Social structure, capital accumulation, and distribution of wealth*

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Abstract

This paper explores a two-period Allow-Debreu production economy where agents cooperatively form firms (coalitions) by jointly determining investments and shareholding rates. The firm structure can be perceived as a social structure based on cooperative decision-making among agents. We introduce the concept of social coalitional general equilibrium, which is regarded as a typical social coalitional equilibrium initially presented by Ichiishi (1981), and characterize all equilibrium states. Our findings demonstrate that an equilibrium social structure is organized with dependence on initial capital accumulation, aiming to generate maximum societal wealth. Furthermore, this paper shows that in equilibrium, society tolerates maldistribution, which is endogenously determined by market mechanisms.

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1 Introduction

Social structure refers to structurally related groups determined by the interrelationships and interactions of agents within society. Naturally, various groups (coalitions) exist for different purposes and benefits, and agents are allowed to belong to groups serving different purposes. Within the theoretical framework of economics, specifically the general equilibrium concept, the structural group organized through these interactions among agents is commonly referred to as a firm. Consequently, social structure can be interpreted as industrial structure, and the process of forming these structural groups can be characterized as a problem of firm formation. Therefore, it is reasonable to address the issue of social structure formation within the general equilibrium framework incorporating firm

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formation.

A firm-formation problem is regarded as a fundamental core solution in the coalition games developed by Aumann and Peleg (1960), Aumann (1961), Scarf (1967), and Scarf (1971). Subsequently, Ichiishi (1981) extended the concept of a coalition game and introduced a generalized cooperative concept known as social coalitional equilibrium. This concept encompasses economic settings (abstract market equilibrium) and cooperative game-theoretic arguments, ultimately aiming to identify a social structure. Simultaneously, the coalition game framework was extended to address core issues within production economies, including general equilibrium economies with coalition production (Boehm, 1974), with labor-managed market (Greenberg 1979; Ichiishi, 1977), and with decentralization issue (Ichiishi, 1981). However, despite the use of these abstract conceptual frameworks to broadly characterize economies (societies), there is still no comprehensive formulation that captures the relationships among exogenous parameters in society, social (firm) structure, and agents' decision-making within organizations. More specifically, the formation of social structure is intricately intertwined with both capital accumulation and wealth distribution. Elaborating on these interrelationships could offer a precise theoretical interpretation of the form of social structure.

Inspired by the above context, we introduce a two-period Arrow-Debreu general equilibrium economy incorporating firm formation. In this economy, there are two (groups of) agents who live in two periods. Each period features a single type of good that can be either consumed or invested in. Each agent forms a firm (coalition) in cooperation with the other agent, jointly determining their investment levels and shareholding rates. Consequently, a firm (social) structure is considered a representative partition of agents. Specifically, a firm formed by two agents is called a cooperative-type firm, whereas a firm organized by a single agent is called a non-cooperative-type firm. Accordingly, a structure comprising cooperative-type firms is labeled a cooperative social structure, while one consisting of non-cooperative-type firms is termed a non-cooperative social structure. Each firm produces goods of period 2 based on the investments received during period 1 so as to maximize its profits. In this paper, the productivity of each type of firm varies, with non-cooperative firms exhibiting higher productivity at lower levels of investment, while cooperative firms prove to be more profitable in the opposite condition.[1]

This paper introduces the concept of social coalition general equilibrium within the established framework and provides a comprehensive characterization of all equilibrium states. The findings demonstrate that the existence of equilibrium is contingent upon initial endowments (capital accumulation). Specifically, a low level of initial endowments in society results in a non-cooperative social structure, whereas a high level of initial endowments leads to the formation of a cooperative social structure. This phenomenon arises because an equilibrium social structure is comprised only of firms that are capable of achieving maximum net profit.[2] This implies that only a social structure capable of generating maximum societal wealth will be established. Unfortunately, variations in

[1] This observation highlights the fact that the production technologies of the two types of firms adhere to the single-crossing property.

[2] Net profit here refers to the difference between firm profit and investment cost.

productivity among different types of firms may internally determine an uneven distribution of firm profits in the case where the cooperative social structure prevails. However, the absence of a social structure that internally alleviates such inequality in wealth distribution among agents compels society to accept such maldistribution in equilibrium. As a result, this paper is anticipated to aid in formulating the relationship between social structure, capital accumulation, and wealth distribution within a dynamic general equilibrium framework.

Section 2 establishes a two-period Arrow-Debreu production economy that incorporates the firm formation problem. In section 3, we present the primary result, which pertains to the characterization of social coalitional general equilibrium. We thoroughly examine the properties exhibited by equilibrium states and delve into a comprehensive discussion on this issue. The proofs for all of the results are provided in Appendix, ensuring a rigorous and thorough treatment of the subject matter.

We use R as the set of real numbers and R^n as the set of n-dimensional vector space. For finite set A, we denote by $\sharp A$ the number of elements of A. The simplex $\Delta^{\sharp A-1}$, a subset of $R^{\sharp A}_{++}$, is defined by the set $\{x \in R^{\sharp A}_{++} \mid x = (x_k)_{k=1}^{\sharp A}, \sum_{k=1}^{\sharp A} x_k = 1\}$.

2 Model

Consider a dynamic Arrow-Debreu production economy with two periods $t = 1, 2$. In each period, there exists a single type of good that can be either consumed by agents or invested to produce goods for the subsequent period. There are two (groups of) agents, denoted by $i \in N = \{1, 2\}$, each endowed with initial endowments of $\omega_i = (\omega_i^1, \omega_i^2) = (\omega, 0)$, $\omega \geqq 0$.

