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# Fixed points and social equilibrium existence without convexity conditions<sup>\*</sup>

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## Abstract

The purpose of this paper is to show the existence of equilibrium for economic models with non-convex constraint correspondences. Our proof is based on a new fixed point theorem (a generalization of Kakutani-Fan-Glicksberg) depending merely on conditions for local directions of correspondences, so that we may obtain a general condition that may not depend on any global continuity and convexity on values of mappings for the existence of economic equilibria. We also apply the result for the existence of competitive equilibrium in economies with default and/or bankruptcy.

JEL Classification: C62; D51; D52; D82; G33

Keywords: Fixed point theorem, Abstract economy, Non-convexity, Competitive equilibrium, Default, Bankruptcy.

## 1 INTRODUCTION

In this paper, we prove the existence of equilibrium for an abstract economy with non-convex constraint correspondences. Moreover, we apply the theorem to the existence of general equilibrium problem in economies with default and/or bankruptcy.

The method we have used here is based on a new fixed point theorem depending merely on the concept of local directions of correspondences (Section 2, Theorem 1), so that we may obtain a general condition which does not depend on any global continuity and/or convexity on values of

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<sup>\*</sup> This paper was originally written by the late Akihiko Yoshimachi and myself, Ken Urai, and submitted to the Journal of Mathematical Economics in 2004. However, about four years later, the manuscript was returned because we could not find a reviewer. Fortunately, through the kindness of Professor Hiromi Murakami, I was given the opportunity to present this paper in the Osaka Economic Papers, a publication commemorating my 60th birthday. I firmly believe that the late Professor Yoshimachi would have shared in our delight. It is with great respect for his memory that I hope this paper to be a valuable addition to this precious collection.

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mappings for the existence of social equilibria (Section 2, Theorem 2). We further generalize the result (Section 2, Corollary 2.1), and apply it to the general equilibrium existence problem for the default economy (Section 3, Theorem 3) as a generalized model for economies with asymmetric information (see, e.g., Dubey-Geanakoplos-Shubik (2000)), which is one of the most important framework under the recent development in the general equilibrium theory.

Under the settings of social equilibrium, our result may be stated as follows: If we select appropriately for each player  $i$  and a strategy profile  $z = (z_j)_{j \in I}$ , a non-empty subset  $M_i(z)$  of (possibly) non-convex constraint set  $K_i(z)$ , the condition for existence of equilibria may be obtained merely as a local condition on directions to the set  $M_i(z)$  and the better set  $P_i(z)$  from  $z_i$  for each  $i$  which is weaker than to assume any convexity and/or global continuity. Hence, in the competitive equilibrium framework (Section 3, Theorem 3), if we identify  $M_i(z)$  with the budget constraint without default and  $K_i(z)$  with the constraint allowing for default, the equilibrium existence condition may directly be characterized as consumer  $i$ 's tastes for keeping promises (moral actions and plans) under exogenously specified default penalties.

It seems somewhat incredible that we have shown the existence of equilibrium without using any convexity condition for budgets. As we shall see in Section 2, mathematically, it is not the convexity but the direction of correspondence that is important for the existence of fixed points, so that, the convexity of the budget is superfluous as long as we use conditions for the direction of better sets and constraint sets.

## 2 EQUILIBRIUM FOR AN ABSTRACT ECONOMY

In this section, we show the existence of an equilibrium for an abstract economy having non-convex constraints. Subsection 2.1 is to develop a fixed point theorem based on the concept of local directions of correspondence. Theorem 2 in subsection 2.2 provides a general result on economic equilibrium existence in which the choice space for the abstract economy is taken to be a subset of locally convex Hausdorff topological vector space  $E$  having topological dual  $E'$ . Constraint correspondences are taken to be (possibly) non-convex. Preferences of agents are also taken to be (possibly) non-ordered and non-convex. They are not even supposed to be continuous in the ordinary sense. What we have assumed here is merely conditions on local directions of correspondences.<sup>1</sup>

### 2.1 A Fixed Point Theorem

Let  $E$  be a locally convex Hausdorff topological vector space having topological dual  $E'$ . We say that a correspondence  $\varphi : E \rightarrow E$  has a *fixed direction*  $p^x \in E'$  near  $x \in E$  if there exists a neighborhood  $U(x)$  of  $x$  such that for all  $z \in U(x)$  and  $y \in \varphi(z)$ ,  $\langle p^x, y - z \rangle > 0$ . We also say that a correspondence  $\varphi : E \rightarrow E$  has a *continuous direction near*  $x$  if there exists a neighborhood  $U(x)$  of  $x$  and a continuous function  $p^x : U(x) \rightarrow E'$  such that  $\langle p^x(z), y - z \rangle > 0$  for all

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<sup>1</sup> The mathematical results in this paper are extensions of theorems in Urai (2000) and Urai-Hayashi (2000). See also these papers for other game theoretic applications and market equilibrium existence results.

$z \in U(x)$  and  $y \in \varphi(z)$ , where  $E'$  is given a topology of compact convergence. Moreover, we say that a correspondence  $\varphi : E \rightarrow E$  has a *compact valued upper semi-continuous direction near  $x$*  if there exists a neighborhood  $U(x)$  of  $x$  and a non-empty compact valued upper semi-continuous correspondence  $p^x : U(x) \rightarrow E'$  such that  $\langle q, y - z \rangle > 0$  for all  $z \in U(x)$ ,  $q \in p^x(z)$  and  $y \in \varphi(z)$ , where the topology on  $E'$  is a topology of compact convergence.

It is clear that if  $\varphi$  has a fixed direction near  $x$ , then  $\varphi$  has a continuous direction near  $x$ , and if  $\varphi$  has a continuous direction near  $x$ , then  $\varphi$  has a compact valued upper semi-continuous direction near  $x$ . The first result of this paper is the following fixed point theorem which, as we shall see below, may be considered as a generalization of Kakutani-Fan-Glicksberg's fixed point theorem.

