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## SOME DECOMPOSITIONS THEOREMS ON ABELIAN GROUPS AND THEIR GENERALISATIONS-II

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In [7] study of those modules  $M_R$  which satisfy the following two conditions was initiated:

- (I) Every finitely generated submodule of every homomorphic image of  $M$  is a direct sum of uniserial modules.
- (II) Given two uniserial submodules  $U$  and  $V$  of a homomorphic image of  $M$ , for any submodule  $W$  of  $U$  any non-zero homomorphism,  $f: W \rightarrow V$  can be extended to a homomorphism  $g: U \rightarrow V$  provided the composition length  $d(U/W) \leq d(V/f(W))$ .

It was shown that some of the well known decomposition theorems for torsion abelian groups, can be generalized to modules satisfying (I) and (II). Here we introduce another condition:

- (III) For any finitely generated submodules  $N$  of  $M$ ,  $R/\text{ann}(N)$  is right artinian.

It can be easily seen that any torsion module over a bounded (*hnp*)-ring satisfies (I), (II) and (III). Let  $M$  be a module satisfying (I) and (II). The concept of *h*-pure submodules of  $M$  was introduced in [7]; if in addition  $M$  satisfies (III) it is shown in section one, that any submodule  $N$  of  $M$  is *h*-pure if and only if it is pure (Theorem (1.3)). Theorem (1.4) shows that any complement of  $H_k(M)$  in  $M$  is a summand of  $M$ . In section 2, the concept of basic submodule is introduced. It is shown that any module  $M$  satisfying (I), (II) and (III) has a basic submodule and any two basic submodules of  $M$  are isomorphic (Theorem (2.7)). This result generalizes the corresponding well known result on basic subgroups of torsion abelian groups. In section 3, a decomposition theorem is proved; which states that given any module  $M$  satisfying (I) and (II), such that  $M/\text{socle}(M)$  is decomposable then  $M$  is decomposable.

Preliminaries: Let  $M$  be a module satisfying (I) and (II). Let us recall some definitions from [6, 7]. An element  $x$  in  $M$  is said to be uniform if  $xR$  is a non-zero uniform (hence uniserial) submodule. For any uniform element  $x$  of  $M$ , its exponent  $e(x)$  is defined to be equal to the composition length  $d(xR)$ ;

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the height of  $x$  is the supremum of all  $d(T/xR)$  where  $T$  is a uniserial submodule of  $M$  containing  $x$ . The height of  $x$  is denoted by  $H_M(x)$  (or simply by  $H(x)$ ). For any  $k \geq 0$ ,  $H_k(M)$  denotes the submodule of  $M$  generated by all those uniform elements  $x$  of  $M$  for which  $H(x) \geq k$ . A submodule  $N$  of  $M$  is said to be an  $h$ -pure submodule if  $N \cap H_k(M) = H_k(N)$  for all  $k$ .  $M$  is said to be bounded if there exists a positive integer  $k$  such that  $H(x) \leq k$  for all uniform elements  $x$  in  $M$ .  $M$  is said to be decomposable if it is a direct sum of uniserial modules. For definition and elementary properties of pure submodules we refer to Stenström [8]. For any ring  $R$ ,  $J(R)$  denotes the Jacobson radical of  $R$ .

**Lemma 1.1.** *Let  $M_R$  be a module satisfying (I), (II) and (III) and  $X$  be a uniserial submodule of  $M$  having*

$$X = X_0 > X_1 > X_2 \cdots > X_t = 0$$

*as its unique composition series. If for  $0 \leq i \leq t-1$ ,  $P_i = \text{ann}(X_i/X_{i+1})$  then  $X_i P_i = X_{i+1}$ .*

Proof. Let  $A = \text{ann}(X)$ . Since  $S = R/A$  is right artinian,  $X_i J(S) = X_{i+1}$  and  $J(S) \subset P_i/A$ , we have  $X_i P_i = X_{i+1}$ .

**Lemma 1.2.** *Let a module  $M_R$  satisfy (I) and (II). If for any finitely many uniform elements  $x_1, x_2, \dots, x_n$  in  $M$*

$$\sum_{i=1}^n x_i R = \bigoplus \sum_{j=1}^m y_j R$$

*where  $y_j R$  are uniserial, then  $m \leq n$ .*

Proof. The result follows by induction on  $n$ .