In period 1, agents form firms (coalitions) to engage in cooperative investments. These firms, established in period 1, exclusively use the investments made by their respective members to produce goods for period 2. Furthermore, we assume that each agent can only belong to a single firm. Thus, a firm structure, denoted by $\mathfrak{T}$, represents a partition of N. To simplify our discussion, we use the following notation to represent the admissible firm structures:

$$\mathfrak{T}_\alpha = \{\{1\}, \{2\}\} \text{ and } \mathfrak{T}_\beta = \{\{1, 2\}\}.$$

For each $S \subset N$, there exists a technology correspondence denoted by $Y^S : R_+ \ni z^{1,S} \mapsto R_+$. Here, $z^{1,S} = \sum_{i \in S} z_i^1$ represents the total amount of the investments made by the agents in coalition S, where z_i^1 denotes the investment made by consumer $i \in N$ in period 1. For a firm $S \in \{\{1\}, \{2\}\}$, we define a production technology as $Y^S(z^{1,S}) = \{y^{2,S} \in R_+ \mid \exists z^{1,S} \in R_+, y^{2,S} \leqq 2^{\alpha-1}(z^{1,S})^\alpha$ and $z^{1,S} = \sum_{i \in S} z_i^1\}$.[3] Similarly, for a coalition $S = \{a(t), b(t)\}$, the production technology is defined as $Y^S(z^{1,S}) = \{y^{2,S} \in R_+ \mid \exists z^{1,S} \in R_+, y^{2,S} \leqq z^{1,S^\beta}$ and $z^{1,S} = \sum_{i \in S} z_i^1\}$. For each firm $S \subset N$, the agents in S collectively determine a shareholding rate $\theta^S = (\theta_i^S)_{i \in S} \in \Delta^{\# S-1}$ for its members. Additionally, we assume that $0 < \alpha < \beta < 1$. It is clear

[3] Due to the symmetry of $Y^S(z^{1,\{1\}})$ and $Y^S(z^{1,\{2\}})$, the optimal inputs and outputs are inherently equal. Given a price p, the aggregate total social optimal inputs and outputs satisfy $\sum_{S \in \mathfrak{T}_\alpha} p^2 2^{\alpha-1}(z^{*1,S})^\alpha - p^1 z^{*1,S} = p^2(2z^{*1,S})^\alpha - p^1 2z^{*1,S}$, and $p^2 2^{\alpha-1}(z^{*1,S})^\alpha - p^1 z^{*1,S} = (p^2(2z^{*1,S})^\alpha - p^1 2z^{*1,S})/2$ holds for every $S \in \mathfrak{T}_\alpha$. Naturally, a generalization of the production technology will not disrupt our discussion.

that $\sum_{S\in\mathfrak{T}_\beta} Y^S(z^{1,S})$ is more efficient than $\sum_{S\in\mathfrak{T}_\alpha} Y^S(z^{1,S})$ when $\sum_{i\in N} z_i^1 \geq 1$ but is more inefficient when $\sum_{i\in N} z_i^1 \in (0,1)$.

Let $p=(p^1,p^2)\in\Delta^2$ be the price of goods, and given the investments $\{z_i^1\}_{i\in S}$, we can define the profit maximization program for firm $S\subset\mathfrak{T}_\kappa$, $\kappa=\alpha,\ \beta$, as follows:

$$\begin{aligned} \max_{y^{2,S}} \quad & \pi^S(p,(z_i^1)_{i\in S}) = p^2 y^{2,S}, \\ \text{subject to} \quad & y^{2,S}\in Y^S(z^{1,S}). \end{aligned} \tag{1}$$

It can be easily verified that for each $z^{1,S}$, the profit of firm S is maximized only when equality holds for all of the inequalities. Therefore, given the investments $(z_i^1)_{i\in N}$, the profit of firm $S\in\mathfrak{T}_\kappa$, $\kappa=\alpha,\ \beta$, is $\pi^S(p,(z_i^1)_{i\in S})=p^2(\sum_{i\in S} z_i^1)^\kappa$.

Each agent $i\in N$ selects consumption (c_i^1,c_i^2) over the two periods, taking into account his/her budget constraint. The consumer's utility function, representing their preferences, is denoted as $u_i(c_i^1,c_i^2)=c_i^1c_i^2$. Consequently, consumer i's maximization problem can be formulated as follows:

$$\begin{aligned} \max_{c_i^1,c_i^2} \quad & u_i(c_i^1,c_i^2) = c_i^1c_i^2, \\ \text{subject to} \quad & p^1c_i^1+p^2c_i^2+p^1z_i^1 = p_1\omega+\theta_i^S\pi^S(p,(z_i^1)_{i\in S}). \end{aligned} \tag{2}$$

Based on the above definition, we establish the following definition of the social coalitional general equilibrium in a two-period Arrow-Debreu production economy.

Definition 1. For a two-period Arrow-Debreu production economy, a *social coalitional general equilibrium* state is represented as a list $\big((c_i^{*1},c_i^{*2},z_i^{*1})_{i\in N}),(\theta^{*S})_{S\in\mathfrak{T}^*},p^*,\mathfrak{T}^*\big)$ satisfying the following conditions:

(Utility Maximization) (c_i^{*1},c_i^{*2}) is a solution of (2) under $\mathfrak{T}^*$, p^* and $(\theta^{*S})_{S\in\mathfrak{T}^*}$ for every $i\in N$.

(Market Feasibility)

$$\sum_{i\in N}(c_i^{*1}+z_i^{*1}) = 2\omega, \tag{3}$$

$$\sum_{i\in N} c_i^{*2} = \sum_{S\in\mathfrak{T}^*} y^{*2,S} \tag{4}$$

where $y^{*2,S}$ is a solution of (1) under $z^{1,S}=\sum_{i\in S} z_i^{*1}$.

(Market Stability) There exists no coalition $D\subset N$, consumptions $(\bar{c}_i^1,\bar{c}_i^2)_{i\in D}$, investment $(\bar{z}_i^1)_{i\in D}$, and shareholding rate $\bar{\theta}^D$ such that $u_i(\bar{c}_i^1,\bar{c}_i^2)>u_i(c_i^{*1},c_i^{*2})$ for all $i\in D$, satisfying the following condition:

$$p^{*1}\bar{c}_i^1 + p^{*2}\bar{c}_i^2 + p^{*1}\bar{z}_i^1 \leqq p^{*1}\omega + \bar{\theta}_i^D \pi^D(p^*, (\bar{z}_i^1)_{i\in D}) \tag{5}$$

for all $i \in D$.

