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String Theory in Curved Space

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Contents

Abstract

The string theory in background fields is firstly reviewed in the context of the non-linear σ-model. The conformal invariance of the σ-model requires equations of motion for background fields. Secondly the geometry independent cubic action of the string field theory is studied. The well known action of the string field theory arises by the expansion around a classical solution to the cubic action. In the main part of the thesis, the solutions that generate the string field theory in curved space are investigated. It is shown that the cubic action theory determines the equations of motion for background fields which are the same as those obtained in the context of the conformal invariant σ-model. A field theory in curved space can be formulated associated with each of these solutions. The cubic action theory is, therefore, qualified as a geometry independent and self-contained field theory of string. In the σ-model analysis the dimensional regularization is used and a consistent extension of antisymmetric tensor ε^{ab} on a curved world sheet from two to n dimensions is shown to exist.

§1. Introduction

Superstring theory[1] is a promising candidate for a consistent quantum theory which unifies all fundamental interactions including gravity. In the framework of local field theory the gravitational field is prepared from the outset as the metric field of space-time. When it is quantized, it inevitably generates unrenormalizable divergences, which destroys the theory. In the string theory the gravitational field arises as a result of quantization of closed string. The theory has much higher symmetries than the former. It is known that the divergences can be eliminated if the symmetries are kept when the system is quantized. This makes the theory mathematically well defined.

The other distinctive feature in string theory is the fact that the consistency of the quantum theory imposes severe restrictions over model constructions and singles out only few possibilities. It is, for instance, the closure property of Lorentz algebra in the light-cone quantization, and the nilpotency of BRST charge operator in the covariant quantization that determine the dimensions of space-time to be 10 for the superstring and 26 for the bosonic string, which are called critical dimensions of respective theories. The latter condition is needed to guarantee the unitarity of theory. Furthermore, the anomalyless condition for the conformal symmetry requires that the internal symmetry of the heterotic string should be either $O(32)$ or $E_8 \times E_8$.

In order to make these theories realistic, one needs to compactify the high dimensional space-time to the one having a four dimensional Lorentz space and a some compact space. The string theory should allow the compactification and tell what sorts of the space-time are realizable. Recent extensive researches enable us to undertake these interesting problems.

Two different approaches have been proposed to study the compactification. One of them has been made in the framework of the first-quantized Polyakov's approach. The Polyakov action can be viewed as a two-dimensional field theory (a linear σ-model) where the D-component string coordinates X^{μ} are regarded as D scalar fields. The action can be extended to a non-linear σ-model field theory which can be explained as a string theory interacting with a set of background fields in the D-dimensional space. The theory is required to be conformal invariant when it is quantized, hence to have vanishing trace anomaly or conformal anomaly. The conformal invariance requires the vanishing of β-functions for the coupling of string with background fields. The vanishing condition of β-functions is able to be shown to derive the equations of motion for background fields, which determine the structure of space-time[2-4].

Those equations of motion are also derivable from the effective action for local component fields which is obtained by the S-matrix calculation in the zero slope limit of the string theory[5]. As a result background fields in the conformal invariant σ-model can be identified with the condensate of massless modes of the string. In the σ-model approach another way of imposing the conformal invariance is to require the nilpotency of BRST charge associated with the non-linear σ-model action[6,7]. The condition derives the same equations of motion for background fields as those obtained from the vanishing condition of β-functions.

Although the σ-model approach is supposed to determine correct geometry of space-time, it cannot be regarded as a self-contained theory. If a solution to the equations of motion required by the conformal invariance is chosen, a σ-model associated with the solution is specified. If another is chosen, a different σ-model defines the theory. The σ-model approach lacks the idea to look for the most stable ground state solution among many allowed solutions and to handle the phase transition from a solution to a different one. This motivates us to consider the second-quantized string field theory.

Another approach to understand the compactification of space-time is indebted to the recent progress and outcome in the covariant string field theory which has been studied by Witten[8], Hata, Itoh, Kugo, Kunitomo and Ogawa(HIKKO)[9,10] and Neveu and West[11]. From the knowledge of the ordinary local field theory, one hopes to be able to deal with a non-perturbative effect in string theory by the use of the covariant string field theory.

The covariant and gauge invariant string field theory can be classified into two types according to how the interaction term is introduced. The one which is based on the mid-point interaction has been formulated by Witten[8]. The theory reproduces open string dual amplitudes and correctly covers the full moduli space of corresponding Riemann surfaces[12]. In spite of the fact that the theory incorporates the closed string exchange in the open string loop amplitudes, the theory consisting of only closed strings is not yet formulated in the framework of Witten's string field theory.

On the other hand, there exists the string field theory consisting of the open and/or closed string formulated by HIKKO[9,10] and Neveu and West[11]. In the formalism the unphysical width parameter is introduced to reproduce correct string tree amplitudes. Although its dependence is decoupled at the string tree level, the infinite covering of moduli space occurs at the loop level due to the unphysical parameter and causes the violation of unitarity. According to Neveu and West and others[13], covariantized light-cone formalism seems to resolve the width parameter problems. We will have to wait further investigations along these lines.

The advantage of HIKKO's formalism, however, is the existence of the closed string field theory that reproduces correct tree amplitudes. Since the closed string contains gravity in its spectrum, one expects that the spontaneous compactification will be caused by the string condensate. The compactification may be studied by finding the solutions to the equations derived from the action in the covariant string field theory. However, the kinetic energy term in the action consists of BRST operator defined on 26-dimensional Minkowski space and hence it seems to be difficult to consider the string in curved space.

On the contrary, the cubic action theory which is independent of background geometry has been proposed by HIKKO[14] and also by Horowitz, Lykken, Rohm and Strominger[15]. The action consists of the only cubic term without the kinetic energy term. The cubic term is essentially given by the overlapping condition of three strings and hence formally geometry independent. The cubic action is an interesting model which provides the string field theory in curved space. The familiar form in the string field theory can arise by the expansion around some classical solution derived from the cubic action. In this approach the solution determines the geometry of space-time. In other words, the space-time is generated by the string field condensation. They looked for general conditions that the solution should satisfy to generate the ordinary form of field theory. Except for some delicate regularization problems about operator products, they found that the crucial point is the existence of BRST operators associated with some background fields. As a special case they demonstrated that a solution constructed with the ordinary BRST charge for a flat Minkowski space does generate the well known string field theory.

In this thesis, we look for what sort of background fields would be allowed as solutions to the equation derived from the cubic action[16]. We find a set of equations of motion for background fields and construct the string field theory in curved space. These equations agree with those obtained by the conformal invariance of the σ-model. The advantage of our method over the σ-model approach consists in the fact that all allowed solutions are contained in a space which a single theory covers. In the σ-model analysis a particular attention is paid to the treatment of antisymmetric tensor $\varepsilon^{ab}(g)$ in the dimensional regularization when the world sheet has a non-vanishing curvature. Although the extension of ε^{ab} from two to n dimensions has been studied, it has been always restricted in the case of that on flat world sheet. It is shown that there exists a consistent extension of ε^{ab} on curved world sheet from two to n dimensions.

The present paper is organized as follows. In sec. 2 the conformal invariance and the nilpotency of BRST charge in the non-linear σ-model are reviewed in the case of the bosonic closed string on a world sheet having spherical topology. In sec. 3 we give a brief review of HIKKO's cubic action theory and prove the general conditions that the solution of the cubic action should satisfy to generate the familiar form of the string field theory. Taking advantage of the σ-model as an auxiliary tool, we look for the solutions that generate the string field theory in curved space in sec. 4. In the σ-model analysis the extension of ε^{ab} on a curved world sheet is determined by the Bianchi identities for background fields so that consistent relations among background fields follow. Sec. 5 is devoted to summary and discussion. In appendix B the effective action in the non-linear σ-model is calculated.

§2. String Theory in Background Fields

§2.1 Conformal Invariance in Non-Linear σ-Model

We will give a review of conformal invariant σ-model. The string theory in curved space is described by a non-linear σ-model in two dimensions. In the σ-model approach, many authors[2-4] have clarified that the condition of conformal invariance of the σ-model requires the equations of motion for background fields that correspond to massless excitations of the string. These equations are equivalent to those obtained by the string S-matrix calculation in zero slope limit[5].

Throughout the present paper we shall consider the closed bosonic string which contains graviton, antisymmetric tensor and dilaton as massless states in its spectrum and restrict ourselves to the string tree level. In a similar way the open string, superstrings and the heterotic string in background fields has been studied by many authors. One of the interesting developments in the σ-model approach is the fact that equations of motion for background fields are modified by a renormalization due to the effect of small handles of string loop amplitudes[17]. On these subjects no discussion will be presented here.

The action of the non-linear σ-model on a fixed curved world sheet is given by

$$S = \frac{1}{4\pi\alpha'}\int d^2\sigma \Big[\sqrt{g}g^{ab}G_{\mu\nu}(X)\partial_a X^\mu \partial_b X^\nu + i\varepsilon^{ab}B_{\mu\nu}(X)\partial_a X^\mu \partial_b X^\nu + \alpha'\sqrt{g}R^{(2)}\Phi(X)\Big], \tag{2.1.1}$$

where $\sigma^a=(\tau,\sigma)$, g_{ab} and $R^{(2)}$ are coordinates, the metric tensor*) and the scalar curvature on the world sheet, respectively. Here X^μ is the string coordinate in space-time M. In (2.1.1) α' is the slope parameter which is the inverse of the string tension. In the context of the σ-model, $G_{\mu\nu}$ is the metric tensor on M, $B_{\mu\nu}$ is the antisymmetric tensor field whose field strength provides the torsion on M and Φ represents the dilaton field.

If we look at the classical action, first two terms in (2.1.1) are invariant under a Weyl (conformal) transformation

$$g_{ab} \to e^{\phi} g_{ab}, \tag{2.1.2}$$

that is a local rescaling of the metric tensor. On the contrary, the last term, which is referred to as dilaton coupling, violates the invariance under (2.1.2) at the classical level. The dilaton coupling, however, is necessary for the renormalizability of the σ-model in two dimensions[2]. Another interpretation of the role of the dilaton coupling is to cancel the conformal anomaly of dilaton vertex which arises from the trace part of the metric tensor of space-time[18]. Thus the dilaton coupling guarantees the conformal invariance of the quantized theory, hence the last term in (2.1.1) is higher by a factor α' in comparison with other terms. If one wants a more general form of the classical action that is allowed by the invariance under two-dimensional reparametrization, one may add the term proportional to $\sqrt{g}\,T(X)$ with $T(X)$ being a scalar function of X^μ. It has been shown that the coupling is a background consisting of a condensate of a tachyonic mode[19].

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*) The metric is taken to be Euclidean here.

We require that the conformal invariance should be kept at the quantum level. The requirement guarantees that the σ-model is consistent in the quantum theory. To discuss the quantum theory of the σ-model, we shall consider the effective action W defined by

$$W = - \ln\left(\int \mathscr{D}X \; e^{-S} \right). \qquad (2.1.3)$$

The invariance of W under the transformation (2.1.2) indicates that

$$-\frac{2}{\sqrt{g}}\frac{\delta W}{\delta \phi} = g_{ab}\frac{2}{\sqrt{g}}\frac{\delta W}{\delta g^{ab}} = \langle T_a{}^a \rangle = 0. \qquad (2.1.4)$$

Hence the conformal invariance is equivalent to the vanishing of the trace anomaly of the energy momentum tensor. The dimensional analysis and the invariance under reparametrization lead us to express as

$$T_a{}^a = \beta^{\Phi}R^{(2)} + \beta^{G}_{\mu\nu}g^{ab}\partial_a X^{\mu}\partial_b X^{\nu} + \beta^{B}_{\mu\nu}\frac{1}{\sqrt{g}}\varepsilon^{ab}\partial_a X^{\mu}\partial_b X^{\nu}. \qquad (2.1.5)$$

As a result, the conformal invariance of the σ-model requires the vanishing of β-functions which are defined by independent coefficients of the trace anomaly. The vanishing condition will impose restrictions on background fields.

Let us find the vanishing condition of β-functions. Assuming that α' is small, we shall first make a perturbative expansion with respect to α'. It is convenient to use the background field expansion[20-22] in Riemann normal coordinate system which will be explained below. The quantum fluctuations Π^{μ} around a classical solution of (2.1.1) are defined by

$$X^\mu = X_B{}^\mu + \Pi^\mu(\xi). \tag{2.1.6}$$

Note that Π^μ is no more a contravariant vector under the general coordinate transformation of space-time. Normal coordinates ξ^μ are defined by the tangent vector at $X_B{}^\mu$ along the geodesic passing through $X_B{}^\mu$ and X^μ. Solving the geodesic equation iteratively, one finds that

$$X^\mu = X_B{}^\mu + \xi^\mu - \frac{1}{2}\Gamma_{\rho\sigma}^{\mu}(X_B)\xi^\rho\xi^\sigma - \ldots , \tag{2.1.7}$$

which defines a transformation from Π^μ to ξ^μ. The advantage by the use of the Riemann normal coordinate ξ^μ is the manifest covariance of the dependence on background fields.

