



Title	Some decomposition theorems on abelian groups and their generalisations. II
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Citation	Osaka Journal of Mathematics. 1979, 16(1), p. 45-55
Version Type	VoR
URL	https://doi.org/10.18910/9335
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SOME DECOMPOSITIONS THEOREMS ON ABELIAN GROUPS AND THEIR GENERALISATIONS-II

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(Received December 1, 1977)

In [7] study of those modules M_R which satisfy the following two conditions was initiated:

- (I) Every finitely generated submodule of every homomorphic image of M is a direct sum of uniserial modules.
- (II) Given two uniserial submodules U and V of a homomorphic image of M , for any submodule W of U any non-zero homomorphism, $f: W \rightarrow V$ can be extended to a homomorphism $g: U \rightarrow V$ provided the composition length $d(U/W) \leq d(V/f(W))$.

It was shown that some of the well known decomposition theorems for torsion abelian groups, can be generalized to modules satisfying (I) and (II). Here we introduce another condition:

- (III) For any finitely generated submodules N of M , $R/\text{ann}(N)$ is right artinian.

It can be easily seen that any torsion module over a bounded (*hnp*)-ring satisfies (I), (II) and (III). Let M be a module satisfying (I) and (II). The concept of *h*-pure submodules of M was introduced in [7]; if in addition M satisfies (III) it is shown in section one, that any submodule N of M is *h*-pure if and only if it is pure (Theorem (1.3)). Theorem (1.4) shows that any complement of $H_k(M)$ in M is a summand of M . In section 2, the concept of basic submodule is introduced. It is shown that any module M satisfying (I), (II) and (III) has a basic submodule and any two basic submodules of M are isomorphic (Theorem (2.7)). This result generalizes the corresponding well known result on basic subgroups of torsion abelian groups. In section 3, a decomposition theorem is proved; which states that given any module M satisfying (I) and (II), such that $M/\text{socle}(M)$ is decomposable then M is decomposable.

Preliminaries: Let M be a module satisfying (I) and (II). Let us recall some definitions from [6, 7]. An element x in M is said to be uniform if xR is a non-zero uniform (hence uniserial) submodule. For any uniform element x of M , its exponent $e(x)$ is defined to be equal to the composition length $d(xR)$;

the height of x is the supremum of all $d(T/xR)$ where T is a uniserial submodule of M containing x . The height of x is denoted by $H_M(x)$ (or simply by $H(x)$). For any $k \geq 0$, $H_k(M)$ denotes the submodule of M generated by all those uniform elements x of M for which $H(x) \geq k$. A submodule N of M is said to be an h -pure submodule if $N \cap H_k(M) = H_k(N)$ for all k . M is said to be bounded if there exists a positive integer k such that $H(x) \leq k$ for all uniform elements x in M . M is said to be decomposable if it is a direct sum of uniserial modules. For definition and elementary properties of pure submodules we refer to Stenström [8]. For any ring R , $J(R)$ denotes the Jacobson radical of R .

Lemma 1.1. *Let M_R be a module satisfying (I), (II) and (III) and X be a uniserial submodule of M having*

$$X = X_0 > X_1 > X_2 \cdots > X_t = 0$$

as its unique composition series. If for $0 \leq i \leq t-1$, $P_i = \text{ann}(X_i/X_{i+1})$ then $X_i P_i = X_{i+1}$.

Proof. Let $A = \text{ann}(X)$. Since $S = R/A$ is right artinian, $X_i J(S) = X_{i+1}$ and $J(S) \subset P_i/A$, we have $X_i P_i = X_{i+1}$.

Lemma 1.2. *Let a module M_R satisfy (I) and (II). If for any finitely many uniform elements x_1, x_2, \dots, x_n in M*

$$\sum_{i=1}^n x_i R = \bigoplus \sum_{j=1}^m y_j R$$

where $y_j R$ are uniserial, then $m \leq n$.

Proof. The result follows by induction on n .

The result that any submodule N of a torsion module over a bounded (hnp)-ring is pure if and only if it is h -pure was proved by M. Khan in [2]. The proof of the following is adapted from [2].