Utility Maximization and Market Feasibility are standard conditions that do not necessitate further elaboration. The Market Stability condition states that no coalition D can improve the utility of its members by dismantling the existing firm structure $\mathfrak{T}$. The inequality in this condition can be interpreted as the budget constraints faced by the members of coalition D when considering a deviation from the current structure. These members are able to freely utilize their investments $(\bar{z}_i^1)_{i\in D}$ and determine their shareholding rate $\bar{\theta}^D$, thereby abandoning their existing relationships within $\mathfrak{T}$.

3 Results

In this section, we present the main results of our paper.

Theorem 2. *For a two-period Arrow-Debreu production economy described in Section 2,*

(1) when $2\omega \leqq \underline{\omega}$, a list $\left((c_i^{*1}, c_i^{*2}, z_i^{*1})_{i\in N}), (\theta^{*S})_{S\in\mathfrak{T}^*}, p^*, \mathfrak{T}^*\right)$ is a social coalitional equilibrium state that satisfies the following conditions: $\mathfrak{T}^* = \mathfrak{T}_\alpha$, (p^{*1}, p^{*2}) fulfills (22), and for every $i \in N$, (c_i^{*1}, c_i^{*2}) fulfills (23), $z_i^{*1} = \frac{\alpha\omega}{1+\alpha}$, $\theta_i^{*\{i\}} = 1$.

(2) when $2\omega \geqq \bar{\omega}$, a list $\left((c_i^{*1}, c_i^{*2}, z_i^{*1})_{i\in N}), (\theta^{*S})_{S\in\mathfrak{T}^*}, p^*, \mathfrak{T}^*\right)$ is a social coalitional general equilibrium state that satisfies the following conditions: $\mathfrak{T}^* = \mathfrak{T}_\beta$, (p^{*1}, p^{*2}) fulfills (30), and for every $i \in N$, (c_i^{*1}, c_i^{*2}) fulfills (31), $(z_1^{*1}, z_2^{*1}), (\theta_1^{*\{1,2\}}, \theta_2^{*\{1,2\}})$ fulfills (33) where $z_i^{*1} \leqq \omega$ for all $i \in N$, $z_1^{*1} + z_2^{*1} = \frac{2\beta\omega}{1+\beta}$ and $\theta_1^{*\{1,2\}} + \theta_2^{*\{1,2\}} = 1$.

(3) if $\omega \in (\underline{\omega}, \bar{\omega})$, no social coalitional general equilibrium exists.

Proof: See Appendix.

The main results indicate that the equilibrium, particularly the equilibrium firm (social) structure, is determined by the initial endowments (capital accumulation) ω. Specifically, although the formation of firms in equilibrium is influenced by the agents' investment behaviors $(z_i^{*1})_{i\in N}$, it is ultimately governed by exogenous parameters α, β, and ω. It is not surprising that equilibrium is conditional on the value of ω, since the convexity assumption of the aggregate total production possibility set, $\sum_{S\subset N}\sum_{z^S\in R_+}(Y^S(z^S) - z^S)$, is not imposed in this economy.[4] Furthermore, it is noteworthy that the production technology functions of the two types of firms intersect at a single point, satisfying the condition of the single-crossing property. Consequently, a cooperative structure, $\mathfrak{T}^* = \{\{1,2\}\}$, is established when the initial endowment level is high, while a non-cooperative structure, $\mathfrak{T}^* = \{\{1\},\{2\}\}$, is implemented in the opposite case.

[4] The convexity assumption of the aggregate total production possibility set is deduced from the Balancedness Condition proposed by Boehm (1974) and Greenberg (1979), which is a necessary condition for the existence of firm-formation equilibrium in a general context.

Corollary 1. Let $z^{*1,S}$ be the solution of $\max_{z^{1,S}} p^{*2}(z^{1,S})^{\kappa} - p^{*1}z^{1,S}$ for the firm $S \in \mathfrak{T}^{*}$, then $\sum_{i \in S} z_i^{*1} = z^{*1,S}$.

This result shows that only a firm structure with high productivity, which maximizes social net profit, is formed thus demonstrating that agents' investment behavior not only yields individual gains but also contributes to the overall improvement of societal wealth. Furthermore, this result is derived exclusively from the Market Stability condition, underscoring the role of market mechanisms in shaping the formation of the firm (social) structure and enhancing social wealth.

Corollary 2. Let a list $\left((c_i^{*1}, c_i^{*2}, z_i^{*1})_{i \in N}), (\theta^{*S})_{S \in \mathfrak{T}_\beta^*}, p^*, \mathfrak{T}_\beta^*\right)$ be a social coalitional general equilibrium state, and let the social coalitional social equilibrium state $\left((c_i^{*1}, c_i^{*2}, z_i^{1})_{i \in N}),\right.$ $\left.(\theta^{S})_{S \in \mathfrak{T}_\beta^*}, p^*, \mathfrak{T}_\beta^*\right)$ be generated through Lemma 2. Then, $\theta_i^S/(1-\theta_i^S)$ may not be equal to 1.

This result suggests that when the (cooperative) firm structure $\mathfrak{T}_\beta^*$ prevails, the presence of income inequality in the distribution of firm profits may arise endogenously. However, from the perspective of each consumer, the firm structure that allows for income inequality is advantageous. In essence, a firm (social) structure formed solely through market mechanisms will tolerate a certain degree of maldistribution.

4 Conclusion

This paper developed a two-period Arrow-Debreu production framework that incorporates firm formation. Agents jointly determine their investments and shareholding rates to form firms. The coalition-deviation within a firm structure allows us to model the social cooperative structure as a standard firm structure. Following this framework, the concept of social coalitional general equilibrium is introduced, and the equilibrium states are characterized. Consequently, our findings demonstrate that the equilibrium social structure, which leads to higher social wealth, is determined by the initial endowments, although this may lead to income inequality. Future research focuses on extending the analysis to dynamic frameworks with infinite periods. As discussed in this study, the formation of the social structure in each period is solely influenced by the initial capital accumulated from the previous period. The introduction of dynamics into the model is expected to provide insights into the relationship between capital accumulation and changes in the social structure.