The background field expansions of relevant quantities are given by

$$\partial_a X^\mu = \partial_a X_B{}^\mu + D_a\xi^\mu + \frac{1}{3}R^\mu{}_{\nu\rho\sigma}(X_B)\xi^\nu\xi^\rho\partial_a X_B{}^\sigma + \mathcal{O}(\xi^3), \tag{2.1.8}$$

$$G_{\mu\nu}(X) = G_{\mu\nu}(X_B) - \frac{1}{3}R_{\mu\rho\nu\sigma}(X_B)\xi^\rho\xi^\sigma + \mathcal{O}(\xi^3), \tag{2.1.9}$$

$$\begin{aligned} B_{\mu\nu}(X) = {} & B_{\mu\nu}(X_B) + \nabla_\rho B_{\mu\nu}(X_B)\xi^\rho + \frac{1}{2}\nabla_\rho\nabla_\sigma B_{\mu\nu}(X_B)\xi^\rho\xi^\sigma \\ & - \frac{1}{6}\left\{R^\lambda{}_{\rho\mu\sigma}(X_B)B_{\lambda\nu}(X_B) - R^\lambda{}_{\rho\nu\sigma}(X_B)B_{\lambda\mu}(X_B)\right\}\xi^\rho\xi^\sigma \\ & + \mathcal{O}(\xi^3), \end{aligned} \tag{2.1.10}$$

$$\Phi(X) = \Phi(X_B) + \nabla_\mu\Phi(X_B)\xi^\mu + \frac{1}{2}\nabla_\mu\nabla_\nu\Phi(X_B)\xi^\mu\xi^\nu + \mathcal{O}(\xi^3), \tag{2.1.11}$$

where

$$D_a\xi^\mu = \partial_a\xi^\mu + \Gamma_{\nu\rho}^{\mu}(X_B)\xi^\nu\partial_a X_B{}^\rho, \tag{2.1.12}$$

is the background field covariant derivative and $R_{\mu\nu\rho\sigma}$ is Riemann tensor*) of space-time. Using (2.1.8-11), one finds the expansion of the action around a classical solution

$$S = \frac{1}{4\pi\alpha'}\int d^2\sigma(\mathcal{L}^G+\mathcal{L}^B+\mathcal{L}^\Phi) \tag{2.1.13}$$

$$\begin{aligned}\mathcal{L}^G = \sqrt{g}g^{ab} \Big\{ & G_{\mu\nu}(X_B)\partial_a X_B{}^\mu \partial_b X_B{}^\nu + G_{\mu\nu}(X_B)D_a\xi^\mu D_b\xi^\nu \\ & - R_{\mu\rho\nu\sigma}(X_B)\xi^\rho\xi^\sigma\partial_a X_B{}^\mu\partial_b X_B{}^\nu \\ & + \frac{4}{3} R_{\mu\lambda\rho\sigma}(X_B)\xi^\lambda\xi^\rho D_a\xi^\mu\partial_b X_B{}^\sigma \\ & - \frac{1}{3} R_{\mu\rho\nu\sigma}(X_B)\xi^\rho\xi^\sigma D_a\xi^\mu D_b\xi^\nu \Big\}\end{aligned}$$

$$\begin{aligned}\mathcal{L}^B = i\varepsilon^{ab} \Big\{ & B_{\mu\nu}(X_B)\partial_a X_B{}^\mu\partial_b X_B{}^\nu - 2S_{\mu\rho\nu}(X_B)\partial_a X_B{}^\mu\xi^\rho D_b\xi^\nu \\ & + \nabla_\rho S_{\sigma\mu\nu}(X_B)\xi^\rho\xi^\sigma\partial_a X_B{}^\mu\partial_b X_B{}^\nu \\ & + \frac{4}{3}\nabla_\sigma S_{\mu\nu\rho}(X_B)\partial_a X_B{}^\mu D_b\xi^\nu\xi^\rho\xi^\sigma \\ & + \frac{2}{3}S_{\mu\nu\rho}(X_B)D_a\xi^\mu D_b\xi^\nu\xi^\rho \\ & + \frac{1}{2}\nabla_\sigma S_{\mu\nu\rho}(X_B)D_a\xi^\mu D_b\xi^\nu\xi^\rho\xi^\sigma \Big\}\end{aligned}$$

$$\mathcal{L}^\Phi = \alpha'\sqrt{g}R^{(2)} \Big\{\Phi(X_B) + \nabla_\mu\Phi(X_B)\xi^\mu + \frac{1}{2}\nabla_\mu\nabla_\nu\Phi(X_B)\xi^\mu\xi^\nu \Big\}$$

where we have only kept terms which are relevant for the perturbative calculation up to $\mathcal{O}(\alpha')$ and the field strength of $B_{\mu\nu}$ is

$$S_{\mu\nu\rho} = \frac{1}{2}(\partial_\rho B_{\mu\nu}+\partial_\nu B_{\rho\mu}+\partial_\mu B_{\nu\rho}), \tag{2.1.14}$$

which is gauge invariant form under $\delta_\varepsilon B_{\mu\nu}=\partial_{[\mu}\varepsilon_{\nu]}$. In order to obtain the background field expansion of the action (2.1.13) in the case of the antisymmetric tensor $B_{\mu\nu}$, we have performed

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*)Notations are $R^\lambda{}_{\mu\nu\rho}=\partial_\nu\Gamma^\lambda{}_{\mu\rho}-\ldots$ and $R_{\mu\nu}=R^\lambda{}_{\mu\lambda\nu}$.

partial integration and dropped non-gauge invariant total derivative terms[22].

Let us now proceed to study the quantum theory of the σ-model. For a while, the ghost contribution to the effective action will be neglected. Since the ghost field does not interact with background fields and remains free, its contribution to the effective action can be dealt with independently. The effective action is given by the path integral

$$W[g_{ab}, X_B] = -\ln\left(\int \mathcal{D}\xi \; e^{-S[g_{ab}, X_B]}\right), \tag{2.1.15}$$

which presents the sum of the contributions of connected graphs. The expectation value of the energy-momentum tensor is given by

$$\langle T_{ab}\rangle = \frac{2}{\sqrt{g}} \frac{\delta W}{\delta g^{ab}} . \tag{2.1.16}$$

The contraction of $\langle T_{ab}\rangle$ with respect to g^{ab} gives rise to the trace anomaly from which β-functions are obtained.

To simplify the discussion, the antisymmetric tensor field $B_{\mu\nu}$ and dilaton Φ are disregarded for the time being. The terms of $\mathcal{O}(\xi^3)$ and higher orders are also disregarded here. We shall use the dimensional regularization and add the mass term to get rid of infrared divergences. The action, up to $\mathcal{O}(\xi^2)$, is

$$S = \frac{1}{2}\int d^n\sigma \; \sqrt{g}g^{ab} \Big[G_{\mu\nu}(X_B)\partial_a X_B^{\mu}\partial_b X_B^{\nu} + D_a\xi_M D_b\xi^M - 2\pi\alpha' R_{\mu M\nu N}(X_B)\, \xi^M\xi^N\partial_a X_B^{\mu}\partial_b X_B^{\nu}\Big] + \frac{1}{2}\int d^n\sigma \; \sqrt{g}m^2 \xi_M\xi^M , \tag{2.1.17}$$

where $n=2+2\varepsilon$ and dimensionless fields have been defined by the replacement $X^{\mu}\rightarrow\sqrt{2\pi\alpha'}X^{\mu}$ in (2.1.13). Here we have introduced the vielbein field $e_{\mu}{}^{M}(X_B)$ and defined $\xi^{M}=e_{\mu}{}^{M}(X_B)\xi^{\mu}$. Since the functional measure in (2.1.15) is invariant under the coordinate transformations, the variables in (2.1.15) can be transformed from ξ^{μ} to ξ^{M} without considering the contribution which arises from the Jacobian factor of the functional measure. The advantage of coordinate transformations from ξ^{μ} to ξ^{M} is to simplify the perturbative calculation, where one can easily extract the free part of the kinetic term in the action.

Instead of the use of Riemann normal coordinate on curved world sheet, we shall here make weak field expansion of $g_{ab}=\delta_{ab}+h_{ab}$ around the flat world sheet. It is of use to introduce

$$\bar{h}_{ab}=h_{ab}-\frac{1}{2}\delta_{ab}h, \tag{2.1.18}$$

$$h=h_{a}{}^{a}, \tag{2.1.19}$$

as independent variables[23]. Using (2.1.18) and (2.1.19), one obtains that

$$\sqrt{g}g^{ab}=\delta^{ab}-\bar{h}^{ab}+\mathcal{O}(h_{ab}^{2}), \tag{2.1.20}$$

$$\sqrt{g}R^{(n)}=\partial_{a}\partial_{b}\bar{h}^{ab}-\frac{1}{2}\partial^{a}\partial_{a}h$$
$$+\frac{1}{4}\bar{h}^{ab}\partial^{c}\partial_{c}\bar{h}_{ab}+\frac{1}{2}\partial_{c}\bar{h}^{ca}\partial_{b}\bar{h}^{b}{}_{a}-\frac{\varepsilon}{8}h\partial^{a}\partial_{a}h+\mathcal{O}(h_{ab}^{3}), \tag{2.1.21}$$

where indices are raised and lowered with Kronecker delta. The weak field expansion of $\sqrt{g}R^{(n)}$ is given here for later use.

The free propagator is given by

$$\Delta(\sigma-\sigma')=\mu^{-2\varepsilon}\int\frac{d^n p}{(2\pi)^n}\,\frac{1}{p^2+m^2}\,e^{ip(\sigma-\sigma')}, \qquad (2.1.22)$$

where μ is an arbitrary mass scale. It is easy to calculate graphs contributing to the effective action. The graphs associated the following amplitudes A, B and C are given in fig. 1. The results are as follows:

$$A = -\frac{D}{4}\,\frac{B(n/2+1,n/2+1)}{(4\pi)^{n/2}\Gamma(2)}\,\Gamma(2-n/2)\int\frac{d^n p}{(2\pi)^n}\left(\frac{\mu^2}{p^2}\right)^{1-n/2}$$
$$\times\Bigg[+\frac{(p_a p_b \bar{h}^{ab}(p)-\frac{1}{2}p^2 h(p))^2}{p^2}$$
$$+\frac{8}{n(2-n)}\Big\{-\frac{1}{4}\bar{h}^{ab}(p)\,p^2\bar{h}_{ab}(-p)$$
$$+\frac{1}{2}\bar{h}^{ab}(p)\,p_a p_c \bar{h}^c{}_b(-p)+\frac{\varepsilon}{8}h(p)\,p^2 h(-p)\Big\}\Bigg]$$

(2.1.23)

$$B = \frac{\alpha'}{4}\int d^n\sigma\sqrt{g}\,g^{ab}\mu^{-2\varepsilon}R_{\mu\nu}(X_B)\,\partial_a X_B{}^{\mu}\partial_b X_B{}^{\nu}\left(\frac{1}{\varepsilon}+\ln(\frac{m^2}{4\pi})-\gamma\right)$$

(2.1.24)

$$C = \pi\alpha'\frac{B(n/2,n/2)}{(4\pi)^{n/2}}\,\Gamma(2-n/2)\int\frac{d^n p}{(2\pi)^n}\left(\frac{\mu^2}{p^2}\right)^{1-n/2}$$
$$\times\frac{(p_a p_b \bar{h}^{ab}(p)-\frac{1}{2}p^2 h(p))}{p^2}\,R_{\mu\nu}(X_B)\,\partial_c X_B{}^{\mu}\partial^c X_B{}^{\nu}(-p)$$

(2.1.25)

where γ is the Euler constant. The σ-model action will be renormalized in the minimal subtraction scheme. The counterterms can be read off from the above results to cancel the one-loop divergences:

$$S_{c.t.} = -\frac{1}{2\varepsilon}\,\frac{D}{24\pi}\int d^n\sigma\sqrt{g}R^{(n)}$$

$$- \frac{\alpha'}{4\varepsilon}\int d^n\sigma\sqrt{g}g^{ab}R_{\mu\nu}(X_B)\partial_a X_B{}^{\mu}\partial_b X_B{}^{\nu} \qquad (2.1.26)$$

Taking account of the contribution of counterterms, one finds the one-loop renormalized effective action

$$W = \int d^2x d^2y\Big[- \frac{D}{96\pi}\sqrt{g(x)}R(x)\,G(x,y)\sqrt{g(y)}R(y)$$
$$- \frac{\alpha'}{4}\sqrt{g}g^{ab}R_{\mu\nu}(X_B)\partial_a X_B{}^{\mu}\partial_b X_B{}^{\nu}(x)\,G(x,y)\sqrt{g(y)}R(y)\Big]$$
$$+ \int d^2x\Big[+ \frac{\alpha'}{4}\left[\ln(\frac{m^2}{4\pi\mu^2})-\psi(1)\right]\sqrt{g}g^{ab}R_{\mu\nu}(X_B)\partial_a X_B{}^{\mu}\partial_b X_B{}^{\nu}$$
$$+ \frac{D}{48\pi}\left[\ln(\frac{-\Box}{\mu^2})-\psi(1)-1\right]R^{(2)}\Big] \qquad (2.1.27)$$

where

$$\frac{1}{\sqrt{g}}\partial_a(\sqrt{g}g^{ab}\partial_b)\,G(x,y) = -\frac{1}{\sqrt{g}}\delta^2(x-y). \qquad (2.1.28)$$

The trace anomaly can be found out from (2.1.27) and (2.1.16) by contracting with respect to g_{ab}:

$$\langle T_a{}^a\rangle = + \frac{D}{24\pi}R^{(2)} + \frac{1}{2}\alpha' R_{\mu\nu}(X_B)\,g^{ab}\partial_a X_B{}^{\mu}\partial_b X_B{}^{\nu} \qquad (2.1.29)$$

Local terms in (2.1.27) do not contribute to (2.1.29) because the variations of W give rise to contributions proportional to $\mathcal{O}(\varepsilon)$. In particular, to find the trace anomaly, non-local terms in the effective action are of importance. Comparing (2.1.5) and (2.1.29), one obtains leading terms of some β-functions.

The ghost contribution to the trace anomaly has been derived by Polyakov[24]. The result is

$$\Big[\langle T_a{}^a\rangle\Big]_{\rm ghost} = - \frac{26}{24\pi}R^{(2)}, \qquad (2.1.30)$$

which arises from the Faddeev-Popov determinant.

In a similar way, one can calculate the contribution to the effective action including higher order terms in ξ and also in the presence of $B_{\mu\nu}$ and Φ. The results calculated by many authors are as follows:

$$\langle T_a{}^a\rangle = \beta^\Phi R^{(2)} + \beta^G_{\mu\nu} g^{ab}\partial_a X_B{}^\mu \partial_b X_B{}^\nu + \beta^B_{\mu\nu}\frac{1}{\sqrt{g}}\varepsilon^{ab}\partial_a X_B{}^\mu \partial_b X_B{}^\nu + \ldots \tag{2.1.31}$$

where

$$\beta^\Phi = \frac{1}{24\pi}\left[D-26-\frac{3}{2}\alpha'(R+4\nabla^\mu\nabla_\mu\Phi-4\nabla^\mu\Phi\nabla_\mu\Phi-\frac{1}{3}S_{\mu\nu\rho}S^{\mu\nu\rho})+\mathcal{O}((\alpha')^2)\right] \tag{2.1.32}$$

$$\beta^G_{\mu\nu} = \frac{\alpha'}{2}(R_{\mu\nu}+2\nabla_\mu\nabla_\nu\Phi-S_{\mu\rho\sigma}S_\nu{}^{\rho\sigma})+\mathcal{O}((\alpha')^2) \tag{2.1.33}$$

$$\beta^B_{\mu\nu} = \frac{\alpha'}{2}(\nabla_\rho S^\rho{}_{\mu\nu}-2S_{\rho\mu\nu}\nabla^\rho\Phi)+\mathcal{O}((\alpha')^2) \tag{2.1.34}$$

Thus the vanishing of β-functions requires a set of equations for background fields. These equations of motion are precisely those of gravity coupled to antisymmetric tensor field and dilaton. In (2.1.31) dots mean non-local terms constructed with powers of $\Box^{-1}$ and $R^{(2)}$. Insofar as we stick to this method followed by calculations of the effective action, non-local terms arise in (2.1.31), so that it is generally complicated to extract the condition over background fields imposed by the requirement of the conformal invariance. On other methods we will comment later.