The result that any submodule  $N$  of a torsion module over a bounded (*hnp*)-ring is pure if and only if it is  $h$ -pure was proved by M. Khan in [2]. The proof of the following is adapted from [2].

**Theorem 1.3.** *Let  $M_R$  be a module satisfying (I), (II) and (III) and  $N$  a submodule of  $M$ . Then  $N$  is  $h$ -pure if and only if it is a pure submodule.*

Proof. Let  $N$  be  $h$ -pure. Consider any finite system of linear equations

$$\sum_i x_i y_{ij} = s_j \in N$$

which admits a solution  $\{x_i\}$  in  $M$ . Let  $K = \sum x_i R + N$ . Then  $K/N$  is a finitely generated module. So by condition (I),

$$K/N = \bigoplus \sum T_\alpha / N$$

where each  $T_\alpha / N$  is uniserial. Then by [7, Lemma 2(i)],  $T_\alpha = y_\alpha R \oplus N$ . Hence

$$K = K_1 \oplus N$$

This gives that the above given system of equations are also solvable in  $N$ . Hence  $N$  is a pure submodule of  $M$ .

Let now  $N$  be a pure submodule. This immediately gives  $MA \cap N = NA$  for all ideals  $A$  of  $R$ . Suppose for some  $k$ ,  $H_k(M) \cap N \neq H_k(N)$ . We choose  $k$  smallest with  $H_k(M) \cap N \neq H_k(N)$ . We can find a uniform element  $x$  of smallest exponent such that  $x \in H_k(M) \cap N$  but  $x \notin H_k(N)$ . Then  $x \in H_{k-1}(N)$ . By definition there exists a uniform element  $y$  in  $M$  such that  $x \in yR$  and  $d(yR/xR) = k$ .

$x \in H_{k-1}(N)$  shows that there exist a uniform element  $u \in N$  such that  $x \in uR$  and  $d(uR/xR) = k-1$ . Let  $zR = \text{socle}(xR)$  and  $m = e(x)$ . Then  $d(uR/zR) = m + k - 2$ , gives  $H_N(z) \geq m + k - 2$ . Suppose  $H_N(z) \geq m + k - 1$ . We can then find a uniform element  $v \in N$  such that  $z \in vR$  and  $d(vR/zR) = m + k - 1$ . By condition (II), we get an isomorphism  $\sigma: yR \rightarrow vR$  which is identity on  $zR$ . Then  $x - \sigma(x)$  is a uniform element with  $e(x - \sigma(x)) < e(x)$ ,  $x - \sigma(x) \in N \cap H_k(M)$ , but  $x - \sigma(x) \notin H_k(N)$ , since  $\sigma(x) \in H_k(N)$ . This contradicts the choice of  $x$ . Hence  $H_N(z) = k + m - 2 = d(uR/zR)$ . So by [7, Lemma 1]

$$N = uR \oplus N_1$$

$uR$  is also a pure submodule. Now  $d(uR/zR) = d(yR/zR) - 1$ . By (1.1) we can find prime ideals  $P_1, P_2, \dots, P_{m+k-1}$  such that  $R/P_i$  is simple artinian for all  $i$  and  $yR > yP_1 > yP_1P_2 > \dots > yP_1P_2 \dots P_{m+k-1} = 0$  with  $zR = yP_1P_2 \dots P_{m+k-2}$ . By condition (II)  $uR \cong yP_1$  and hence  $uP_2P_3 \dots P_{m+k-1} = 0$ . However  $yP_2P_3 \dots P_{m+k-1} \neq 0$ . Thus  $z \in MP_2P_3 \dots P_{m+k-1} \cap uR = uP_2P_3 \dots P_{m+k-1} = 0$ . This is a contradiction. Hence  $N$  is an  $h$ -pure submodule of  $M$ .

The following theorem generalizes Erdelyi's theorem [1, Theorem (24.8)].

