *weighted diagram* corresponding to $f$. If $c$ is a virtual crossing of a weighted diagram $D_f$, there exist two types of virtual crossings on $D_f$. If the product of weights of two edges which intersect at $c$ is $+1$ (resp. $-1$), $c$ is called a *regular crossing* (resp. *irregular crossing*).

Let $D$ be a pure VMG diagram and $f$ a weight map of $D$. Let $c$ be an irregular virtual crossing of $D_f$. Suppose that $c$ is formed with two edges $e_1$ and $e_{-1}$ whose
weights are +1 and −1, respectively. Then c can be replaced with a real crossing \( \hat{c} \) so that the edges \( e_1 \) and \( e_{-1} \) are changed into the overpath and the underpath at \( \hat{c} \), respectively. Such a replacement is called a raise of an irregular crossing. The raised diagram of \( D \) with respect to \( f \), which is denoted by \( \hat{D}_f \), is defined to be the VMG diagram obtained from the weighted diagram \( D_f \) by doing raises of all irregular crossings of \( D_f \). For example, the raised diagram derived from the weighted diagram in Fig. 6 is given in Fig. 7.

For a pure VMG diagram \( D \), let \( F_D \) be a map from \( \text{WM}(D) \) to \( \mathbb{Z} \) defined by

\[
F_D(f) = \omega(\hat{D}_f)
\]

for all weight map \( f \) of \( D \). If we put \( \text{WM}_n(D) = \{ f \in \text{WM}(D) \mid F_D(f) = n \} \) for any integer \( n \), then we have

**Lemma 2.1.** For a pure VMG diagram \( D \) and an integer \( n \), there exists a one-to-one correspondence between \( \text{WM}_n(D) \) and \( \text{WM}_{-n}(D) \).

**Proof.** For a weight map \( f \) of \( D \), we define a map \( \tilde{f} \) from \( E(D) \) to \( \{+1, -1\} \) by

\[
\tilde{f}(e) = -f(e), \quad \text{for all } e \in E(D).
\]
Then $\tilde{f}$ is also a weight map of $D$. Let $c$ be a real crossing of the raised diagram $\hat{D}_f$ and $\tilde{c}$ the real crossing of the raised diagram $\hat{D}_{\tilde{f}}$ corresponding to $c$. Then $\text{sign}(\tilde{c}) = -\text{sign}(c)$ and hence $w(\hat{D}_{\tilde{f}}) = -w(\hat{D}_f)$. It follows that $\tilde{f} \in \text{WM}_{-n}(D)$ if $f \in \text{WM}_n(D)$. Now we define a map $\phi_n$ from $\text{WM}_n(D)$ to $\text{WM}_{-n}(D)$ by

$$\phi_n(f) = \tilde{f}, \quad \text{for all } f \in \text{WM}_n(D).$$

Then $\phi_n$ is well-defined. Since $\phi_{-n} \circ \phi_n$ and $\phi_n \circ \phi_{-n}$ are the identity maps, $\phi_n$ is a one-to-one correspondence between $\text{WM}_n(D)$ and $\text{WM}_{-n}(D)$. □

Let $g$ be a map from $\mathbb{Z}$ to a Laurent polynomial ring $\mathbb{Z}[h^{\pm 1}]$. The double bracket polynomial $\llangle D \rrangle_g$ of a pure VMG diagram $D$ associated to $g$ is a Laurent polynomial in $\mathbb{Z}[2^{-1}, h^{\pm 1}]$ defined by

$$\llangle D \rrangle_g = 2^{-h(D)} \sum_{f \in \text{WM}(D)} (g \circ F_D)(f).$$

If $c$ is a real crossing of $D$, then there are two kinds of splices at $c$, which are called 0-splice and $\infty$-splice at $c$ as in Fig. 8. A state of $D$ is a pure VMG diagram obtained from $D$ by doing 0-splice or $\infty$-splice at each real crossing of $D$. We denote the set of states of $D$ by $S(D)$. For a state $s$ of $D$, let $C_0(D; s)$ (resp. $C_\infty(D; s)$) be the set of real crossings of $D$ where 0-splices (resp. $\infty$-splices) are applied to obtain $s$ from $D$. We put

$$P(D; s) = \sum_{c \in C_0(D; s)} \text{sign}(c) - \sum_{c \in C_\infty(D; s)} \text{sign}(c),$$

where $\text{sign}(c)$ is the crossing sign of $c$.