Theorem 1.3. *Let M_R be a module satisfying (I), (II) and (III) and N a submodule of M . Then N is h -pure if and only if it is a pure submodule.*

Proof. Let N be h -pure. Consider any finite system of linear equations

$$\sum_i x_i \gamma_{ij} = s_j \in N$$

which admits a solution $\{x_i\}$ in M . Let $K = \sum x_i R + N$. Then K/N is a finitely generated module. So by condition (I).

$$K/N = \bigoplus \sum T_\alpha/N$$

where each T_α/N is uniserial. Then by [7, Lemma 2(i)], $T_\alpha = y_\alpha R \oplus N$. Hence

$$K = K_1 \oplus N$$

This gives that the above given system of equations are also solvable in N . Hence N is a pure submodule of M .

Let now N be a pure submodule. This immediately gives $MA \cap N = NA$ for all ideals A of R . Suppose for some k , $H_k(M) \cap N \neq H_k(N)$. We choose k smallest with $H_k(M) \cap N \neq H_k(N)$. We can find a uniform element x of smallest exponent such that $x \in H_k(M) \cap N$ but $x \notin H_k(N)$. Then $x \in H_{k-1}(N)$. By definition there exists a uniform element y in M such that $x \in yR$ and $d(yR/xR) = k$.

$x \in H_{k-1}(N)$ shows that there exist a uniform element $u \in N$ such that $x \in uR$ and $d(uR/xR) = k-1$. Let $zR = \text{socle}(xR)$ and $m = e(x)$. Then $d(uR/zR) = m + k - 2$, gives $H_N(z) \geq m + k - 2$. Suppose $H_N(z) \geq m + k - 1$. We can then find a uniform element $v \in N$ such that $z \in vR$ and $d(vR/zR) = m + k - 1$. By condition (II), we get an isomorphism $\sigma: yR \rightarrow vR$ which is identity on zR . Then $x - \sigma(x)$ is a uniform element with $e(x - \sigma(x)) < e(x)$, $x - \sigma(x) \in N \cap H_k(M)$, but $x - \sigma(x) \notin H_k(N)$, since $\sigma(x) \in H_k(N)$. This contradicts the choice of x . Hence $H_N(z) = k + m - 2 = d(uR/zR)$. So by [7, Lemma 1]

$$N = uR \oplus N_1$$

uR is also a pure submodule. Now $d(uR/zR) = d(yR/zR) - 1$. By (1.1) we can find prime ideals $P_1, P_2, \dots, P_{m+k-1}$ such that R/P_i is simple artinian for all i and $yR > yP_1 > yP_1P_2 > \dots > yP_1P_2 \dots P_{m+k-1} = 0$ with $zR = yP_1P_2 \dots P_{m+k-2}$. By condition (II) $uR \cong yP_1$ and hence $uP_2P_3 \dots P_{m+k-1} = 0$. However $yP_2P_3 \dots P_{m+k-1} \neq 0$. Thus $z \in MP_2P_3 \dots P_{m+k-1} \cap uR = uP_2P_3 \dots P_{m+k-1} = 0$. This is a contradiction. Hence N is an k -pure submodule of M .

The following theorem generalizes Erdelyi's theorem [1, Theorem (24.8)].

Theorem 1.4. *Let M be a module satisfying (I), (II) and (III) then for any $k \geq 1$, any complement of $H_k(M)$ is a summand of M .*

Proof. Let N be a complement of $H_k(M)$. Then N is bounded. If we show that N is a pure submodule, the result follows from [7, Theorem 3]. In view of (1.2) it is equivalent to showing $H_n(M) \cap N = H_n(N)$ for every n . Since $H_k(M) \cap N = 0 = H_k(N)$, the result holds for $n \geq k$. To apply induction we suppose that for some n with $0 \leq n < k$, $H_n(M) \cap N = H_n(N)$, we prove that same for $n+1$. Let the contrary hold. Then there exists a uniform element $x \in H_{n+1}(M) \cap N$ such that $x \notin H_{n+1}(N)$. Then $H_N(x) = n$. Now there exists a uniform element y in M such that $d(yR/xR) = n+1$. Let $\text{socle}(yR/xR) = x_1R/xR$. If $x_1 \in N$, we get $x_1 \in N \cap H_n(M) = H_n(N)$ and hence $x \in H_{n+1}(N)$. This is a contradiction. Consequently $x_1 \notin N$ and $(N + x_1R) \cap H_k(M) \neq 0$. Thus there exists a uniform element $z \in H_k(M)$ such that $z = u + x_1s$ for some $u \in N$ and $s \in R$. If $x_1sR \neq x_1R$, then $x_1sR \subset x_1R$ and $z \in N$; this is a contradiction to the fact that

