Egalitarian equivalence and implementability imply equal division*

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Abstract

This paper shows that any Nash-implementable social choice correspondence satisfying egalitarian equivalence must select equal division when the number of individuals is greater than the number of goods. It also shows that any strategy-proof and non-bossy social choice function satisfying egalitarian equivalence must select equal division, under the same assumption.

JEL Classification: D61, D63, D71, D82

Keywords: Egalitarian equivalence, Nash implementability, strategy-proofness, non-bossiness

1 Introduction

Egalitarian equivalence, a concept of equity proposed by Pazner and Schmeidler [23], requires that an allocation should be associated with some common reference bundle which is equally preferable as his/her own consumption bundle for each individual. It is seen as an ordinal counterpart of the concept of welfare egalitarianism and allows a wider set of allocations than equal division.[1] It is compatible with efficiency, in contrast to equal division that is typically inefficient. It is compatible with efficiency even in economies in which envy-freeness, another prominent concept of equity (Foley [10]), fails to be compatible with efficiency (Pazner and Schmeidler [22], Varian [32], Maniquet [14]), as shown for example in Fleurbaey and Maniquet [8, 9]. Moreover, it is compatible with or characterized by various solidarity conditions (Moulin [16], Dutta and Vohra [6], Fleurbaey and Maniquet [8, 9]).

An egalitarian-equivalent solution is manipulable in general, however. To illustrate, consider that there are two individuals, A and B, and two private goods 1 and 2. Consider for simplicity that reference bundles are taken from the 45-degree line $\{(v,v)\in\mathbb{R}^2_+ : v\geq 0\}$, and we represent each individual i's preference by $u_i(x_{i1},x_{i2})=v_i$ where $(x_{i1},x_{i2})\sim(v_i,v_i)$, for $i=A,B$. Then, maximizing the egalitarian social welfare function $\min\{u_A,u_B\}$ yields an egalitarian-equivalent and

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* I thank François Maniquet for comments to the previous version.

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[1] See Moulin [17, 18] and Thomson [31] for comprehensive illustrations.

efficient allocation, uniquely up to Pareto-indifference.[2]

Suppose that A and B have preferences with constant marginal rates of substitution, where A's MRS is 2 and B's MRS is 1. Then A's preference is represented in the form $u_A(x_{A1}, x_{A2}) = \frac{2x_{A1}+x_{A2}}{3}$, B's preference is represented in the form $u_B(x_{B1}, x_{B2}) = \frac{x_{B1}+x_{B2}}{2}$. Suppose there are 2 units of Good 1 and 1 unit of Good 2. Then the solution yields $(x_{A1}, x_{A2}) = \left(\frac{9}{7}, 0\right)$ and $(x_{B1}, x_{B2}) = \left(\frac{5}{7}, 1\right)$.

Now suppose that A misreports his/her preference, saying that MRS is still constant but it is $1+\epsilon$, where ϵ is positive but close to 0. Then the representation to be used is changed to $\widetilde{u}_A(x_{A1}, x_{A2}) = \frac{(1+\epsilon)x_{A1}+x_{A2}}{2+\epsilon}$, and the allocation is changed to $(x_{A1}, x_{A2}) = \left(\frac{6+3\epsilon}{4+3\epsilon}, 0\right)$ and $(x_{B1}, x_{B2}) = \left(\frac{2-3\epsilon}{4+3\epsilon}, 1\right)$. Hence A gains by misreporting his/her preference.

One might think that the above problem is rather due to efficiency, since we know that dictatorship or something close is the only solution satisfying strategy-proofness and efficiency (see Gibbard [11] and Satterthwaite [24] for the results on the abstract domain, and Zhou [34], Serizawa [26], Serizawa and Weymark [27] for the results in exchange economies). If we give up efficiency, there is a straightforward solution that is strategy-proof (also Nash-implementable) and egalitarian-equivalent: equal division.

But can we have anything nicer, because equal division is typically severely inefficient? In other words, is there an implementable/non-manipulable selection of egalitarian-equivalent allocations that still leaves a room for other nice features? The current paper examines this question.[3]

We show that any Nash-implementable social choice correspondence satisfying egalitarian equivalence must select equal division, when the number of individuals is greater than the number of goods. Also, we show that any strategy-proof and non-bossy social choice function satisfying egalitarian equivalence must select equal division, under the same assumption.[4] This contrasts to and is parallel to the argument by Fleurbaey and Maniquet [7], who showed that envy-freeness is rather a consequence of implementability plus a mild horizontal equity condition. Also it is parallel to the finding by Thomson [29] that an *equilibrium* allocation in the game of reporting preference under egalitarian equivalence must weakly Pareto-dominate the equal division.

