An infinite-dimensional extension of the von Neumann model*

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Abstract

The equilibrium existence theorem of Kemeny et al. (1956) treats one of the most general extensions of the von Neumann model. Their method, however, crucially depends on the theorems of finite-dimensional dual-linear systems, and it has great difficulty in generalization for infinite-dimensional situations. In this paper, the equilibrium existence theorem is extended to cases where the index sets of commodities and activities are not finite but compact metric spaces having a certain kind of countable dense subset. This extension enables us to treat models with continuously many kinds of activities and countably infinite commodities, which is essentially equivalent to considering continuous production functions homogeneous of degree 1 and differentiated commodities including intermediate goods for capital accumulation.

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1 Introduction

The model of von Neumann (1937) brought remarkable progress to economic theories such as input-output analysis, fixed-point arguments ensuring the existence of general economic equilibrium, the minimax saddle-point problem for a zero-sum matrix game, theory of capital accumulation, and fundamental concepts and settings for balanced growth theory. In this model, every action in the economy is treated as a process called "input and output" with an elementary positive time length. As Morishima studied and strongly emphasized in his books and papers (see, for example, Morishima1964; Morishima1969), the von Neumann model gives us a general perspective in describing capital goods under multi-sectoral economic growth and capital accumulation settings. By treating *capital goods at different stages of wear and tear as different goods*, an economic situation

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can be observed where the length of time under which a capital good stays in the same condition depends on the *intensity* of its *use*. Morishima called this the *von Neumann revolution*.

From a purely mathematical view, extensions of von Neumann's results are abundant in the literature, such as fixed-point theorems, minimax arguments, and saddle-point problems. Above all, the models based on the settings of Kemeny et al. (1956) are the most general kind of extension of the von Neumann model. However, the equilibrium existence theorem of Kemeny et al. (1956) crucially depends on the finite-dimensional matrix concepts and tools like the central solutions and theorems on finite-dimensional dual-linear systems (see, e.g., Tucker1955).[1]

In this paper, the equilibrium existence theorem of Kemeny et al. (1956) is extended to the case with continuously many kinds of activities and countably infinite commodities under a boundary condition on a set of activity or commodity indices.[2] The theorem enables us to treat models with infinitely many kinds of activities and commodities, which is essentially equivalent to considering differentiated commodities, including intermediate goods for capital accumulation or continuous production functions homogeneous of degree 1.

2 Extension of von Neumann Model

Denote by N the set of non-negative integers and by R the set of real numbers. For each $n \in N$, denote by R^n the n-dimensional Euclidean space. On R^n, consider three types of order relations, $\leqq$, $<$, and $\ll$, such that for each $x = (x_1, \dots, x_n)$ and $y = (y_1, \dots, y_n)$ in R^n, we have $(x \leqq y) \iff (\forall i = 1, 2, \dots, n,\ x_i \leqq y_i)$, $(x < y) \iff (x \leqq y \text{ and } x \neq y)$, and $(x \ll y) \iff (\forall i = 1, 2, \dots, n,\ x_i < y_i)$. Moreover, by R^n_+ and R^n_{++}, we denote $\{x \in R^n \,|\, 0 \leqq x\}$ and $\{x \in R^n \,|\, 0 \ll x\}$, respectively.

Let I be a *compact metric space* and J be a *countable compact metric space*.[3] Set $I = \{i \mid i \in I\}$ is the index set of *production processes*, and $J = \{j \mid j \in J\}$ is the index set of *goods*. Function $A(i, j)$ represents *input* and $B(i, j)$ represents *output*. Both are assumed to be non-negative real-valued on $I \times J$. Variable x represents an *intensity of production*, a Borel probability measure on I, and y represents a *level of prices*, which is also taken as a Borel probability measure on J.

Two variables, $\alpha = (1 + \mathit{growth\ rate})$ and $\beta = (1 + \mathit{interest\ rate})$, are used to look for a solution (x, y, α, β) satisfying the following system of inequalities.

$$\int (B(i,j) - \alpha A(i,j))dx(i) = \langle x, B - \alpha A \rangle \geqq 0, \tag{1}$$

$$\int (B(i,j) - \beta A(i,j))dy(j) = \langle B - \beta A, y \rangle \leqq 0, \tag{2}$$

[1] Nikaido (1968) proved the existence theorem of Kemeny et al. (1956) under the settings of dual-linear systems in finite-dimensional spaces without using the *central solution* or any game theoretic term. The linear programming in infinite-dimensional spaces was discussed by Anderson and Nash (1987) in detail.

[2] There is an abundant literature on the extension of the von Neuaman model with coordinate-free input-output structures. See, for example, Fujimoto (1986). For models with non coordinate-free matrices, see Urai and Kageyama (2012).

[3] A compact metric space has a countable dense subset, so I and J are Polish spaces (separable metric spaces).

$$\iint (B(i,j) - \alpha A(i,j)) dx(i) dy(j) = \langle\langle x, B - \alpha A\rangle, y\rangle = 0, \tag{3}$$

$$\iint (B(i,j) - \beta A(i,j)) dy(j) dx(i) = \langle x, \langle B - \beta A, y\rangle\rangle = 0. \tag{4}$$

Equation (1) implies that at any time period, no more goods can be used than were produced during the preceding time period. Equation (2) shows that there exist no residual profits, i.e., no process can yield a return in the given time period greater than that which would have been obtained by the going interest rate. Equation (3) says that the prices of overproduced goods are zero, and (4) shows that the intensity of deficit processes must be zero.