It is worthwhile to emphasize that the linear combinations of these equations are derivable from a single effective action for background fields. In fact, the effective action turns out to be

$$S = \int d^{26}X\sqrt{G}e^{-2\Phi}\beta^\Phi \ , \tag{2.1.35}$$

where we have set D=26. The equations of motion for background fields can be derived from (2.1.35). These linear combinations are nothing but the vanishing conditions of β-functions, i.e.,

$$\frac{\delta S}{\delta\Phi} = 2e^{-2\Phi}\beta^{\Phi} , \tag{2.1.36}$$

$$\frac{\delta S}{\delta B^{\mu\nu}} = e^{-2\Phi}\beta^{B}_{\mu\nu} , \tag{2.1.37}$$

$$\frac{\delta S}{\delta G^{\mu\nu}} - \frac{1}{4}G_{\mu\nu}\frac{\delta S}{\delta\Phi} = - e^{-2\Phi}\beta^{G}_{\mu\nu} . \tag{2.1.38}$$

Let us summarize the above discussion. In the σ-model approach, one requires that the conformal invariance should be kept at the quantum level. The condition of conformal invariance requires the vanishing of β-functions which are defined by the independent coefficients of trace anomaly. The vanishing condition of β-functions requires equations of motion for background fields. These equations are derivable from an effective action for background fields. Thus, the conformal invariant σ-model provides a consistent string theory in curved space.

On the other hand, the effective action for local component fields and equations derived from it are also obtained by the string S-matrix calculation in zero slope limit[5]. Comparing the equations of the former with the latter, one finds that the equations of motion obtained by the conformal invariance of the σ-model are equivalent to those for massless excitations of the string.

The method reviewed here is able to be extended to higher orders in α'. In particular, it should be noted that the effective action in (2.1.35) is proportional to β^{Φ} in the first two orders in α'. It is expected that (2.1.35) may be true to all orders in α'[25].

In another method of perturbative calculation[26] one might make use of a weak field expansion of background fields around flat space-time instead of α' expansion. Although in the weak field expansion the results includes information of all orders in α' and the effects of massive modes in addition to massless modes can be treated, the covariance of space-time is not manifest. The connection between the conformal invariance and the equations of motion which are derivable from the effective action in the string S-matrix calculation will be more transparent in weak field expansion than α' expansion.

Finally, we remark on renormalization group(RG) β-functions. Since the renormalized one-loop effective action was found in (2.1.27), the renormalized metric tensor of space-time results in

$$G^R_{\mu\nu} = G_{\mu\nu} + \frac{\alpha'}{2}\left[\ln\left(\frac{m^2}{4\pi\mu^2}\right)+\gamma\right]R_{\mu\nu} . \tag{2.1.39}$$

The RG β-function is given by

$${}^{(RG)}\beta^G_{\mu\nu} = \mu\frac{\partial}{\partial\mu}G^R_{\mu\nu} = -\alpha' R_{\mu\nu} , \tag{2.1.40}$$

which is also derivable from $\frac{1}{\varepsilon}$ divergent term in (2.1.23-25) by the use of renormalization group equation. In particular, ${}^{(RG)}\beta^G_{\mu\nu}$ and ${}^{(RG)}\beta^B_{\mu\nu}$ are calculated with ease because it is enough to find ultraviolet properties in the σ-model on flat world sheet. Since renormalization group equation for effective action gives a relation between RG β-functions, ${}^{(RG)}\beta^\Phi$ can be more easily obtained without calculating non-local terms in the effective action on curved world sheet. The results[4,20,27,28] already known for RG β-functions are as follows:

$$^{(\mathrm{RG})}\beta^{\Phi} = \frac{1}{6}\left[D-26-\frac{3}{2}\alpha'(2\nabla^{\mu}\nabla_{\mu}\Phi+\frac{1}{3}S_{\mu\nu\rho}S^{\mu\nu\rho})+\mathcal{O}((\alpha')^2)\right] \quad (2.1.41)$$

$$^{(\mathrm{RG})}\beta^{G}_{\mu\nu} = -\alpha'(R_{\mu\nu}-S_{\mu\rho\sigma}S_{\nu}{}^{\rho\sigma})+\mathcal{O}((\alpha')^2) \quad (2.1.42)$$

$$^{(\mathrm{RG})}\beta^{B}_{\mu\nu} = \frac{1}{2}\alpha'\nabla_{\rho}S^{\rho}{}_{\mu\nu}+\mathcal{O}((\alpha')^2) \quad (2.1.43)$$

The RG β-functions are ambiguous because they depend on the choice of coordinates of space-time as is seen from its tensor structure. In general, they are affected by the diffeomorphism in such a way as

$$^{(\mathrm{RG})}\beta^{\Phi} \rightarrow {}^{(\mathrm{RG})}\beta^{\Phi} + v^{\mu}\nabla_{\mu}\Phi, \quad (2.1.44)$$

$$^{(\mathrm{RG})}\beta^{G}_{\mu\nu} \rightarrow {}^{(\mathrm{RG})}\beta^{G}_{\mu\nu} + \nabla_{\mu}v_{\nu}+\nabla_{\nu}v_{\mu}, \quad (2.1.45)$$

$$^{(\mathrm{RG})}\beta^{B}_{\mu\nu} \rightarrow {}^{(\mathrm{RG})}\beta^{B}_{\mu\nu} + 2v_{\rho}S_{\mu\nu}{}^{\rho}, \quad (2.1.46)$$

if string coordinates transform as $X^{\mu}\rightarrow X^{\mu}+v^{\mu}$. In the leading order of α' the vanishing of β-functions in (2.1.32-34) is equivalent to the vanishing of $^{(\mathrm{RG})}\beta$ in (2.1.41-43) up to diffeomorphism ambiguity.

The relation between the conformal invariance and RG β-functions has been studied in refs.[4,23,27,29]. One might renormalize composite operators in the σ-model by the use of normal products developed for quantum field theories in curved space[30]. In this method the conformal anomaly in (2.1.5) is regarded as the operator equation in terms of normal products. The implications of the conformal invariance in the σ-model and the relation to RG β-functions will be more transparent by the use of normal products rather than effective action approach reviewed here.

§2.2 Nilpotency of BRST Charge

Another way of imposing the conformal invariance in the σ-model is to require the nilpotency of BRST charge which guarantees the existence of the Virasoro algebra. We give a review of the nilpotency of BRST charge studied by Banks, Nemeschansky and Sen[6]. It is interpreted as a generalization from the nilpotency constraint studied by Kato and Ogawa[31] in flat space-time to that in curved space-time. In two dimensions one can choose a coordinate system on world sheet at least locally such that the metric takes the form $g_{ab}=e^{\phi}\delta_{ab}$ with ϕ being a conformal factor. Since we are interested in local properties, it is the most general metric. In the following analysis of the nilpotency, ϕ is taken to be zero from the outset because the conformal factor is decoupled when the nilpotency of BRST charge is satisfied.

When the conformal factor vanishes, the action in the σ-model is given by

$$S_x + S_{gh} = \frac{1}{4\pi\alpha'}\int d^2\sigma\Big[\delta^{ab}G_{\mu\nu}(X)\partial_a X^\mu \partial_b X^\nu + i\varepsilon^{ab}B_{\mu\nu}(X)\partial_a X^\mu \partial_b X^\nu + 2b_{++}\partial_- c^+ + 2b_{--}\partial_+ c^- \Big], \qquad (2.2.1)$$

where complex coordinates

$$z^+ = \frac{1}{\sqrt{2}}(\sigma^0 + i\sigma^1), \quad z^- = \frac{1}{\sqrt{2}}(\sigma^0 - i\sigma^1) \qquad (2.2.2)$$

have been introduced. In the following the replacement $X^\mu \to \sqrt{2\pi\alpha'}\,X^\mu$ will be made. The action is invariant under BRST transformations

$$\delta_B X^\mu = -(c^+\partial_+ X^\mu + c^-\partial_- X^\mu), \qquad (2.2.3)$$

$$\delta_B c^+ = -(c^+\partial_+ c^+ + c^-\partial_- c^+), \tag{2.2.4}$$

$$\delta_B b_{++} = T^X_{++} + T^{gh}_{++}, \tag{2.2.5}$$

where

$$T_{++} = \frac{2}{\sqrt{g}} \frac{\delta S}{\delta g^{++}} \tag{2.2.6}$$

is the energy-momentum tensor which is given by

$$T^X_{++} = G_{\mu\nu}\partial_+ X^\mu \partial_+ X^\nu - \alpha' \partial_+\partial_+\Phi, \tag{2.2.7}$$

and

$$T^{gh}_{++} = b_{++}\partial_+ c^+ + \frac{1}{2}\partial_+ b_{++} c^+. \tag{2.2.8}$$

The superscripts X and gh of T_{++} in (2.2.5) stand for the string coordinate and ghost parts of T_{++}, respectively. By Noether's method, BRST current turns out to be

$$J_+ = c^+(T^X_{++} + \frac{1}{2}T^{gh}_{++}), \tag{2.2.9}$$

where the complex conjugate of J_+ is also conserved. The other independent conserved current is obtained by the replacement of $z^+ \leftrightarrow z^-$.

BRST charge is defined by

$$Q = \oint \frac{dz}{2\pi i} J_+ , \tag{2.2.10}$$

where the contour is taken around the origin circling counterclockwise. Instead of computing anti-commutators of BRST operator Q, we shall make use of the method of the operator product expansion[32]. Consider the operator product of BRST current $J_+(z)J_+(z')$ where the operator product is defined by the radial ordering. When Q^2 is evaluated, the contour integral in

(2.2.10) picks up the singularities of $J_+(z)J_+(z')$ in the limit $z'\to z$. To calculate the nilpotency of BRST charge is, therefore, equivalent to picking up the singularities of $J_+(z)J_+(z')$ in the limit $z'\to z$.

The operator product expansion of $T^X_{++}(z)T^X_{++}(z')$ satisfies

$$T^X_{++}(z)T^X_{++}(z') = -\frac{1}{2\pi}\left[\frac{c}{(z^+-z'^+)^4} + \frac{2}{(z^+-z'^+)^2}T^X_{++}\left(\frac{z+z'}{2}\right) + \text{non-singular terms}\right] \tag{2.2.11}$$

where c is the central charge of the Virasoro algebra. Let us look at a matrix element of the operator product of BRST current as foll●ws;

$$\begin{aligned} J_+(z)J_+(z') &= c^+(z)c^+(z')T^X_{++}(z)T^X_{++}(z') \\ &+ \frac{1}{2}c^+(z)c^+(z')T^{gh}_{++}(z')T^X_{++}(z) + \frac{1}{2}c^+(z)T^{gh}_{++}(z)c^+(z')T^X_{++}(z') \\ &+ \frac{1}{4}c^+(z)T^{gh}_{++}(z)c^+(z')T^{gh}_{++}(z') \\ &= c^+(z)c^+(z')\left[\langle T^X_{++}(z)T^X_{++}(z')\rangle - \frac{1}{4\pi^2}\frac{13}{(z^+-z'^+)^4}\right] \end{aligned} \tag{2.2.12}$$

where (2.2.9) and the operator product expansion of ghost fields

$$\langle c^+(z)b_{++}(z')\rangle = \frac{1}{\pi}\frac{1}{z^+-z'^+} \tag{2.2.13}$$

have been used. Substituting (2.2.11) into (2.2.12), one obtains

$$J_+(z)J_+(z') = \left(-\frac{c}{2\pi} - \frac{13}{4\pi^2}\right)\frac{1}{(z^+-z'^+)^4}\, c^+(z)c^+(z'). \tag{2.2.14}$$

In D-dimensional Minkowski space, $\langle T^X_{++}(z)T^X_{++}(z')\rangle$ yields

$$\langle T^X_{++}(z)T^X_{++}(z')\rangle = \frac{D}{8\pi^2}\frac{1}{(z^+-z'^+)^4} \tag{2.2.15}$$

where $T^X_{++}=\partial_+X^\mu\partial_+X_\mu$ and

$$\langle X(z)X(z')\rangle = -\frac{1}{4\pi}\ln|z-z'|^2 \qquad (2.2.16)$$

have been used. Substituting (2.2.15) into (2.2.12), one finds that

$$J_+(z)J_+(z') = \frac{1}{8\pi^2}\frac{D-26}{(z^+-z'^+)^4}c^+(z)c^+(z'). \qquad (2.2.17)$$

The nilpotency of BRST charge requires that the singularities in (2.2.17) should vanish, so that the dimension of space-time is restricted to being $D=26$.

Next let us calculate the singularities of $\langle T_{++}(z)T_{++}(z')\rangle$ in the limit $z'\to z$ in the presence of background fields. The bracket indicates path integral average with the weight of the action (2.2.1) discarding ghost part. Assuming that α' is small, we use the background field expansion. The background field expansion of the bosonic part of action is given by (2.1.13) with the world sheet metric being flat. The background field expansion of the energy-momentum tensor is given by

$$\begin{aligned} T^X_{++} = {} & 2G_{\mu\nu}\partial_+X_B{}^\mu D_+\xi^\nu + D_+\xi_M D_+\xi^M - 2\pi\alpha' R_{\mu\rho\nu\sigma}\xi^\rho\xi^\sigma\partial_+X_B{}^\mu\partial_+X_B{}^\nu \\ & - \frac{8\pi\alpha'}{3}R_{\mu\rho\nu\sigma}\xi^\rho\xi^\sigma\partial_+X_B{}^\mu D_+\xi^\nu - \frac{2\pi\alpha'}{3}R_{\mu\rho\nu\sigma}\xi^\rho\xi^\sigma D_+\xi^\mu D_+\xi^\nu \\ & -\alpha'\left\{2\nabla_\mu\nabla_\nu\Phi\partial_+X_B{}^\mu D_+\xi^\nu + \nabla_\mu\Phi D_+D_+\xi^\mu + \frac{1}{2}\nabla_\mu\nabla_\nu\Phi D_+D_+(\xi^\mu\xi^\nu)\right\} \end{aligned} \qquad (2.2.18)$$

where (2.2.7) and (2.1.8-11) have been used and irrelevant terms have been disregarded.

Straightforward calculation of graphs which contribute to the terms proportional to $\partial_+X_B\partial_-X_B$ in the matrix element of

$\langle T_{++}(z)T_{++}(z')\rangle$ results in

$$\frac{\alpha'}{2\pi}\,\partial_+X_B{}^\mu\partial_-X_B{}^\nu\frac{z^--z'^-}{(z^+-z'^+)^3}$$
$$\times\left[\,R_{\mu\nu}+2\nabla_\mu\nabla_\nu\Phi-S_{\mu\rho\sigma}S_\nu{}^{\rho\sigma}+\nabla_\rho S^\rho{}_{\mu\nu}-2S_{\rho\mu\nu}\nabla^\rho\Phi\,\right], \qquad (2.2.19)$$

which should vanish by the requirement of the nilpotency of BRST charge. In the course of the calculation no divergences arise, so that the renormalization of the action is not needed. The vanishing of (2.2.19) gives rise to

$$R_{\mu\nu}+2\nabla_\mu\nabla_\nu\Phi-S_{\mu\rho\sigma}S_\nu{}^{\rho\sigma} = 0, \qquad (2.2.20)$$

$$\nabla_\rho S^\rho{}_{\mu\nu}-2S_{\rho\mu\nu}\nabla^\rho\Phi = 0, \qquad (2.2.21)$$

corresponding to the symmetric and antisymmetric part of (2.2.19), respectively.