$N \cap H_k(M) = 0$. So $x_1sR = x_1R$ and $x_1 = x_1ss'$, $s' \in R$. Then $zs' = us' + x_1 \in (N + x_1R) \cap H_k(M)$ and $zs' \neq 0$. So we can suppose that $x_1s = x_1$. Thus $z = u + x_1$.

Let $P = \text{ann}(x_1R/xR)$. By (1.1) $xR = x_1P$. So for any $r \in P$, $zr = 0$ and $ur = -x_1r$. Now $H(z) \geq k > n$, $H(x_1) \geq n$, gives $u \in H_n(M) \cap N = H_n(N)$. For some $r_0 \in P$, $x = x_1r_0 = -ur_0$. If uR is uniform and $ur_0R < uR$, then $H_N(ur_0) \geq H_N(u) + 1 \geq n + 1$ and hence $H_N(x) \geq n + 1$; this is a contradiction. Hence the following two cases arise.

Case I: uR is uniform and $ur_0R = uR$. In this case $u = x_1b_0 = xrb$ for some $b \in R$ and $z = x_1r_0b + x_1 = x_1c$, $c \in R$. Thus $zR = x_1R$ and $x_1 \in H_k(M)$. This shows that $H(x_1) \geq k$ and hence $H(x) \geq k + 1$. This contradicts the fact that $N \cap H_{k+1}(N) = 0$.

Case II: uR is not uniform. The fact that $u \in zR + x_1R$ and that zR, x_1R are uniform together with (1.2) yields $uR = u_1R \oplus u_2R$ with u_1, u_2 both uniform. Further we can take $u = u_1 + u_2$. Then $xR = uP = u_1P \oplus u_2P$. So $u_1P = 0$ or $u_2P = 0$. To be definite let $u_2P = 0$. Then u_2R is a simple R -module and $x_1R = u_1P$. Let $u_1P < u_1R$ then $xR = u_1P = v_1R$ for some $v_1 \in u_1R$. Now $H(u_1) \geq \min(H(x_1), H(x)) \geq n$. So by induction hypothesis $u_1 \in H_n(N)$ and hence $H_N(v_1) \geq n + 1$. Consequently $xR = v_1R$ gives $H_N(x) \geq n + 1$. This is a contradiction. Thus $u_1R = u_1P = xR$. Hence $u_1 = xa$, $a \in R$. Consequently $z = u_2 + xa + x_1 = u_2 + x_1s$, $s \in R$. This reduces to case I and hence again gives a contradiction. Hence N is a pure submodule. This proves the theorem.

2. Basic submodules

DEFINITION 2.1. Let M be a module satisfying (I) and (II). A subset $\{x_\lambda : \lambda \in \Lambda\}$ of uniform elements of M is called h -pure independent if it is independent in the sense that $\sum x_\lambda R$ is direct, and $\sum x_\lambda R$ is an h -pure submodule of M .

The following Lemma generalizes [1, Lemma (29.1)].

Lemma 2.2. Let a module M_R satisfy (I) and (II). An h -pure independent subset $\{x_\lambda : \lambda \in \Lambda\}$ is maximal if and only if M/L , where $L = \sum x_\lambda R$, is a direct sum of infinite length uniform submodules.

Proof. The result follows from [7, Lemma 2 and Theorem 5].

This motivates the following:

DEFINITION 2.3. Let M be a module satisfying (I) and (II). A submodule B of M is called a basic submodule of M if it satisfies the following:

- (i) B is an h -pure submodule.
- (ii) B is a direct sum of uniserial modules.

(iii) M/B is a direct sum of uniform modules of infinite lengths.

[7, Lemma 2 and Theorem 5] and the fact that union of any chain of h -pure submodules is an h -pure submodule gives the following:

Lemma 2.4. *Any module satisfying (I) and (II) has a basic submodule.*

The main purpose of this section is to prove that any two basic submodules of a module M satisfying (I), (II) and (III) are isomorphic. The following theorem generalizes [1, Theorem (29.3)]. Since the proof is on similar lines it is omitted.