In general, without further assumptions about $A(i,j)$ and $B(i,j)$, there is no solution satisfying these conditions. In von Neumann (1937), $A(i,j)$ and $B(i,j)$ are assumed to be finite matrices, and the following additional assumption is used.

$$a(i,j) + b(i,j) > 0,\ a(i,j) \in A(i,j) \text{ and } b(i,j) \in B(i,j),\ \text{for all } i \text{ and } j. \tag{5}$$

Intuitively, this means that every production process must consume or produce a positive amount of every good. As pointed out in Kemeny et al. (1956), this assumption is highly restrictive.[4] We use the following weaker and economically plausible conditions, which are based on the idea of Kemeny et al. (1956).

$$\iint B(i,j) dx(i) dy(j) > 0, \tag{6}$$

(i) every row of A has at least one positive entry,

(ii) every column of B has at least one positive entry.

The condition of equation (6) means that the total value of all goods produced in the economy must be positive. Assumption (i) implies that every process uses some inputs, and (ii) says that every good can be produced in at least one process. More precisely, since I and J are metric spaces, we must write conditions (i) and (ii) as follows.

(i′) $\forall i \in I, \exists j \in J, A(i,j) > 0$,

(ii′) $\forall j \in J, \exists i \in I, B(i,j) > 0$.

Note that here we use conditions (i) and (ii) with the meanings of (i′) and (ii′).

If (x, y, α, β) satisfies equations (1)–(6) with assumptions (i) and (ii), this will provide a solution to the economic model that holds in *every time period*. This is the *equilibrium* for the von Neumann Model.

Following the same approach in Kemeny et al. (1956; Lemma 1), we can show that $\alpha = \beta$ under our settings. Suppose that we have a solution (x, y, α, β) to (1)–(6). From (6) we see that $\iint B(i,j) dx(i) dy(j) > 0$, and thus by (3) and (4) we have

$$\iint B(i,j) dx(i) dy(j) = \alpha \iint A(i,j) dx(i) dy(j) = \beta \iint A(i,j) dy(j) dx(i) > 0. \tag{7}$$

4 See, e.g., Takayama (1985).

From the last equation, $\iint A(i,j)dy(j)dx(i) > 0$, and it follows that $\alpha = \beta = \frac{\iint B(i,j)dx(i)dy(j)}{\iint A(i,j)dx(i)dy(j)} > 0$. Hence, we need to look only for the solutions in which $\alpha = \beta$, i.e., the case where the *interest rate* is equal to the *growth rate*.

Since $\alpha = \beta$, we see that (3) and (4) become identical as the consequences of the conditions in equations (1) and (2). Thus, our equations can be written simply as

$$\int (B(i,j) - \alpha A(i,j))dx(i) = \langle x, B - \alpha A \rangle \geqq 0, \tag{8}$$

$$\int (B(i,j) - \alpha A(i,j))dy(j) = \langle B - \alpha A, y \rangle \leqq 0, \tag{9}$$

$$\iint B(i,j)dx(i)dy(j) > 0. \tag{10}$$

For technical reasons, we give the next operation to $B(i,j) - \alpha A(i,j)$. Divide $B(i,j) - \alpha A(i,j)$ by $1 + \alpha$ and define $p = \alpha/(1+\alpha)$ (note that $0 < p < 1$). Then, we obtain the function $(1-p)B(i,j)$ $+\, p(-A(i,j))$, which is a convex combination of B and $-A$. Let us define $M_p(i,j)$ as $M_p(i,j) = (1-p)B(i,j) + p(-A(i,j))$, and our system of equations becomes

$$\int M_p(i,j)dx(i) = \langle x, M_p \rangle \geqq 0, \tag{11}$$

$$\int M_p(i,j)dy(j) = \langle M_p, y \rangle \leqq 0, \tag{12}$$

$$\iint B(i,j)dx(i)dy(j) = \iint Bdydx > 0. \tag{13}$$

3 Infinite-Dimensional Zero-Sum Game

It is well known that we can interpret the equilibrium of the von Neumann Model as the *saddle-point solution* of a *zero-sum matrix game*. In this section, we extend the game theoretic concepts and method to our infinite-dimensional settings.

Lemma 1: *For continuous function M_p on $I \times J$, the extended zero-sum matrix game defined by $M_p(i,j)$ is strictly determined, and the value of game, $v(M_p)$, is a continuous function of p.*[5]

Proof: Let $M_p(i,j)$ be the real-valued continuous function on $I \times J$. We assume that x (resp. y) is a probability measure on the Borel σ-algebra $\mathfrak{B}_I$ (resp.$\mathfrak{B}_J$) of I (resp. J), and let $\mathfrak{M}(I)$ (resp. $\mathfrak{M}(J)$) be the whole space of the measure $x(i)$ (resp. $y(j)$). Here, $\mathfrak{M}(I)$ (resp. $\mathfrak{M}(J)$) becomes a compact

[5] Note that this result also implies the existence of the solution of such a zero-sum matrix game defined by $M_p(i,j)$ for each p. See also Miyazawa (1958) as a reference to the continuity of $v(M)$ with respect to p for a general version of the zero-sum minimax theorem.

metric space under the *topology of weak convergence*.[6] Then,

$$F(x) \stackrel{\text{def}}{=} \min_{y \in \mathfrak{M}(J)} \iint M_p(i,j)dx(i)dy(j) = \min_{j \in J} \int M_p(i,j)dx(i), \tag{14}$$

$$G(y) \stackrel{\text{def}}{=} \max_{x \in \mathfrak{M}(I)} \iint M_p(i,j)dy(j)dx(i) = \max_{i \in I} \int M_p(i,j)dy(j). \tag{15}$$