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Appendix

Lemma 1. *Let the list* $\big((c_i^{*1}, c_i^{*2}, z_i^{*1})_{i\in N}), (\theta^{*S})_{S\in\mathfrak{T}^*}, p^*, \mathfrak{T}^*\big)$ *be a social coalitional general equilibrium state, then the following conditions are satisfied:*

(i) for any $S \in \mathfrak{T}^*$*, it holds that* $\pi^S(p^*, (z_i^{*1})_{i\in S}) - p^{*1}\sum_{i\in S} z_i^{*1} = \max p^{*2}(z^{1,S})^\kappa - p^{*1}z^{1,S}$*, where* $z^{1,S} \in R^+$*;*

(ii) for any $\mathfrak{T}_\kappa$*,* $\kappa = \alpha,\ \beta$*, and for any* $z_i^1 \in R^+$*,* $i \in N$*, the inequality*

$$\sum_{D\in\mathfrak{T}_\kappa} (\pi^D(p^*, (z_i^1)_{i\in D}) - p^{*1}\sum_{i\in D} z_i^1) \leqq \sum_{S\in\mathfrak{T}^*} (\pi^S(p^*, (z_i^{*1})_{i\in S}) - p^{*1}\sum_{i\in S} z_i^{*1}) \tag{6}$$

is satisfied.

Proof: (i) Clearly, according to the definition of the maximization program, inequality $\pi^S(p^*, (z_i^{*1})_{i\in S}) - p^{*1}\sum_{i\in S} z_i^{*1} \leqq \max p^{*2}(z^{1,S})^\kappa - p^{*1}z^{1,S}$ must hold. Thus, it suffices to demonstrate that $\pi^S(p^*, (z_i^{*1})_{i\in S}) - p^{*1}\sum_{i\in S} z_i^{*1} < \max p^{*2}(z^{1,S})^\kappa - p^{*1}z^{1,S}$ is false. Suppose, as a contradiction, that there exists $z^{1,S}$ such that $\pi^S(p^*, (z_i^{*1})_{i\in S}) - p^{*1}\sum_{i\in S} z_i^{*1} < \max p^{*2}(z^{1,S})^\kappa - p^{*1}z^{1,S}$ holds. Let $\pi^{*S}(p^*) = \max p^{*2}(z^{1,S})^\kappa - p^{*1}z^{1,S}$ and let $\bar{z}^{1,S}$ be the solution to $\max p^{*2}(z^{1,S})^\kappa - p^{*1}z^{1,S}$. We now define a new shareholding rate $\bar{\theta}^S$ as

$$\bar{\theta}_i^S = \frac{\theta_i^{*S}\pi^S(p^*, (z_i^{*1})_{i\in S}) - p^{*1}z_i^{*1}}{\pi^S(p^*, (z_i^{*1})_{i\in S}) - p^{*1}\sum_{i\in S} z_i^{*1}} \tag{7}$$

for all $i \in S$. Then, for every $i \in S$, we have $\theta_i^{*S}\pi^S(p^*, (z_i^{*1})_{i\in S}) - p^{*1}z_i^{*1} < \bar{\theta}_i^S\pi^{*S}(p^*)$. In other words, for every $i \in S$, the condition

$$p^{*1}c_i^{*1} + p^{*2}c_i^{*2} = p^{*1}\omega + \theta_i^{*S}\pi^S(p^*, (z_i^{*1})_{i\in S}) - p^{*1}z_i^{*1} < p^{*1}\omega + \bar{\theta}_i^S\pi^{*S}(p^*) \tag{8}$$

holds. This condition states that for every $i \in S$, there exists an alternative consumption and investment triple $(\bar{c}_i^1, \bar{c}_i^2, \bar{z}_i^1)_{i\in S}$ such that $u_i(c_i^{*1}, c_i^{*2}) < u_i(\bar{c}_i^1, \bar{c}_i^2)$, where $\bar{z}_i^1 = \bar{\theta}_i^S\bar{z}^1$ and

$$p^{*1}\bar{c}_i^1 + p^{*2}\bar{c}_i^2 = p^{*1}\omega + \bar{\theta}_i^S\pi^{*S}(p^*). \tag{9}$$

This fact follows from the strict concavity of u_i. Consequently, this constitutes a deviation, contradicting the condition of Market Stability. The proof of (i) is thus completed.

(ii) From (i), we have $\sum_{S\in\mathfrak{T}^*}(\pi^S(p^*,(z_i^1)_{i\in S})-p^{*1}\sum_{i\in S}z_i^1)\leqq\sum_{S\in\mathfrak{T}^*}(\pi^S(p^*,(z_i^{*1})_{i\in S})$ $-p^{*1}\sum_{i\in S}z_i^{*1})$ for any $z_i^1\in R_+$ and $i\in N$. Therefore, it is sufficient to verify that for a partition, $\mathfrak{T}_\kappa\neq\mathfrak{T}^*$, for any $z_i^1\in R_+$ and $i\in N$ such that inequality (6) holds. It is evident that the result is true if both $\mathfrak{T}_\alpha^*$ and $\mathfrak{T}_\beta^*$ are equilibrium firm structures. Thus, we consider the case where only one type of structure is an equilibrium structure. Let $\mathfrak{T}^*=\mathfrak{T}_\beta$. Suppose that there exists $(\bar{z}_i^1)_{i\in N}$ such that

$$\sum_{D\in\mathfrak{T}_\alpha}(\pi^D(p^*,(\bar{z}_i^1)_{i\in D})-p^{*1}\sum_{i\in D}\bar{z}_i^1)>\sum_{S\in\mathfrak{T}_\beta}(\pi^S(p^*,(z_i^{*1})_{i\in S})-p^{*1}\sum_{i\in S}z_i^{*1}). \tag{10}$$

That is, there must exist a coalition $\{i\}\in\mathfrak{T}_\alpha$ such that $\pi^{\{i\}}(p^*,\bar{z}_i^1)-p^{*1}\bar{z}_i^1>\theta_i^*\pi^S(p^*,(z_i^{*1})_{i\in S})$ $-p^{*1}z_i^{*1}$. For agent i, there exists alternative consumption and investment triple $(\bar{c}_i^1,\bar{c}_i^2,\bar{z}_i^1)_{i\in S}$ such that $u_i(c_i^{*1},c_i^{*2})<u_i(\bar{c}_i^1,\bar{c}_i^2)$, where

$$p^{*1}\bar{c}_i^1+p^{*2}\bar{c}_i^2=p^{*1}\omega+\pi^{\{i\}}(p^*,\bar{z}_i^1)-p^{*1}\bar{z}_i^1. \tag{11}$$

Note that $\theta_i^{\{i\}}=1$. Hence, we obtain a contradiction. The case where $\mathfrak{T}^*=\mathfrak{T}_\alpha$ can be proved using the same logic. Thus, the proof of (ii) is completed.