Our results tell that we must accept severe inefficiency if we insist on egalitarian equivalence under the condition of implementability. Or, our results explain at least why a multi-stage mechanism

[2] This is a special case of r-egalitrian-equivalent solution, which is a refinement of egalitarian equivalence, in which reference points are taken from the line spanned by vector r. It allows us to avoid the problem with general egalitarian-equivalent and efficient allocation that somebody's consumption bundle is dominated by another's. See Thomson [31].

[3] Thomson [29] instead considered a *positive* question about what the equilibrium consequence of adopting a particular class of equity criteria including egalitarian equivalence is, by considering a game of reporting preferences in which the agents play a generalized version of Nash equilibrium.

[4] In the domain of allocating indivisible objects with monetary transfers, Ohseto [21] shows that there is no strategy-proof, budgets-balanced and egalitarian-equivalent mechanism, and also that there is no Nash-implementable and egalitarian-equivalent mechanism, which may be attributed to the fact that there is no such thing as equal division in that domain. In this sense we provide a tighter characterization in the domain of exchange economies.

inducing subgame-perfection (Moore and Repullo [19]) is necessary for implementing egalitarian-equivalent and efficient allocations, as in Crawford [3] and Demange [5], or why a specific domain has to be considered for implementation with normal-form mechanisms as in Chun and Mutuswami [2], Yengin [33].

Finally, let us note two reservations. One is that we assume the number of individuals is greater than the number of goods. It is left as an open question whether we still obtain the same theorem or there is a counterexample when the commodity space is richer compared to the number of individuals. Second is that we assume the domain of *all* the complete, transitive, continuous, convex and *strongly monotone* preferences. It excludes Leontief-type preferences and includes all the strongly monotone preferences arbitrarily closer to them. On the other hand, there are studies since Nicoló [20], in which positive results are obtained in the restricted domain of Leontief-type preferences.Ghodsi, Zaharia, Hindman, Konwinski, Shenker and Stoica [12], Li and Xue [13] show that egalitarian equivalence and efficiency are compatible with implementability in the domain of Leontief-type preferences. Note also that such possibility results are not obtained in the domain of linear preferences (Schummer [25]), which suggests the importance of Leontief-type preferences. It is left as an open question whether we still obtain the same theorem or there is a counterexample when we consider the domain of all the complete, transitive, continuous, convex and weakly monotone preferences.

2 Nash-implementable social choice correspondences

There are l private goods in the economy, and let $\Omega \in \mathbb{R}^l_{++}$ be the social endowment.[5]

There are n individuals. Each individual's consumption set is the non-negative quadrant $\mathbb{R}^l_+$. Let $\mathcal{R}$ be the set of preference relations over $\mathbb{R}^l_+$, which are complete, transitive, continuous, convex and strongly monotone, although further domain restriction is possible following Fleurbaey and Maniquet [7].[6]

For each $i \in \{1, \cdots, n\}$, $R_i \in \mathcal{R}$ and $x_i \in \mathbb{R}^l_+$, let $U(x_i, R_i) = \{y_i \in \mathbb{R}^l_+ : y_i R_i x_i\}$, $L(x_i, R_i) = \{y_i \in \mathbb{R}^l_+ : x_i R_i y_i\}$ and $I(x_i, R_i) = \{y_i \in \mathbb{R}^l_+ : x_i I_i y_i\}$.

Let $F = \left\{x \in \mathbb{R}^{nl}_+ : \sum_{i=1}^n x_i \leq \Omega\right\}$ be the set of feasible allocations. Then, let $\Phi : \mathcal{R}^n \to 2^F \setminus \{\emptyset\}$ denote a social choice correspondence.

The primary normative criterion we consider is Egalitarian Equivalence (Pazner and Schmeidler [23]).

Egalitarian Equivalence: For all $R \in \mathcal{R}^n$ and $x \in \Phi(R)$ there is $e \in \mathbb{R}^l_+$ such that $x_i I_i e$ for all $i \in \{1, \cdots, n\}$.

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5 Our arguments can be readily extended to production economies and economies with public goods, as long as we can define equal division as an element of feasible set.

6 Preference relation R over $\mathbb{R}^l_+$ is strongly monotone if $x_k \geq y_k$ for all $k \in \{1, \cdots, l\}$ and $x_k > y_k$ for at least one k imply xPy.