Let $D$ be a virtual link diagram of a virtual link $L$ and $g : \mathbb{Z} \to \mathbb{Z}[h^{\pm 1}]$. In [8], Miyazawa gave a Laurent polynomial $H_{D, g}(A, h)$ (or briefly, $H(D, g)$) of $D$ associated with $g$ in $\mathbb{Z}[2^{-1}, A^{\pm 1}, h^{\pm 1}]$ defined by

$$H_{D, g}(A, h) = \sum_{s \in S(D)} A^{P(D; s)} d^{\mu(s)-1} \llangle s \rrangle,$$

where $d = -A^2 + A^{-2}$ and $\mu(s)$ is the number of components of $s$. The Miyazawa polynomial $R_{L, g}(A, h)$ (or briefly, $R(L, g)$) of $L$ associated with $g$ is a Laurent polynomial.
in $\mathbb{Z}[2^{-1}, A^{\pm 1}, h^{\pm 1}]$ defined by

$$R_{L,g}(A, h) = R_{D,g}(A, h) = (-A^{3})^{-w(D)}H_{D,g}(A, h).$$

In [8], Miyazawa showed that $R_{L,g}(A, h)$ is a virtual link invariant and gave some properties.

**Proposition 2.2 ([8]).** (1) If $g: \mathbb{Z} \rightarrow \mathbb{Z}[h^{\pm 1}]$ is defined by $g(n) = 1$, then $R(L, g)$ is identical with the Jones-Kauffman polynomial of $L$.

(2) If $g: \mathbb{Z} \rightarrow \mathbb{Z}[h^{\pm 1}]$ is defined by $g(n) = n$ and $L$ is a classical link, then $R(L, g)$ is equal to zero.

(3) If $g: \mathbb{Z} \rightarrow \mathbb{Z}[h^{\pm 1}]$ is defined by $g(n) = h(n)$ and $L$ is the virtual crossing number of $L$, then $v(L) \geq \max \deg h R(L, g)$.

(4) If $g: \mathbb{Z} \rightarrow \mathbb{Z}[h^{\pm 1}]$ is defined by $g(n) = h(n)$ and $L$ is the virtual crossing number of $L$, then $v(L) \geq \max \deg h R(L, g)$.

**Remark 2.3.** In [8], Miyazawa used an arbitrary Laurent polynomial ring $\Gamma$ over $\mathbb{Q}$ as the range of $g$. If $\Gamma = \mathbb{Q}[h^{\pm 1}]$, then $\langle D \rangle g \in \mathbb{Q}[h^{\pm 1}]$ and $R(L, g) \in \mathbb{Q}[h^{\pm 1}, A^{\pm 1}]$. Since the ideal of $\mathbb{Q}[h^{\pm 1}, A^{\pm 1}]$ generated by a non-zero integer is itself, our theorems in Section 3 are meaningless for $g: \mathbb{Z} \rightarrow \mathbb{Q}[h^{\pm 1}]$. On the other hand, the range of $g$ in propositions of [8] can be restricted in $\mathbb{Z}[h^{\pm 1}]$. Thus we can use the Laurent polynomial ring $\mathbb{Z}[h^{\pm 1}]$ as the range of $g$. Since the ideals in Section 3 are proper, our theorems are meaningful.

3. Periodic virtual links

An oriented virtual link $L$ is said to have period $p \geq 2$ if it admits an oriented virtual link diagram $D$ in $\mathbb{R}^2 \setminus \{0\}$ that invariant under the rotation $\zeta$ of $\mathbb{R}^2$ about the origin $0$ through $2\pi/p$. The virtual link $L_\zeta$ represented by the quotient $D/\langle \zeta \rangle$ is called the factor link of $L$. The diagram described in Fig. 1 is a virtual link diagram of a virtual link having period 3.