Lemma 2. *Let the list $\big((c_i^{*1},c_i^{*2},z_i^{*1})_{i\in N}),(\theta^{*S})_{S\in\mathfrak{T}^*},p^*,\mathfrak{T}^*\big)$ be a social coalitional general equilibrium state, then an alternative social coalitional general equilibrium state $\big((c_i^{*1},c_i^{*2},z_i^1)_{i\in N}),$ $(\theta^S)_{S\in\mathfrak{T}^*},p^*,\mathfrak{T}^*\big)$ exists in the same economy. In this new equilibrium state, for every $i\in N$, the following conditions are satisfied:*

$$\theta_i^S=\frac{\theta_i^{*S}\pi^S(p^*,(z_i^{*1})_{i\in S})-p^{*1}z_i^{*1}}{\pi^S(p^*,(z_i^{*1})_{i\in S})-p^{*1}\sum_{i\in S}z_i^{*1}} \tag{12}$$

and $z_i^1=\theta_i^S\sum_{i\in S}z_i^{*1}$.

Proof: According to the newly defined $(\theta^S)_{S\in\mathfrak{T}^*}$ and $(z_i^1)_{i\in N}$, we can observe that $\theta_i^S(\pi^S(p^*,$ $(z_i^{*1})_{i\in S})-p^{*1}\sum_{i\in S}z_i^{*1})=\theta_i^{*S}\pi^S(p^*,(z_i^{*1})_{i\in S})-p^{*1}z_i^{*1}$. This demonstrates that the budget constraint remains unchanged for every $i\in N$ under $(\theta^S)_{S\in\mathfrak{T}^*}$ and $(z_i^1)_{i\in N}$. Hence, (c_i^{*1},c_i^{*2}) maximizes agent i's utility under $(\theta^S)_{S\in\mathfrak{T}^*}$ and $(z_i^1)_{i\in N}$, satisfying the Market Stability condition. Furthermore, the condition $z_i^1=\theta_i^S\sum_{i\in S}z_i^{*1}$ implies

$$\sum_{i\in N}z_i^1=\sum_{S\in\mathfrak{T}^*}\sum_{i\in S}\theta_i^S\sum_{i\in S}z_i^{*1}=\sum_{i\in N}\sum_{i\in S}z_i^{*1}. \tag{13}$$

The final equality holds because $\sum_{i\in S}\theta_i^S=1$ for all $S\in\mathfrak{T}^*$. Thus, the Market Feasibility condition also holds under the state $\big((c_i^{*1},c_i^{*2},z_i^1)_{i\in N}),(\theta^S)_{S\in\mathfrak{T}^*},p^*,\mathfrak{T}^*\big)$. Hence, $\big((c_i^{*1},c_i^{*2},z_i^1)_{i\in N}),$ $(\theta^S)_{S\in\mathfrak{T}^*},p^*,\mathfrak{T}^*\big)$ is a social coalitional general equilibrium state. This completes the proof.

According to Lemma 1, for a given coalition $\mathfrak{T}_\alpha$ and p, the optimal consumptions under $\mathfrak{T}_\alpha$ satisfy the following condition:

$$\max_{z^{1,\{i\}}} \qquad p^2 2^{\alpha-1}(z^{1,\{i\}})^{\alpha} - p^1 z^{1,\{i\}}. \tag{14}$$

Solving this program, we obtain $p^1 = \alpha p^2 2^{\alpha-1}(z^{*1,\{i\}})^{\alpha-1}$, and hence, $p^2 2^{\alpha-1}(z^{*1,\{i\}})^{\alpha} - p^1 z^{*1,\{i\}} = (1-\alpha)p^2 2^{\alpha-1}(z^{*1,\{i\}})^{\alpha}$. Since $\theta_i^{\{i\}} = 1$, we have $z_i^{*1} = z^{*1,\{i\}}$. Therefore, given $\mathfrak{T}_\alpha$, p, and $(\theta^S)_{S\in\mathfrak{T}_\alpha}$, agent i's utility maximization program is given by

$$\begin{aligned} &\max_{c_i^1, c_i^2} && u_i(c_i^1, c_i^2) = c_i^1 c_i^2, \\ &\text{subject to} && p^1 c_i^1 + p^2 c_i^2 + p^1\theta_i^{\{i\}} z^{*1,\{i\}} = p_1\omega + \theta_i^{\{i\}}(1-\alpha)p^2 2^{\alpha-1}(z^{*1,\{i\}})^{\kappa}. \end{aligned} \tag{15}$$

Solving the above program, the optimal consumption satisfies the following conditions:

$$c_i^{*1} = \frac{p^1\omega + \theta_i^{\{i\}}(1-\alpha)p^2 2^{\alpha-1}(z^{*1,\{i\}})^{\alpha}}{2p^1}, \tag{16}$$

$$c_i^{*2} = \frac{p^1\omega + \theta_i^{\{i\}}(1-\alpha)p^2 2^{\alpha-1}(z^{*1,\{i\}})^{\alpha}}{2p^2}. \tag{17}$$

Furthermore, since $\theta_i^{\{i\}} = 1$, the optimal consumption can be rewritten as follows:

$$c_i^{*1} = \frac{p^1\omega + (1-\alpha)p^2 2^{\alpha-1}(z^{*1,\{i\}})^{\alpha}}{2p^1}, \tag{18}$$

$$c_i^{*2} = \frac{p^1\omega + (1-\alpha)p^2 2^{\alpha-1}(z^{*1,\{i\}})^{\alpha}}{2p^2}. \tag{19}$$