**Theorem 1.4.** *Let  $M$  be a module satisfying (I), (II) and (III) then for any  $k \geq 1$ , any complement of  $H_k(M)$  is a summand of  $M$ .*

**Proof.** Let  $N$  be a complement of  $H_k(M)$ . Then  $N$  is bounded. If we show that  $N$  is a pure submodule, the result follows from [7, Theorem 3]. In view of (1.2) it is equivalent to showing  $H_n(M) \cap N = H_n(N)$  for every  $n$ . Since  $H_k(M) \cap N = 0 = H_k(N)$ , the result holds for  $n \geq k$ . To apply induction we suppose that for some  $n$  with  $0 \leq n < k$ ,  $H_n(M) \cap N = H_n(N)$ , we prove that same for  $n+1$ . Let the contrary hold. Then there exists a uniform element  $x \in H_{n+1}(M) \cap N$  such that  $x \notin H_{n+1}(N)$ . Then  $H_N(x) = n$ . Now there exists a uniform element  $y$  in  $M$  such that  $d(yR/xR) = n+1$ . Let  $\text{socle}(yR/xR) = x_1R/xR$ . If  $x_1 \in N$ , we get  $x_1 \in N \cap H_n(M) = H_n(N)$  and hence  $x \in H_{n+1}(N)$ . This is a contradiction. Consequently  $x_1 \notin N$  and  $(N + x_1R) \cap H_k(M) \neq 0$ . Thus there exists a uniform element  $z \in H_k(M)$  such that  $z = u + x_1s$  for some  $u \in N$  and  $s \in R$ . If  $x_1sR \neq x_1R$ , then  $x_1sR \subset xR$  and  $z \in N$ ; this is a contradiction to the fact that

$N \cap H_k(M) = 0$ . So  $x_1sR = x_1R$  and  $x_1 = x_1ss'$ ,  $s' \in R$ . Then  $zs' = us' + x_1 \in (N + x_1R) \cap H_k(M)$  and  $zs' \neq 0$ . So we can suppose that  $x_1s = x_1$ . Thus  $z = u + x_1$ .

Let  $P = \text{ann}(x_1R/xR)$ . By (1.1)  $xR = x_1P$ . So for any  $r \in P$ ,  $zr = 0$  and  $ur = -x_1r$ . Now  $H(z) \geq k > n$ ,  $H(x_1) \geq n$ , gives  $u \in H_n(M) \cap N = H_n(N)$ . For some  $r_0 \in P$ ,  $x = x_1r_0 = -ur_0$ . If  $uR$  is uniform and  $ur_0R < uR$ , then  $H_N(ur_0) \geq H_N(u) + 1 \geq n + 1$  and hence  $H_N(x) \geq n + 1$ ; this is a contradiction. Hence the following two cases arise.

Case I:  $uR$  is uniform and  $ur_0R = uR$ . In this case  $u = x_1b_0 = xrb$  for some  $b \in R$  and  $z = x_1r_0b + x_1 = x_1c$ ,  $c \in R$ . Thus  $zR = x_1R$  and  $x_1 \in H_k(M)$ . This shows that  $H(x_1) \geq k$  and hence  $H(x) \geq k + 1$ . This contradicts the fact that  $N \cap H_{k+1}(N) = 0$ .

Case II:  $uR$  is not uniform. The fact that  $u \in zR + x_1R$  and that  $zR$ ,  $x_1R$  are uniform together with (1.2) yields  $uR = u_1R \oplus u_2R$  with  $u_1$ ,  $u_2$  both uniform. Further we can take  $u = u_1 + u_2$ . Then  $xR = uP = u_1P \oplus u_2P$ . So  $u_1P = 0$  or  $u_2P = 0$ . To be definite let  $u_2P = 0$ . Then  $u_2R$  is a simple  $R$ -module and  $x_1R = u_1P$ . Let  $u_1P < u_1R$  then  $xR = u_1P = v_1R$  for some  $v_1 \in u_1R$ . Now  $H(u_1) \geq \min(H(x_1), H(x)) \geq n$ . So by induction hypothesis  $u_1 \in H_n(N)$  and hence  $H_N(v_1) \geq n + 1$ . Consequently  $xR = v_1R$  gives  $H_N(x) \geq n + 1$ . This is a contradiction. Thus  $u_1R = u_1P = xR$ . Hence  $u_1 = xa$ ,  $a \in R$ . Consequently  $z = u_2 + xa + x_1 = u_2 + x_1s$ ,  $s \in R$ . This reduces to case I and hence again gives a contradiction. Hence  $N$  is a pure submodule. This proves the theorem.