**THEOREM 1.** *Let  $X$  be a non-empty compact convex subset of a locally convex Hausdorff topological vector space  $E$  having topological dual,  $E'$ . Let  $\varphi : X \rightarrow X$  be a non-empty valued correspondence. Suppose that  $\varphi$  has a compact valued upper semi-continuous direction near  $x$  for each  $x$  such that  $x \notin \varphi(x)$ . Then,  $\varphi$  has a fixed point.*

**PROOF.** Suppose that  $\varphi$  does not have a fixed point. Then, since  $X$  is compact, we have points  $x^1, \dots, x^n \in X$  and their open neighborhoods  $U(x^1), \dots, U(x^n)$  in  $X$  such that  $\bigcup_{t=1}^n U(x^t) = X$ , and for each  $t = 1, \dots, n$ , there is a non-empty compact valued upper semi-continuous correspondence  $p^{x^t} : U(x^t) \rightarrow E'$  satisfying that  $\forall z \in U(x^t), \forall q \in p^{x^t}(z), \forall v \in \varphi(z) - z, \langle q, v \rangle > 0$ . Let  $\beta_t : X \rightarrow [0, 1], t = 1, \dots, n$ , be a partition of unity subordinated to  $\{U(x^1), \dots, U(x^n)\}$ . For each  $z \in X$ , define a set  $q(z) \subset E'$  as  $q(z) = \sum_{t=1}^n \beta_t(z)p^{x^t}(z)$ . The set  $q(z)$  is non-empty and compact, and the correspondence  $q : X \rightarrow E'$  is upper semi-continuous. Let  $\Phi(z) = \{y \in X | \langle q, y - z \rangle > 0 \text{ for all } q \in q(z)\}$  for each  $z \in X$ . Then, for all  $y \in \varphi(z)$ , and for all  $t$  such that  $z \in U(x^t)$ , we have  $\langle q^t, y - z \rangle > 0$  for all  $q^t \in p^{x^t}(z)$ , so that  $\sum_{t=1}^n \beta_t(z)\langle q^t, y - z \rangle = \langle \sum_{t=1}^n \beta_t(z)q^t, y - z \rangle > 0$  for all  $q^t \in p^{x^t}(z)$  for all  $t$ . That is, for all  $y \in \varphi(z)$ , we have  $\langle q, y - z \rangle > 0$  for all  $q \in q(z)$ . Hence,  $\Phi : X \rightarrow X$  is a non-empty convex valued correspondence having no fixed point. On the other hand, for each  $x \in X$  and for an arbitrary element  $y^x \in \varphi(x)$ , there is an open neighborhood  $V(x)$  of  $x$  in  $X$  satisfying  $\forall z \in V(x), y^x \in \Phi(z)$ . (Indeed, since  $y^x \in \Phi(x)$ , we have  $\langle q, y^x - x \rangle > 0$  for all  $q \in q(x)$ . Since  $q(x)$  is compact, we have neighborhood  $V$  of  $x$  and  $U$  of  $q(x)$  such that for all  $z \in V$  and  $q \in U, q, y^x - z > 0$  by the definition of the topology of compact convergence. Moreover, since  $q(x)$  is upper semi-continuous, we may obtain a neighbourhood  $V(x) \subset V$  of  $x$  such that  $\forall z \in V(x), q(z) \subset U$ .) Hence,  $\Phi$  has a fixed point by Browder's fixed point theorem (Browder (1968)), a contradiction. ■

Note that the above theorem is a generalization of well known Kakutani-Fan-Glicksberg's fixed point theorem<sup>2</sup> since if  $\varphi : X \rightarrow X$  is a non-empty compact convex valued upper semi-continuous

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<sup>2</sup> Let  $X$  be a non-empty compact subset of a locally convex Hausdorff topological vector space. Every non-empty closed convex valued upper semi-continuous correspondence  $\varphi : X \rightarrow X$  has a fixed point  $x^* \in \varphi(x^*)$ . (c.f. Glicksberg (1952).)

correspondence,  $\varphi$  satisfies all conditions in Theorem 1. Indeed, if  $\varphi(x)$  is compact and convex and if  $x \notin \varphi(x)$ , there is a  $p \in E'$  such that  $\langle p, z - x \rangle > 0$  for all  $z \in \varphi(x)$  by the separation theorem. Then, the upper semi-continuity of  $\varphi$  means that  $\varphi$  has a fixed direction  $p$  near  $x$ .

In view of economics, the better-set correspondence for a continuously differentiable quasi-concave utility function may be considered as a typical mapping having a continuous direction near every point. It is also easy to check that a better-set correspondence which may be represented (at least locally) by quasi-concave utility functions have a compact valued upper semi-continuous direction near every point. Since the budget correspondence is ordinarily assumed to be non-empty compact convex valued upper semi-continuous, the generalized fixed point theorem enables us to treat these two mappings (the better-set correspondence and the budget constraint correspondence) in a unified manner so that we may characterize the existence of economic equilibrium merely through conditions on local directions for these two mappings (see Theorem 2).

### 2.2 Existence of Equilibrium for an Abstract Economy

Let us define an *abstract economy* as a list,  $(X_i, P_i, K_i)_{i \in I}$ , where  $I$  is the finite index set of *agents*,  $X_i$  is the *choice set* of  $i$  for each  $i \in I$ ,  $K_i : X = \prod_{j \in I} X_j \rightarrow X_i$  is the *constraint correspondence* of  $i$ , and  $P_i : X = \prod_{j \in I} X_j \rightarrow X_i$  is the *preference correspondence* of  $i$ . For each  $i \in I$  and  $x \in X$ , we say that  $x_i$  is a *maximal point for  $P_i$  under  $K_i$*  iff  $x_i \in K_i(x)$  and  $P_i(x) \cap K_i(x) = \emptyset$ .<sup>3</sup> An *equilibrium* of the abstract economy  $(X_i, P_i, K_i)_{i \in I}$  is a point  $\bar{x} \in X = \prod_{i \in I} X_i$  such that for each  $i \in I$ ,  $\bar{x}_i \in K_i(\bar{x})$  and  $P_i(\bar{x}) \cap K_i(\bar{x}) = \emptyset$ , i.e., a maximal point for  $P_i$  under  $K_i$  for all  $i \in I$ .