**Theorem 3.1** (Fermat’s little theorem, [2]). If $p$ is a prime and $a$ an integer relatively prime to $p$, then

$$a^{p-1} \equiv 1 \mod p.$$

**Theorem 3.2.** Let $p$ be an odd prime and $L$ a virtual link that has period $p^r$ ($r \geq 1$). Let $g$ be a map from $\mathbb{Z}$ to $\mathbb{Z}[h^{\pm 1}]$.

(1) If $g: (\mathbb{Z}, +) \rightarrow (\mathbb{Z}[h^{\pm 1}], \cdot)$ is a homomorphism, then

$$R(L, g) \equiv [R(L_\zeta, g)]^{p^r} \mod (p, (-A^{2} - A^{-2})^{p^r} - 1).$$
(2) If $g : \mathbb{Z} \to \mathbb{Z}[h^{\pm 1}]$ is defined by $g(n) = h^{(1 - (-1)^n)/2}$, then

$$R(L, g) = [R(L_s, g)] = (p, (-A^2 - A^{-2})^{p-1} - 1, h^{p-1} - 1).$$

Proof. It suffices to prove the theorem for $r = 1$ (the theorem for $r > 1$ is proved by applying the argument for $r = 1$ repeatedly). Let $D$ be a virtual link diagram of $L$ in $\mathbb{R}^2 \setminus \{0\}$ that invariant under the rotation $\zeta$ of $\mathbb{R}^2$ about the origin 0 through $2\pi/p$. Then $D$ can be divided into $p$ pieces $D_0, D_1, \ldots, D_{p-1}$ such that $\zeta(D_i) = D_{i+1} (i = 0, 1, \ldots, p - 1)$ and $D_p = D_0$. Let $I(0, 2\pi/p)$ be the closed domain bounded by two half lines $\theta = 0$ and $\theta = 2\pi/p$ in the polar coordinate system. We may assume that $D_0 = D \cap I(0, 2\pi/p)$. Let $A_1, A_2, \ldots, A_l$ be the points of intersection of $D_0$ and the line $\theta = 0$ and let $\zeta(A_i) = B_i (i = 1, 2, \ldots, l)$. By joining $A_i$ and $B_i$ on $\mathbb{R}^2 \setminus I(0, 2\pi/p)$ by circle $C_i$ centered 0, we obtain a diagram $D_s$ of the factor link $L_s$. For example, see Fig. 9. For simplicity, we write $D_s = D/\zeta$. We note that the rotation $\zeta : \mathbb{R}^2 \to \mathbb{R}^2$ maps $D$ onto itself preserving the sign of each crossing. If $s$ is a state in $\mathcal{S}(D)$, then either $\zeta(s) \neq s$ or $\zeta(s) = s$.

If $\zeta(s) \neq s$, then $s$, $\zeta(s)$, $\zeta^2(s)$, $\ldots$, $\zeta^{p-1}(s)$ are all distinct. Since any two of these are isomorphic, we have $p$ identical terms in $H(D, g)$, and they vanish by reducing modulo $p$.

If $\zeta(s) = s$, then $s$ defines a unique quotient state $s_a (= s/\zeta)$. Let $\alpha$ and $\alpha_a$ be the terms in $H(D, g)$ and $H(D_a, g)$ which are associated with $s$ and $s_a$, respectively. Since $\sum_{C_i(D,s)} \text{sign}(c) = p \cdot \sum_{C_i(D,s_a)} \text{sign}(c)$ and $\sum_{C_i(D,s)} \text{sign}(c) = p \cdot \sum_{C_i(D_a,s_a)} \text{sign}(c)$, we have

$$P(D; s) = p \cdot P(D_a; s_a).$$

Then we have that

$$\alpha = A^{p-P(D_a,s_a)} d^{p(s)-1} \langle s \rangle, \quad \alpha_a = A^{P(D_a,s_a)} d^{p(s)-1} \langle s_a \rangle.$$
We will compare $\mu(s) - 1$ and $\mu(s_s) - 1$. Let $G = \{id, \zeta, \ldots, \zeta^{p-1}\}$ and $C = \{C \mid C$ is a component of $s\}$, where $id$ is the identity of $\mathbb{R}^2$. Then $G$ acts on $C$ by $\zeta^i \cdot C = \zeta^i(C)$. We put $C_G = \{C \in C \mid gC = C, \forall g \in G\}$ and $C/G = \{G(C) \mid G(C)$ is the orbit of $C \in C\}$. For a set $S$, we denote by $|S|$ the number of elements in $S$. If $\zeta^i(C) = C$ for some $i$ ($1 \leq i \leq p - 1$), then $\zeta^i(C) = C$ for all $j$ because $p$ is prime. Thus $|G(C)| = p$ or 1. We note that $|G(C)| = 1$ if and only if $C \in C_G$. Since $\mu(s_s) = |C/G|$, we calculate that