For (14), the existence of $\min_{y \in M(J)} \iint M_p(i,j)dx(i)dy(j)$ is assured by the weak compactness of $M(J)$ and continuity of $\int M_p(i,j)dx(i)$. Inequality $\min_{y \in \mathfrak{M}(J)} \iint M_p(i,j)dx(i)dy(j) \leqq \min_{j \in J} \int M_p(i,j)dx(i)$ holds, since $\{j\}$ is a Borel set for each $j \in J$. To show the contrary, let us define $\delta = \min_{j \in J} \int M_p(i,j)dx(i)$. We see that $\int M_p(i,j)dx(i) \geqq \delta$. Then, $\min_{y \in \mathfrak{M}(J)} \iint M_p(i,j)dx(i)dy(j) \geqq \min_{j \in J} \int M_p(i,j)dx(i)$ follows from the fact that $\iint M_p(i,j)dx(i)dy(j) \geqq \int \mathrm{Const}_\delta dy(j) = \delta$ for all $y(j) \in \mathfrak{M}(J)$.[7] In the same way, for (15), the existence of $\max_{x \in \mathfrak{M}(I)} \iint M_p(i,j)dy(j)dx(i)$ is assured by the weak compactness of $\mathfrak{M}(I)$ and the continuity of $\int M_p(i,j)dy(j)$. Here, $\max_{x \in \mathfrak{M}(I)} \iint M_p(i,j)dy(j)dx(i) \geqq \max_{i \in I} \int M_p(i,j)dy(j)$ holds, since $\{i\}$ is a Borel set for each $i \in I$. To show the contrary, let $\delta' = \max_{i \in I} \int M_p(i,j)dy(j)$. Then, $\int M_p(i,j)dy(j) \leqq \delta'$ and $\max_{x \in \mathfrak{M}(I)} \iint M_p(i,j)dy(j)dx(i) \leqq \max_{i \in I} \int M_p(i,j)dy(j)$ follow from the fact that $\iint M_p(i,j)dy(j)dx(i) \leqq \int \mathrm{Const}_{\delta'} dx(i) = \delta'$ for all $x \in \mathfrak{M}(I)$.

Furthermore, we define

$$v_1(M_p) \stackrel{\text{def}}{=} \max_{x \in \mathfrak{M}(I)} F(x), \tag{16}$$

$$v_2(M_p) \stackrel{\text{def}}{=} \min_{y \in \mathfrak{M}(J)} G(y). \tag{17}$$

The continuity of F and G follows from Parthasarathy (1967; Lemma 1.1, p.57) and Berge's maximum theorem (see, for example, Ichiishi 1983). Hence, we have the maximum value of v_1 and the minimum value of v_2. From (14) and (15), $F(x) \leqq G(y)$ for all (x, y), and we have $v_1(M_p) \leqq v_2(M_p)$.

Next, we must show that $v_1(M_p) \geqq v_2(M_p)$. Let us define $K(x,y) = \iint M_p(i,j)dx(i)dy(j)$ and elaborate correspondence $\varphi : \mathfrak{M}(I) \times \mathfrak{M}(J) \to \mathfrak{M}(I) \times \mathfrak{M}(J)$ as follows:

$$\varphi : \mathfrak{M}(I) \times \mathfrak{M}(J) \ni (x,y) \mapsto$$
$$\{x' \mid K(x',y) \geqq K(\hat{x},y) \text{ for all } \hat{x} \in \mathfrak{M}(I)\} \times \{y' \mid K(x,y') \leqq K(x,\hat{y}) \text{ for all } \hat{y} \in \mathfrak{M}(J)\}. \tag{18}$$

Then φ has a graph with non-empty, closed, and convex values (see Parthasarathy 1967; Theorem 5.7, p. 35). From the fixed-point theorem of Glicksberg (Glicksberg 1952), the correspondence φ has a

[6] Consider the case of x. When f is any continuous function on I, since I is compact, f is bounded, so that $x^\nu \to x^*$ iff $\int f dx^\nu \to \int f dx^*$. See, for example, Parthasarathy (1967; Theorem 6.4, p. 45).

[7] See, for example, Fubini's Theorem in Halmos (1974; Theorem 36.B, p. 147). Note that Const_δ is a constant mapping with the value δ.

fixed point (x^*, y^*).[8] Hence, we have $v_1(M_p) \geqq v_2(M_p)$ as follows:

$$v_2(M_p) = \min_{y \in \mathfrak{M}(J)} G(y) \leqq G(y^*) \leqq K(x^*, y^*) = \min_{y \in \mathfrak{M}(J)} K(x^*, y) = F(x^*) \leqq v_1(M_p). \tag{19}$$

Thus we establish $v_1(M_p) = v_2(M_p)$, or minimax=maximin.

Now we confirm that the minimax becomes a continuous function of p. Since $v_1(M_p) = \max_{x \in \mathfrak{M}(I)} F(x)$ and $F(x) = \min_{j \in J} \int M_p(i,j)dx(i)$, we can combine them as

$$v_1(M_p) = \max_{x \in \mathfrak{M}(I)} \min_{j \in J} \int M_p(i,j)dx(i). \tag{20}$$

Assume that $j^\nu \to j^*$, $x^\nu \to x^*$ and $p^\nu \to p^*$, then we have to show that $|\int M_{p^*}(i,j^*)dx^*(i) - \int M_{p^\nu}(i,j^\nu)dx^\nu(i)|$ tends to 0. Let us consider the following inequation.

$$\left|\int M_{p^*}(i,j^*)dx^*(i) - \int M_{p^\nu}(i,j^\nu)dx^\nu(i)\right| \leqq$$
$$\left|\int M_{p^*}(i,j^*)dx^*(i) - \int M_{p^\nu}(i,j^\nu)dx^*(i)\right| + \left|\int M_{p^\nu}(i,j^\nu)dx^*(i) - \int M_{p^\nu}(i,j^\nu)dx^\nu(i)\right|. \tag{21}$$

The first term of the right-hand side of (21), $|\int M_{p^*}(i,j^*)dx^*(i) - \int M_{p^\nu}(i,j^\nu)dx^*(i)|$, tends to 0 under the topology of weak convergence. On the other hand, the second term of the right-hand side of (21), $|\int M_{p^\nu}(i,j^\nu)dx^*(i) - \int M_{p^\nu}(i,j^\nu)dx^\nu(i)|$, also tends to 0, since $M_{p^\nu}(i,j^\nu)$ uniformly converges to $M_{p^*}(i,j^*)$. Hence, we see that $|\int M_{p^*}(i,j^*)dx^*(i) - \int M_{p^\nu}(i,j^\nu)dx^\nu(i)| \to 0$ and confirm that minimax value $v_1(M_p)$ becomes a continuous function of p. ■