It is immediate to see that Egalitarian Equivalence is stronger than Equal Treatment of Equals, which states that individuals with identical preferences must be treated equally in welfare.

Equal Treatment of Equals: For all $R \in \mathcal{R}^n$ and $i, j \in \{1, \cdots, n\}$, if $R_i = R_j$ then for all $x \in \Phi(R)$ it holds $x_i I_i x_j$ and $x_j I_j x_i$.

Envy-Freeness is another prominent concept of equity.

Envy-Freeness: For all $R \in \mathcal{R}^n$ and $x \in \Phi(R)$ it holds $x_i R_i x_j$ for all $i, j \in \{1, \cdots, n\}$.

We examine the consequence of imposing Egalitarian Equivalence under an implementability condition. Here we consider Monotonicity (Maskin [15]), which is necessary and sufficient for implementability in Nash equilibria in the current economic environment.

Monotonicity: For all $R \in \mathcal{R}^n$, $x \in \Phi(R)$ and $\widetilde{R} \in \mathcal{R}^n$, if $L(x_i, \widetilde{R}_i) \supset L(x_i, R_i)$ for all $i \in \{1, \cdots, n\}$ then $x \in \Phi(\widetilde{R})$.

The proposition below has been shown by Fleurbaey and Maniquet [7].

Proposition 1 If a social choice correspondence satisfies Monotonicity and Equal Treatment of Equals, then it satisfies Envy-Freeness.

Since Egalitarian Equivalence immediately implies Equal Treatment of Equals, we obtain the following proposition.[7]

Proposition 2 If a social choice correspondence satisfies Monotonicity and Egalitarian Equivalence, then it satisfies Envy-Freeness.

A social choice correspondence Φ is said to select equal division if for all $R \in \mathcal{R}^n$ and $x \in \Phi(R)$ there is $e \in \mathbb{R}^l_+$ with $ne \leq \Omega$ such that $x_i = e$ for all $i \in \{1, \cdots, n\}$.

Theorem 1 Suppose $n \geq l + 1$. Then, if a social choice correspondence satisfies Monotonicity and Egalitarian Equivalence then it must select equal division.

Proof. Let Φ be any social choice correspondence satisfying Monotonicity and Egalitarian Equivalence.

7 This contrasts to the result by Thomson [30] that egalitarian equivalence and envy-freeness are generally incompatible in the problem of allocating indivisible objects with monetary transfers.

Pick any $R \in \mathcal{R}^n$. Suppose that $x \in \Phi(R)$ is not an equal division.

Case 1: Suppose there exist j, k with $x_j = x_k$. Since x is not equal division, there is $h \neq j, k$ such that $x_h \neq x_j = x_k$. Note that since x is envy-free it holds $x_h R_h x_j = x_k$.

Then one can take $\widetilde{R}_k$ so that $L(x_k, \widetilde{R}_k) \supset L(x_k, R_k)$ and $I(x_j, R_j) \cap I(x_k, \widetilde{R}_k) = \{x_j\} = \{x_k\}$, by taking $\widetilde{R}_k$ so that $U(x_k, \widetilde{R}_k)$ is sufficiently close to $\{x_k\} + \mathbb{R}^l_+$.

By Monotonicity, we have $x \in \Phi(\widetilde{R}_k, R_{-k})$.

Let $\widetilde{R}_h \in \mathcal{R}$ be such that $L(x_h, \widetilde{R}_h) \supset L(x_h, R_h)$ and $x_h \widetilde{P}_h x_j = x_k$.

By Monotonicity, we have $x \in \Phi(\widetilde{R}_h, \widetilde{R}_k, R_{-hk})$.

Since $x_h \widetilde{P}_h x_j = x_k$, x cannot be an egalitarian equivalent allocation since $x_j = x_k$ is the only possible candidate for reference point.

Case 2: Suppose we have $x_j \neq x_k$ for all j, k.

Then, since x is envy-free and preferences are strongly monotone, for all j, k it holds $x_j - x_k$, $x_k - x_j \notin \mathbb{R}^l_+$.

Since $n \geq l + 1$, it holds $\cap_{i=1}^n \left(\{x_i\} + \partial\mathbb{R}^l_+\right) = \emptyset$.