Substituting the optimal consumption and optimal investment into the conditions in Market Feasibility, we have

$$\begin{aligned} 2\omega &= \frac{p^1\omega + (1-\alpha)p^2 2^{\alpha-1}(z^{*1,\{1\}})^{\alpha}}{2p^1} + \frac{p^1\omega + (1-\alpha)p^2 2^{\alpha-1}(z^{*1,\{2\}})^{\alpha}}{2p^1} + z^{*1,\{1\}} + z^{*1,\{2\}} \\ &= \frac{2p^1\omega + (1-\alpha)p^2(2z^{*1,\{i\}})^{\alpha}}{2p^1} + 2z^{*1,\{i\}} \end{aligned} \tag{20}$$

and

$$\begin{aligned} (2z^{*1,\{i\}})^{\alpha} &= 2^{\alpha-1}(z^{*1,\{1\}})^{\alpha} + 2^{\alpha-1}(z^{*1,\{2\}})^{\alpha} \\ &= \frac{p^1\omega + (1-\alpha)p^2 2^{\alpha-1}(z^{*1,\{1\}})^{\alpha}}{2p^2} + \frac{p^1\omega + (1-\alpha)p^2 2^{\alpha-1}(z^{*1,\{2\}})^{\alpha}}{2p^2} \\ &= \frac{2p^1\omega + (1-\alpha)p^2(2z^{*1,\{i\}})^{\alpha}}{2p^2}. \end{aligned} \tag{21}$$

Solving equation (21), we find $(2z^{*1,\{i\}})^{\alpha} = \frac{2p^1\omega}{(1+\alpha)p^2}$. Substituting $(2z^{*1,\{i\}})^{\alpha}$ into equation (20), we obtain $z_i^{*1} = z^{*1,\{i\}} = \frac{\alpha\omega}{(1+\alpha)}$. Substituting $z^{*1,\{i\}} = \frac{\alpha\omega}{(1+\alpha)}$ into $(2z^{*1,\{i\}})^{\alpha} = \frac{2p^1\omega}{(1+\alpha)p^2}$, we have $p^1 = \frac{\alpha^{\alpha}(2\omega)^{\alpha-1}}{(1+\alpha)^{\alpha-1}}p^2$. Since $p \in \Delta^2$, we can express the price under $\mathfrak{T}_\alpha$ as

$$(p^{*1}, p^{*2}) = (\frac{\alpha^{\alpha}(2\omega)^{\alpha-1}}{(1+\alpha)^{\alpha-1} + \alpha^{\alpha}(2\omega)^{\alpha-1}}, \frac{(1+\alpha)^{\alpha-1}}{(1+\alpha)^{\alpha-1} + \alpha^{\alpha}(2\omega)^{\alpha-1}}). \tag{22}$$

Given a price (p^{*1}, p^{*2}) that satisfies condition (22), the optimal consumptions under $\mathfrak{T}_\alpha$ are given by

$$(c_i^{*1}, c_i^{*2}) = (\frac{\omega}{1+\alpha}, 2^{\alpha-1}(\frac{\alpha\omega}{1+\alpha})^{\alpha}). \tag{23}$$

Similarly, for a given coalition $\mathfrak{T}_\beta$ and p, we can calculate the optimal investments $z_i^{*1,\{1,2\}}$

through the firm $\{1,2\}$'s maximum program

$$\max_{z^{1,\{1,2\}}} \quad p^2(z^{1,\{1,2\}})^\beta - p^1 z^{1,\{1,2\}}. \tag{24}$$

Solving this program, we have $p^1 = \beta p^2(z^{*1,\{1,2\}})^{\beta-1}$, and hence, $p^2(z^{*1,\{1,2\}})^\beta - p^1 z^{*1,\{1,2\}} = (1-\beta)p^2(z^{*1,\{1,2\}})^\beta$. According to Lemma 2, it is sufficient to use an arbitrary pair $((z_i^1)_{i\in N}, (\theta_i^{\{1,2\}})_{i\in\{1,2\}})$ converted according to the logic of Lemma 2. Therefore, given $\mathfrak{T}_\beta$, p, and $(\theta^S)_{S\in\mathfrak{T}_\beta}$, agent i's utility maximization program is given by

$$\begin{aligned} \max_{c_i^1, c_i^2} \quad & u_i(c_i^1, c_i^2) = c_i^1 c_i^2, \\ \text{subject to} \quad & p^1 c_i^1 + p^2 c_i^2 + p^1\theta_i^{\{1,2\}} z^{*1,\{1,2\}} = p_1\omega + \theta_i^{\{1,2\}}(1-\beta)p^2(z^{*1,\{1,2\}})^\kappa. \end{aligned} \tag{25}$$

Solving the above program, the optimal consumption satisfies the following conditions:

$$c_i^{*1} = \frac{p^1\omega + \theta_i^{\{1,2\}}(1-\beta)p^2(z^{*1,\{1,2\}})^\beta}{2p^1}, \tag{26}$$

$$c_i^{*2} = \frac{p^1\omega + \theta_i^{\{1,2\}}(1-\beta)p^2(z^{*1,\{1,2\}})^\beta}{2p^2}. \tag{27}$$

Substituting the optimal consumption and optimal investment into the Market Feasibility conditions, we have

$$\begin{aligned} 2\omega &= \frac{p^1\omega + \theta_1^{\{1,2\}}(1-\beta)p^2(z^{*1,\{1,2\}})^\beta}{2p^1} + \frac{p^1\omega + \theta_2^{\{1,2\}}(1-\beta)p^2(z^{*1,\{1,2\}})^\beta}{2p^1} + z_1^{*1,\{1,2\}} + z_2^{*1,\{1,2\}} \\ &= \frac{2p^1\omega + (1-\beta)p^2(z^{*1,\{i\}})^\beta}{2p^1} + z^{*1,\{1,2\}} \end{aligned} \tag{28}$$

and

$$\begin{aligned} (z^{*1,\{1,2\}})^\beta &= \frac{p^1\omega + \theta_1^{\{1,2\}}(1-\beta)p^2(z^{*1,\{1,2\}})^\beta}{2p^2} + \frac{p^1\omega + \theta_2^{\{1,2\}}(1-\beta)p^2(z^{*1,\{1,2\}})^\beta}{2p^2} \\ &= \frac{2p^1\omega + (1-\beta)p^2(z^{*1,\{1,2\}})^\beta}{2p^2}. \end{aligned} \tag{29}$$