With the property of the fixed point (x^*, y^*) of φ, $K(x, y^*) \leqq K(x^*, y^*) \leqq K(x^*, y)$ for all $(x, y) \in \mathfrak{M}(I) \times \mathfrak{M}(J)$, we call (x^*, y^*) the *saddle-point solution* of the zero-sum matrix game. Consider two different players U and V, and interpret $M_p(i,j)$ as the profit of U as well as the loss of V for their pure strategy (i,j). Then, we could summarize the above result as follows:

> *Value of game* of the *extended zero-sum matrix game*, in which $M_p(i,j)$ represents the profit-matrix, is $v(M_p) = v_1(M_p) = v_2(M_p)$, and it is *strictly determined*. In addition, the value is $v(M_p) = K(x^*, y^*)$ as the *saddle-point* (x^*, y^*) of $K(x,y) = \iint M_p(i,j)dx(i)dy(j)$, which represents U's profit (resp. V's loss) under the mixed strategy $x \in \mathfrak{M}(I)$ and $y \in \mathfrak{M}(J)$.

If we interpret $B - \alpha A$ as an extended matrix game, then equations (11) and (12) assert that x and y become the optimal strategies for this game and α must be chosen such that the value of the game is

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[8] Since $\mathfrak{M}(I)$ is a subset of the vector space generated by whole measures on I under the topology of weak convergence, it depends on the duality with the vector space $M_p(I)$ generated by a real-valued, bounded, and continuous function on I.

zero.[9] The condition of equation (13) remains as an economic condition.

The above economic assumptions (i) and (ii) can also be reformulated in the zero-sum game theoretic notation. We consider B and $-A$ to be extended matrix games, where the maximizing player controls the I-side and the minimizing player controls the J-side. Let $v(B)$ and $v(-A)$ be the values of each of these games. Since A and B are non-negative, we can easily confirm that the conditions (i′) and (ii′) are equivalent to

(i″) $v(-A) < 0$,

(ii″) $v(B) > 0$.

4 Boundary Condition

To consider the existence theorem, we introduce the next two assumptions, "Boundary Condition" and "Minimum Positive Amount Condition for Input-Output Coefficients."

Boundary Condition

There exist a countable dense subset I^* of I, finite points $i^*(1)$, $i^*(2)$, $\cdots$, $i^*(\bar{m})$ of I^*, and points $j^*(1)$, $j^*(2)$, $\cdots$, $j^*(\bar{n})$ of J, such that each $i^*(m)$ (resp. $j^*(n)$) is associated with an open neighborhood U_m of $i^*(m)$ (resp. V_n of $j^*(n)$), for $m = 1, 2, \cdots, \bar{m}$ (resp. $n = 1, 2, \cdots, \bar{n}$), satisfying the following conditions on the restriction M^* of M_p on $I^* \times J$:

(B-1) For each m and $j \in J$, we have $M^*(i, j) \leqq M^*(i_{m^*}, j)$ for all $i \in \{i \mid i \in U_m$ or $i \notin U = \bigcup_{m'=1}^{\bar{m}} U_{m'}\}$.

(B-2) For each n and $i \in I^*$, we have $M^*(i, j_{\hat{n}}) \leqq M^*(i, j) \leqq M^*(i, j_{n^*})$ or $M^*(i, j_{n^*}) \leqq M^*(i, j) \leqq M^*(i, j_{\hat{n}})$ for all $j \in \{j \mid j \in V_n$ or $j \notin V = \bigcup_{n'=1}^{\bar{n}} V_{n'}\}$.

One can easily confirm that if B^* and $-A^*$ (which are restrictions of B and $-A$ on $I^* \times J$) satisfy the boundary condition, then M^* also satisfies exactly the same requirements in this condition.

Minimum Positive Amount Condition for Input-Output Coefficients

Let a_{ij} and b_{ij} be the values of functions $A(i, j)$ and $B(i, j)$, respectively, at $(i, j) \in I \times J$. For any $i \in I$ and $j \in J$, $a_{ij} = b_{ij} = 0$ or $|a_{ij}| + |b_{ij}| > \epsilon$ holds, where $\epsilon > 0$ is fixed.

Under these assumptions, we show the existence of an economic solution of equations (11), (12) and (13) under (i) and (ii) using the game-theoretic framework. Before moving on to the proof of our main theorem, we prove the following lemma.

Lemma 2: *Let M^* satisfy the boundary condition and $v(M^*) = 0$. Then, the dual systems $\langle x, M^* \rangle \geqq 0$, $x \geqq 0$, and $\langle M^*, y \rangle \leqq 0$, $y \geqq 0$ possess solutions $x^* \neq 0, y^* \neq 0$ such that $\langle x^*, M^* \rangle + y^* \gg 0$. Pair (x^*, y^*) is called a central solution for matrix game M^*.*

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9 A game with the value of 0 is called a *fair game*.