## 2. Basic submodules

**DEFINITION 2.1.** Let  $M$  be a module satisfying (I) and (II). A subset  $\{x_\lambda : \lambda \in \Lambda\}$  of uniform elements of  $M$  is called  $h$ -pure independent if it is independent in the sense that  $\sum x_\lambda R$  is direct, and  $\sum x_\lambda R$  is an  $h$ -pure submodule of  $M$ .

The following Lemma generalizes [1, Lemma (29.1)].

**Lemma 2.2.** Let a module  $M_R$  satisfy (I) and (II). An  $h$ -pure independent subset  $\{x_\lambda : \lambda \in \Lambda\}$  is maximal if and only if  $M/L$ , where  $L = \sum x_\lambda R$ , is a direct sum of infinite length uniform submodules.

Proof. The result follows from [7, Lemma 2 and Theorem 5].

This motivates the following:

**DEFINITION 2.3.** Let  $M$  be a module satisfying (I) and (II). A submodule  $B$  of  $M$  is called a basic submodule of  $M$  if it satisfies the following:

- (i)  $B$  is an  $h$ -pure submodule.
- (ii)  $B$  is a direct sum of uniserial modules.

(iii)  $M/B$  is a direct sum of uniform modules of infinite lengths.

[7, Lemma 2 and Theorem 5] and the fact that union of any chain of  $h$ -pure submodules is an  $h$ -pure submodule gives the following:

**Lemma 2.4.** *Any module satisfying (I) and (II) has a basic submodule.*

The main purpose of this section is to prove that any two basic submodules of a module  $M$  satisfying (I), (II) and (III) are isomorphic. The following theorem generalizes [1, Theorem (29.3)]. Since the proof is on similar lines it is omitted.

**Theorem 2.5.** *Let  $M$  be a module satisfying (I), (II) and (III) and  $B$  be a submodule of  $M$  such that  $B = \bigoplus_{n=1}^{\infty} B_n$ , where each  $B_n$  is a direct sum of uniserial modules each of length  $n$ . Then  $B$  is a basic submodule of  $M$ , if and only if*

$$M = (B_1 + \cdots + B_n) \oplus (B_n^* + H_n(M)) \quad \text{where} \quad B_n^* = \sum_{i>n} B_i.$$

The following theorem generalizes Szele's theorem [1, Theorem (29.4)].

**Theorem 2.6.** *Let  $M$  and  $B$  be as in (2.5).  $B$  is a basic submodule if and only if  $B_1 + \cdots + B_n$  is a summand of  $M$  and is maximal with respect to the property  $(B_1 + \cdots + B_n) \cap H_n(M) = 0$ .*

Proof. Let  $B$  be a basic submodule of  $M$ . From (2.5)  $(B_1 + \cdots + B_n) \cap H_n(M) = 0$ . Let  $N$  be a complement of  $H_n(M)$  containing  $B_1 + \cdots + B_n$ .  $N$  is a summand of  $M$  by (1.4). By [7, Corollary 1],  $N$  is a direct sum of uniserial modules. Suppose  $N \neq B_1 + \cdots + B_n$ . Then we can find a uniform element  $y \in N$  such that  $B_1 \oplus \cdots \oplus B_n \oplus yR$  is a summand of  $M$ . By using (2.5), we can suppose that  $yR \subset B_n^* + H_n(M)$ . Let  $zR = \text{socle}(yR)$ . Since  $yR \cap H_n(M) = 0$ , and  $yR$  is a pure submodule, we get  $H(z) \leq n-1$ . Let

$$M' = B_n^* + H_n(M) \quad (\text{i})$$

If for every  $i \geq n+1$ ,

$$C_i = \sum_{j=n+1}^i B_j \quad (\text{ii})$$

each  $C_i$  being pure and bounded, is a summand of  $M'$ . Further

$$M' = U_i(C_i + H_n(M)).$$

Consequently for some  $i$ ,

$$z \in C_i + H_n(M)$$

Again

$$C_i = \bigoplus \sum_{\alpha} y_{\alpha} R \quad (\text{iii})$$

where  $y_{\alpha}R$  are uniserial. Also

$$M' = C_i \oplus D \quad (\text{iv})$$

Now  $z=c+x$ ,  $c \in C_i$ ,  $x \in H_n(M)$ .