Theorem 2.5. *Let M be a module satisfying (I), (II) and (III) and B be a submodule of M such that $B = \bigoplus_{n=1}^{\infty} B_n$, where each B_n is a direct sum of uniserial modules each of length n . Then B is a basic submodule of M , if and only if*

$$M = (B_1 + \cdots + B_n) \oplus (B_n^* + H_n(M)) \quad \text{where} \quad B_n^* = \sum_{i>n} B_i.$$

The following theorem generalizes Szele's theorem [1, Theorem (29.4)].

Theorem 2.6. *Let M and B be as in (2.5). B is a basic submodule if and only if $B_1 + \cdots + B_n$ is a summand of M and is maximal with respect to the property $(B_1 + \cdots + B_n) \cap H_n(M) = 0$.*

Proof. Let B be a basic submodule of M . From (2.5) $(B_1 + \cdots + B_n) \cap H_n(M) = 0$. Let N be a complement of $H_n(M)$ containing $B_1 + \cdots + B_n$. N is a summand of M by (1.4). By [7, Corollary 1], N is a direct sum of uniserial modules. Suppose $N \neq B_1 + \cdots + B_n$. Then we can find a uniform element $y \in N$ such that $B_1 \oplus \cdots \oplus B_n \oplus yR$ is a summand of M . By using (2.5), we can suppose that $yR \subset B_n^* + H_n(M)$. Let $zR = \text{socle}(yR)$. Since $yR \cap H_n(M) = 0$, and yR is a pure submodule, we get $H(z) \leq n-1$. Let

$$M' = B_n^* + H_n(M) \tag{i}$$

If for every $i \geq n+1$,

$$C_i = \sum_{j=n+1}^i B_j \tag{ii}$$

each C_i being pure and bounded, is a summand of M' . Further

$$M' = U_i(C_i + H_n(M)).$$

Consequently for some i ,

$$z \in C_i + H_n(M)$$

Again

$$C_i = \bigoplus \sum_{\alpha} y_{\alpha} R \quad (\text{iii})$$

where $y_{\alpha} R$ are uniserial. Also

$$M' = C_i \oplus D \quad (\text{iv})$$

Now $z = c + x$, $c \in C_i$, $x \in H_n(M)$.

Using (iii) and (iv) we get

$$\begin{aligned} c &= \sum_{\alpha} u_{\alpha}, \quad u_{\alpha} \in y_{\alpha} R \\ x &= \sum_{\alpha} v_{\alpha} + d, \quad v_{\alpha} \in y_{\alpha} R, \quad d \in D. \end{aligned}$$

Each $u_{\alpha} + v_{\alpha}$ being a homomorphic image of z must be either zero or be such that $(u_{\alpha} + v_{\alpha})R$ is the minimal submodule of $y_{\alpha} R$. However as

$$C_i = \sum_{j=n+1}^t B_j,$$

C_i is a direct sum of uniserial modules of lengths at least $n+1$. Consequently $H(u_{\alpha} + v_{\alpha}) \geq n$. Hence, as also $d \in H_n(M)$, we get $z \in H_n(M)$, by [6, Lemma 4]. This contradicts the fact that $H(z) \leq n-1$. This proves the necessity.

Conversely let B satisfy the given conditions. Then $B_1 + \dots + B_n$ is a pure submodule of M , gives B is a pure submodule of M . If B is not a basic submodule in M , we can find a uniform element $u \in M$ such that $B \cap uR = 0$ and $B \oplus uR$ is a pure submodule (use [7, Lemma 2 and Theorem 5]). Let $d(uR) = n$. Then $(B_1 \oplus \dots \oplus B_n \oplus uR) \cap H_n(M) = 0$. This contradicts the hypothesis. Hence the result follows.