The calculation of graphs which lead to the terms proportional to $\partial_+X_B\partial_+X_B$ turns out to be

$$-\frac{\alpha'}{\pi}S_{\mu\rho\sigma}S_\nu{}^{\rho\sigma}\partial_+X_B{}^\mu\partial_+X_B{}^\nu\frac{1}{(z^+-z'^+)^2}\ . \qquad (2.2.22)$$

which is the anomalous term in the operator product expansion in (2.2.11). The anomalous term arises when the antisymmetric tensor field $B_{\mu\nu}$ exists. The term can be removed by a finite renormalization to the energy-momentum tensor, i.e. ,

$$T'_{++} = T_{++} - \alpha' S_{\mu\rho\sigma}S_\nu{}^{\rho\sigma}\partial_+X^\mu\partial_+X^\nu. \qquad (2.2.23)$$

The redefinition does not affect the results of (2.2.19).

Finally, we focus on the contribution to the central charge. In Appendix B the effective action W is calculated in detail. Note that

$$\langle T_{ab}(z)T_{cd}(z')\rangle = -\frac{2}{\sqrt{g}}\frac{\delta}{\delta g^{ab}(z)}\frac{2}{\sqrt{g}}\frac{\delta}{\delta g^{cd}(z')}W . \qquad (2.2.24)$$

To find the matrix element $\langle T_{++}(z)T_{++}(z')\rangle$, we can use the results of the calculation given in Appendix B. Taking account of the relation (2.2.24), we set $\bar{h}^{ab}=\delta^{a+}\delta^{b+}$, $h=0$ and multiply it by -8 and finally take a Fourier component of $e^{ip(z-z')}$ in the calculation in Appendix B, so that $\langle T_{++}(z)T_{++}(z')\rangle$ is obtained. In the case of the vanishing background fields, for example, one finds that

$$\langle T_{++}(z)T_{++}(z')\rangle = \frac{D}{12\pi}\int\frac{d^2p}{(2\pi)^2}\frac{p_+^4}{p^2}e^{ip(z-z')}, \qquad (2.2.25)$$

which is read from the result in (B.12). In a similar way, the calculation shows that

$$\langle T'_{++}(z)T'_{++}(z')\rangle = \frac{1}{8\pi^2}\frac{1}{(z^+-z'^+)^4} \times\left[D-\frac{3}{2}\alpha'(R+4\nabla^{\mu}\nabla_{\mu}\Phi-4\nabla^{\mu}\Phi\nabla_{\mu}\Phi-\frac{1}{3}S_{\mu\nu\rho}S^{\mu\nu\rho})\right] \qquad (2.2.26)$$

Note that the finite renormalization in (2.2.23) affects the coefficient proportional to $S_{\mu\nu\rho}S^{\mu\nu\rho}$. Taking account of the ghost contribution in (2.2.12), one eventually finds that the nilpotency of BRST charge requires

$$D-26-\frac{3}{2}\alpha'(R+4\nabla^{\mu}\nabla_{\mu}\Phi-4\nabla^{\mu}\Phi\nabla_{\mu}\Phi-\frac{1}{3}S_{\mu\nu\rho}S^{\mu\nu\rho})=0. \qquad (2.2.27)$$

Summarizing the above argument, by the requirement of the nilpotency of BRST charge one finds the equations of motion for background fields. These equations are the same as those obtained from the vanishing condition of β-functions.

§3. Cubic Action in String Field Theory

In this section we give a brief review of the cubic action in the string field theory. At the present stage the string field theory formulated by HIKKO[10] and also Neveu and West[11] is the only available covariant theory for the closed string which provides correct tree amplitudes. Here we adopt HIKKO's closed string model.

The cubic action in the bosonic closed string theory[14] is given by*)

$$S = \frac{2}{3g^2}\,\Psi\cdot(\Psi*\Psi), \tag{3.1}$$

which consists of the only cubic term without the kinetic energy term. The cubic action is formally invariant under a coordinate transformations $X\to Y=Y(X)$, if a string Ψ transforms as a half-density[33]. Here Ψ is the closed string field which is a functional of bosonic coordinate X, ghost(anti-ghost) coordinate $C(\bar{C})$ and the string width parameter α. The Faddeev-Popov ghost number of Ψ is $N_{FP}=-1$. Although the formalism contains the unphysical width parameter and makes loop amplitudes problematical, the discussions are restricted to the string tree level, hence no problem happen. In (3.1) the *-product is defined by using the three string vertex operator $|V\rangle$ as follows;

$$|(\Psi_1*\Psi_2)[3]\rangle = \int \langle\Psi_1(1)|\langle\Psi_2(2)||V(1,2,3)\rangle d1d2, \tag{3.2}$$

where

$$dr = d\alpha_r d\bar{c}_0^{(r)}\frac{d\alpha_r}{2\pi} \tag{3.3}$$

*)Notations are the same as those used in ref.[14]. They are listed in Appendix A for convenience.

is the integration with respect to zero-mode variables of r-th string. The vertex is given by the overlapping condition and is expressed in terms of oscillator modes with Neumann function coefficients. The oscillator representation of V is given in Appendix A. The $\cdot$-product stands for the inner product.

One finds that the cubic action (3.1) is invariant under the local gauge transformation

$$\delta\Psi=2\Psi*\Lambda, \tag{3.4}$$

with $N_{\rm FP}\Lambda=-2\Lambda$, where (i) the Jacobi identity

$$\Phi*(\Psi*\Lambda)+(-)^{\Phi(\Psi+\Lambda)}\Psi*(\Lambda*\Phi)+(-)^{\Lambda(\Phi+\Psi)}\Lambda*(\Phi*\Psi)=0, \tag{3.5}$$

which holds only when $D=26$, (ii) cyclicity

$$\Phi\cdot(\Psi*\Lambda)=(-)^{\Phi(\Psi+\Lambda)}\Psi\cdot(\Lambda*\Phi), \tag{3.6}$$

and (iii) commutativity

$$\Phi*\Psi=(-)^{\Phi\Psi+1}\Psi*\Phi \tag{3.7}$$

have been used. Here $(-)^{\Phi}=1(0)$ when Φ is Grassmann odd(even).

The equation of motion which follows (3.1) is

$$\Psi*\Psi = 0. \tag{3.8}$$

Assuming that a classical solution Ψ_0 which satisfies (3.8) is found, one expands Ψ around Ψ_0:

$$\Psi=\Psi_0+g\Phi, \tag{3.9}$$

where Φ is a quantum fluctuation. Substituting (3.9) back into (3.1) and (3.4), one finds that the familiar form of the string field theory which consists of both the kinetic energy term and interaction term

$$S = \Phi\cdot Q\Phi + \frac{2}{3}g\Phi\cdot(\Phi*\Phi), \tag{3.10}$$

and its local gauge transformation

$$\delta\Phi = \frac{1}{g}Q\Lambda+2\Phi*\Lambda, \tag{3.11}$$

where Q is a linear operator defined by

$$\Psi_0*\Phi = \frac{1}{2}Q\Phi \tag{3.12}$$

for an arbitrary field Φ. Here (3.8) with Ψ being Ψ_0 has been used. It should be emphasized that a classical solution to the equation of motion (3.8) determines a geometry of space-time which is specified by the operator Q in (3.12). As a result the action (3.10) defines a new string action for certain background fields whose information is supposed to be contained in Q.

Let us now proceed to find a solution to (3.8). As was constructed by HIKKO, one can find a string field Γ which obeys the equations

$$\Gamma*\Phi = \left(N_{\mathrm{FP}}+1-\frac{\alpha}{|\alpha|} - \alpha\frac{\partial}{\partial\alpha}\right)\Phi \tag{3.13}$$

and

$$N_{\mathrm{FP}}\Gamma = -2\Gamma \tag{3.14}$$

for arbitrary Φ, where α is the width parameter of Φ. The construction of Γ is given in Appendix A. Note that an explicit form of Γ is independent of background geometry. Suppose that there exists an operator Q which satisfies all the properties that BRST operator does, i.e., (i) the nilpotency

$$Q^2 = 0 \tag{3.15}$$

and (ii) the distribution law

$$Q(\Phi * \Psi) = Q\Phi * \Psi + (-)^{\Phi}\Phi * Q\Psi. \tag{3.16}$$

If the operator Q which satisfies (3.15) and (3.16) is found, the solution to (3.8) is given by

$$\Psi_0 = -\frac{1}{2}Q\Gamma. \tag{3.17}$$

It is rather easy to show that (3.17) satisfies (3.8) and (3.12);

$$\Psi_0 * \Psi_0 = \frac{1}{4}Q(\Gamma * Q\Gamma) = \frac{1}{4}Q\left(N_{FP}+1-\frac{\alpha}{|\alpha|} - \alpha\frac{\partial}{\partial\alpha}\right)Q\Gamma$$

$$\propto Q^2\Gamma = 0, \tag{3.18}$$

where the nilpotency (3.15) the distribution law (3.16) have been used, and

$$\Psi_0 * \Phi = -\frac{1}{2}\left\{Q(\Gamma * \Phi) - \Gamma * Q\Gamma\right\}$$

$$= \frac{1}{2}[N_{FP}, Q]\Phi = \frac{1}{2}Q\Phi, \tag{3.19}$$

where $[N_{FP}, Q] = Q$ and (3.16) have been used.

Although there are many Γ's which satisfy (3.13) and (3.14), the solution formed by (3.17) is unique as will be seen below. For a given Γ, Ψ_0 given by (3.17) satisfies (3.19) for an arbitrary Φ. Since (3.19) is satisfied for an arbitrary Φ, one chooses another string field $\hat{\Gamma}$ which satisfies (3.13) and (3.14) as Φ in (3.19), i.e. ,

$$\Psi_0 * \hat{\Gamma} = \frac{1}{2}Q\hat{\Gamma}. \tag{3.20}$$

The left hand side of (3.20) is rewritten as

$$\Psi_0 * \hat{\Gamma} = \frac{1}{2}\hat{\Gamma} * Q\Gamma$$

$$= \frac{1}{2}\left(N_{\mathrm{FP}} + 1 - \frac{\alpha}{|\alpha|} - \alpha\frac{\partial}{\partial\alpha} \right) Q\Gamma$$

$$= \frac{1}{2}Q\Gamma, \qquad (3.21)$$

where commutativity (3.7) and (3.13) for $\hat{\Gamma}$ have been used. In passing the last equality in (3.21), $[N_{\mathrm{FP}}, Q] = Q$ and $\Gamma * \Gamma = 0$ have been used. Comparing (3.21) with (3.20), one finds that $Q\Gamma$ is unique.

Let us summarize above arguments. If an operator Q that satisfies (3.15) and (3.16) is found, one can construct a solution Ψ_0 to (3.8). The solution formed by (3.17) defines a string field theory (3.10) associated with the geometry specified by the operator Q.

If Q happens to be the usual BRST operator defined on 26-dimensional Minkowski space, BRST operator Q_{B} satisfies (3.15) and (3.16) with $Q = Q_{\mathrm{B}}$. In this case the solution $\Psi_0 = -\frac{1}{2}Q_{\mathrm{B}}\Gamma$ generates the well-known string field theory in flat space-time.

§4. Curved Space-Time Solutions in Cubic Action Theory

We look for the solutions that generate the string field theory in curved space[16]. As was reviewed in sec. 3, if an operator Q which satisfies the nilpotency (3.15) and distribution law (3.16) is found, one can construct a string field theory associated with the geometry specified by Q. Thus instead of solving the equation (3.8) derived from the cubic action, we solve (3.15) and (3.16).

To seek the operator Q associated with a non-trivial curved space, we first prepare a test operator $Q[G_{\mu\nu},B_{\mu\nu},\Phi]$ as a function of yet unspecified fields $G_{\mu\nu}$, $B_{\mu\nu}$, and Φ. In the following we look for the operator $Q[G_{\mu\nu},B_{\mu\nu},\Phi]$ which satisfies (i) nilpotency

$$Q^2[G_{\mu\nu},B_{\mu\nu},\Phi]=0, \tag{4.1}$$

and (ii) distribution law

$$\sum_{r=1}^{3} Q^{(r)}[G_{\mu\nu},B_{\mu\nu},\Phi]\,|V(1,2,3)\rangle=0, \tag{4.2}$$

where the superscript r refers the channels 1, 2 and 3. In (4.2) the vertex operator $|V\rangle$ is given from the outset by the overlapping condition, hence (4.2) requires conditions over background fields $G_{\mu\nu}$, $B_{\mu\nu}$, Φ. The relation (4.2) means that the vertex should transform as a conformal tensor under the operation Q.

In choosing the test operator Q we take advantage of the non-linear σ-model as an auxiliary tool:

$$S = S_X + S_{gh}, \tag{4.3}$$

where

$$S_X = \int d^2\sigma\sqrt{g}g^{ab}\Big[\ \frac{1}{2}\partial_a\xi_M\partial_b\xi^M - \frac{1}{3}\pi\alpha' R_{MKNL}\xi^K\xi^L\partial_a\xi^M\partial_b\xi^N\ \Big]$$

$$+\int d^2\sigma\sqrt{g}R^{(2)}\Big[\ \frac{\sqrt{2\pi\alpha'}}{4\pi}\nabla_M\Phi\xi^M + \frac{1}{4}\alpha'\nabla_M\nabla_N\Phi\xi^M\xi^N\Big]$$

$$+\int d^2\sigma\Big[i\frac{1}{3}\sqrt{2\pi\alpha'}S_{LMN}\varepsilon^{ab}\xi^L\partial_a\xi^M\partial_b\xi^N$$

$$+\ i\frac{1}{2}\pi\alpha'\nabla_K S_{LMN}\varepsilon^{ab}\xi^K\xi^L\partial_a\xi^M\partial_b\xi^N\Big],$$

$$S_{gh} = \frac{1}{2}\int d^2\sigma\Big[\overline{C}(\sigma)\partial_{\overline{\sigma}}C(\sigma)+\overline{C}(\overline{\sigma})\partial_{\sigma}C(\overline{\sigma})\Big],$$

where $\xi^M = e_\mu{}^M(x)\xi^\mu$ with $e_\mu{}^M$ being a vielbein. Assuming that α' is small, we have made the normal coordinate expansion of $X^\mu = x^\mu + \sqrt{2\pi\alpha'}\,\xi^\mu$ around a constant classical solution x^μ. The test operator Q can be read from (4.3) and is given by

$$Q^{(r)}[G,B,\Phi] = \int_0^{\pi\alpha_r} d\sigma\, j(\sigma)$$

$$= \int_0^{\pi\alpha_r} d\sigma\Big\{C(\sigma)(T^X_{++}+\frac{1}{2}T^{gh}_{++})+C(\overline{\sigma})(T^X_{--}+\frac{1}{2}T^{gh}_{--})\Big\}, \qquad (4.4)$$

where $C(\overline{C})$ stands for the ghost(anti-ghost) field and T_{ab} is the two dimensional stress tensor defined by

$$T_{ab}(\sigma) = \frac{2}{\sqrt{g}}\ \frac{\delta S}{\delta g^{ab}(\sigma)}\ . \qquad (4.5)$$

The superscripts X and gh of T in (4.4) represent the string coordinate and ghost parts of T, respectively.