Using (iii) and (iv) we get

$$\begin{aligned} c &= \sum_{\alpha} u_{\alpha}, \quad u_{\alpha} \in y_{\alpha} R \\ x &= \sum_{\alpha} v_{\alpha} + d, \quad v_{\alpha} \in y_{\alpha} R, \quad d \in D. \end{aligned}$$

Each  $u_{\alpha} + v_{\alpha}$  being a homomorphic image of  $z$  must be either zero or be such that  $(u_{\alpha} + v_{\alpha})R$  is the minimal submodule of  $y_{\alpha}R$ . However as

$$C_i = \sum_{j=n+1}^i B_j,$$

$C_i$  is a direct sum of uniserial modules of lengths at least  $n+1$ . Consequently  $H(u_{\alpha} + v_{\alpha}) \geq n$ . Hence, as also  $d \in H_n(M)$ , we get  $z \in H_n(M)$ , by [6, Lemma 4]. This contradicts the fact that  $H(z) \leq n-1$ . This proves the necessity.

Conversely let  $B$  satisfy the given conditions. Then  $B_1 + \cdots + B_n$  is a pure submodule of  $M$ , gives  $B$  is a pure submodule of  $M$ . If  $B$  is not a basic submodule in  $M$ , we can find a uniform element  $u \in M$  such that  $B \cap uR = 0$  and  $B \oplus uR$  is a pure submodule (use [7, Lemma 2 and Theorem 5]). Let  $d(uR) = n$ . Then  $(B_1 \oplus \cdots \oplus B_n \oplus uR) \cap H_n(M) = 0$ . This contradicts the hypothesis. Hence the result follows.

**Theorem 2.7.** *Let a module  $M$  satisfy (I), (II) and (III). Then  $M$  has a basic submodule. Any two basic submodules of  $M$  are isomorphic.*

Proof. Existence follows from (2.4). Let  $B'$  and  $B$  be two basic submodules of  $M$ . We have

$$B = B_1 \oplus B_2 \oplus \cdots \oplus B_n \oplus \cdots \quad (\text{i})$$

$$B' = B'_1 \oplus B'_2 \oplus \cdots \oplus B'_n \oplus \cdots \quad (\text{ii})$$

where  $B_n$ ,  $B'_n$  are direct sums of uniserial modules, each of length  $n$ . By (2.6)

$$M = (B_1 + \cdots + B_n) \oplus N_1 \quad (\text{iii})$$

$$M = (B'_1 + \cdots + B'_n) \oplus N'_1 \quad (\text{iv})$$

for some submodules  $N_1$ ,  $N'_1$  of  $M$ , containing  $H_n(M)$  such that  $H_n(M)$  is an essential submodule of each of them. Let  $p: M \rightarrow B_1 + \cdots + B_n$  be projection given by (iii). By (2.6),  $B'_n \cap N_1 = 0$  and hence  $B'_n \cong p(B'_n)$ . For each  $i=1$ ,

2,  $\dots$ ,  $n$ , let

$$p_i: B_1 + \dots + B_n \rightarrow B_i$$

be natural projections. We claim that  $q$ , the restriction of  $p_n p$  to  $B'_n$  is a monomorphism. Suppose  $\ker q \neq 0$ . As  $p$  restricted to  $B'_n$  is a monomorphism, we can find a minimal submodule  $xR$  of  $B'_n$  such that  $p_n p(xR) = 0$ ; clearly  $p(xR) \neq 0$ . Now  $H(x) = n-1$ . So there exists a uniform element  $z \in B'_n$ , such that  $x \in zR$  and  $d(zR) = n$ . For some  $i < n$ ,  $p_i p(x) \neq 0$ , since  $p_n p(x) = 0$ . Then from socle  $(zR) = xR \cong p_i p(xR)$ , we get  $zR \cong p_i p(zR) \subset B_1 + \dots + B_{n-1}$ . But  $d(zR) = n$ , and  $B_1 + \dots + B_{n-1}$  has no uniserial submodule of length exceeding  $n-1$ . Thus we get, a contradiction. Hence  $q: B'_n \rightarrow B_n$  is a monomorphism. In particular we get a monomorphism;