Theorem 2.7. *Let a module M satisfy (I), (II) and (III). Then M has a basic submodule. Any two basic submodules of M are isomorphic.*

Proof. Existence follows from (2.4). Let B' and B be two basic submodules of M . We have

$$B = B_1 \oplus B_2 \oplus \dots \oplus B_n \oplus \dots \quad (\text{i})$$

$$B' = B'_1 \oplus B'_2 \oplus \dots \oplus B'_n \oplus \dots \quad (\text{ii})$$

where B_n, B'_n are direct sums of uniserial modules, each of length n . By (2.6)

$$M = (B_1 + \dots + B_n) \oplus N_1 \quad (\text{iii})$$

$$M = (B'_1 + \dots + B'_n) \oplus N'_1 \quad (\text{iv})$$

for some submodules N_1, N'_1 of M , containing $H_n(M)$ such that $H_n(M)$ is an essential submodule of each of them. Let $p: M \rightarrow B_1 + \dots + B_n$ be projection given by (iii). By (2.6), $B'_n \cap N_1 = 0$ and hence $B'_n \cong p(B'_n)$. For each $i=1$,

2, ..., n, let

$$p_i: B_1 + \dots + B_n \rightarrow B_i$$

be natural projections. We claim that q , the restriction of $p_n p$ to B'_n is a monomorphism. Suppose $\ker q \neq 0$. As p restricted to B'_n is a monomorphism, we can find a minimal submodule xR of B'_n such that $p_n p(xR) = 0$; clearly $p(xR) \neq 0$. Now $H(x) = n-1$. So there exists a uniform element $z \in B'_n$, such that $x \in zR$ and $d(zR) = n$. For some $i < n$, $p_i p(x) \neq 0$, since $p_n p(x) = 0$. Then from $\text{socle}(zR) = xR \cong p_i p(xR)$, we get $zR \cong p_i p(zR) \subset B_1 + \dots + B_{n-1}$. But $d(zR) = n$, and $B_1 + \dots + B_{n-1}$ has no uniserial submodule of length exceeding $n-1$. Thus we get, a contradiction. Hence $q: B'_n \rightarrow B_n$ is a monomorphism. In particular we get a monomorphism;

$$\lambda: \text{socle}(B'_n) \rightarrow \text{socle}(B_n)$$

Similarly we get a monomorphism:

$$\mu: \text{socle}(B_n) \rightarrow \text{socle}(B'_n)$$

Consequently $\text{socle}(B_n) \cong \text{socle}(B'_n)$

$$\text{Now } B_n = \bigoplus_{i \in \Lambda} A_i$$

$$\text{and } B'_n = \bigoplus_{j \in \Gamma} A'_j$$

where A_i and A'_j are uniserial modules, each of length n . Then

$$\begin{aligned} \text{socle}(B_n) &= \bigoplus_i \text{socle}(A_i) \\ \text{socle}(B'_n) &= \bigoplus_j \text{socle}(A'_j), \end{aligned}$$

we get a one-to-one mapping σ of Λ onto Γ such that $\text{socle}(A_i) = \text{socle}(A'_{\sigma(i)})$. By condition (II), $A_i \cong A'_{\sigma(i)}$. Hence $B_n \cong B'_n$. This in turn gives $B \cong B'$. This proves the theorem.

3. A decomposition theorem

Main purpose of this section is to prove the following:

Theorem 3.1. *If a module M satisfying (I) and (II), is such that for its socle S , M/S is decomposable, then M is also decomposable.*

We state the following without proof, since its proof is verbatim same as of Corollary 1 in [6].

Theorem 3.2. *Let M be a module satisfying (I) and (II), and P be its socle. M is a direct sum of uniserial modules if and only if P is a union of ascending sequence $P_n (n=1, 2, 3, \dots)$ of submodules such that for each n , there exists a positive*

integer k_n with the property that $H(x) \leq k_n$ for every uniform element x of P_n .

Lemma 3.3. *If a module M satisfying (I), (II) is such that for some $k \geq 0$, $H_k(M)$ is decomposable then M is decomposable.*

Proof. Let N be an h -pure submodule of M , maximal with respect to the property that $N \cap H_k(M) = 0$. N is bounded and decomposable. Further by [7, Theorem 3].