$$
(3.2) \quad \mu(s) = |C| = |C_G| + p(|C/G| - |C_G|) = p \cdot \mu(s_s) - (p - 1)|C_G|.
$$

Since $\mu(s) - 1 = p(\mu(s_s) - 1) - (p - 1)(|C_G| - 1)$, we have that

$$
(3.3) \quad d^{\mu(s) - 1} = d^{p(\mu(s_s) - 1)} \mod (d^{p-1} - 1).
$$

By Theorem 3.1 and (3.2), it follows that

$$
(3.4) \quad 2^{-\mu(s)} \equiv 2^{p(-\mu(s_s))} \mod p.
$$

Let $f$ be a weight map of $s$. We define a weight map $\zeta(f)$ of $s$ by, for each edge $e$ of $s$,

$$
\zeta(f)(e) = f(e') \quad \text{whenever} \quad \zeta(e') = e.
$$

If $\zeta(f) \neq f$, then $f, \zeta(f), \ldots, \zeta^{p-1}(f)$ are all distinct but $\hat{s}_f, \hat{s}_{\zeta(f)}, \ldots, \hat{s}_{\zeta^{p-1}(f)}$ are equivalent. Thus $w(\hat{s}_f) = w(\hat{s}_{\zeta(f)}) = \cdots = w(\hat{s}_{\zeta^{p-1}(f)})$. If $\zeta(f) = f$, then $f$ defines a unique weight map $f_s = f/\zeta$ of $s_s$. Let $WD(s)$ denote the set of weighted diagram of $s$, that is, $WD(s) = \{s_f \mid f \in WM(s)\}$. Then $G$ acts on $WD(s)$ by

$$
\zeta(s_f) = s_{\zeta(f)}.
$$

We can put that $WD(s) = \{s_{f_1}, s_{f_2}, \ldots, s_{f_n}\} \cup \{s_{f_{1,0}}, s_{f_{1,1}}, \ldots, s_{f_{1,p-1}}\} \cup \cdots \cup \{s_{f_{m,0}}, s_{f_{m,1}}, \ldots, s_{f_{m,p-1}}\}$ where $\zeta(s_{f_i}) = s_{f_i}$ for all $i$ ($1 \leq i \leq m$) and $f_{j,k} = \zeta^k(f_{j,0})$ for each $k = 1, 2, \ldots, p - 1, j = 1, 2, \ldots, n$. We set that $w_i = w(\hat{s}_{f_i})$ and $w_{j,k} = w(\hat{s}_{f_{j,k}})$ for each $i = 1, 2, \ldots, m, k = 1, 2, \ldots, p - 1, j = 1, 2, \ldots, n$ and set $w^*_i = w(s_{f_{(i,0)}})$ for each $i = 1, 2, \ldots, m$. For each $i = 1, 2, \ldots, m$, we have

$$
(3.5) \quad w_i = p \cdot w_i^*.
$$

For any map $g : \mathbb{Z} \to \mathbb{Z}[h^{\pm 1}]$, we have that

$$
\langle s_s \rangle = 2^{-\mu(s_s)}[g(w_1^*) + \cdots + g(w_m^*)]
$$

and

$$
\langle s \rangle = 2^{-\mu(s)}[g(w_1) + \cdots + g(w_m) + p \cdot g(w_{1,0}) + \cdots + p \cdot g(w_{m,0})].
$$
By (3.4) and (3.5), it follows that

$$\langle s \rangle \equiv 2^{p(-\mu(s))}[g(p \cdot w_i^p) + \cdots + g(p \cdot w_m^p)] \mod p.$$  