$$\lambda: \text{socle}(B'_n) \rightarrow \text{socle}(B_n)$$

Similarly we get a monomorphism:

$$\mu: \text{socle}(B_n) \rightarrow \text{socle}(B'_n)$$

Consequently  $\text{socle}(B_n) \cong \text{socle}(B'_n)$

Now  $B_n = \bigoplus_{i \in \Lambda} A_i$

and  $B'_n = \bigoplus_{j \in \Gamma} A'_j$

where  $A_i$  and  $A'_j$  are uniserial modules, each of length  $n$ . Then

$$\begin{aligned} \text{socle}(B_n) &= \bigoplus_i \text{socle}(A_i) \\ \text{socle}(B'_n) &= \bigoplus_j \text{socle}(A'_j), \end{aligned}$$

we get a one-to-one mapping  $\sigma$  of  $\Lambda$  onto  $\Gamma$  such that  $\text{socle}(A_i) = \text{socle}(A'_{\sigma(i)})$ . By condition (II),  $A_i \cong A'_{\sigma(i)}$ . Hence  $B_n \cong B'_n$ . This in turn gives  $B \cong B'$ . This proves the theorem.

### 3. A decomposition theorem

Main purpose of this section is to prove the following:

**Theorem 3.1.** *If a module  $M$  satisfying (I) and (II), is such that for its socle  $S$ ,  $M/S$  is decomposable, then  $M$  is also decomposable.*

We state the following without proof, since its proof is verbatim same as of Corollary 1 in [6].

**Theorem 3.2.** *Let  $M$  be a module satisfying (I) and (II), and  $P$  be its socle.  $M$  is a direct sum of uniserial modules if and only if  $P$  is a union of ascending sequence  $P_n$  ( $n=1, 2, 3, \dots$ ) of submodules such that for each  $n$ , there exists a positive*

integer  $k_n$  with the property that  $H(x) \leq k_n$  for every uniform element  $x$  of  $P_n$ .

**Lemma 3.3.** *If a module  $M$  satisfying (I), (II) is such that for some  $k \geq 0$ ,  $H_k(M)$  is decomposable then  $M$  is decomposable.*

Proof. Let  $N$  be an  $h$ -pure submodule of  $M$ , maximal with respect to the property that  $N \cap H_k(M) = 0$ .  $N$  is bounded and decomposable. Further by [7, Theorem 3].

$$M = N \oplus K.$$

Let  $T$  be a complement of  $H_k(M)$  containing  $N$ . If  $T \neq N$ , we can find a uniform element  $z \in \text{socle}(T)$  such that  $z \in K$ . Now  $H(z) = t \leq k-1$ . If  $u$  is a uniform element in  $K$  with  $z \in uR$  and  $d(uR/zR) = t-1$ , we get from [7, Lemma 1], that  $K = uR \oplus K_1$ . Then

$$M = N \oplus uR \oplus K_1$$

and  $N + uR$  is an  $h$ -pure submodule of  $M$  containing  $N$  properly, having zero intersection with  $H_k(M)$ . This contradicts the choice of  $N$ . Hence  $N$  is a complement of  $H_k(M)$ . Further

$$H_k(M) = H_k(N) + H_k(K) = H_k(K)$$

Thus  $H_k(M) \subset 'K$ . Hence to prove that  $M$  is decomposable we only need to show that  $K$  is decomposable. So without loss of generality we may suppose that  $H_k(M) \subset 'M$ . So  $S = \text{socle}(M) = \text{socle}(H_k(M))$ . By hypothesis  $H_k(M)$  is decomposable. So by (3.1),  $S = \bigcup_n S_n$ , where  $S_n$  ( $n = 1, 2, \dots$ ) is an ascending sequence of submodules, such that for each  $n$ , we have a positive integer  $l_n$  such that the height of any uniform element  $x$  of  $S_n$  taken in  $H_k(M)$  does not exceed  $l_n$ . Then the height of any uniform element  $x$  in  $S_n$  taken in  $M$  does not exceed  $l_n + k$ . So by (3.2)  $M$  is decomposable.