It is important to note that $\sum_{i\in\{1,2\}} \theta_i^{\{1,2\}} = 1$. Calculating the equation (29), we find $(z^{*1,\{1,2\}})^\beta = \frac{2p^1\omega}{(1+\beta)p^2}$. Substituting $(z^{*1,\{1,2\}})^\beta$ into equation (28), we obtain $z^{*1,\{1,2\}} = \frac{2\beta\omega}{(1+\beta)}$ and $z_i^{*1} = \theta_i^{\{1,2\}} z^{*1,\{1,2\}} = \theta_i^{\{1,2\}} \frac{2\beta\omega}{(1+\beta)}$. Substituting $z^{*1,\{1,2\}} = \frac{2\beta\omega}{(1+\beta)}$ into $(z^{*1,\{1,2\}})^\beta = \frac{2p^1\omega}{(1+\beta)p^2}$, we have $p^1 = \frac{\beta^\beta(2\omega)^{\beta-1}}{(1+\beta)^{\beta-1}}p^2$. Since $p\in\Delta^2$, we can express the price under $\mathfrak{T}_\beta$ as

$$(p^{*1}, p^{*2}) = (\frac{\beta^\beta(2\omega)^{\beta-1}}{(1+\beta)^{\beta-1} + \beta^\beta(2\omega)^{\beta-1}}, \frac{(1+\beta)^{\beta-1}}{(1+\beta)^{\beta-1} + \beta^\beta(2\omega)^{\beta-1}}). \tag{30}$$

Given a price (p^{*1}, p^{*2}) that satisfies this condition (30), the optimal consumptions under $\mathfrak{T}_\beta$ satisfy the following condition:

$$(c_i^{*1}, c_i^{*2}) = (\frac{(1+\beta+2\theta_i^{\{1,2\}} - 2\theta_i^{\{1,2\}}\beta)\omega}{2(1+\beta)}, 2^{\beta-2}(1+\beta+2\theta_i^{\{1,2\}} - 2\theta_i^{\{1,2\}}\beta)(\frac{\beta\omega}{1+\beta})^\beta). \tag{31}$$

Now, we examine the Market Stability condition for a given firm structure $\mathfrak{T}_\kappa$, $\kappa = \alpha,\ \beta$. Since $Y^{\{1\}} = Y^{\{2\}}$, we only need to determine under what conditions agents do not deviate to $\{i\}$ from $\{1,2\}$. Let $\theta = \theta_i^{\{1,2\}}$. Given a price p^*, the optimal investment for coalition $\{i\}$ is $z^{*1,\{i\}} = \frac{\alpha\omega}{(1+\alpha)}$, and that for coalition $\{1,2\}$ is $z^{*1,\{1,2\}} = \frac{2\beta\omega}{(1+\beta)}$. By the definition of θ^S, we have $\theta_i^{\{i\}} = 1$ for every $\{i\}$, $i \in N$, which implies $z_i^{1,\{i\}} = \theta_i^{\{i\}} z^{*1,\{i\}} = \frac{\alpha\omega}{(1+\alpha)}$. Similarly, $z_i^{1,\{1,2\}} = \theta_i^{\{1,2\}} z^{*1,\{1,2\}} = \theta \frac{2\beta\omega}{(1+\beta)}$.

Lemma 2 shows that every equilibrium investment and shareholding rate pair $((z_i^{*1})_{i\in N}, (\theta^{*S})_{S\in\mathfrak{T}^*})$ is equivalent to the pair $((z_i^1)_{i\in N}, (\theta^S)_{S\in\mathfrak{T}^*})$, where θ_i^S satisfies condition (12) and $z_i^1 = \theta_i^S \sum_{i\in S} z_i^{*1}$. Therefore, it is sufficient to verify the Market Stability condition using the pair $((z_i^1)_{i\in N}, (\theta^S)_{S\in\mathfrak{T}^*})$. Furthermore, as previously shown, the maximum profit of coalition S is independent of the investment level. Thus, according to the Market Stability condition, the condition under which agents do not deviate from $\{1,2\}$ to $\{i\}$ is given by $\theta(\frac{2(1-\beta)p^{*1}\omega}{1+\beta}) \geqq \frac{(1-\alpha)p^{*1}\omega}{1+\alpha}$. That is, $\theta \geqq \frac{(1-\alpha)(1+\beta)}{2(1+\alpha)(1-\beta)}$. Since $\beta > \alpha$, we have

$$\theta_i^{\{1,2\}} = \theta^* \in [\frac{(1-\alpha)(1+\beta)}{2(1+\alpha)(1-\beta)}, \frac{1+3\alpha-3\beta-\alpha\beta}{2(1+\alpha)(1-\beta)}]. \tag{32}$$

It is evident that $\theta_i^{\{1,2\}} \in [\frac{(1-\alpha)(1+\beta)}{2(1+\alpha)(1-\beta)}, \frac{1+3\alpha-3\beta-\alpha\beta}{2(1+\alpha)(1-\beta)}]$ also implies $\theta_j^{\{1,2\}} \in [\frac{(1-\alpha)(1+\beta)}{2(1+\alpha)(1-\beta)}, \frac{1+3\alpha-3\beta-\alpha\beta}{2(1+\alpha)(1-\beta)}]$, where $i \neq j$. Moreover, the equation (28) demonstrates that the Market Feasibility condition is independent of the value of $\theta_i^{\{1,2\}}$. Therefore, such θ^* holds for every p. Recalling Lemma 2, agents with an investment and shareholding rate pair $((z_i^{*1})_{i\in N}, (\theta^{*S})_{S\in\mathfrak{T}^*})$ will not deviate if there exists $\theta_i^{\{1,2\}} = \theta^*$ for an agent i such that

$$\theta_i^{*S}\pi^S(p^*, (z_i^{*1})_{i\in S} - p^{*1}z_i^{*1} = \theta^*(p^{*2}(z^{*1,\{1,2\}})^\beta - p^{*1}z^{*1,\{1,2\}}) = \frac{2(1-\beta)p^{*1}\omega}{1+\beta}. \tag{33}$$