Proof : [First Step] Under the boundary condition, we consider the next subsets of I^* or J:

$$\bar{I} = \{i_{1^*}, i_{2^*}, \cdots, i_{\bar{m}^*}\}, \tag{22}$$
$$\bar{J} = \{j_{1^*}, j_{2^*}, \cdots, j_{\bar{n}^*}, j_{\hat{1}}, j_{\hat{2}}, \cdots, j_{\hat{n}}\}, \tag{23}$$
$$\bar{J}_1 = \{j \in J \mid j \in \bar{J} \text{ or } j \notin \textstyle\bigcup_{n=2}^{\bar{n}} V_n\}, \tag{24}$$
$$\bar{J}_n = \{j \in J \mid j \in \bar{J}_{n-1} \text{ or } j \in V_n\}, n = 2, 3, \cdots, \bar{n}. \tag{25}$$

Note that $\bar{J}_{\bar{n}} = J$. Let $M^{\bar{m}}$ be the restriction of M^* on $\bar{I} \times J$, $M_{\bar{n}}$ be the restriction of M^* on $I^* \times \bar{J}$, and $M_{\bar{n}}^{\bar{m}}$ be the restriction of M^* on $\bar{I} \times \bar{J}$. We also define $\bar{M}_n$ as the restriction of M^* on $I^* \times \bar{J}_n$ for $n = 1, 2, \cdots, \bar{n}$.

We apply Theorem 5 of Tucker (1956) (see also, Tucker 1955; Theorem 6 and Nikaido 1968: Corollary 3, p. 39) to the skew-symmetric matrix T of order $\bar{m} + 2\bar{n}$ $(= \sharp\bar{I} + \sharp\bar{J})$ defined by

$$T = \begin{bmatrix} 0 & M_{\bar{n}}^{\bar{m}\prime} \\ -M_{\bar{n}}^{\bar{m}} & 0 \end{bmatrix}, \tag{26}$$

where the upper left 0 is the zero matrix of order $\bar{m}$ and the lower right 0 is the zero matrix of order $2\bar{n}$.[10] We have $w = (w_1, \ldots, w_{\bar{m}+2\bar{n}})'$ satisfying $\langle T, w\rangle \geqq 0$, $w \geqq 0$, $\langle T, w\rangle + w \gg 0$. If we let $\bar{y} = (w_1, \ldots, w_{2\bar{n}})'$ and $\bar{x} = (w_{2\bar{n}+1}, \ldots, w_{\bar{m}+2\bar{n}})'$, the above relations yield the following results: $\bar{x} \geqq 0$, $\bar{y} \geqq 0$, $\bar{x} \neq 0$, $\bar{y} \neq 0$, $\langle \bar{x}, M_{\bar{n}}^{\bar{m}}\rangle \geqq 0$, $\langle M_{\bar{n}}^{\bar{m}}, \bar{y}\rangle \leqq 0$, $\langle \bar{x}, M_{\bar{n}}^{\bar{m}}\rangle + \bar{y} \gg 0$, and $-\langle M_{\bar{n}}^{\bar{m}}, \bar{y}\rangle + \bar{x} \gg 0$.

[Second Step] We observe that the solution x^* of (x^*, y^*) for the dual system of the lemma is constructed from the solution $\bar{x}$ of $(\bar{x}, \bar{y})$ for the dual system with respect to $M_{\bar{n}}^{\bar{m}}$. From the boundary condition, for each n and for each $i \in I^*$, $M^*(i, j_{\hat{n}}) \leqq M^*(i, j) \leqq M^*(i, j_{n^*})$ or $M^*(i, j_{n^*}) \leqq M^*(i, j) \leqq M^*(i, j_{\hat{n}})$ for all $j \in \{j \mid j \in V_n \text{ or } j \notin V\}$. From $\langle \bar{x}, M_{\bar{n}}^{\bar{m}}\rangle \geqq 0$, we have $\langle \bar{x}, M^{\bar{m}}(\cdot, j_{\hat{n}})\rangle \geqq 0$ and $\langle \bar{x}, M^{\bar{m}}(\cdot, j_{n^*})\rangle \geqq 0$, since $j_{\hat{n}}, j_{n^*} \in \bar{J}$, $n = 1, 2, \cdots, \bar{n}$. Hence, $\langle \bar{x}, M^{\bar{m}}(\cdot, j)\rangle \geqq 0$ for all $j \in J$, and we have $\langle \bar{x}, M^{\bar{m}}\rangle \geqq 0$. Thus, we define x^* on I^* such that $x_i^* = \bar{x}_i$ when $i \in \bar{I}$ and $x_i^* = 0$ when $i \notin \bar{I}$. Consequently, we obtain $\langle x^*, M^*\rangle \geqq 0$.

[**Third Step**] Then, we will also define y^* by using $\bar{y}$. For this, we extend the above result to $\bar{M}_1$. From the boundary condition, for each m and for each $j \in J$, $M^*(i, j) \leqq M^*(i_{m^*}, j)$ for all $i \in \{i \mid i \in U_m \text{ or } i \notin U\}$. In addition, from $\langle M_{\bar{n}}^{\bar{m}}, \bar{y}\rangle \leqq 0$, we see $\langle M_{\bar{n}}(i_{m^*}, \cdot), \bar{y}\rangle \leqq 0$ for $m = 1, 2, \cdots, \bar{m}$, since $i_{m^*} \in \bar{I}$. Hence, we have $\langle M_{\bar{n}}, \bar{y}\rangle \leqq 0$. Now we show that there exist $y_{(1)}^*$ on $\bar{J}_1$ such that $\langle \bar{M}_1, y_{(1)}^*\rangle \leqq 0$. From $\langle x^*, M^*\rangle \geqq 0$, we can consider the next three cases.

(Case 1) $\langle x^*, \bar{M}_1(\cdot, j_{1^*})\rangle = \langle x^*, \bar{M}_1(\cdot, j_{\hat{1}})\rangle = 0$.

(Case 2) $\langle x^*, \bar{M}_1(\cdot, j_{1^*})\rangle > 0$ and $\langle x^*, \bar{M}_1(\cdot, j_{\hat{1}})\rangle > 0$.

(Case 3) $\min_{j \in \{j_{1^*}, j_{\hat{1}}\}} \langle x^*, \bar{M}_1(\cdot, j)\rangle = 0$ and $\max_{j \in \{j_{1^*}, j_{\hat{1}}\}} \langle x^*, \bar{M}_1(\cdot, j)\rangle > 0$.