The second result of this paper is the following equilibrium existence theorem. Assume that for each  $i \in I$ ,  $X_i$  is a subset of locally convex Hausdorff topological vector space  $E$  having topological dual  $E'$ . We say that a correspondence  $\varphi_i : X \rightarrow X_i$  has a *compact valued upper semi-continuous direction near  $x \in X$*  if there exist a neighborhood  $U(x)$  of  $x \in X$  and a non-empty compact valued upper semi-continuous correspondence  $p_i^x : U(x) \rightarrow E'$  such that  $\langle q, y_i - z_i \rangle > 0$  for all  $z \in U(x)$ ,  $q \in p_i^x(z)$ , and  $y_i \in \varphi_i(z)$ .<sup>4</sup> The next theorem characterizes the existence of equilibrium merely through conditions on local directions of correspondences,  $P_i$ 's and  $K_i$ 's. The condition for  $P_i$  includes all cases such that preferences are locally represented by continuous utility functions. The condition for  $K_i$  includes all cases with continuous closed convex valued upper semi-continuous constraint correspondences.

**THEOREM 2.** *Suppose that abstract economy  $(X_i, P_i, K_i)_{i \in I}$  satisfies the following conditions:*

(A1) *For each  $i \in I$ ,  $X_i$  is a non-empty compact convex subset of locally convex Hausdorff topological vector space  $E$  having a topological dual  $E'$ .*

<sup>3</sup> In the following, given a vector  $x \in X = \prod_{i \in I} X_i$ , we denote by  $x_i$  the  $i$ -th coordinate of  $x$  as long as there is no ambiguity.

<sup>4</sup> The definition in the previous section may be identified with the special case such that  $I = \{i\}$  and  $X = X_i$ .

(A2) For each  $i \in I$ ,  $P_i : X \rightarrow X_i$  is a (possibly empty valued) correspondence having a compact valued upper semi-continuous direction  $p_i^x$  near every  $x \in X$ .

(A3) For each  $i \in I$ ,  $K_i : X \rightarrow X_i$  is a non-empty valued correspondence having a compact valued upper semi-continuous direction  $q_i^x$  near every  $x \in X$  such that  $x_i \notin K_i(x)$ .

(A4) For each  $i \in I$ , the set  $\{x \in X \mid P_i(x) \cap K_i(x) \neq \emptyset\}$  is open in  $X$ .<sup>5</sup>

Then,  $(X_i, P_i, K_i)_{i \in I}$  has an equilibrium.

The theorem says that as long as the feasible set and the better set at a certain action level have continuous directions, (i) the convexity of these sets and the global continuity for these mappings are not necessary, and (ii) merely the continuous local direction property of each mapping completely characterize the existence of equilibrium.

We provide a proof of the theorem under more general settings in the next corollary. In corollary 2.1, if we take  $M_i(x)$  as  $M_i(x) = K_i(x)$ , condition (B5) is nothing but a tautology, hence, we have Theorem 2. The corollary is used in the next section, where the existence of the default leads to more serious non-convexity for  $K_i$ .

**COROLLARY 2.1.** *Suppose that abstract economy  $(X_i, P_i, K_i)_{i \in I}$  satisfies the following conditions:*

(B1) For each  $i \in I$ ,  $X_i$  is a non-empty compact convex subset of locally convex Hausdorff topological vector space  $E$  having a topological dual  $E'$ .

(B2) For each  $i \in I$ ,  $P_i : X \rightarrow X_i$  is a (possibly empty valued) correspondence having a compact valued upper semi-continuous direction  $p_i^x$  near every  $x \in X$ .

(B3) For each  $i \in I$ ,  $K_i : X \rightarrow X_i$  is a correspondence such that there is a non-empty valued correspondence  $M_i : X \rightarrow X_i$ ,  $M_i(z) \subset K_i(z)$  for all  $z \in X$ , having a compact valued upper semi-continuous direction  $q_i^x$  near every  $x \in X$  such that  $x_i \notin K_i(x)$ .

(B4) For each  $i \in I$ , the set  $\{x \in X \mid P_i(x) \cap K_i(x) \neq \emptyset\}$  is open in  $X$ .

(B5) For each  $x \in X$  and  $i \in I$  such that  $x_i \notin K_i(x)$  and  $P_i(x) \cap K_i(x) = \emptyset$ , there exists an open neighborhood  $U(x)$  of  $x$  such that for all  $z \in U(x)$ ,  $P_i(z) \cap K_i(z) \neq \emptyset$  implies  $P_i(z) \cap M_i(z) \neq \emptyset$ .

Then,  $(X_i, P_i, K_i)_{i \in I}$  has an equilibrium.

**PROOF.** In this proof, we denote by  $V(z_i, Q)$  the set  $\{y_i \in X_i \mid q, y_i - z_i > 0 \text{ for all } q \in Q\}$ , the set of direction  $Q$  at  $z_i$  for each  $z_i \in X_i \subset E$  and  $Q \subset E'$ . Assume that there is no equilibrium for

<sup>5</sup> Under (A2) and (A3), condition (A4) may be dropped if “for each  $x \in X$ ,  $K_i(x)$  is a subset of the closure of the interior of  $K_i$ ” and “for each  $x \in X$ , the value,  $K_i(x)$ , is not equal to  $X_i$  and does not depend on the  $i$ -th component,  $x_i$ , of  $x$ .” In view of economics, these settings are quite natural. Of course, (A4) is automatically satisfied if  $K_i$  is lower semi-continuous and  $P_i$  has an open graph.