For each $i \in \{1, \cdots, n\}$, let $\widetilde{R}_i$ be such that $L(x_i, \widetilde{R}_i) \supset L(x_i, R_i)$ and $I(x_i, \widetilde{R}_i)$ is sufficiently close to $\{x_i\} + \partial\mathbb{R}^l_+$.

By Monotonicity, $x \in \Phi(\widetilde{R})$, but it cannot be an egalitarian equivalent allocation since $\cap_{i=1}^n I(x_i, \widetilde{R}_i) = \emptyset$.

Remark 1 When $n = 2$, Envy-Freeness implies Egalitarian Equivalence, because if an allocation is not egalitarian-equivalent it must be the case that one's corresponding indifference surface is strictly above the other's corresponding indifference surface, which must cause an envy. Since the correspondence that selects the entire set of envy-free allocations is monotonic, the above theorem does not hold when $n = 2$.

We do not have a proof of the theorem or a counterexample, unfortunately, for the case $3 \leq n \leq l$, in which the commodity space is richer compared to the set of individuals. We leave it as an open question.

The above result implies that under the restriction of Nash-implementability any social choice correspondence satisfying egalitarian equivalence is Pareto-dominated by another one, for example the (constrained) Walrasian solution in which the initial endowments are taken to be the equal division $\left(\frac{\Omega}{n}, \cdots, \frac{\Omega}{n}\right)$, regardless of which allocation is selected from the prescribed set.

3 Strategy-proof and non-bossy social choice functions

Let $\phi : \mathcal{R}^n \to F$ denote a social choice function.

We consider the following two non-manipulability conditions. One is that reporting own true preference truthfully is a dominant strategy under the direct mechanism.

Strategy-Proofness: For all $R \in \mathcal{R}^n$, $i \in \{1, \cdots, n\}$ and $R'_i \in \mathcal{R}$, it holds $\phi_i(R) R_i \phi_i(R'_i, R_{-i})$.

The other is that if one cannot change own consumption allocation by reporting different preference of own, it cannot change others' allocations either, since otherwise another individual has an incentive to bribe him to report a different preference.

Non-Bossiness: For all $R \in \mathcal{R}^n$, $i \in \{1, \cdots, n\}$ and $R'_i \in \mathcal{R}$, if $\phi_i(R) = \phi_i(R'_i, R_{-i})$ then $\phi(R) = \phi(R'_i, R_{-i})$.

The proposition below has been shown by Fleurbaey and Maniquet [7].

Proposition 3 If a social choice function satisfies Strategy-Proofness, Non-Bossiness and Equal Treatment of Equals, then it satisfies Envy-Freeness.

Since Egalitarian Equivalence immediately implies Equal Treatment of Equals, we obtain the following proposition.

Proposition 4 If a social choice correspondence satisfies Strategy-Proofness, Non-Bossiness and Egalitarian Equivalence, then it satisfies Envy-Freeness.

Theorem 2 Suppose $n \geq l + 1$. Then, if a social choice function satisfies Strategy-Proofness, Non-Bossiness and Egalitarian Equivalence then it must select equal division.

Proof. Let ϕ be any social choice function satisfying Strategy-Proofness, Non-Bossiness and Egalitarian Equivalence.

By following the standard argument (see Dasgupta, Hammond and Maskin [4] for example), one can establish a version of monotonicity property: For all $R \in \mathcal{R}^n$ and $\widetilde{R} \in \mathcal{R}^n$, if $L(\phi_i(R), R_i) \cap U(\phi_i(R), \widetilde{R}_i) = \{\phi_i(R)\}$ for all $i \in \{1, \cdots, n\}$, then $\phi(\widetilde{R}) = \phi(R)$.

Under the monotonicity property, we can prove the assertion in the same way as in the previous theorem.

Remark 2 When $n = 2$, Envy-Freeness implies Egalitarian Equivalence. Hence any strategy-proof rule satisfying Envy-Freeness serves as a counterexample to the theorem, as Non-Bossiness is vacuously met as far as resource constraint is met with equality. For example, when we draw a line passing through the equal division point which lies on a hyperplane with some positive normal vector, the individuals have single-peaked preferences over the line and we can apply the uniform rule there (Sprumont [28]).

We do not have a proof of the theorem or a counterexample, unfortunately, for the case $3 \leq n \leq l$ in which the commodity space is richer compared to the number of goods.

The above result implies that under the restriction of strategy-proofness and non-bossiness any social choice function satisfying egalitarian equivalence is Pareto-dominated by another one, for example the fixed proportion trading away from the equal division (Barbera and Jackson [1]).

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