$$M = N \oplus K.$$

Let T be a complement of $H_k(M)$ containing N . If $T \neq N$, we can find a uniform element $z \in \text{socle}(T)$ such that $z \in K$. Now $H(z) = t \leq k-1$. If u is a uniform element in K with $z \in uR$ and $d(uR/zR) = t-1$, we get from [7, Lemma 1], that $K = uR \oplus K_1$. Then

$$M = N \oplus uR \oplus K_1$$

and $N + uR$ is an h -pure submodule of M containing N properly, having zero intersection with $H_k(M)$. This contradicts the choice of N . Hence N is a complement of $H_k(M)$. Further

$$H_k(M) = H_k(N) + H_k(K) = H_k(K)$$

Thus $H_k(M) \subset K$. Hence to prove that M is decomposable we only need to show that K is decomposable. So without loss of generality we may suppose that $H_k(M) \subset M$. So $S = \text{socle}(M) = \text{socle}(H_k(M))$. By hypothesis $H_k(M)$ is decomposable. So by (3.1), $S = \bigcup_n S_n$, where S_n ($n=1, 2, \dots$) is an ascending sequence of submodules, such that for each n , we have a positive integer l_n such that the height of any uniform element x of S_n taken in $H_k(M)$ does not exceed l_n . Then the height of any uniform element x in S_n taken in M does not exceed $l_n + k$. So by (3.2) M is decomposable.

Proof of (3.1). In view of (3.3) it is enough to prove that $H_1(M)$ is decomposable. Now by hypothesis

$$\bar{M} = M/S = \bigoplus \sum_{\alpha} \bar{y}_{\alpha} R$$

where $\bar{y}_{\alpha} R$ are uniserial.

As seen in the proof of (3.3), without loss of generality we can suppose that $H_1(M) \subset M$. In view of the condition (I) we take each y_{α} to be uniform in M . Now $d(y_{\alpha} R) \geq 2$. Let $x_{\alpha} R < y_{\alpha} R$ with $d(y_{\alpha} R/x_{\alpha} R) = 1$. Then $x_{\alpha} \in H_1(M)$. We claim,

$$H_1(M) = \bigoplus \sum_{\alpha} x_{\alpha} R$$

and this will prove the result. Suppose

$$x_\alpha R \cap \left(\sum_{i=1}^n x_i R \right) \neq 0$$

with $x_\alpha R \neq x_i R$ for $1 \leq i \leq n$. Then

$$z_\alpha R = x_\alpha R \cap \left(\sum_{i=1}^n x_i R \right) = \text{socle}(x_\alpha R) = y_\alpha R \cap \left(\sum_{i=1}^n y_i R \right)$$

$$\text{Now } \sum_{i=1}^n y_i R = \oplus \sum_{j=1}^m u_j R$$

where $u_j R$ are uniserial and by (1.2) $m \leq n$. If for some j , $d(u_j R) = 1$, we have

$$\oplus \sum_{i=1}^n \bar{y}_i R = \oplus \sum_{j=1}^m u_j R \pmod{S}$$

and the right hand side has less than n terms. This is a contradiction. Therefore $d(u_j R) \geq 2$ for every j and $m = n$. We write

$$z_\alpha = v_1 + v_2 + \cdots + v_n, \quad v_i \in u_i R$$

We can find $t_\alpha \in y_\alpha R$ such that $d(t_\alpha R / z_\alpha R) = 1$. By condition (II), we get homomorphisms

$$\sigma_j: t_\alpha R \rightarrow u_j R$$

such that $\sigma_j(z_\alpha) = v_j$. Define

$$\sigma: t_\alpha R \rightarrow \oplus \sum_{j=1}^n u_j R$$

$$\text{by } \sigma(y) = \sum_j \sigma_j(y), \quad y \in t_\alpha R$$

Then σ is identity on $z_\alpha R$. Let

$$A = \{r \in R: t_\alpha r \in z_\alpha R\}$$

Then A is a maximal right ideal of R with $z_\alpha R = t_\alpha A$. So for $r \in A$, $t_\alpha r = \sigma(t_\alpha r) = \sum_{j=1}^n \sigma_j(t_\alpha r)$. Consequently $t_\alpha - \sigma(t_\alpha)$ is a uniform element such that

$$(t_\alpha - \sigma(t_\alpha))R \cong R/A$$

This gives $t_\alpha - \sigma(t_\alpha) \in S$. Hence,

$$t_\alpha \equiv \sigma(t_\alpha) \pmod{S}$$

Consequently $\bar{t}_\alpha \in \bar{y}_\alpha R \cap \left(\sum_{j=1}^n \bar{y}_j R \right) = \bar{0}$. Hence $t_\alpha \in S$. This is a contradiction.