Finally, since $0 < \alpha < \beta < 1$, we can observe that $(2z^{1,\{i\}})^\alpha > (z^{1,\{1,2\}})^\beta$ for $2z^{1,\{i\}} = z^{1,\{1,2\}} < 1$ and that $(2z^{1,\{i\}})^\alpha < (z^{1,\{1,2\}})^\beta$ for $2z^{1,\{i\}} = z^{1,\{1,2\}} > 1$. This implies that the aggregate total production technologies satisfy the single-crossing property. Moreover, since the optimal investment level depends only on the price and endowment level, following the single-crossing property, there must exist a price p for which the optimal investment levels $2z^{*1,\{i\}}$ under $\mathfrak{T}_\alpha$ and $z^{*1,\{1,2\}}$ under $\mathfrak{T}_\beta$ satisfy the following condition:

$$(1-\alpha)p^2(2z^{*1,\{i\}})^\alpha = (1-\beta)p^2(z^{*1,\{1,2\}})^\beta. \tag{34}$$

Furthermore, since the price is the same, the marginal productivity of the optimal investment level must also be the same for each coalition structure:

$$\alpha(2z^{*1,\{i\}})^{\alpha-1} = \beta(z^{*1,\{1,2\}})^{\beta-1}. \tag{35}$$

Combining the equations (34) and (35), we find that

$$(z_i^{*1,\{i\}}, z^{*1,\{1,2\}}) = (\frac{1}{2}(\frac{1-\alpha}{1-\beta})^{\frac{\beta-1}{\alpha-\beta}}(\frac{\beta}{\alpha})^{\frac{\beta}{\alpha-\beta}}, (\frac{(1-\beta)}{(1-\alpha)})^{\frac{\alpha-1}{\beta-\alpha}}(\frac{\alpha}{\beta})^{\frac{\beta}{\beta-\alpha}}). \tag{36}$$

This condition suggests that the firm structure $\mathfrak{T}_\alpha$ is an equilibrium structure if $2z_i^{*1,\{i\}} \leqq (\frac{1-\alpha}{1-\beta})^{\frac{\beta-1}{\alpha-\beta}}(\frac{\beta}{\alpha})^{\frac{\beta}{\alpha-\beta}}$ and that $\mathfrak{T}_\beta$ is an equilibrium structure if $z^{*1,\{1,2\}} \geqq (\frac{(1-\beta)}{(1-\alpha)})^{\frac{\alpha-1}{\beta-\alpha}}(\frac{\alpha}{\beta})^{\frac{\beta}{\beta-\alpha}}$. Moreover, since the optimal investment levels $z_i^{*1,\{i\}}$ and $z^{*1,\{1,2\}}$ are determined solely by the initial endowments, we can express the conditions as follows:

$$\frac{2\alpha\omega}{1+\alpha} \leqq (\frac{1-\alpha}{1-\beta})^{\frac{\beta-1}{\alpha-\beta}}(\frac{\beta}{\alpha})^{\frac{\beta}{\alpha-\beta}} \quad \Longrightarrow \quad \underline{\omega} \leqq \frac{1}{2}(\frac{1-\alpha}{1-\beta})^{\frac{\beta-1}{\alpha-\beta}}(\frac{(1+\alpha)\beta^{\frac{\beta}{\alpha-\beta}}}{\alpha^{\frac{\alpha}{\alpha-\beta}}}) \tag{37}$$

and

$$\frac{2\beta\omega}{1+\beta} \geqq (\frac{(1-\beta)}{(1-\alpha)})^{\frac{\alpha-1}{\beta-\alpha}}(\frac{\alpha}{\beta})^{\frac{\beta}{\beta-\alpha}} \quad \Longrightarrow \quad \bar{\omega} \geqq \frac{1}{2}(\frac{1-\beta}{1-\alpha})^{\frac{\alpha-1}{\beta-\alpha}}(\frac{(1+\beta)\alpha^{\frac{\alpha}{\beta-\alpha}}}{\alpha^{\frac{\beta}{\beta-\alpha}}}). \tag{38}$$

Since $0 < \alpha < \beta < 1$, it follows that $\underline{\omega} < \bar{\omega}$. Hence, the firm structure $\mathfrak{T}_\alpha$ is an equilibrium structure if $\omega \leqq \underline{\omega}$, and $\mathfrak{T}_\beta$ is an equilibrium structure when $\omega \geqq \bar{\omega}$.

In summary, the social coalitional general equilibrium state $\big((c_i^{*1}, c_i^{*2}, z_i^{*1})_{i\in N}), (\theta^{*S})_{S\in\mathfrak{T}^*}, p^*, \mathfrak{T}^*\big)$ can be formulated as follows:

(1) if $\omega \leqq \underline{\omega}$, then the following conditions are holding: $\mathfrak{T}^* = \mathfrak{T}_\alpha$, (p^{*1}, p^{*2}) satisfies condition (22), and for every $i \in N$, (c_i^{*1}, c_i^{*2}) satisfies (23), $z_i^{*1} = \frac{\alpha\omega}{1+\alpha}$, $\theta_i^{*\{i\}} = 1$.
(2) if $\omega \geqq \bar{\omega}$, then the following conditions are holding: $\mathfrak{T}^* = \mathfrak{T}_\beta$, (p^{*1}, p^{*2}) satisfies condition (30), and for every $i \in N$, (c_i^{*1}, c_i^{*2}) satisfies (31), (z_1^{*1}, z_2^{*1}) and $(\theta_1^{*\{1,2\}}, \theta_2^{*\{1,2\}})$ satisfy condition (33), where $z_1^{*1} + z_2^{*1} = \frac{2\beta\omega}{1+\beta}$ and $\theta_1^{*\{1,2\}} + \theta_2^{*\{1,2\}} = 1$.
(3) if $\omega \in (\underline{\omega}, \bar{\omega})$, no social coalitional general equilibrium exists.