$(X_i, P_i, K_i)$ . Then, for each  $x = (x_i)_{i \in I} \in X$ , there is at least one  $i \in I$  satisfying one and only one of the following three conditions:

- (a)  $K_i(x) \cap P_i(x) \neq \emptyset$ .
- (b)  $x$  is an interior point of  $\{z \in X | K_i(z) \cap P_i(z) = \emptyset \text{ and } z_i \notin K_i(z)\}$ .
- (c)  $x$  is a boundary point of  $\{z \in X | K_i(z) \cap P_i(z) = \emptyset \text{ and } z_i \notin K_i(z)\}$ .

In the following, we fix such an agent  $i(x) \in I$  (satisfying one and only one of the conditions, (a), (b), and (c)) for each  $x \in X$ . For each  $x$  and case (a)–(c) of  $i(x)$ , define, locally, on a certain open neighborhood  $U(x)$  of  $x$ , a correspondence  $\varphi^x : U(x) \rightarrow X_{i(x)}$  as

$$\begin{aligned} \varphi^x(z) &= V(z_{i(x)}, p_{i(x)}^x(z)) \text{ for case (a),} \\ \varphi^x(z) &= V(z_{i(x)}, q_{i(x)}^x(z)) \text{ for case (b), and} \\ \varphi^x(z) &= V(z_{i(x)}, q_{i(x)}^x(z)) \text{ for case (c),} \end{aligned}$$

where  $p_{i(x)}^x$  for case (a) is the local direction for  $P_i$  on a certain open set  $U(x)$  such that  $z \in U(x)$  implies  $K_i(z) \cap P_i(z) \neq \emptyset$  (use (B4)),  $q_{i(x)}^x$  for case (b) is the local direction for  $M_i$  on a certain open set  $U(x)$  such that  $z \in U(x)$  implies  $K_i(z) \cap P_i(z) \neq \emptyset$  and  $z_i \notin K_i(z)$ , and  $q_{i(x)}^x$  for case (c) is the local direction for  $M_i$  on a certain open neighborhood  $U(x)$  of  $x$ . Under (B2) and (B4), we may chose  $U(x)$  so that  $\varphi^x(z) \supset K_{i(x)}(z) \cap P_{i(x)}(z) \neq \emptyset$  for all  $z \in U(x)$  for case (a). Moreover, under (B3), we may chose  $U(x)$  so that  $\varphi^x(z) \supset M_{i(x)}(z) \neq \emptyset$  for all  $z \in U(x)$  for cases (b) and (c). Note also that for each  $x \in X$ ,  $\varphi^x : U(x) \rightarrow X_{i(x)}$  is convex valued and  $z_{i(x)} \notin \varphi^x(z)$  for each  $z \in U(x)$ . Since  $X$  is compact, there is a finite sub-covering of the covering  $\{U(x)\}_{x \in X}$  of  $X$ . Let  $U(x^1), \dots, U(x^n)$  be such a sub-covering, and let  $i^1, \dots, i^n$  be names of agents fixed respectively for  $x^1, \dots, x^n$  in defining  $U(x^1), \dots, U(x^n)$  and  $\varphi^{x^1}, \dots, \varphi^{x^n}$ . Moreover, for each  $t = 1, \dots, n$ , let correspondence  $p^t : U(x^t) \rightarrow E'$  be  $p_{i(x^t)}^{x^t}$  for case (a) and be  $q_{i(x^t)}^{x^t}$  for cases (b) and (c). Let  $\beta_t : X \rightarrow [0, 1], t = 1, \dots, n$ , be a partition of unity subordinated to  $U(x^1), \dots, U(x^n)$ , satisfying that  $(\beta_t(x) > 0) \iff (x \in U(x^t))$ . For each  $x$  and  $i$ , denote by  $T_i$  and  $S_x$  the set of indices  $\{t | i = i^t\}$  and  $\{t | x \in U(x^t)\}$ , respectively, and define a correspondence  $\Phi : X \rightarrow X$  as  $\Phi(x) = \prod_{i \in I} \Phi_i(x)$ , where  $\Phi_i(x) = V_i(x_i, \sum_{t \in S_x \cap T_i} \frac{\beta_t(x)}{\sum_{s \in S_x \cap T_i} \beta_s(x)} p^t(x))$  if  $S_x \cap T_i \neq \emptyset$ , and  $\Phi_i(x) = X_i$  if  $S_x \cap T_i = \emptyset$ . Note that since for each  $x$ , there is at least one  $i$  such that  $x \in U(x^t)$  and  $i = i^t$ , (i.e.,  $S_x \cap T_i \neq \emptyset$ ), the mapping  $\Phi$  has no fixed point. On the other hand, the mapping  $\Phi$  is clearly convex valued. Moreover,  $\Phi$  is non-empty valued since every  $\varphi^{x^t}$  is non-empty valued on  $U(x^t)$  and since  $\Phi_i(x) \supset \bigcap_{t \in S_x \cap T_i} V_i(x_i, p^t(x))$  for each  $i$  and  $x$  such that  $S_x \cap T_i \neq \emptyset$ . (Indeed,  $\bigcap_{t \in S_x \cap T_i} V_i(x_i, p^t(x))$  is clearly non-empty when all elements of  $T_i \cap S_x$  are type (b) or (c). We may also check the set to be non-empty when the elements of  $T_i \cap S_x$  are type (a) or (c), under condition (B5). Note that  $U(x^t)$ 's of type (a) and type (b) never intersect.) Furthermore,  $\Phi$  has a compact valued upper semi-continuous direction near each  $x \in X$  since  $S_x \cap T_i \neq \emptyset$  implies that  $\Phi_i(z) \subset V_i(z_i, \sum_{t \in S_x \cap T_i} \frac{\beta_t(z)}{\sum_{s \in S_x \cap T_i} \beta_s(z)} p^t(z))$  for all  $z \in \bigcap_{t \in S_x \cap T_i} U(x^t)$ . Hence,  $\Phi$  has a fixed

point by Theorem 1, so we have a contradiction. ■

### 3 APPLICATION FOR THE DEFAULT ECONOMY

In the previous section, we have shown the theorem on the existence of equilibria for an abstract economy under non-convex constraint correspondence. We apply in this section the theorem to prove the existence of equilibria allowing for default and/or bankruptcy.