Proof of (3.1). In view of (3.3) it is enough to prove that  $H_1(M)$  is decomposable. Now by hypothesis

$$\bar{M} = M/S = \bigoplus \sum_{\alpha} \bar{y}_{\alpha} R$$

where  $\bar{y}_{\alpha} R$  are uniserial.

As seen in the proof of (3.3), without loss of generality we can suppose that  $H_1(M) \subset 'M$ . In view of the condition (I) we take each  $y_{\alpha}$  to be uniform in  $M$ . Now  $d(y_{\alpha} R) \geq 2$ . Let  $x_{\alpha} R < y_{\alpha} R$  with  $d(y_{\alpha} R/x_{\alpha} R) = 1$ . Then  $x_{\alpha} \in H_1(M)$ . We claim,

$$H_1(M) = \bigoplus \sum_{\alpha} x_{\alpha} R$$

and this will prove the result. Suppose

$$x_\alpha R \cap \left( \sum_{i=1}^n x_i R \right) \neq 0$$

with  $x_\alpha R \neq x_i R$  for  $1 \leq i \leq n$ . Then

$$z_\alpha R = x_\alpha R \cap \left( \sum_{i=1}^n x_i R \right) = \text{socle}(x_\alpha R) = y_\alpha R \cap \left( \sum_{i=1}^n y_i R \right)$$

$$\text{Now } \sum_{i=1}^n y_i R = \bigoplus_{j=1}^m u_j R$$

where  $u_j R$  are uniserial and by (1.2)  $m \leq n$ . If for some  $j$ ,  $d(u_j R) = 1$ , we have

$$\bigoplus_{i=1}^n \bar{y}_i R = \bigoplus_{j=1}^m \bar{u}_j R \pmod{S}$$

and the right hand side has less than  $n$  terms. This is a contradiction. Therefore  $d(u_j R) \geq 2$  for every  $j$  and  $m = n$ . We write

$$z_\alpha = v_1 + v_2 + \cdots + v_n, \quad v_i \in u_i R$$

We can find  $t_\alpha \in y_\alpha R$  such that  $d(t_\alpha R / z_\alpha R) = 1$ . By condition (II), we get homomorphisms

$$\sigma_j: t_\alpha R \rightarrow u_j R$$

such that  $\sigma_j(z_\alpha) = v_j$ . Define

$$\sigma: t_\alpha R \rightarrow \bigoplus_{j=1}^n u_j R$$

$$\text{by } \sigma(y) = \sum_j \sigma_j(y), \quad y \in t_\alpha R$$

Then  $\sigma$  is identity on  $z_\alpha R$ . Let

$$A = \{r \in R: t_\alpha r \in z_\alpha R\}$$

Then  $A$  is a maximal right ideal of  $R$  with  $z_\alpha R = t_\alpha A$ . So for  $r \in A$ ,  $t_\alpha r = \sigma(t_\alpha r) = \sum_{j=1}^n \sigma_j(t_\alpha) r$ . Consequently  $t_\alpha - \sigma(t_\alpha)$  is a uniform element such that

$$(t_\alpha - \sigma(t_\alpha))R \cong R/A$$

This gives  $t_\alpha - \sigma(t_\alpha) \in S$ . Hence,

$$t_\alpha \equiv \sigma(t_\alpha) \pmod{S}$$

Consequently  $\bar{t}_\alpha \in \bar{y}_\alpha R \cap \left( \sum_{j=1}^n \bar{y}_j R \right) = \bar{0}$ . Hence  $t_\alpha \in S$ . This is a contradiction.