Therefore

$$\sum_\alpha x_\alpha R = \oplus \sum_\alpha x_\alpha R$$

This also yields $\sum_\alpha y_\alpha R = \oplus \sum_\alpha y_\alpha R$, since each $y_\alpha R$ is an essential extension

of $x_\alpha R$. Consider any uniform element $x \in H_1(M)$ such that $x \notin S$. We can find a uniform element $y \in M$ such that, $x \in yR$ and $d(yR/xR)=1$. If

$$A = \{r \in R; yr \in xR\}$$

then A is a maximal right ideal of R and $xR = yA$. Now

$$\bar{y} = y + S = \bar{\xi}_1 + \bar{\xi}_2 + \cdots + \bar{\xi}_n, \quad \bar{\xi}_i \in \bar{y}_i R$$

for some $\bar{y}_1, \bar{y}_2, \dots, \bar{y}_n$ among \bar{y}_α 's, $\alpha \in \Lambda$. We take $\xi_i \in y_i R$. If for any i , $\bar{\xi}_i \neq 0$, then the natural homomorphism $\eta_i: \bar{y}R \rightarrow \bar{\xi}_i R$ is non-zero; since $\bar{y}R$ is uniserial, it follows that $\text{Ker } \eta_i \subset \bar{x}R = \bar{y}A$ and so $\xi_i R / \xi_i A \cong \bar{y}R / \bar{x}R \neq 0$. Thus $\bar{\xi}_i \neq 0$ implies $\bar{\xi}_i A \subset x_i R$. Consequently $\bar{x} \in \sum x_i R$ and hence

$$x \in \sum_\alpha x_\alpha R + S.$$

$$H_1(M) = \sum x_\alpha R + S$$

This proves:

We claim: $S \subset \sum x_\alpha R$. If not we can find a uniform element $x \in S$ such that $x \notin \sum x_\alpha R$. Then $xR \cap \sum x_\alpha R = 0$. We can find a uniform element $y \in M$ such that $x \in yR$ and $d(yR/xR)=1$. Now let $N' = yR + \sum y_\alpha R = yR \oplus (\sum y_\alpha R)$. Then

$$\begin{aligned} M/S &= (\sum y_\alpha R + S)/S = (N' + S)/S \cong N'/\text{socle}(N') \\ &\cong yR/xR \oplus \sum (y_\alpha R)/\text{socle}(y_\alpha R) \end{aligned}$$

Therefore

$$\oplus \sum_\alpha y_\alpha R / \text{socle}(y_\alpha R) \cong (yR/xR) \oplus \sum_\alpha y_\alpha R / \text{socle}(y_\alpha R)$$

This isomorphism is natural. Hence $yR/xR = 0$. This is a contradiction. Hence $S \subset \sum_\alpha x_\alpha R$. This yields

$$H_1(M) = \oplus \sum_\alpha x_\alpha R$$

Hence the result follows.

We end this paper with a few remarks.

- (1) Any module M over a commutative ring R satisfying (I) and (II) must satisfy (III). However, a simple faithful module over a nonartinian primitive ring trivially satisfies (I) and (II), but not (III).
- (2) If a module M satisfies (I) and (II), then (II) gives that any uniserial submodule xR of M is quasi-injective. The example on page 362 in [3] is of a uniserial module which is not quasi-injective. This shows that although a uniserial module always satisfies (I), but it need not satisfy (II).
- (3) If a commutative ring R , admits a faithful finitely generated module satisfying (I) and (II), then R is a principal ideal ring with d.c.c. It will be interesting to investigate the structure of noncommutative rings admitting faithful, finitely

generated modules satisfying conditions (I), (II) and (III).

(4) Consider a local ring R , with maximal ideal W , such that $W^2=0$, $Q=R/W$, a field with the property that $\dim_Q W=1$, and $\dim W_Q=2$. R is not a right principal ideal ring. However for any $x \neq 0$ in W , R/xR is a uniserial, injective, faithful, right R -module of length two, so it satisfies (I), (II) and (III). (See V. Dlab, and C. M. Ringel, Math. Ann. 195, (1972) Proposition 2)

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