Under general equilibrium settings, problems with asymmetric information (including default and/or bankruptcy as well as adverse selection and moral hazard) are closely related to non-convex agents' budget constraints since we may settle the problem into the situation that the constraint (the price) as a buyer is different from the constraint from the view point of a seller. Especially, the non-convexity problem in economies with default and/or bankruptcy is classical and has been treated in many literature (see, e.g., Green (1974), Grandmont (1982), Eichberger (1989)).

Recent several authors (see, e.g., Zame (1993), Dubey-Geanakoplos-Shubik (2000)) have avoided the problem by considering a completely anonymous asset market in which the average default rate (for each asset in the market) is perfectly expected and each agent is allowed simultaneously to buy and sell (go long and short in) the same asset. It is interesting that they have changed the problem about moral hazard into the choice problem under the taste for morals of each agent. Such an approach, however, (1) fails to describe an accidental or unintended default and/or bankruptcy, (2) implicitly changes the concept of standard economic equilibrium where for each commodity, demand and supply are treated in a total net amount, and, especially, (3) they have used the concept of the "complete anonymity" as a "necessary" condition for the existence of equilibria. Of course, it is more desirable to show an existence of equilibrium directly with non-convex budget constraints with or without the complete anonymity and perfect foresights.

To overcome the non-convexity problem on budget constraints in economies with default or bankruptcy, many authors use the smoothing method under continuum of traders or the methods of Starr (1969). An exceptional approach (in temporary general equilibrium model) with money and the central bank under a special bankruptcy rule is Eichberger (1989).<sup>6</sup> Also, Green (1974) proved the existence of an approximate equilibrium, following the Starr method, by introducing a bankruptcy rule and disutility from the extent of bankruptcy. In the following, we treat the problem by using the concept we have used in sections 1 and 2, the direction of correspondences.

#### 3.1 The Default Economy

Now let us consider a two-period economy, period 1 (present) and 2 (future). There are  $m$  types of agents indexed by  $i = 1, \dots, m$ . In periods 1 and 2 there are, respectively,  $\ell$  types of real commodities. We also assume that there are  $K$  types of assets indexed by  $k = 1, \dots, K$  that is sold and purchased in the asset market in each period.<sup>7</sup> Asset  $k$ ,  $A^k \in \mathcal{R}^\ell$ , specifies bundles of goods to

<sup>6</sup> A further development of the model in Eichberger (1989) is treated in Yoshimachi (1999).

<sup>7</sup> We suppose that the number of states in period 2 is one for the sake of notational simplicity.



be delivered in period 2. We assume that the payoff of each asset is specified to a certain commodity. Hence, in each period, two types of markets, the *spot market* for  $\ell$  types of real commodities and the *asset market* for  $K$  types of assets, exist.

Prices for commodities in the spot market and the asset market in period 1 are denoted by  $p^1 = (p^{11}, \dots, p^{1\ell}) \in R_+^\ell$  and  $q^1 = (q^{11}, \dots, q^{1K}) \in R_+^K$ , respectively. The *set of prices* in period 1 is defined by  $\Delta = \{(p^1, q^1) \in R_+^\ell \times R_+^K : p^{11} + \dots + p^{1\ell} + q^{11} + \dots + q^{1K} = 1\}$  be the set of prices.

Agent  $i$  has an *initial endowment*,  $\omega_i = (\omega_i^1, \omega_i^2) \in R^\ell \times R^\ell$ , of real commodities, and a *consumption set*,  $X_i \subset R^\ell \times R^\ell$ . We assume that  $X_i$  is a closed convex subset bounded from below such that  $X_i = X_i + R_+^{2\ell}$  for each  $i$ . The preference of agent  $i$  which may possibly be non-ordered is represented at each  $x \in \prod_{i=1}^m X_i$  by a continuous concave utility function  $u_i^x : X_i \rightarrow R$ . We also assume that at each  $x \in X$ , the value of utility function  $u_i^x$  at  $y_i \in X_i$ , i.e.,  $u_i^x(y_i)$  is continuous with respect to  $(x, y_i) \in X \times X_i$ . In order to determine his plan, agent  $i$  forecast the prices of commodities and the default rate of each asset in period 2. We assume that agent  $i$  has continuous *expectation functions* for prices,

$$p_i : \Delta \times \Delta \times [0, 1]^K \rightarrow \Delta \times \Delta \times [0, 1]^K, \tag{1}$$

$$q_i : \Delta \times \Delta \times [0, 1]^K \rightarrow \Delta \times \Delta \times [0, 1]^K, \tag{2}$$

and for *delivery rates*,

$$\rho_i : \Delta \times \Delta \times [0, 1]^K \rightarrow \Delta \times \Delta \times [0, 1]^K. \tag{3}$$

For a list of other person's consumption plans,  $x_j, j = 1, \dots, m, j \neq i$ , agent  $i$  chooses  $x_i = (x_i^1, x_i^2) \in X_i, \theta_i = (\theta_i^1, \theta_i^2) \in R_+^K \times R_+^K$ , and  $\varphi_i = (\varphi_i^1, \varphi_i^2) \in R_+^K \times R_+^K$  so that  $x_i$  maximize his utility function  $u_i^{(x_1, \dots, x_i, \dots, x_m)}$  under his budget constraints, where  $\theta_i$  and  $\varphi_i$  indicate the quantity of assets *purchased* and *sold*, respectively. Notice that for each agent  $i$ , each asset  $k$  has a *limit on the quantity for sales*,  $Q_i^{tk}, \varphi_i^{tk} \leq Q_i^{tk}, t = 1, 2$ . We now define the *budget set* of agent  $i, K_i(p, q, \rho)$ . (An element  $(p, q, \rho) = ((p^1, p^2), (q^1, q^2), \rho)$  of  $\Delta \times \Delta \times [0, 1]^K$  will be denoted by  $s$  in the following.) In period 1, agent  $i$  should satisfy

$$p^1 \cdot x_i^1 + q^1 \cdot \theta_i^1 \leq p^1 \cdot \omega_i^1 + q^1 \cdot \varphi_i^1. \tag{4}$$