As far as the nilpotency condition (4.1) is concerned, Banks et al.[6] studied its implication in the context of non-linear σ-model and obtained the same equations of motion for background fields as those required by the conformal invariance of the σ-model(see sec. 2.2).

Here we address ourselves to the condition (4.2), which has a straightforward connection with the cubic theory. In the Hamiltonian operator formalism, the test operator Q should be written in terms of the normal coordinate $\xi^N(\sigma)$ and its canonical momentum $p_N(\sigma)=\delta S/\delta\dot{\xi}^N(\sigma)$ together with ghost fields. The formula (4.2) implies that Q should be operated at $\tau=0$ on each channel of the vertex shown in fig. 2. In order to make the operation definite, we operate the charge density $j(\rho=\tau+i\sigma)$ along the contour C_0 in fig. 2., and then take the limit $\delta\to 0$. The effect of this operation can be equivalently estimated by the use of Lagrangian formalism, i.e., by calculating

$$\int d\sigma\ j(\sigma)\,|V\rangle = \lim_{\delta\to 0}\int \mathcal{D}X\mathcal{D}C\mathcal{D}\bar{C}\int_{C_0} d\rho\ j(\rho)\,e^{-S_D}|V\rangle, \qquad (4.6)$$

where S_D stands for the action (4.3) defined over the strip domain D in fig. 2. The equivalence in (4.6) is known as Matthews's theorem[34].

Readers might wonder that the shift of j-operation by $\tau=\pm\delta$ with the background dependent action S_D could not be legitimate in the cubic theory because all information about background fields should be contained in j only. This is, however, allowed. As one sees in the following calculation, the corrections due to S_D appears only along the contour C_0. The role of $\exp[-S_D]$, therefore, can be considered to take care of background corrections to the canonical momenta, which no more be $\dot{\xi}$, as it should be.

The calculation of (4.6) is performed on the complex z-plane instead of ρ-plane after making the Mandelstam mapping

$$\rho(z)=\sum_{r=1}^{3}\alpha_r \ln(z-z_r) \quad . \tag{4.7}$$

It is important to note the role of the vertex operator $|V\rangle$. As is well known, $|V\rangle$ is determined from the overlapping condition and is best defined in terms of oscillators with Neumann function coefficients(see Appendix A). In particular in the zero-th approximation where the space is flat, the following relation holds in operator formalism,

$$ {}_{1,2,3}\langle 0|\prod_{i=1}^{2n}\xi(\rho_i)|V_0\rangle = \sum\prod_{r=1}^{n} N(z_{r_1},z_{r_2}), \tag{4.8}$$

where $|V_0\rangle$ is $|V\rangle$ in which ghost parts are factored out, $N(z_{r_1},z_{r_2})$ is the Neumann function on z-plane, and the summation in the right-hand-side covers all possible partitions of $2n$ indices into n pairs of (r_1,r_2).

The relation (4.8) seems to keep the conformal invariance under the transformation $\rho\to z$. If, however, some of ρ's coincide, the conformal factor dependence remains unvanished in the right hand side when a certain regularization is made. The dimensional regularization will be used later. In our calculation, j is a composite field with some derivatives, and it transforms no more as a conformal tensor under (4.7) and some conditions over background fields are required to recover the covariance. This is the essential feature in our calculation.

Let us look at typical matrix elements of (4.2) as follows;

$$
{}_{1,2,3}\langle 0|\sum_{r=1}^{3} Q^{(r)}|V(1,2,3)\rangle
$$

$$
= -\frac{i\sqrt{\pi}}{2}\int dz\left(\frac{d\rho(z)}{dz}\right)^{-1} {}_{1,2,3}\langle 0|C(z)\left[-T^{X}_{++}(z)+2\partial_z C(z)\overline{C}(z)\right]|V\rangle
$$

$$
\begin{aligned}
= -\frac{i\sqrt{\pi}}{2}\int dz\left(\frac{d\rho(z)}{dz}\right)^{-1}\Big[& -\langle T^{X}_{++}(z)\rangle C(z)\,C(z_0) \\
& +2\langle\partial_z C(z)\overline{C}(z)\rangle C(z)\,C(z_0) \\
& -2\langle C(z)\overline{C}(z)\rangle\partial_z C(z)\,C(z_0) \\
& +2\langle\overline{C}(z)\,C(z_0)\rangle C(z)\partial_z C(z)\Big],
\end{aligned} \tag{4.9}
$$

and

$$
{}_{1,2,3}\langle 0|\xi_\mu(1)\xi_\nu(2)\sum_{r=1}^{3} Q^{(r)}|V(1,2,3)\rangle
$$

$$
= +\frac{i\sqrt{\pi}}{2}\int dz\left(\frac{d\rho(z)}{dz}\right)^{-1}\langle\xi_\mu(1)\xi_\nu(2)\,T^{X}_{++}(z)\rangle C(z)\,C(z_0), \tag{4.10}
$$

where (4.6) and (4.8) have been used. In passing the first equality in (4.9) and (4.10), we transformed the variable ρ to z, hence the integration contour C_0 to C_0' in fig. 3. Here the ghost fields C and $\overline{C}$ are

$$
C(z)=C(\rho), \tag{4.11}
$$

$$
\overline{C}(z)=\frac{d\rho(z)}{dz}\overline{C}(\rho), \tag{4.12}
$$

which are those defined in refs.[9,10]. To see the relation to the ordinary conformal ghost fields, see ref.[35].

The bracket in (4.9) and (4.10) indicates path integral average with the weight of action in (4.3). It is crucial to examine the dependence of the conformal factor:

$$
\phi=\ln\left|\frac{d\rho}{dz}\right|^2 \tag{4.13}
$$

in $g_{ab}=e^{\phi}\delta_{ab}$ in z-plane. To study the path integral we must prepare a renormalized σ-model action on a fixed curved world sheet. The σ-model action is renormalized and the effective action up to two-loop order is calculated in Appendix B. In the course of the σ-model analysis, we have made weak field expansion of $g_{ab}=\delta_{ab}+h_{ab}$ around flat world sheet and

$$\bar{h}_{ab}=h_{ab}-\frac{1}{2}\delta_{ab}h, \tag{4.14}$$

$$h=h_a{}^a, \tag{4.15}$$

have been introduced as independent variables. A difficulty arises due to the ambiguity of definition of antisymmetric tensor $\varepsilon^{ab}(g)$ in n-dimensional space. In the presence of $B_{\mu\nu}$ we look for a consistent dimensional regularization in the σ-model where antisymmetric tensor $\varepsilon^{ab}(g)$ is extended to that on n-dimensional curved world sheet. The weak field expansion of $\varepsilon^{ab}(g)$ will be represented as

$$\varepsilon^{ab}(g)=\varepsilon^{ab}+c\frac{\varepsilon}{2}h\varepsilon^{ab}+d\frac{1}{2}(\bar{h}^a{}_c\varepsilon^{bc}-\bar{h}^b{}_c\varepsilon^{ac})+\mathcal{O}(h_{ab}^2), \tag{4.16}$$

where c and d are some constants and will be determined later.

The renormalized action to two-loop level becomes

$$S = S_0+S_{\text{int}}+S_{\text{c.t.}}+\Delta S+S_{\text{non-cov}}+S_{\text{c.t.}}^{(2)}, \tag{4.17}$$

where

$$S_0=\frac{1}{2}\int d^n\sigma(\delta^{ab}\partial_a\xi_M\partial_b\xi^M+m^2\xi_M\xi^M),$$

$$S_{\text{int}} = \int d^n\sigma(\sqrt{g}g^{ab}-\delta^{ab})(\frac{1}{2}\partial_a\xi_M\partial_b\xi^M)+\int d^n\sigma(\sqrt{g}-1)\frac{1}{2}m^2\xi_M\xi^M$$

$$+\int d^n\sigma\sqrt{g}g^{ab}\Big[-\frac{1}{3}\pi\alpha' R_{MKNL}\xi^K\xi^L\partial_a\xi^M\partial_b\xi^N\Big]$$

$$+\int d^n\sigma\sqrt{g}R^{(n)}\left[\frac{\sqrt{2\pi\alpha'}}{4\pi}\nabla_M\Phi\xi^M+\frac{1}{4}\alpha'\nabla_M\nabla_N\Phi\xi^M\xi^N\right]$$

$$+\int d^n\sigma\left[i\frac{1}{3}\sqrt{2\pi\alpha'}\,S_{LMN}\varepsilon^{ab}(g)\,\xi^L\partial_a\xi^M\partial_b\xi^N\right.$$

$$\left.+\ i\frac{1}{2}\pi\alpha'\nabla_K S_{LMN}\varepsilon^{ab}(g)\,\xi^K\xi^L\partial_a\xi^M\partial_b\xi^N\right],$$

$$S_{\mathrm{c.t.}}=-\frac{1}{2\varepsilon}\frac{D}{24\pi}\int d^n\sigma\sqrt{g}R^{(n)}+\frac{1}{16\pi\varepsilon}\alpha'\nabla_M\nabla^M\Phi\int d^n\sigma\sqrt{g}R^{(n)}$$

$$+\int d^n\sigma\sqrt{g}g^{ab}\left[-\frac{1}{12\varepsilon}\alpha' R_{MN}\partial_a\xi^M\partial_b\xi^N\right]$$

$$+\int d^n\sigma\sqrt{g}\left[+\frac{1}{12\varepsilon}\alpha' R_{MN}m^2\xi^M\xi^N\right]$$

$$+\int d^n\sigma\left[+\frac{1}{4\varepsilon}\alpha' S_{MIJ}S_N{}^{IJ}\sqrt{g}g^{ab}\partial_a\xi^M\partial_b\xi^N\right.$$

$$\left.+\ i\frac{1}{8\varepsilon}\alpha'\nabla_K S^K{}_{MN}\varepsilon^{ab}(g)\,\partial_a\xi^M\partial_b\xi^N\right],$$

$$\Delta S=\frac{1}{2}\int d^n\sigma\alpha' S_{MIJ}S_N{}^{IJ}\bar{h}^{ab}\partial_a\xi^M\partial_b\xi^N,$$

$$S_{\mathrm{non\text{-}cov}}=-\left(\frac{1}{48}-\frac{c^2}{72}\right)\frac{\alpha'}{16\pi}S_{IJK}S^{IJK}\int d^n\sigma h\partial^a\partial_a h$$

$$-\frac{1}{6}(c-1)\frac{\alpha'}{16\pi}S_{IJK}S^{IJK}\int d^n\sigma\left(\bar{h}^{ab}\partial_a\partial_b h-\frac{1}{2}h\partial^a\partial_a h\right),$$

$$S_{\mathrm{c.t.}}^{(2)}=+\frac{1}{2\varepsilon}\frac{\alpha'}{16\pi}S_{IJK}S^{IJK}\int d^n\sigma\sqrt{g}R^{(n)}.$$

The calculation of fig. 4.d4 generates non-local divergent terms as well as infrared divergent terms. Assuming that the counterterms should be local, we conclude that d in (4.16) should be vanishing. The finite renormalization term ΔS has been added to eliminate the anomalous term given by (2.2.22). The extension of $\varepsilon^{ab}(g)$ given by (4.16) generally violates the invariance under the reparametrization. We have added the term in $S_{\text{non-cov}}$ which is finite local counterterm as it should be. The reparametrization non-invariant terms disappear in the effective action as will be seen in Appendix B. The counterterms $S_{\text{c.t.}}$ and $S_{\text{c.t.}}^{(2)}$ are local and invariant, which are needed at one and two-loop level, respectively.

The effective action defined by (2.1.15) with $X_B = x$ becomes

$$W = -\frac{1}{96\pi}\left\{ D - \frac{3}{2}\alpha'\left(R + 4\nabla^{\mu}\nabla_{\mu}\Phi - 4\nabla^{\mu}\Phi\nabla_{\mu}\Phi - \frac{1}{3}S_{\mu\nu\rho}S^{\mu\nu\rho}\right)\right\} \times \int d^2x d^2y \sqrt{g(x)}R(x)\,G(x,y)\sqrt{g(y)}R(y) + \ldots \quad (4.18)$$

where

$$\frac{1}{\sqrt{g}}\partial_a(\sqrt{g}g^{ab}\partial_b)\,G(x,y) = -\frac{1}{\sqrt{g}}\delta^2(x-y). \quad (4.19)$$

Here ellipses mean some finite terms proportional to $R^{(2)}$ with finite coefficient and are irrelevant for our calculations. No infrared divergences appear in the final expression.

The trace anomaly can be found from (4.18) by taking derivative and contracting with respect to g_{ab}, i.e.,

$$\langle T_a{}^a\rangle = +\frac{1}{24\pi}\left\{ D - \frac{3}{2}\alpha'\left(R + 4\nabla^{\mu}\nabla_{\mu}\Phi - 4\nabla^{\mu}\Phi\nabla_{\mu}\Phi - \frac{1}{3}S_{\mu\nu\rho}S^{\mu\nu\rho}\right)\right\}\sqrt{g}R^{(2)}, \quad (4.20)$$

which is proportional to the β^{Φ} without ghost contribution as is expected. Note, however, (4.20) is independent of the constant c which has been introduced in (4.16).

To calculate the conformal factor dependence of $\langle T_{++}\rangle$ and

$\langle\xi_\mu(1)\xi_\nu(2)T_{++}\rangle$, we set

$$\bar{h}_{ab}=-\varepsilon\delta_{ab}\phi, \tag{4.21}$$

and

$$h=(2+2\varepsilon)\phi. \tag{4.22}$$

The energy-momentum tensor defined by (4.5) results in

$$T_{++}=T_{++}{}^{0}+T_{++}{}^{\mathrm{int}}+T_{++}{}^{\mathrm{c.t.}}+\Delta T_{++}+T_{++}{}^{\mathrm{non\text{-}cov}}+T_{++}{}^{(2)\mathrm{c.t.}}, \tag{4.23}$$

where

$$T_{++}{}^{0}=\partial_+\xi^M\partial_+\xi_M,$$

$$\begin{aligned}T_{++}{}^{\mathrm{int}}=&-\frac{1}{3}2\pi\alpha' R_{MKNL}\xi^K\xi^L\partial_+\xi^M\partial_+\xi^N\\&+\frac{\sqrt{\alpha'}}{\sqrt{2\pi}}\nabla_M\Phi(-\varepsilon\partial_+\partial_+\phi\xi^M-\partial_+\partial_+\xi^M+\partial_+\phi\partial_+\xi^M)\\&+\frac{\alpha'}{2}\nabla_M\nabla_N\Phi(-\varepsilon\partial_+\partial_+\phi\xi^M\xi^N-\partial_+\partial_+(\xi^M\xi^N)+2\partial_+\phi\partial_+\xi^M\xi^N),\end{aligned}$$

$$\begin{aligned}T_{++}{}^{\mathrm{c.t.}}=&\frac{D}{24\pi}\partial_+\partial_+\phi-\frac{1}{6\varepsilon}\alpha' R_{MN}\partial_+\xi^M\partial_+\xi^N-\frac{\alpha'}{8\pi}\nabla_M\nabla^M\Phi\partial_+\partial_+\phi\\&+\frac{\alpha'}{2\varepsilon}S_{MIJ}S_N{}^{IJ}\partial_+\xi^M\partial_+\xi^N,\end{aligned}$$

$$\Delta T_{++}=-\alpha' S_{MIJ}S_N{}^{IJ}\partial_+\xi^M\partial_+\xi^N,$$

$$T_{++}{}^{\mathrm{non\text{-}cov}}=+\frac{(c-1)\alpha'}{24\pi}S_{IJK}S^{IJK}\partial_+\partial_+\phi,$$

$$T_{++}{}^{(2)\mathrm{c.t.}}=-\frac{\alpha'}{16\pi}S_{IJK}S^{IJK}\partial_+\partial_+\phi,$$

where $T_{++}{}^{0}$, $T_{++}{}^{\mathrm{int}}$, $T_{++}{}^{\mathrm{c.t.}}$, ΔT_{++}, $T_{++}{}^{\mathrm{non\text{-}cov}}$ and $T_{++}{}^{(2)\mathrm{c.t.}}$ are terms associated with S_0, S_{int}, $S_{\mathrm{c.t.}}$, ΔS, $S_{\mathrm{non\text{-}cov}}$ and $S_{\mathrm{c.t.}}{}^{(2)}$, respectively. In (4.23) $\mathcal{O}(\phi^2)$ terms are ignored.