We use $j_{\min}$ and $j_{\max}$ in the sense that $\langle x^*, \bar{M}_1(\cdot, j_{\min})\rangle = \min_{j \in \{j_{1^*}, j_{\hat{1}}\}} \langle x^*, \bar{M}_1(\cdot, j)\rangle$ and $\langle x^*, \bar{M}_1(\cdot, j_{\max})\rangle = \max_{j \in \{j_{1^*}, j_{\hat{1}}\}} \langle x^*, \bar{M}_1(\cdot, j)\rangle$.

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[10] For a finite set A, we denote by $\sharp A$ the number of elements in A.

(Case 1) From equation $\langle x^*, \bar{M}_1(\cdot, j_{1^*})\rangle = \langle x^*, \bar{M}_1(\cdot, j_{\hat{1}})\rangle = 0$ and assumption (B-2) in the boundary condition,[11] we have $\langle x^*, \bar{M}_1(\cdot, j)\rangle = 0$ for all $j \in \{j \mid j \in V_n$ or $j \notin V\}$. Since $\langle \bar{x}, M_{\bar{n}}^{\bar{m}}\rangle + \bar{y} \gg 0$, we have $\bar{y}_{j_{1^*}} > 0$ and $\bar{y}_{j_{\hat{1}}} > 0$. Let us define ι and τ as $\iota = \bar{y}_{j_{1^*}}$ and $\tau = \bar{y}_{j_{\hat{1}}}$, and κ as $\min\{\iota, \tau\}$. In addition, we consider an index set, $\{\tilde{1}, \tilde{2}, \cdots\}$, of $\bar{J}_1 \setminus \{j_{2^*}, j_{3^*}, \cdots, j_{\bar{n}^*}, j_{\hat{2}}, j_{\hat{3}}, \cdots, j_{\hat{\bar{n}}}\}$. Then, we introduce $\tilde{y}$ on $\bar{J}_1$ such that $\tilde{y}_j = \bar{y}_j$ when $j \in \bar{J} \setminus \{j_{1^*}, j_{\hat{1}}\}$ and $\tilde{y}_j = \frac{1}{2^\nu}\kappa$ when j corresponds to $\tilde{\nu} \in \{\tilde{1}, \tilde{2}, \cdots\}$. Let $Const_1$ represent a constant mapping with the value 1 on $\bar{J}_1$. Then, we define $y^*_{(1)_j} = \frac{\tilde{y}_j}{\int Const_1 d\tilde{y}}$ for all $j \in \bar{J}_1$.

(Case 2) From $\langle x^*, \bar{M}_1(\cdot, j_{1^*})\rangle > 0$, $\langle x^*, \bar{M}_1(\cdot, j_{\hat{1}})\rangle > 0$, and the boundary condition, we have $\langle x^*, \bar{M}_1(\cdot, j)\rangle > 0$ for all $j \in \{j \mid j \in V_n$ or $j \notin V\}$. Thus we define $y^*_{(1)}$ on $\bar{J}_1$ such that $y^*_{(1)_j} = \bar{y}_j$ when $j \in \bar{J}$ and $y^*_{(1)_j} = 0$ when $j \notin \bar{J}$.

(Case 3) Since $\langle x^*, \bar{M}_1(\cdot, j_{\min})\rangle = 0$ and $\langle x^*, \bar{M}_1(\cdot, j_{\max})\rangle > 0$, we can confirm whether $\langle x^*, \bar{M}_1(\cdot, j)\rangle$ is equal to zero or positive for each $j \in \bar{J}_1 \setminus \{j_{2^*}, \cdots, j_{\bar{n}^*}, j_{\hat{2}}, \cdots, j_{\hat{\bar{n}}}\}$. Then, we find the first j corresponding to $\nu \in \{\tilde{1}, \tilde{2}, \cdots\}$ such that $\langle x^*, \bar{M}_1(\cdot, j)\rangle = 0$, and we rename the j as $\tilde{j}_1$. In addition, we continue to confirm whether $\langle x^*, \bar{M}_1(\cdot, j)\rangle$ is equal to zero or positive for all $j \in \bar{J}_1 \setminus \{j_{2^*}, \cdots, j_{\bar{n}^*}, j_{\hat{2}}, \cdots, j_{\hat{\bar{n}}}\}$ corresponding to some $\tilde{\nu} \in \{\tilde{1}, \tilde{2}, \cdots\}$. At every time we find that $\langle x^*, \bar{M}_1(\cdot, j)\rangle = 0$, we rename the j as $\tilde{j}_2$, $\tilde{j}_3$, $\cdots$. We call the set $\{\tilde{j}_1, \tilde{j}_2, \cdots\}$ as $\tilde{J}$. From $\langle x^*, \bar{M}_1(\cdot, j_{\min})\rangle = 0$, we have $\bar{y}_{j_{\min}} > 0$. Let $\tau = \bar{y}_{j_{\min}}$, and we introduce $\tilde{y}$ on $\bar{J}_1$ such that $\tilde{y}_j = \bar{y}_j$ when $j \in \bar{J} \setminus \{j_{1^*}, j_{\hat{1}}\}$, $\tilde{y}_j = \frac{1}{2^k}\tau$ when j corresponds to $\tilde{j}_k$, $k = 1, 2, \cdots$, and $\tilde{y}_j = 0$ when $j \in \bar{J}_1 \setminus \{j_{2^*}, \cdots, j_{\bar{n}^*}, j_{\hat{2}}, \cdots, j_{\hat{\bar{n}}}\}$ and is not assigned any $\tilde{j}_k$. Let $Const_1$ represent a constant mapping with the value 1 on $\bar{J}_1$. Then, we define $y^*_{(1)_j} = \frac{\tilde{y}_j}{\int Const_1 d\tilde{y}}$ for all $j \in \bar{J}_1$.

For all three cases, we have $\langle \bar{M}_1, y^*_{(1)}\rangle \leqq 0$ and $\langle x^*, \bar{M}_1\rangle + y^*_{(1)} \gg 0$.