In the same way, we may suppose an expected budget constraint at period 2 for  $i$  to be

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<sup>8</sup> In the domains and ranges of these functions, the first  $\Delta$  denotes the set of prices in period 1, the second  $\Delta$  denotes the set of prices in period 2, and  $[0, 1]^K$  denotes the set of delivery rates. Of course, the case with rational expectations is treated as the special (identity function) case of these expectation functions. For the concept of delivery rates, see Zame (1993) and Dubey-Geanakoplos-Shubik (2000).

$$p_i^2(s) \cdot (x_i^2)^+ + q_i^2(s) \cdot \theta_i^2 \leq \max \left\{ 0, p_i^2(s) \cdot (\omega_i^2 + (x_i^2)^-) + q_i^2(s) \cdot \varphi_i^2 + \sum_{k=1}^K p_i^2(s) A^{1k} (\rho_i^k(s) \theta_i^{1k} - \varphi_i^{1k}) \right\}, \quad (5)$$

where  $(x_i^2)^+ = \sup\{0, x_i^2\}$ ,  $(x_i^2)^- = \sup\{0, -x_i^2\}$ , and the max operation allows for the possibility of default. Also, we have to allow for the bound on sales for asset  $k$  for agent  $i$  and a natural condition on the quantity of assets purchased and sold, respectively.

$$\forall t, \forall k, \quad \varphi_i^{tk} \leq Q_i^{tk}, \quad (6)$$

$$\forall t, \forall k, \quad \theta_i^{tk} \cdot \varphi_i^{tk} = 0. \quad (7)$$

Thus, the budget set of agent  $i$  is

$$K_i(p, q, \rho) = \{(x_i, \theta_i, \varphi_i) | (x_i, \theta_i, \varphi_i) \text{ satisfies (4), (5), (6) and (7) under } (p, q, \rho)\}. \quad (8)$$

As in Dubey, Geanakoplos, and Shubik (2000), we define for each agent  $i$  a constant,  $\lambda_i \in R_+$ , the *real default penalty on  $i$* . We assume that the default penalties are assessed directly in terms of utility of agent  $i$  and are proportional to the size of the default. Hence, under  $s = (p, q, \rho)$ , the utility level of  $i$  for  $(y_i, \theta_i, \varphi_i)$  at  $x \in X$ ,  $W_i^x(y_i, \theta_i, \varphi_i, s)$  is as follows:

$$W_i^x(y_i, \theta_i, \varphi_i, s) = u_i^x(y_i) + U_i^x(\theta_i^2, \varphi_i^2) - \lambda_i \max \left\{ 0, \sum_{k=1}^K p_i^2(s) A^{1k} (\varphi_i^{1k} - \rho_i^k(s) \theta_i^{1k}) - p_i^2(s) \cdot (\omega_i^2 + (x_i^2)^-) - q_i^2(s) \cdot \varphi_i^2 \right\}, \quad (9)$$

where constraints (4)–(7) are not taken into consideration and the term  $U_i^x(\theta_i^2, \varphi_i^2)$  denotes an appropriately defined *indirect utility* for assets in the second period. In the following, we neglect every terms on  $\theta_i^2$  and  $\varphi_i^2$  in equation (9) for the sake of simplicity. When condition (4) is satisfied, the second entry in the last term in (9) (i.e.,  $\sum_{k=1}^K p_i^2(s) A^{1k} (\varphi_i^{1k} - \rho_i^k(s) \theta_i^{1k}) - p_i^2(s) \cdot (\omega_i^2 + (x_i^2)^-) - q_i^2(s) \cdot \varphi_i^2$ ), represents the size of the default of  $i$  associated with the action and plan  $(x_i, \theta_i, \varphi_i)$  under  $s$ . We denote the amount by  $D_i(x_i, \theta_i, \varphi_i, s)$ .

Denote by  $\mathcal{E} = (X_i, \omega_i, u_i, (A^k)_{k=1}^K, \lambda_i, (Q_i^k)_{k=1}^K)_{i=1}^m$  the economy stated in the above. An *equilibrium* for economy  $\mathcal{E}$  is  $(x, \theta, \varphi, p^1, q^1, p^2, q^2, \rho) \in \prod_{i=1}^m X_i \times (R_+^K)^2 \times (R_+^K)^2 \times \Delta \times \Delta \times [0, 1]^K$  such that for  $i \in \{1, \dots, m\}$ ,  $(x_i, \theta_i, \varphi_i) \in \arg \max W_i^x(y_i, \theta_i, \varphi_i)$  over  $i$ 's budget under  $(p, q, \rho)$ , and *market clearing conditions* together with a *specification of expectation functions* such that

$$\sum_{i=1}^m (x_i^1 - \omega_i^1) = 0 \text{ and } \sum_{i=1}^m (\theta_i^1 - \varphi_i^1) = 0, \quad (10)$$

$$\forall i = 1, \dots, m, p_i^1 \text{ and } q_i^1 \text{ are identity functions,} \quad (11)$$

(for the *short run equilibrium*), or

$$\sum_{i=1}^m (x_i - \omega_i) = 0 \text{ and } \sum_{i=1}^m (\theta_i - \varphi_i) = 0, \tag{12}$$

$$\forall k = 1, \dots, K, 1 - \rho^k = \left( \sum_{i=1}^m D_i(x_i, \theta_i, \varphi_i, s) \right) / \left( \sum_{i=1}^m p_i^2(s) A^{1k} \varphi_i^{1k} \right), \tag{13}$$

$$\forall i = 1, \dots, m, p_i, q_i, \rho_i \text{ are identity functions}, \tag{14}$$

(for the *long run equilibrium* with perfect foresights). As in Eichberger (1989) and Dubey, Geanakoplos, and Shubik (2000), consumers or agents having budget constraints like the above  $K_i(p, q, \rho)$  have a common problem that the budget set may not be convex as long as we suppose the natural setting (4).