Graphs contributing to $\langle T_{++}\rangle$ at $\mathcal{O}(\phi)$ are given in fig. 5. The straightforward calculations of the individual graphs are given in Appendix C. All of contributions amount to

$$\frac{1}{24\pi}\left\{ D- \frac{3}{2}\alpha' (R+4\nabla^{\mu}\nabla_{\mu}\Phi-4\nabla^{\mu}\Phi\nabla_{\mu}\Phi- \frac{1}{3}S_{\mu\nu\rho}S^{\mu\nu\rho})\right\}\partial_z\partial_z\phi, \qquad (4.24)$$

which has turned out to be c independent. Graphs connected with at most two ϕ's contribute to $\langle T_{++}\rangle$. Taking account of them, we eventually find

$$\langle T^X_{++}(z)\rangle = \frac{1}{24\pi}\left\{D- \frac{3}{2}\alpha' (R+4\nabla^{\mu}\nabla_{\mu}\Phi-4\nabla^{\mu}\Phi\nabla_{\mu}\Phi- \frac{1}{3}S_{\mu\nu\rho}S^{\mu\nu\rho})\right\}$$
$$\times(\partial_z\partial_z\phi- \frac{1}{2}\partial_z\phi\partial_z\phi), \qquad (4.25)$$

which is also derivable from the variation with respect to the metric of the effective action given by (4.18)(see Appendix C).

Another matrix element which is necessary for our calculation is $\langle\xi_{\mu}(1)\xi_{\nu}(2)T_{++}\rangle$. The relevant graphs are listed in fig. 6 and the result of calculation of each graph is given in Appendix D. Summing up contributions of all graphs in fig. 6, we arrive at

$$\langle\frac{1}{2}[\xi_I(1)\xi_J(2)+\xi_J(1)\xi_I(2)]T_{++}(z)\rangle$$
$$= \frac{1}{2}\alpha' \left\{R_{IJ}+2\nabla_I\nabla_J\Phi-(3- \frac{4}{3}c)S_{IKL}S_J{}^{KL}\right\}$$
$$\times \int d^2x\Delta(1-x)\partial_z\Delta(x-z)\partial^c\partial_c\phi(x)\partial_z\Delta(2-z)$$
$$+ \frac{1}{2}\alpha' \left\{R_{IJ}+2\nabla_I\nabla_J\Phi-(3- \frac{4}{3}c)S_{IKL}S_J{}^{KL}\right\}\Delta(1-z)\partial_z\Delta(2-z)\partial_z\phi(z)$$
$$+ \frac{1}{6}\alpha' \left\{R_{IJ}-(\frac{3}{2}-c)S_{IKL}S_J{}^{KL}\right\}\partial_z\Delta(1-z)\partial_z\Delta(2-z)\phi(1) \qquad (4.26)$$
$$+ (1\longleftrightarrow 2)$$

and

$$\langle \tfrac{1}{2}[\xi_I(1)\xi_J(2)-\xi_J(1)\xi_I(2)]T_{++}(z)\rangle$$

$$= \tfrac{1}{2}\alpha'\left\{\nabla^K S_{KIJ}-2\nabla^K\Phi S_{KIJ}\right\}\int d^2x\Delta(1-x)\partial_z\Delta(x-z)\partial^C\partial_C\phi(x)\partial_z\Delta(2-z)$$

$$+ \tfrac{1}{2}\alpha'\left\{\nabla^K S_{KIJ}-2\nabla^K\Phi S_{KIJ}\right\}\Delta(1-z)\partial_z\Delta(2-z)\partial_z\phi(z)$$

$$+ \tfrac{1}{2}\alpha'\left\{\nabla^K S_{KIJ}-2\nabla^K\Phi S_{KIJ}\right\}\ \partial_z\Delta(1-z)\partial_z\Delta(2-z)\phi(1)$$

$$- (1\longleftrightarrow 2), \tag{4.27}$$

where $\Delta(x-y)$ is the Green's function defined on a flat world sheet. The matrix elements including the ghost are given by

$$\langle C(z)\overline{C}(z)\rangle = -\frac{1}{4\pi}\frac{1}{2}\partial_z\phi, \tag{4.28}$$

$$\langle \partial_z C(z)\overline{C}(z)\rangle = \frac{1}{4\pi}\frac{1}{6}(\partial_z\partial_z\phi-\frac{1}{2}\partial_z\phi\partial_z\phi), \tag{4.29}$$

$$\langle \overline{C}(z)C(z_0)\rangle = \frac{1}{4\pi}\frac{1}{z-z_0}. \tag{4.30}$$

One substitutes (4.25-30) into (4.9) and (4.10) and integrates along the contour $C'_\bullet$.

It should be noted that the perturbation corrections due to interactions in (4.17) occur only on the line C_0 in ρ-plane or C'_0 in z-plane. If we look at (4.9), there are potential sources of non-local corrections, for instance, coming from the diagram shown in fig. 5. In the amplitude, however, some of propagators which link $\mathscr{L}_{\rm int}(w)$ with $T^{(\bullet)}_{++}(z)$ and $\phi(u)$ collapse into delta functions due to derivatives on the propagator. Thus we have obtained (4.25), leaving a local correction to $T^{(\bullet)}_{++}(z)$. A similar situation also happens in fig. 6 which occurs in the calculation of (4.10). Careful inspection of the integrand, in particular (4.25-27), shows that no singularities appear except at the image of interaction point ρ_0, so that the integration is able to be shrunken to a small circle C_1' around z_0. Singularities at the image z_0 of vertex point appear from two sources, one from $(d\rho/dz)^{-1}$ in (4.9) and (4.10) which comes from the Jacobian and the transformation factor of j, and the other from the conformal factor in $g_{ab}=e^{\phi}\delta_{ab}$ in z-plane ($g_{ab}=\delta_{ab}$ in ρ-plane). Collecting those singular terms and performing the Cauchy integral, we obtain

$$
{}_{1,2,3}\langle 0|\sum_{r=1}^{3} Q^{(r)}|V(1,2,3)\rangle
$$

$$
= \frac{\sqrt{\pi}}{64}\left[\; D-26-\frac{3}{2}\alpha'\,(R+4\nabla^{\mu}\nabla_{\mu}\Phi-4\nabla^{\mu}\Phi\nabla_{\mu}\Phi-\frac{1}{3}S_{\mu\nu\rho}S^{\mu\nu\rho})\right]
$$

$$
\times\left[\frac{1}{a}C''(z_0)C(z_0)-\frac{b}{a^2}C'(z_0)C(z_0)\right], \qquad (4.31)
$$

where (4.13) and the expansion of $C(z)$ and $\rho(z)$ around the z_0;

$$
C(z)=C(z_0)+C'(z_0)(z-z_0)+\frac{1}{2}C''(z_0)(z-z_0)^2+\ldots \qquad (4.32)
$$

and

$$
\rho(z)=\sum_{i=1}^{3}\alpha_i\ln(z-z_i)=\rho_0-a(z-z_0)^2-b(z-z_0)^3-\ldots \qquad (4.33)
$$

have been used with

$$
\rho_0=\rho(z_0),\quad a=-\frac{1}{2}\frac{d^2\rho}{dz^2}(z_0)\quad\text{and}\quad b=-\frac{1}{6}\frac{d^3\rho}{dz^3}(z_0).
$$

(4.31) should vanish according to the condition (4.2). Thus the distribution law requires the equation for background fields which agrees with the vanishing condition of β^{Φ}.

Another matrix element of (4.2) was given by (4.10). We substitute (4.26) and (4.27) into (4.10) and carry out the contour integration. Picking up some poles around z_0, one finds

$$
{}_{1,2,3}\langle 0|\xi_I(1)\xi_J(2)\sum_{r=1}^{3}Q^{(r)}|V(1,2,3)\rangle
$$

$$
=\frac{(\pi)^{3/2}}{8a}\alpha'\left[R_{IJ}+2\nabla_I\nabla_J\Phi-(3-\frac{4}{3}c)S_{IKL}S_J{}^{KL}\right]
$$

$$
\times\ \Delta(1-z_0)\partial_{z_0}\Delta(2-z_0)C'(z_0)C(z_0)
$$

$$
+\ (1\longleftrightarrow 2)
$$

$$
+\ \frac{(\pi)^{3/2}}{8a}\alpha'\left[\nabla^K S_{KIJ}-2S_{IJK}\nabla^K\Phi\right]\Delta(1-z_0)\partial_{z_0}\Delta(2-z_0)C'(z_0)C(z_0)
$$

$$
-\ (1\longleftrightarrow 2). \qquad (4.34)
$$

The last terms in (4.26) and (4.27) do not contribute to (4.34) because they are regular near $z=z_0$. The vanishing condition of (4.10) requires that both the symmetric and antisymmetric factors in the square brackets in (4.34) should be zero.

All matrix elements other than $\langle T_{++}\rangle$ and $\langle\xi_\mu(1)\xi_\nu(2)T_{++}\rangle$ are trivially zero. The reason is as follows. Consider the contraction of the energy-momentum tensor with more than three ξ's. Up to the $\mathcal{O}(\alpha')$, the graphs which contribute to $\langle\xi_{\mu_1}(1)\xi_{\mu_2}(2)\ldots\xi_{\mu_n}(n)T_{++}(z)\rangle$ $(n\geqq 4)$ do not include loop diagrams, so that no divergent terms arise. Each graph gives rise to a factor $\varepsilon\phi$ and vanishes in tw● dimensions($\varepsilon\to 0$).

Summarizing the above argument, we find that the distribution law of BRST charge imposes following conditions over background fields:

$$
D-26-\frac{3}{2}\alpha'(R+4\nabla^\mu\nabla_\mu\Phi-4\nabla^\mu\Phi\nabla_\mu\Phi-\frac{1}{3}S_{\mu\nu\rho}S^{\mu\nu\rho})=0\ , \qquad (4.35)
$$

$$R_{\mu\nu}+2\nabla_\mu\nabla_\nu\Phi-(3-\frac{4}{3}c)S_{\mu\rho\sigma}S_\nu{}^{\rho\sigma}=0, \qquad (4.36)$$

$$\nabla_\rho S^\rho{}_{\mu\nu}-2S_{\rho\mu\nu}\nabla^\rho\Phi=0 \ . \qquad (4.37)$$

In (4.36) the constant c remains undetermined. These three equations, however, should not be independent but be related with each other by the Bianchi identities which ensure the consistency of non-vanishing background fields. Taking the covariant derivative of (4.36), one can show that

$$0=\nabla^\mu\left\{R_{\mu\nu}+2\nabla_\mu\nabla_\nu\Phi-(3-\frac{4}{3}c)S_{\mu\rho\sigma}S_\nu{}^{\rho\sigma}\right\}$$

$$=\frac{1}{2}\nabla_\nu\left\{R+4\nabla^\mu\nabla_\mu\Phi-4\nabla^\mu\Phi\nabla_\mu\Phi-(1-\frac{4}{9}c)S_{\mu\rho\sigma}S^{\mu\rho\sigma}\right\}, \qquad (4.38)$$

where (4.37) and the Bianchi identities

$$\nabla^\mu R_{\mu\nu}=\frac{1}{2}\nabla_\nu R, \qquad (4.39)$$

$$S_{\mu\rho\sigma}\nabla^\mu S_\nu{}^{\rho\sigma}=\frac{1}{6}\nabla_\nu(S^{\mu\rho\sigma}S_{\mu\rho\sigma}), \qquad (4.40)$$

have been used. Comparing (4.38) with (4.35), one can conclude that the parameter c must be $\frac{3}{2}$. Thus the extension of the anti-symmetric tensor $\varepsilon^{ab}(g)$ on a curved world sheet is uniquely determined by the consistency condition among the equations for background fields. Substituting $c=\frac{3}{2}$ into (4.36), we find that these equations are the same as those obtained from the vanishing conditions of β-functions. Banks et al. studied the nilpotency (4.1) in $\mathcal{O}(\alpha')$ and found that those equations for $G_{\mu\nu}$, $B_{\mu\nu}$ and Φ are necessary. It is worthwhile to emphasize that same equations have been deduced from a new requirement which is linear in Q.*)

Thus the nilpotency (4.1) and the distribution law (4.2) require the equations of motion for background fields which agree with those obtained in the context of the conformal invariant σ-model.

Summing up the arguments in secs. 3 and 4, one can conclude that the equation of motion (3.8) derived from the cubic action determines the equations of motion for background fields. Those equations are the same as those required by the conformal invariance of the non-linear σ-model. Thus the cubic action theory generates all allowed conformal independent background fields as the solutions of the cubic action. The cubic theory, therefore, can be qualified as a non-trivial, self-contained and geometry independent field theory of string.

The method developed above is able to be extended to higher orders in α'. Using the sigma model as an auxiliary tool one constructs a test operator $Q[G_{\mu\nu}, B_{\mu\nu}, \Phi]$, then derives the equations for background fields to any order in α' from conditions of the nilpotency (4.1) and the distribution law (4.2). Substituting a set of solutions that obey the equations one determines a BRST operator

$$Q = Q^{(0)} + \alpha' Q^{(1)} + (\alpha')^2 Q^{(2)} + \ldots \tag{4.41}$$

The string action

$$S = \Phi \cdot Q\Phi + \frac{2}{3} g \Phi \cdot (\Phi * \Phi) \tag{4.42}$$

---

*)The conformal covariance of the tachyon emission vertex was studied in ref.[7] and they obtained the same equations for background fields.

associated with the geometry specified by (4.41) then provides the S-matrix theory in the curved space. A practical method for small α' is to develop a perturbation theory in (4.42) where $\Phi Q^{(0)}\Phi$ is treated as an unperturbed term.