(1) If \( g: (Z, +) \to (\mathbb{Z}[h^{\pm 1}], \cdot) \) is a homomorphism, then

$$\langle s \rangle^p = 2^{p(-\mu(s))}[g(w_1^p) + \cdots + g(w_m^p)]^p$$

$$\equiv 2^{p(-\mu(s))}[g(w_1^p)^p + \cdots + g(w_m^p)^p] \mod p$$

$$\equiv 2^{p(-\mu(s))}[g(p \cdot w_1^p) + \cdots + g(p \cdot w_m^p)] \mod p$$

$$\equiv \langle s \rangle \mod p.$$  

By (3.1), (3.3) and (3.6), it follows that

$$\alpha_s^p \equiv \alpha \mod (p, (-A^2 - A^{-2})^{p-1} - 1).$$

Hence we have

$$H(D, g) \equiv [H(D_s, g)]^p \mod (p, (-A^2 - A^{-2})^{p-1} - 1).$$

Since \( w(D) = p \cdot w(D_s), (-A^3)^{-w(D')} = [(-A^3)^{-w(D'_s)}]^p \). Therefore we have

$$R(L, g) \equiv [R(L_s, g)]^p \mod (p, (-A^2 - A^{-2})^{p-1} - 1).$$

(2) Suppose that \( g: \mathbb{Z} \to \mathbb{Z}[h^{\pm 1}] \) is defined by \( g(n) = h^{(1-(-1)^n)/2} \). Since \( w_i = p \cdot w_i^p \) and \( p \) is an odd prime, \( w_i \) and \( w_i^p \) have the same parity and hence \( g(w_i) = g(w_i^p) \). Since \( g(w_i^p) \) is either \( h \) or 1, we have

$$g(w_i^p)^p \equiv g(w_i^p) \mod (h^{p-1} - 1).$$

Then we know that

$$\langle s \rangle^p = 2^{p(-\mu(s))}[g(w_1^p) + \cdots + g(w_m^p)]^p$$

$$\equiv 2^{p(-\mu(s))}[g(w_1^p)^p + \cdots + g(w_m^p)^p] \mod p$$

$$\equiv 2^{p(-\mu(s))}[g(p \cdot w_1^p) + \cdots + g(p \cdot w_m^p)] \mod (p, h^{p-1} - 1)$$

$$\equiv \langle s \rangle \mod (p, h^{p-1} - 1).$$

By (3.1), (3.3) and (3.7), it follows that

$$\alpha_s^p \equiv \alpha \mod (p, (-A^2 - A^{-2})^{p-1} - 1, h^{p-1} - 1).$$
Thus we have
\[ H(D, g) \equiv [H(D_s, g)]^p \mod (p, (-A^2 - A^{-2})^{p-1} - 1, h^{p-1} - 1). \]
Hence we get
\[ R(L, g) \equiv [R(L_s, g)]^p \mod (p, (-A^2 - A^{-2})^{p-1} - 1, h^{p-1} - 1). \]
This completes the proof.

**Example 3.3.** Let \( L_1 \) be a virtual knot as in Fig. 10. Then the Jones-Kauffman polynomial of \( L_1 \) is equal to 1 [8]. Let \( g: \mathbb{Z} \to \mathbb{Z}[h^{\pm 1}] \) be a map given by \( g(n) = h^n \). It is known [8] that
\[ R_{L_1}^g = \frac{1}{2}(1 + A^{-8}) + \frac{1}{4}(1 - A^{-8})(h^2 + h^{-2}). \]
Suppose that \( L_1 \) has period 3. From Theorem 3.2, it follows that
\[ \frac{1}{4}(1 - A^{-8}) \equiv 0 \mod (3, (-A^2 - A^{-2})^2 - 1). \]
Since \((-A^2 - A^{-2})^2 - 1 = A^4 + 1 + A^{-4} = A^{-4}(A^8 + A^4 + 1) \) and \( 1 - A^{-8} = (A^{-4} - A^{-8})(A^8 + A^4 + 1) + (1 - A^4), \)
\[ \frac{1}{4}(1 - A^{-8}) \equiv 1 + 2A^4 \mod (3, A^8 + A^4 + 1). \]
Let \( \mathcal{I} \) be the ideal of \( \mathbb{Z}[2^{-1}, A^{\pm 1}] \) generated by 3. We note that the quotient ring of \( \mathbb{Z}[2^{-1}, A^{\pm 1}] \) by \( \mathcal{I} \) is isomorphic to the ring \( \mathbb{Z}_3[A^{\pm 1}] \). So it is not true that \( 1 + 2A^4 \equiv 0 \mod (3, A^8 + A^4 + 1) \). Hence \( L_1 \) does not have period 3.