### 3.2 Existence of Equilibrium

For each  $\epsilon > 0$  sufficiently small, we define an abstract economy  $\mathcal{E}_\epsilon$  for the economy  $\mathcal{E}$  as follows.

Let  $\Delta_\epsilon$  be the set of all vectors in  $\Delta$  whose all coordinates are greater than or equal to  $\epsilon > 0$ , and denote by  $S$  the set  $\Delta \times \Delta \times [0, 1]^K$ . If we restrict the set of prices for economy  $\mathcal{E}$  on the compact set  $S_\epsilon = \Delta_\epsilon \times \Delta_\epsilon \times [\epsilon, 1]^K$ , then the set of all consumption and asset holding levels that are not associated with default is also confined in a certain compact cube,  $C_\epsilon \subset (R^\ell \times R_+^K) \times (R^\ell \times R_+^K)$ . Under the default penalty, we may take such a  $C_\epsilon$  so large that for each  $x \in \prod_{i=1}^m X_i$ , every action  $(y_i, \theta_i, \varphi_i)$  which maximizes  $W_i^x$  in  $C_\epsilon$  maximizes  $W_i^x$  in  $(X_i \times R_+^K \times R_+^K) \times (X_i \times R_+^K \times R_+^K)$ .<sup>9</sup> For  $i = 1, \dots, m$ , let  $Z_i = (X_i \times R_+^K \times R_+^K) \times (X_i \times R_+^K \times R_+^K)$ , and let the choice set  $Z_{i\epsilon}$  of abstract economy  $\mathcal{E}_\epsilon$  as  $C_\epsilon \cap Z_i$ .

For each agent  $i = 1, \dots, m$ , define the preference correspondence  $P_{i\epsilon}$  as the better set correspondence at  $x \in \prod_{i=1}^m X_i$  under  $W_i^x$  on  $Z_{i\epsilon}$ , and define the constraint correspondence  $K_{i\epsilon}$  as (8) on  $Z_{i\epsilon}$ .

For 0-th agent, an auctioneer, define a utility (payoff) function,  $W_0(x, \theta, \varphi, p, q, \rho)$ , as

$$W_0(x, \theta, \varphi, p, q, \rho) = p^1 \sum_{i=1}^m (x_i^1 - \omega_i^1) + q^1 \sum_{i=1}^m (\theta_i^1 - \varphi_i^1), \tag{15}$$

for the short run equilibrium, and as

$$W_0(x, \theta, \varphi, p, q, \rho) = p \sum_{i=1}^m (x_i - \omega_i) + q \sum_{i=1}^m (\theta_i - \varphi_i), \tag{16}$$

for the long run equilibrium with perfect foresights. For agent 0, define the choice set as  $Z_{0\epsilon} = S_\epsilon$ , ( $Z_0 = S$ ), and the constraint correspondence as  $K_{0\epsilon}(x, \theta, \varphi, p, q, \rho) = Z_{0\epsilon}$ .

Let the set of players for  $\mathcal{E}_\epsilon$  be  $I = \{0, 1, \dots, m\}$ . Then, the following two additional structures

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<sup>9</sup> As stated before, the last two terms  $R_+^K \times R_+^K$  (the amount of assets purchased and sold in the second period) together with their all entries in  $W_i^x$  will be neglected in the following for the sake of simplicity.

(the minimum wealth structure and the minimum delivery rate structure) enable us to show that for a sufficiently small  $\epsilon > 0$ , an equilibrium for  $\mathcal{E}_\epsilon$  is an equilibrium for the default economy  $\mathcal{E}$ .

**ASSUMPTION (Minimum Wealth) :** *There is a continuous wealth transfer mechanism (a function  $\tau^w : \Delta \times \Delta \times [0, 1]^K \rightarrow R^m$ ) which assures for each agent  $i$  and  $(p, q, \rho)$ , a wealth level which is strictly greater than the minimum level under  $(p, q, \rho)$ .*

**ASSUMPTION (Minimum Delivery Rate) :** *There is a continuous insurance mechanism (a function  $\tau^d : \Delta \times \Delta \times [0, 1]^K \rightarrow [0, 1]^K$  financed by the transfer of wealth) which assures for each asset  $k$  and  $(p, q, \rho)$ , a strictly positive minimum delivery rate under  $(p, q, \rho)$ .*

It is true that in considering these two mechanisms, we have to rewrite the budget for  $i$  (equations (4) and (5)) and reconsider all related assertions. We shall omit, however, the process since (i) it is a routine task (e.g., use a continuous transfer function on endowments  $\tau : \Delta \times \Delta \times [0, 1]^K \rightarrow R^\ell \times R^\ell$ ), and (ii) they are not indispensable structures.<sup>10</sup>

We now state the main theorem in this section. Let  $M_i(s)$  be the set of elements of  $K_i(s)$  such that  $D(x_i, \theta_i, \varphi_i, s) = 0$  for each  $s = (p, q, \rho)$ . It is clear that  $M_i(s)$ , the set of *moral actions and plans*, is convex. It is also easy to verify that for each  $i = 1, \dots, m$ ,  $W_i^x(y_i, \theta_i, \varphi_i, s)$  is continuous with respect to  $x, y_i, \theta_i, \varphi_i$ , and  $s$ , (see equation (9) for the definition of  $W_i^x$ ), and at each  $x$  and  $s$ , the better set correspondence defined by  $W_i^x$  on  $Z_\epsilon = \prod_{i=0}^m Z_{i\epsilon}$  into  $Z_{i\epsilon}$  is convex valued. Hence, each  $P_{i\epsilon}$ ,  $i = 1, \dots, m$ , is a convex valued correspondence having continuous local utility representation, so that has a compact valued upper semi-continuous direction at every point.