§5. Summary and Discussion

The cubic action theory has been shown to determine the equations of motion for background fields which agree with those required by the conformal invariance of the non-linear σ-model. Taking advantage of the σ-model as an auxiliary tool, we solved the equation of motion derived from the cubic action and constructed the string field theory in curved space. Thus the cubic action theory generates all conformal independent background fields as the solutions of the cubic action. The advantage of our method over the σ-model consists in the fact that all allowed solutions are contained in a space which a single theory covers. The cubic action theory, therefore, can be qualified as a non-trivial, self-contained and geometry independent field theory of string.

In the course of the analysis, it was shown that there exists a consistent dimensional regularization in the σ-model where antisymmetric tensor $\varepsilon^{ab}(g)$ is extended to that on n-dimensional curved world sheet. The extension of $\varepsilon^{ab}(g)$ from two to n dimensions was determined by the consistency of the equations of motion for background fields.

To guarantee that the cubic action theory is independent of the space-time metric, further investigation should be made. In solving the equation of motion of the cubic action, we chose BRST operator as a test operator by the use of the non-linear σ-model as an auxiliary tool. The BRST operator we have used is expressed in terms of the normal coordinate variable $\xi^M(x)$, which is the tangent vector at the center of mass x along the geodesics, and its canonical momentum. The positions of the center of mass are not on the three string overlap, so that the operation of BRST charge at the vertex may depend on the background geometry. In order to avoid the geometry dependence, we change variable $\xi^M(x)$ at the center of mass into that at the interaction point[36].

We remark on a renormalization correction to the background field equations due to string loop amplitudes. As was discussed by Fischler and Susskind and many others[17], the divergences in amplitudes having non-trivial topology require new counter terms in the action of the σ-model on spherical topology and hence the equations for background fields are to be modified thereby. Insofar as we stick to HIKKO's formalism for closed string, our method should not be applied straightforwardly to this problem, because the unphysical width parameter seems to violate the modular invariance in loop amplitudes. The covariantized light-cone field theory which has been proposed by Neveu and West and others[13] seems to resolve the width parameter problems. It is interesting to include string loop corrections in the new formalism.

To practice our program in the framework of Witten's theory is of extreme interest. As has been studied by Horowitz and Strominger[37], however, special care must be paid about the middle point of string when operators Q_L and Q_R are multiplied. Moreover, the closed string state seems not to be included in the physical Hilbert space which is annihilated by the BRST charge for the open string. The closed string field theory has been proposed by Strominger[38]. We hope that our method developed here will be so extended that be applicable to Witten's string field theory.

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Appendix A

The construction of Γ is reviewed here[14]. Notations followed by HIKKO are listed below:

Oscillator modes:

$$A_{\pm}{}^{\mu} \equiv \eta^{\mu\nu}P_{\nu} \mp X^{\mu\prime} = \frac{1}{\sqrt{\pi}} \sum_{n=-\infty}^{\infty} \alpha_n^{\mu(\pm)} e^{\pm in\sigma}$$

$$C_{\pm} \equiv i\pi_{\bar{c}} \mp c = \frac{1}{\sqrt{\pi}} \sum_{n=-\infty}^{\infty} c_n^{(\pm)} e^{\pm in\sigma}$$

$$\bar{C}_{\pm} \equiv \bar{c} \mp i\pi_{c} = \frac{1}{\sqrt{\pi}} \sum_{n=-\infty}^{\infty} \bar{c}_n^{(\pm)} e^{\pm in\sigma} \tag{A.1}$$

(anti-)Commutation relations:

$$[\alpha_n^{\mu(i)},\ \alpha_m^{\nu(j)}] = n\delta_{n+m,0}\eta^{\mu\nu}\delta^{ij} \qquad (i,j=\pm)$$

$$\{c_n^{(i)},\ \bar{c}_m^{(j)}\} = \delta_{n+m,0}\delta^{ij}$$

$$\alpha_{-n}^{\mu(i)} = \alpha_n^{\mu(i)\dagger},\quad c_{-n}^{(i)} = c_n^{(i)\dagger},\quad \bar{c}_{-n}^{(i)} = \bar{c}_n^{(i)\dagger}$$

$$\alpha_0^{\mu(+)} = \alpha_0^{\mu(-)} - \frac{1}{2}p^{\mu},\quad \bar{c}_0^{(\pm)} = \frac{1}{2}\bar{c}_0 \mp \frac{\partial}{\partial c_0},\quad c_0^{(\pm)} = \frac{\partial}{\partial \bar{c}_0} \mp \frac{1}{2}c_0$$

$$\alpha_n^{\mu(\pm)}|0\rangle=0,\quad c_n^{(\pm)}|0\rangle=0,\quad \bar{c}_n^{(i)}|0\rangle=0 \qquad (n\geq 1) \tag{A.2}$$

The oscillator representation of three string vertex is given by

$$|V(1,2,3)\rangle = \mathscr{P}^{(1)}\mathscr{P}^{(2)}\mathscr{P}^{(3)}\prod_{r=1}^{3}\left[1-\bar{c}_0^{(r)}\frac{1}{\sqrt{2}}w_I^{(r)}\right]e^{F(1,2,3)}|0\rangle_{123}\delta(1,2,3), \tag{A.3}$$

where $\mathscr{P}$ is the projection operator into the same number of (+) and (-) modes and

$$F(1,2,3) = \frac{1}{2}\sum_{\pm}\sum_{\substack{n,m\geq 1\\ r,s}} \bar{N}_{nm}^{rs}\left[\alpha_{-n}^{(\pm)(r)}\cdot\alpha_{-m}^{(\pm)(s)} + 2i\gamma_{-n}^{(\pm)(r)}\bar{\gamma}_{-m}^{(\pm)(s)}\right]$$

$$+ \frac{1}{2}\sum_{\substack{n\geqq 1\\ r}} \bar{N}^r_n\left[\alpha^{(+)(r)}_{-n}+\alpha^{(-)(r)}_n\right]\cdot P + \tau_0\sum_{r=1}^{3}\frac{1}{\alpha_r}\left(\tfrac{1}{4}p_r^2-2\right)$$

$$\gamma^{(r)}_n = in\alpha_r c^{(r)}_n, \quad \bar{\gamma}^{(r)}_n = \frac{\bar{c}^{(r)}_n}{\alpha_r}$$

$$w^{(r)}_I = \frac{i}{\sqrt{2}}\sum_{\substack{n\geqq 1\\ s}}\left[\chi^{rs}\bar{N}^s_n + \frac{1}{\alpha_r}\sum_{m=1}^{n-1}\bar{N}^{s\ s}_{n-m,m}\right](\gamma^{(+)(s)}_{-n}+\gamma^{(-)(s)}_{-n})$$

$$\chi^{rs} = \delta^{rs}\frac{\alpha_{r-1}-\alpha_{r+1}}{\alpha_r} + \sum_{t=1}^{3}\varepsilon^{rst}, \qquad \varepsilon^{123}=1$$

$$\bar{N}^{rs}_{nm} = -\alpha_1\alpha_2\alpha_3\left(\frac{\alpha_r}{n}+\frac{\alpha_s}{m}\right)^{-1}\bar{N}^r_n\bar{N}^s_m \quad (n,m\geqq 1)$$

$$\bar{N}^{rs}_{n0} = -c_s\frac{\alpha_1\alpha_2}{\alpha_s}\bar{N}^r_n, \quad (c_1,c_2,c_3)=(1,-1,0) \quad (n\geqq 1)$$

$$\bar{N}^{rs}_{00} = \tau_0\left(\frac{\delta_{rs}}{\alpha_r} - \frac{\delta_{r3}}{\alpha_3} - \frac{\delta_{s3}}{\alpha_3}\right)$$

$$\tau_0 = \sum_{r=1}^{3}\alpha_r\ln|\alpha_r| \quad , \quad P=\alpha_1p_2-\alpha_2p_1, \quad \delta(1,2,3)=(2\pi)^{D+1}\delta(\sum_r\alpha_r)\delta(\sum_r p_r)$$

$$\bar{N}^r_n = \frac{1}{\alpha_r}f_n\left(-\frac{\alpha_{r+1}}{\alpha_r}\right)e^{n\tau_0/\alpha_r} \qquad (\alpha_4=\alpha_1, \alpha_0=\alpha_3)$$

$$f_n(x) = \frac{\Gamma(nx)}{n!\,\Gamma(nx-n+1)} = (-)^{n+1}f_n(1-x) \tag{A.4}$$

Let us construct a string field Γ which satisfies

$$\Gamma*\Phi = \left(N_{\mathrm{FP}}+1-\frac{\alpha}{|\alpha|} - \alpha\frac{\partial}{\partial\alpha}\right)\Phi \tag{A.5}$$

for arbitrary Φ and α is the width parameter of Φ and

$$N_{\mathrm{FP}}\Gamma = -2\Gamma, \tag{A.6}$$

where N_{FP} is the Faddeev-Popov ghost number operator. Eq.(A.5) is more explicitly represented as

$$\int d1 \langle \Gamma(1) | | V(1,2,3) \rangle = \left(N_{\mathrm{FP}}^{(3)} + 1 - \frac{\alpha_3}{|\alpha_3|} - \alpha_3 \frac{\partial}{\partial \alpha_3} \right) | \tilde{R}(2,3) \rangle , \quad (A.7)$$

where

$$d1 = \frac{d^D p_1}{(2\pi)^D} d\bar{c}_0^{(1)} \frac{d\alpha_1}{2\pi}$$

is the integration measure with respect to zero-mode variables and

$$|\tilde{R}(2,3)\rangle = (2\pi)^D \delta(p_2+p_3) 2\pi\delta(\alpha_2+\alpha_3)(\bar{c}_0^{(2)} - \bar{c}_0^{(3)}) | r(2,3) \rangle$$

with

$$|r(2,3)\rangle$$

$$\equiv \exp\left[-\sum_{\pm} \sum_{n\geqq 1} (-)^n \left\{ \frac{1}{n} \alpha_{-n}^{(\pm)(2)} \cdot \alpha_{-n}^{(\pm)(3)} - c_{-n}^{(\pm)(2)} \bar{c}_{-n}^{(\pm)(3)} + \bar{c}_{-n}^{(\pm)(2)} c_{-n}^{(\pm)(3)} \right\} \right] |0\rangle_{23}$$

$$N_{FP} = \sum_{\pm} \sum_{n\geqq 1} (c_{-n}^{(\pm)} \bar{c}_n^{(\pm i)} - \bar{c}_{-n}^{(\pm)} c_n^{(\pm)}) - \bar{c}_0 \frac{\partial}{\partial \bar{c}_0} . \quad (A.8)$$

We look for a string field such that $|\Gamma\rangle = \bar{c}_0 |\tilde{\Gamma}\rangle$. To find $\tilde{\Gamma}$, the following formula is of great use;

$$\lim_{\substack{\alpha_1 = \varepsilon \to 0 \\ p_1 \to 0}} {}_1\langle 0 | \exp\left\{ \sum_{\pm} \sum_{n\geqq 1} (j_n^{(\pm)} \cdot \alpha_n^{(\pm)(1)} + \bar{\lambda}_n^{(\pm)} c_n^{(\pm)(1)} + \lambda_n^{(\pm)} \bar{c}_n^{(\pm)(1)}) \right\}$$

$$\times \mathscr{P}^{(2)} \mathscr{P}^{(3)} \prod_{r=2,3} \left[1 - \bar{c}_0^{(r)} \frac{1}{\sqrt{2}} w_I^{(r)} \right] e^{F(1,2,3)} |0\rangle_{123}$$

$$= \frac{1}{2} e^2 \mathscr{P}^{(2)} \mathscr{P}^{(3)} \left[\left(\sum_{\pm} \sum_{n\geqq 1} g_n \lambda_n^{(\pm)} \right) \left(N_{\mathrm{FP}}^{(3)} + 1 - \frac{\alpha_3}{|\alpha_3|} - \frac{\alpha_3}{\varepsilon} \right) + \sum_{\pm} \sum_{n\geqq 1} h_n \lambda_n^{(\pm)} u^{(\pm)} \right]$$

$$\times \delta(\bar{c}_0^{(2)} - \bar{c}_0^{(3)}) | r(2,3) \rangle \exp\left[\sum_{\pm} \sum_{n,m\geqq 1} \frac{h_n h_m}{n+m} \left(\frac{1}{2} nm j_m^{(\pm)} \cdot j_m^{(\pm)} + n \lambda_n^{(\pm)} \bar{\lambda}_m^{(\pm)} \right) \right] \quad (A.9)$$

where (A.2) and (A.4) have been used and

$$h_n=\frac{1}{n!}\left(\frac{n}{e}\right)^n[\mathrm{sgn}(\varepsilon\alpha_3)]^n,\quad g_n=\sum_{m=1}^{n-1}h_{n-m}h_m \tag{A.10}$$

and j, λ_n and $\bar{\lambda}_n$ are c-number sources. Here only the terms which contain one more λ than $\bar{\lambda}$'s have been retained. Since $N_{\mathrm{FP}}\tilde{\Gamma}=-\tilde{\Gamma}$, other terms do not contribute to necessary matrix element to find $\tilde{\Gamma}$. For instance, we choose $\bar{\lambda}_1^{(+)}$, $\lambda_1^{(+)}$, $\lambda_2^{(\pm)}$ and $\lambda_3^{(-)}$ as non-vanishing sources and set

$$\lambda_2^{(-)}=\lambda_3^{(-)},$$

$$\lambda_1^{(+)}=-\frac{6}{e^2}\lambda_2^{(+)}. \tag{A.11}$$

In that case, the second term in the square bracket in the right hand side of the formula (A.9) vanishes. It is easy to show that

$$\langle\Gamma(1)|=\bar{c}_0^{(1)}{}_1\langle 0|\left[c_1^{(+)(1)}\bar{c}_1^{(+)(1)}\bar{c}_2^{(-)(1)}-\frac{1}{6}e^2c_1^{(+)(1)}\bar{c}_2^{(+)(1)}\bar{c}_3^{(-)(1)}\right]$$
$$\times\ \delta(p_1)\lim_{\varepsilon\to 0}\left[\delta(\alpha_1-\varepsilon)+\delta(\alpha_1+\varepsilon)\right] \tag{A.12}$$

satisfies (A.7). There are many other Γ's which satisfy (A.5) and (A.6).