[Fourth Step] In a similar way to the third step, we also define $y^*_{(2)}$ on $\bar{J}_2$ by using $y^*_{(1)}$. From $\langle x^*, M^*\rangle \geqq 0$, we can consider the following three cases.

(Case 1′) $\langle x^*, \bar{M}_2(\cdot, j_{2^*})\rangle = \langle x^*, \bar{M}_2(\cdot, j_{\hat{2}})\rangle = 0$.

(Case 2′) $\langle x^*, \bar{M}_2(\cdot, j_{2^*})\rangle > 0$ and $\langle x^*, \bar{M}_2(\cdot, j_{\hat{2}})\rangle > 0$.

(Case 3′) $\min_{j\in\{j_{2^*}, j_{\hat{2}}\}}\langle x^*, \bar{M}_2(\cdot, j)\rangle = 0$ and $\max_{j\in\{j_{2^*}, j_{\hat{2}}\}}\langle x^*, \bar{M}_2(\cdot, j)\rangle > 0$.

We use $j'_{\min}$ and $j'_{\max}$ in the sense that $\langle x^*, \bar{M}_1(\cdot, j'_{\min})\rangle = \min_{j\in\{j_{2^*}, j_{\hat{2}}\}}\langle x^*, \bar{M}_2(\cdot, j)\rangle$ and $\langle x^*, \bar{M}_2(\cdot, j'_{\max})\rangle = \max_{j\in\{j_{2^*}, j_{\hat{2}}\}}\langle x^*, \bar{M}_2(\cdot, j)\rangle$.

(Case 1′) From equation $\langle x^*, \bar{M}_2(\cdot, j_{2^*})\rangle = \langle x^*, \bar{M}_2(\cdot, j_{\hat{2}})\rangle = 0$ and assumption (B-2) in the boundary condition, we have $\langle x^*, \bar{M}_2(\cdot, j)\rangle = 0$ for all $j \in V_2 = \bar{J}_2 \setminus \bar{J}_1$. Since $\langle \bar{x}, M_{\bar{n}}^{\bar{m}}\rangle + \bar{y} \gg 0$, we have $\bar{y}_{j_{2^*}} > 0$ and $\bar{y}_{j_{\hat{2}}} > 0$. Let us define ι' and τ' as $\iota' = \bar{y}_{j_{2^*}}$ and $\tau' = \bar{y}_{j_{\hat{2}}}$, and κ' as min $\{\iota', \tau'\}$. In addition, we consider an index set, $\{\tilde{1}', \tilde{2}', \cdots\}$, of $V_2 \cup \{j_{\hat{2}}, j_{2^*}\}$. Then, we introduce $\tilde{y}'$ on $\bar{J}_2$ such that $\tilde{y}'_j = y^*_{(1)_j}$ when $j \in \bar{J}_1 \setminus \{j_{\hat{2}}, j_{2^*}\}$ and $\tilde{y}'_j = \frac{1}{2^\nu}\kappa'$ when j corresponds to $\tilde{\nu}' \in \{\tilde{1}', \tilde{2}', \cdots\}$. Let $Const_1$ represent a constant mapping with the value 1 on $\bar{J}_2$. Then, we define $y^*_{(2)_j} = \frac{\tilde{y}'_j}{\int Const_1 d\tilde{y}'}$ for all $j \in \bar{J}_2$.

(Case 2′) From $\langle x^*, \bar{M}_2(\cdot, j_{2^*})\rangle > 0$, $\langle x^*, \bar{M}_2(\cdot, j_{\hat{2}})\rangle > 0$, and the boundary condition, we have $\langle x^*, \bar{M}_2(\cdot, j)\rangle > 0$ for all $j \in V_2$. Thus we define $y^*_{(2)}$ on $\bar{J}_2$ such that $y^*_{(2)_j} = y^*_{(1)_j}$ when $j \in \bar{J}_1$ and

[11] Read assumption (B-2) with $n = 1$.

${y^*_{(2)}}_j = 0$ when $j \notin \bar{J}_1$.

(Case 3′) Since $\langle x^*, \bar{M}_2(\cdot, j'_{\min})\rangle = 0$ and $\langle x^*, \bar{M}_2(\cdot, j'_{\max})\rangle > 0$, we can confirm whether $\langle x^*, \bar{M}_2(\cdot, j)\rangle$ is equal to zero or positive for each $j \in V_2 \cup \{j_{\hat{2}}, j_{2^*}\}$. Then, we find the first j corresponding to $\nu \in \{\tilde{1}', \tilde{2}', \cdots\}$ such that $\langle x^*, \bar{M}_2(\cdot, j)\rangle = 0$, and we rename the j as $\tilde{j}'_1$. In addition, we continue to confirm whether $\langle x^*, \bar{M}_2(\cdot, j)\rangle$ is equal to zero or positive for all $j \in V_2 \cup \{j_{\hat{2}}, j_{2^*}\}$ corresponding to some $\nu \in \{\tilde{1}', \tilde{2}', \cdots\}$. At every time we find that $\langle x^*, \bar{M}_2(\cdot, j)\rangle = 0$, we rename the j as $\tilde{j}'_2$, $\tilde{j}'_3$, $\cdots$. We call the set $\{\tilde{j}'_1, \tilde{j}'_2, \cdots\}$ as $\tilde{J}'$. From $\langle x^*, \bar{M}_2(\cdot, j'_{\min})\rangle = 0$, we have $\bar{y}_{j'_{\min}} > 0$. Let $\tau' = \bar{y}_{j'_{\min}}$, and we introduce $\tilde{y}'$ on $\bar{J}_2$ such that $\tilde{y}'_j = {y^*_{(1)}}_j$ when $j \in \bar{J}_1 \setminus \{j_{\hat{2}}, j_{2^*}\}$, $\tilde{y}'_j = \frac{1}{2^k}\tau'$ when j corresponds to $\tilde{j}'_k$, $k = 1, 2, \cdots$, and $\tilde{y}'_j = 0$ when $j \in V_2 \cup \{j_{\hat{2}}, j^*_2\}$ and is not assigned any $\tilde{j}'_k$. Let $Const_1$ represent a constant mapping with the value 1 on $\bar{J}_2$. Then, we define ${y^*_{(2)}}_j = \frac{\tilde{y}'_j}{\int Const_1 d\tilde{y}'}$ for all $j \in \bar{J}_2$.