**Theorem 3.4.** Let \( p \) be a prime and \( L \) a virtual link that has period \( p' \) (\( r \geq 1 \)). Let \( g \) be a map from \( \mathbb{Z} \) to \( \mathbb{Z}[h^{\pm 1}] \). Then
\[ R_{L,g}(A, h) \equiv R_{L,g}(A^{-1}, h) \mod (p, A^{p'} - 1). \]
Proof. Let $D$ be a virtual link diagram of $L$ in $\mathbb{R}^2 \setminus \{0\}$ that invariant under the rotation $\zeta$ of $\mathbb{R}^2$ about the origin $0$ through $2\pi/p'$ and $D_s = D/\zeta$. Let $s$ be a state of $D$.

If $\zeta(s) \neq s$, then there exist $p^n$ distinct but equivalent states $s, \zeta(s), \ldots, \zeta^{p^n-1}(s)$ for some $n$ ($1 \leq n \leq r$). Contribution of these states to the polynomial vanishes by reducing modulo $p$.

If $\zeta(s) = s$, then $s$ defines a unique quotient states $s_s (= s/\zeta)$. Since $P(D; s) = p' \cdot P(D_s; s_s)$, we get

$$A^{P(D; s)} = A^{p' \cdot P(D_s; s_s)} \equiv 1 \mod (A^{p'} - 1).$$

Since $d = -A^2 - A^{-2}$ is symmetric and $\langle s \rangle \in \mathbb{Z}[2^{-1}, h^{\pm 1}]$, we obtain

$$H_{D,g}(A, h) \equiv H_{D,g}(A^{-1}, h) \mod (p, A^{p'} - 1).$$

Since $w(D) = p' \cdot w(D_s)$,

$$(-A^3)^{-w(D)} \equiv (-A^{-3})^{-w(D)} \mod (A^{p'} - 1).$$

Hence we have

$$R_{L,g}(A, h) \equiv R_{L,g}(A^{-1}, h) \mod (p, A^{p'} - 1).$$

This completes the proof.

\begin{corollary}
Let $p$ be a prime and $L$ a virtual link that has period $p'$. Let $g: (\mathbb{Z}, +) \rightarrow (\mathbb{Z}[h^{\pm 1}], \cdot)$ be a homomorphism. Then

$$R_{L,g}(A, h) \equiv R_{L,g}(A^{-1}, h^{-1}) \mod (p, A^{p'} - 1).$$

\end{corollary}

Proof. Let $D$ be a virtual link diagram of $L$ in $\mathbb{R}^2 \setminus \{0\}$ that invariant under the rotation $\zeta$ of $\mathbb{R}^2$ about the origin $0$ through $2\pi/p'$ and $D_s = D/\zeta$.

Let $s$ be a state of $D$. Since $g: (\mathbb{Z}, +) \rightarrow (\mathbb{Z}[h^{\pm 1}], \cdot)$ is a homomorphism, we have $\langle s \rangle(h) = \langle s \rangle(h^{-1})$ by Lemma 2.1. By the similar argument to Theorem 3.4, we have

$$R_{L,g}(A, h) \equiv R_{L,g}(A^{-1}, h^{-1}) \mod (p, A^{p'} - 1).$$

This completes the proof.

\begin{example}
Let $L_1$ be a virtual knot as in Fig. 10. Then

$$R_{L_1}^g(A, h) - R_{L_1}^g(A^{-1}, h) = \frac{1}{2}(A^{-8} - A^8) + \frac{1}{4}(A^8 - A^{-8})(h^2 + h^{-2}).$$

\end{example}
We observe that
\[ \frac{1}{2}(A^{-8} - A^8) \equiv 2A + A^2 \not\equiv 0 \mod (3, A^3 - 1). \]

Hence this is an another proof to show that \( L_1 \) does not have period 3.

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**References**

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