Therefore

$$\sum_\alpha x_\alpha R = \bigoplus_\alpha x_\alpha R$$

This also yields  $\sum_\alpha y_\alpha R = \bigoplus_\alpha y_\alpha R$ , since each  $y_\alpha R$  is an essential extension

of  $x_\alpha R$ . Consider any uniform element  $x \in H_1(M)$  such that  $x \notin S$ . We can find a uniform element  $y \in M$  such that,  $x \in yR$  and  $d(yR/xR) = 1$ . If

$$A = \{r \in R; yr \in xR\}$$

then  $A$  is a maximal right ideal of  $R$  and  $xR = yA$ . Now

$$\bar{y} = y + S = \bar{\xi}_1 + \bar{\xi}_2 + \cdots + \bar{\xi}_n, \quad \bar{\xi}_i \in \bar{y}_i R$$

for some  $\bar{y}_1, \bar{y}_2, \dots, \bar{y}_n$  among  $\bar{y}_\alpha$ 's,  $\alpha \in \Lambda$ . We take  $\xi_i \in y_i R$ . If for any  $i$ ,  $\bar{\xi}_i \neq 0$ , then the natural homomorphism  $\eta_i: \bar{y}_i R \rightarrow \bar{\xi}_i R$  is non-zero; since  $\bar{y}_i R$  is uniserial, it follows that  $\text{Ker } \eta_i \subset \bar{x}_i R = \bar{y}_i A$  and so  $\xi_i R / \xi_i A \cong \bar{y}_i R / \bar{x}_i R \neq 0$ . Thus  $\bar{\xi}_i \neq 0$  implies  $\bar{\xi}_i A \subset x_i R$ . Consequently  $\bar{x} \in \sum \bar{x}_i R$  and hence

$$x \in \sum_\alpha x_\alpha R + S.$$

$$H_1(M) = \sum x_\alpha R + S$$

This proves:

We claim:  $S \subset \sum x_\alpha R$ . If not we can find a uniform element  $x \in S$  such that  $x \notin \sum x_\alpha R$ . Then  $xR \cap \sum x_\alpha R = 0$ . We can find a uniform element  $y \in M$  such that  $x \in yR$  and  $d(yR/xR) = 1$ . Now let  $N' = yR + \sum y_\alpha R = yR \oplus (\sum y_\alpha R)$ . Then

$$\begin{aligned} M/S &= (\sum y_\alpha R + S)/S = (N' + S)/S \cong N' / \text{socle}(N') \\ &\cong yR/xR \oplus (\sum y_\alpha R) / \text{socle}(y_\alpha R) \end{aligned}$$

Therefore

$$\oplus \sum_\alpha y_\alpha R / \text{socle}(y_\alpha R) \cong (yR/xR) \oplus \sum_\alpha y_\alpha R / \text{socle}(y_\alpha R)$$

This isomorphism is natural. Hence  $yR/xR = 0$ . This is a contradiction. Hence  $S \subset \sum x_\alpha R$ . This yields

$$H_1(M) = \oplus \sum_\alpha x_\alpha R$$

Hence the result follows.

We end this paper with a few remarks.

- (1) Any module  $M$  over a commutative ring  $R$  satisfying (I) and (II) must satisfy (III). However, a simple faithful module over a nonartinian primitive ring trivially satisfies (I) and (II), but not (III).
- (2) If a module  $M$  satisfies (I) and (II), then (II) gives that any uniserial submodule  $xR$  of  $M$  is quasi-injective. The example on page 362 in [3] is of a uniserial module which is not quasi-injective. This shows that although a uniserial module always satisfies (I), but it need not satisfy (II).
- (3) If a commutative ring  $R$ , admits a faithful finitely generated module satisfying (I) and (II), then  $R$  is a principal ideal ring with d.c.c. It will be interesting to investigate the structure of noncommutative rings admitting faithful, finitely

generated modules satisfying conditions (I), (II) and (III).

(4) Consider a local ring  $R$ , with maximal ideal  $W$ , such that  $W^2=0$ ,  $Q=R/W$ , a field with the property that  $\dim_Q W=1$ , and  $\dim W_Q=2$ .  $R$  is not a right principal ideal ring. However for any  $x \neq 0$  in  $W$ ,  $R/xR$  is a uniserial, injective, faithful, right  $R$ -module of length two, so it satisfies (I), (II) and (III). (See V. Dlab, and C. M. Ringel, Math. Ann. 195, (1972) Proposition 2)

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