For all three cases, we have $\langle \bar{M}_2, y^*_{(2)}\rangle \leqq 0$ and $\langle x^*, \bar{M}_2\rangle + y^*_{(2)} \gg 0$.

[Fifth Step] By repeating the same discussion in the fourth step, we can extend the above $y^*_{(2)}$ for $\bar{M}_2$ to $y^*_{(3)}$ for $\bar{M}_3$. Then we have $\langle \bar{M}_3, y^*_{(3)}\rangle \leqq 0$ and $\langle x^*, \bar{M}_3\rangle + y^*_{(3)} \gg 0$. In the same way, we can get step by step $y^*_{(4)}$ for $\bar{M}_4$, $y^*_{(5)}$ for $\bar{M}_5$, $\cdots$, $y^*_{(\bar{n}-1)}$ for $\bar{M}_{\bar{n}-1}$, and y^* for M^*. Hence, finally, we have $\langle M^*, y^*\rangle \leqq 0$ and $\langle x^*, M^*\rangle + y^* \gg 0$. ■

5 Existence Theorem

Now we can prove the equilibrium existence theorem with a method similar to Thompson (1956), which treats the von Neumann model with finite matrices. As stated above, we can assume that the boundary condition holds on M_p for all p.

Theorem 1: *If p is the largest value that makes M_p a fair game, then there exists a pair (x, y) of optimal strategies for M_p such that $\iint Bdydx > 0$.*

Proof: Let p be the largest value that makes $v(M_p) = 0$. By assumption, $v(M_0) = v(B) > 0$ and $v(M_1) = v(-A) < 0$. Hence, from the continuity, there is at least one positive p such that $v(M_p) = 0$.

Assume that there is no solution to M_p with $\iint Bdydx > 0$ for all p, i.e., all such solutions have $\iint Bdydx = 0$. Then, if (x, y) is a pair of optimal strategies for M_p, we have $\iint((1-p)B - pA)\, dydx = 0$, and since $\iint Bdydx = 0$ and $p > 0$, we also have $\iint Adydx = 0$. Let x and y run over all optimal strategies for the game M_p and $(i, j) \in I \times J$. We define O and S as

$$O \stackrel{\text{def}}{=} \{(i,j) \mid \text{There is an optimal solution } (x,y) \text{ that satisfies } x(\hat{I}) > 0 \text{ and } y(j) > 0 \text{ for any open set } \hat{I} \supset \{i\}\}, \quad (27)$$

$$S \stackrel{\text{def}}{=} \{(i,j) \mid \exists \hat{j}, (i, \hat{j}) \in O, (i,j) \notin O\}. \quad (28)$$

We now confirm that $O \neq \emptyset$. For an optimal solution (x^*, y^*), since J is countable, we can find $\hat{j} \in J$ such that $y^*(\hat{j}) > 0$. Then, if we assume that there is no $i \in I$ such that $x^*(\hat{I}) > 0$, where $\hat{I}$ is some open set $\hat{I} \supset \{i\}$, we have $x^* = 0$, and this is a contradiction. Thus, there exists $\hat{i} \in I$ such that $x^*(\hat{I}) > 0$ for some open set $\hat{I} \supset \{i\}$. This $(\hat{i}, \hat{j})$ belongs to O and $O \neq \emptyset$. We can also say $S \neq \emptyset$, since every row of A has at least one positive entry. If $S = \emptyset$, $y(j)$ must be positive for all $j \in J$ from $O \neq \emptyset$. Since there exists some $\hat{i} \in I$ such that $(\hat{i}, \hat{j}) \in O$, we have $x(\hat{I}) > 0$ for any open set $\hat{I} \ni \hat{i}$. By considering the fact that $\iint Adydx = 0$ and $y(j) > 0$ for all $j \in J$, we must have $A(\hat{i}, j) = 0$ for all $j \in J$. Again, this is a contradiction.

Hence, now we consider $S \neq \emptyset$. Take any $j' \in J$ such that $(i, j') \in S$. We define $S_{j'}$ as

$$S_{j'} \overset{\text{def}}{=} \{(i, j') \mid (i, j') \in S\}. \tag{29}$$

Let us define (x^{**}, y^{**}) as follows:

$$x^{**}(\{i\}) = \begin{cases} x^*(\{i\}) & \text{if } i \in I^*, \\ 0 & \text{if } i \notin I^*, \end{cases}$$

$$y^{**}(j) = y^*(j) \quad \text{for all } j \in J,$$

where (x^*, y^*) is an optimal solution assured by Lemma 8.2 for M^* on $I^* \times J$. From the lemma, we must have either $\langle x^{**}, M_p(\cdot, j)\rangle > 0$ or $y^{**}(j) > 0$ for all $j \in J$. For any $(i, j') \in S$, $y^{**}(j') = 0$. Hence, $\langle x^{**}, M_p(S_{j'})\rangle$ must be strictly positive. From the minimum positive amount condition for input-output coefficients, $\langle x^{**}, M_p(S_{j'})\rangle$ has a minimum value $\delta > 0$, and $\langle x^{**}, M_p(S_{j'})\rangle \geqq \delta > 0$.

This implies that even if we take a larger p, the optimal solution (x^{**}, y^{**}) is still the solution of the game M_p with $v(M_p) = 0$. This contradicts the maximality of p. ■

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