**THEOREM 3.** *Assume the minimum wealth condition and the minimum delivery rate condition. Then, the economy  $\mathcal{E} = (X_i, \omega_i, u_i, (A^k)_{k=1}^K, \lambda_i, (Q_i^k)_{k=1}^K)_{i=1}^m$  has a long run (perfect foresight) equilibrium if the following condition is satisfied.*

(Local Directions for Morals) *At each  $(x_i, \theta_i, \varphi_i)$ ,  $i = 1, \dots, m$ , and  $s = (p, q, \rho)$  such that  $(x_i, \theta_i, \varphi_i) \notin K_i(s)$  and  $P_{i\epsilon}(x, \theta, \varphi) \cap K_i(s) = \emptyset$ , there is an open neighbourhood  $U(s, x, \theta, \varphi)$  of  $(s, x, \theta, \varphi) \in Z = \prod_{i=0}^m Z_i$  such that for all  $z \in U(s, x, \theta, \varphi)$ ,  $P_{i\epsilon}(z_1, \dots, z_m) \cap K_i(z_0) \neq \emptyset$  implies  $P_{i\epsilon}(z_1, \dots, z_m) \cap M_i(z_0) \neq \emptyset$ .*

*Adding to the above condition, if we suppose for each  $i$  and  $k$ , that the range of  $\rho_i^k$  is a subset of strictly positive reals, then a short run equilibrium exists.*

<sup>10</sup> It is always possible to replace these two mechanisms with more individualistic conditions on endowments, weaker boundary conditions, and standard limit arguments. The mechanism on default penalties, however, is rather essential in this paper. The ordinary cubic truncation method fails to assure demands under truncation to be true maximal elements under the non-convexity of constraints.

PROOF. The compactness of  $\Delta \times \Delta \times [0, 1]^K$  assures that there are positive numbers  $\delta > 0$  and  $\mu > 0$  such that all agents have an ability to pay (without default) greater than  $\delta$  and all delivery rates are greater than  $\mu$ . For the case with short run equilibrium, we may suppose without loss of generality that the range of  $\rho_i^k$  is a subset of  $[\mu, 1]$ . Since utility functions are strictly positive and  $X_i \supset X_i + R_+^{2\ell}$ , and since feasible utility levels are bounded, the existence of  $\delta$  and  $\mu$  means that all equilibrium prices, if such exist, should be in  $\Delta_\epsilon$  for an  $\epsilon > 0$  sufficiently small. Moreover, since the aggregate supply in economy  $\mathcal{E}$  has an upper bound, and since the existence of an upper bound for the aggregate demand means the existence of an upper bound for the utility levels (without default) for all  $i$ , we may take such an  $\epsilon$  so small that under any boundary price  $(p, q) \in \Delta_\epsilon \times \Delta_\epsilon$  ( $(p^1, q^1) \in \Delta_\epsilon$ , for the short run case) and a delivery rate  $\rho^k \geq \mu$ , the summation of excess demands of all commodities (in period 1, resp., for the short run case) under  $(p, q, \rho)$  is positive. (Indeed, by taking  $\epsilon$  arbitrarily small, since  $C_\epsilon$  includes all non default actions and plans, the utility level under boundary prices can be taken arbitrarily large. Hence, the total excess demand under such boundary prices cannot be bounded.) That is, the price system  $\bar{p} = (\frac{1}{\ell}, \dots, \frac{1}{\ell})$  and  $\bar{q} = (\frac{1}{K}, \dots, \frac{1}{K})$  ( $\bar{p}^1$  and  $\bar{q}^1$ , resp., for the short run) appreciates all boundary excess demands positively. This fact together with the Walras' law in all periods (in period 1, resp., for the short run) implies that the maximal point for player 0 in abstract economy  $\mathcal{E}_\epsilon$  for such an  $\epsilon$  is the market clearing price. Moreover, since we have chosen the bound  $C_\epsilon$  for  $\mathcal{E}_\epsilon$  so that for each  $i$ , every solution to the utility maximization in  $\mathcal{E}_\epsilon$  to be a solution in  $\mathcal{E}$ , we have shown that an equilibrium for  $\mathcal{E}_\epsilon$  is an equilibrium for  $\mathcal{E}$ .

Next, we have to show that abstract economy  $(Z_{i\epsilon}, P_{i\epsilon}, K_{i\epsilon})_{i \in I}$  for economy  $\mathcal{E}$  satisfies all conditions in Corollary 2.1. Condition (B1) is clearly satisfied. For consumers, the minimum wealth condition assures the continuity of  $K_i$  and  $M_i$  together with the non-emptiness for the value of  $M_i$ . By the convexity and closedness for the value of  $M_i$ , (B3) is automatically satisfied. Moreover, as stated before, the better set correspondence under  $W_i^x$  has a compact valued upper semi-continuous direction near every point, so that (B2) is also satisfied. Furthermore, the continuity of  $W_i^x(y_i, \theta_i, \varphi_i, s)$  with respect to  $x, y_i, \theta_i, \varphi_i$ , and  $s$  means that  $P_{i\epsilon}$  has an open graph. This, together with the continuity of  $K_i$ , assures (B4). It is clear that the condition on Local Directions for Morals assures condition (B5). For the auctioneer, the upper semi-continuity and closed convex non-empty valuedness of price transformation correspondence satisfies condition (B2). Moreover, by considering  $M_i(s) = K_i(s) = Z_{i\epsilon}$  for all  $s$ , (B3), (B4), and (B5) are automatically satisfied. Hence,  $\mathcal{E}_\epsilon$  satisfies all conditions in Theorem 2. ■

A typical example for preferences assuring the condition of local directions in Theorem 3 is the family of CES utility functions on the positive orthant consumption sets. For any utility function, however, we may expect that an appropriate level of utility punishment always assures the condition since at each  $x$  such that  $x_i \notin K_i(x)$ , the better set at  $x_i$  eventually include a point in  $M_i(x)$  for all  $\lambda_i$  sufficiently large.

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