Appendix B

The effective action in the non-linear σ-model is calculated here. The σ-model action is given by

$$S = \frac{1}{4\pi\alpha'}\int d^2\sigma\left[\sqrt{g}g^{ab}G_{\mu\nu}(X)\partial_a X^\mu\partial_b X^\nu + i\varepsilon^{ab}B_{\mu\nu}(X)\partial_a X^\mu\partial_b X^\nu + \alpha'\sqrt{g}R^{(2)}\Phi(X)\right]. \tag{B.1}$$

By making Riemann's normal coordinate expansion of $X^\mu = x^\mu + \sqrt{2\pi\alpha'}\,\xi^\mu$ around a constant solution, the action, up to $\mathcal{O}(\alpha')$, is

$$\begin{aligned} S = &\int d^2\sigma\sqrt{g}g^{ab}\left[\frac{1}{2}\partial_a\xi_M\partial_b\xi^M - \frac{1}{3}\pi\alpha' R_{MKNL}\xi^K\xi^L\partial_a\xi^M\partial_b\xi^N\right] \\ &+\int d^2\sigma\sqrt{g}R^{(2)}\left[\frac{\sqrt{2\pi\alpha'}}{4\pi}\nabla_M\Phi\xi^M + \frac{1}{4}\alpha'\nabla_M\nabla_N\Phi\xi^M\xi^N\right] \\ &+\int d^2\sigma\left[i\frac{1}{3}\sqrt{2\pi\alpha'}\,S_{LMN}\varepsilon^{ab}\xi^L\partial_a\xi^M\partial_b\xi^N \right.\\ &\left.+ i\frac{1}{2}\pi\alpha'\nabla_K S_{LMN}\varepsilon^{ab}\xi^K\xi^L\partial_a\xi^M\partial_b\xi^N\right], \end{aligned} \tag{B.2}$$

where $R_{\mu\nu\rho\sigma}$ is the Riemann tensor of space-time and $S_{\mu\nu\rho}$ is the field strength of $B_{\mu\nu}$. They are all evaluated at x^μ. The vielbein field $e_\mu{}^M(x)$ is introduced. The effective action is defined by

$$W[g_{ab},x] = -\ln\left(\int\mathcal{D}\xi\; e^{-S[g_{ab},x]}\right). \tag{B.3}$$

The weak field expansion of relevant quantities are given by

$$g_{ab}=\delta_{ab}+h_{ab}, \tag{B.4}$$

$$\sqrt{g}g^{ab}=\delta^{ab}-\bar{h}^{ab}+\mathcal{O}(h_{ab}^2), \tag{B.5}$$

$$\begin{aligned}\sqrt{g}R^{(n)}=&\partial_a\partial_b\bar{h}^{ab}-\frac{1}{2}\partial^a\partial_a h \\ &+\frac{1}{4}\bar{h}^{ab}\partial^c\partial_c\bar{h}_{ab}+\frac{1}{2}\partial_c\bar{h}^{ca}\partial_b\bar{h}^b{}_a-\frac{\varepsilon}{8}h\partial^a\partial_a h+\mathcal{O}(h_{ab}^3),\end{aligned} \tag{B.6}$$

$$\varepsilon^{ab}(g)=\varepsilon^{ab}+ c\frac{\varepsilon}{2}h\varepsilon^{ab}+ d\frac{1}{2}(\bar{h}^{a}{}_{c}\varepsilon^{bc}-\bar{h}^{b}{}_{c}\varepsilon^{ac})+\mathcal{O}(h_{ab}^{2}), \tag{B.7}$$

where $n=2+2\varepsilon$ and

$$\bar{h}_{ab}=h_{ab}- \frac{1}{2}\delta_{ab}h, \tag{B.8}$$

$$h=h_{a}{}^{a}, \tag{B.9}$$

are introduced as independent variables. Indices are raised and lowered with Kronecker delta.

The renormalized action at one-loop level is

$$S = S_0+ S_{\rm int}+ S_{\rm c.t.}+ \Delta S, \tag{B.10}$$

where

$$S_0= \frac{1}{2} \int d^{n}\sigma(\delta^{ab}\partial_{a}\xi_{M}\partial_{b}\xi^{M} + m^2\xi_{M}\xi^{M}),$$

$$S_{\rm int} = \int d^{n}\sigma(\sqrt{g}g^{ab}-\delta^{ab})(\frac{1}{2}\partial_{a}\xi_{M}\partial_{b}\xi^{M}) +\int d^{n}\sigma(\sqrt{g}-1)\frac{1}{2}m^2\xi_{M}\xi^{M}$$

$$+\int d^{n}\sigma\sqrt{g}g^{ab}\left[- \frac{1}{3}\pi\alpha' R_{MKNL}\xi^{K}\xi^{L}\partial_{a}\xi^{M}\partial_{b}\xi^{N} \right]$$

$$+\int d^{n}\sigma\sqrt{g}R^{(n)}\left[\frac{\sqrt{2\pi\alpha'}}{4\pi}\nabla_{M}\Phi\xi^{M} + \frac{1}{4}\alpha'\nabla_{M}\nabla_{N}\Phi\xi^{M}\xi^{N}\right]$$

$$+\int d^{n}\sigma\left[i\frac{1}{3}\sqrt{2\pi\alpha'}S_{LMN}\varepsilon^{ab}(g)\xi^{L}\partial_{a}\xi^{M}\partial_{b}\xi^{N}\right.$$

$$\left. + i\frac{1}{2}\pi\alpha'\nabla_{K}S_{LMN}\varepsilon^{ab}(g)\xi^{K}\xi^{L}\partial_{a}\xi^{M}\partial_{b}\xi^{N}\right],$$

$$S_{\rm c.t.} = - \frac{1}{2\varepsilon} \frac{D}{24\pi}\int d^{n}\sigma\sqrt{g}R^{(n)} + \frac{1}{16\pi\varepsilon}\alpha'\nabla_{M}\nabla^{M}\Phi \int d^{n}\sigma\sqrt{g}R^{(n)}$$

$$+\int d^{n}\sigma\sqrt{g}g^{ab}\left[- \frac{1}{12\varepsilon}\alpha' R_{MN}\partial_{a}\xi^{M}\partial_{b}\xi^{N}\right]$$

$$+\int d^n\sigma\sqrt{g}\left[+\ \frac{1}{12\varepsilon}\ \alpha'\, R_{MN}m^2\,\xi^M\xi^N\right]$$

$$+\int d^n\sigma\left[+\ \frac{1}{4\varepsilon}\alpha'\, S_{MIJ}S_N{}^{IJ}\sqrt{g}g^{ab}\partial_a\xi^M\partial_b\xi^N\right.$$

$$\left.+\ i\frac{1}{8\varepsilon}\alpha'\,\nabla_K S^K{}_{MN}\varepsilon^{ab}(g)\,\partial_a\xi^M\partial_b\xi^N\right],$$

$$\Delta S = \frac{1}{2}\int d^n\sigma\alpha'\, S_{MIJ}S_N{}^{IJ}\bar{h}^{ab}\partial_a\xi^M\partial_b\xi^N .$$

The finite renormalization term ΔS which is necessary to prove the nilpotency of BRST charge has been added(see (2.2.23)).

Graphs contributing to the effective action is given in fig. 4. The straightforward calculation of fig. 4.a1 shows that

$$\text{a1} = -\frac{D}{4}\frac{B(n/2+1,n/2+1)}{(4\pi)^{n/2}\Gamma(2)}\Gamma(2-n/2)\int\frac{d^np}{(2\pi)^n}\left(\frac{\mu^2}{p^2}\right)^{1-n/2}$$

$$\times\left[+\ \frac{(p_ap_b\bar{h}^{ab}(p)-\frac{1}{2}p^2h(p))^2}{p^2}\right.$$

$$+\ \frac{8}{n(2-n)}\left\{-\ \frac{1}{4}\bar{h}^{ab}(p)\,p^2\bar{h}_{ab}(-p)\right.$$

$$\left.\left.+\ \frac{1}{2}\bar{h}^{ab}(p)\,p_ap_c\bar{h}^c{}_b(-p)\ +\ \frac{\varepsilon}{8}h(p)\,p^2h(-p)\right\}\right]$$

$$= -\frac{D}{96\pi}\int\frac{d^2p}{(2\pi)^2}\frac{(p_ap_b\bar{h}^{ab}(p)-\frac{1}{2}p^2h(p))^2}{p^2}$$

$$+\ \frac{1}{2\varepsilon}\frac{D}{24\pi}\int\frac{d^np}{(2\pi)^n}\left(-\ \frac{1}{4}\bar{h}^{ab}(p)\,p^2\bar{h}_{ab}(-p)\right.$$

$$\left.+\ \frac{1}{2}\bar{h}^{ab}(p)\,p_ap_c\bar{h}^c{}_b(-p)\ +\ \frac{\varepsilon}{8}h(p)\,p^2h(-p)\right)$$

$$+\ \ldots\ldots \qquad \text{(B.11)}$$

The ellipses are terms proportional to $R^{(n)}$ with finite coefficient and the terms of $\mathcal{O}(m^2)$. These terms are irrelevant for our calculations. They will be disregarded hereafter also. Adding the counterterm in (B.10)(see fig. 4.a2), one obtains the effective action in flat space-time:

$$\mathrm{a1+a2} = -\frac{D}{96\pi}\int\frac{d^2p}{(2\pi)^2}\,\frac{(p_ap_b\bar{h}^{ab}(p)-\frac{1}{2}p^2h(p))^2}{p^2} \qquad (B.12)$$

Next consider the contributions to the effective action in curved backgrounds. The results of calculations of graphs related to the space-time metric $G_{\mu\nu}$(see fig. 4.b1-b5) turn out to be

$$\mathrm{b1} = +\frac{\alpha'}{64\pi}R\int\frac{d^2p}{(2\pi)^2}\,\frac{(p_ap_b\bar{h}^{ab}(p)-\frac{1}{2}p^2h(p))^2}{p^2} \qquad (B.13)$$

$$\mathrm{b2+b3} = \mathcal{O}(m^2) \qquad (B.14)$$

$$\mathrm{b4+b5} = \mathcal{O}(m^2) \qquad (B.15)$$

In fig. 4.b1 the only finite non-local term contributes to the effective action without divergent terms. As given by (B.14) and (B.15), all other graphs cancel out to give the terms of $\mathcal{O}(m^2)$.

The dilaton contributions(see fig. 4.c1-c4) to the effective action are given by

$$\mathrm{c2} = +\frac{\alpha'}{16\pi}\nabla^M\nabla_M\Phi\int\frac{d^2p}{(2\pi)^2}\,\frac{(p_ap_b\bar{h}^{ab}(p)-\frac{1}{2}p^2h(p))^2}{p^2} \qquad (B.16)$$

$$\mathrm{c3+c4} = \mathcal{O}(\varepsilon) \qquad (B.17)$$

$$\mathrm{c1} = -\frac{\alpha'}{16\pi}\nabla^M\Phi\nabla_M\Phi\int\frac{d^2p}{(2\pi)^2}\,\frac{(p_ap_b\bar{h}^{ab}(p)-\frac{1}{2}p^2h(p))^2}{p^2} \qquad (B.18)$$

Let us calculate the effective action in the presence of $B_{\mu\nu}$(see fig. 4.d1-d7). If d in (B.7) is non-vanishing, the calculation of fig. 4.d4 generates non-local divergent terms as well as infrared divergent terms. These terms do not cancel even if contributions of any other graphs are added. Assuming that the counterterms should be local, we conclude that d in (B.7) should be vanishing. With this condition each graph will be calculated below. The two-loop integral of fig. 4.d1 takes the following form;

$$\mathrm{d1} = +\ 2\pi\alpha' S_{IJK}S^{IJK} \int\frac{d^n p}{(2\pi)^n}\bar{h}^{ab}(p)\bar{h}^{cd}(-p)\ \ \varepsilon^{ef}\varepsilon^{gh}$$
$$\times \int\frac{d^n q}{(2\pi)^n}\ \frac{d^n k}{(2\pi)^n}\ \frac{q_a(p+q)_b q_e(q-k)_f(p+q)_g(q-k)_h k_c(p+k)_d}{q^2(p+q)^2(q-k)^2k^2(p+k)^2} \tag{B.19}$$

where mass terms are neglected because no infrared divergences arise in the course of the calculation. Using the rule:

$$\varepsilon^{ab}\varepsilon^{cd}=\delta^{ac}\delta^{bd}-\delta^{ad}\delta^{bc} \tag{B.20}$$

and making transformations of momentum variables, we rewrite (B.19) as

$$\mathrm{d1} = +\ \pi\alpha' S_{IJK}S^{IJK} \int\frac{d^n p}{(2\pi)^n}\bar{h}^{ab}(p)\bar{h}^{cd}(-p)$$
$$\times \int\frac{d^n q}{(2\pi)^n}\ \frac{d^n k}{(2\pi)^n}\ \frac{q_a(p+q)_b k_c(p+k)_d}{q^2(p+q)^2(q-k)^2k^2(p+k)^2}$$
$$\times \left\{ -(q-k)^2p^2 + (q-k)^2 q\cdot k - 2(q+p)^2(q-k)\cdot k \right\} \tag{B.21}$$

Performing the momentum integral, one finds that

$$+ \frac{\pi\alpha'}{(4\pi)^n} S_{IJK}S^{IJK}\int\frac{d^n p}{(2\pi)^n} \left\{ -\frac{3}{4}\left[B(n/2,n/2)\Gamma(2-n/2)\right]^2 \frac{(p_a p_b \bar{h}^{ab} - \frac{1}{2}p^2 h)^2}{p^2} \right.$$

$$+ B(n,1+n/2)\Gamma(3-n)B(n/2,n/2)$$

$$\times \left[\frac{1+4\varepsilon}{8(1+2\varepsilon)} h(p)p^2 h(-p) + \frac{1+\varepsilon}{4\varepsilon^2(1+2\varepsilon)} p^2 \bar{h}_{ab}(p)\bar{h}^{ab}(-p) \right.$$

$$- \frac{1+4\varepsilon}{2\varepsilon} p_a p_b \bar{h}^{ab}(p)h(-p) - \frac{2-\varepsilon}{4\varepsilon^2} p_a \bar{h}^a{}_c(p) p_b \bar{h}^{bc}(-p)$$

$$\left.\left. + \frac{1+4\varepsilon}{\varepsilon} \frac{(p_a p_b \bar{h}^{ab})^2}{p^2} \right] \right\}$$

(B.22)

In the above result, $\frac{1}{\varepsilon^2}$ divergences arise. The fig. 4.d2 gives the momentum integral of the form;

$$d2 = + 2\pi\alpha' S_{IJK}S^{IJK} \int\frac{d^n p}{(2\pi)^n} \bar{h}^{ab}(p)\bar{h}^{cd}(-p) \ \varepsilon^{ef}\varepsilon^{gh}$$

$$\times \int\frac{d^n q}{(2\pi)^n} \frac{d^n k}{(2\pi)^n} \frac{q_a(p+q)_b q_e(q-k)_f q_g(q-k)_h q_c(p+q)_d}{q^2 q^2 (p+q)^2 (q-k)^2 k^2}$$

(B.23)

By using the antisymmetric property with respect to the suffix of ε^{ab}, the momentum integral is easily carried out to give

$$+ \frac{\pi\alpha'}{(4\pi)^n} S_{IJK}S^{IJK} \int\frac{d^n p}{(2\pi)^n} B(n,n/2+1)\Gamma(3-n)B(n/2,n/2)$$

$$\times \left[-\frac{1}{8} h(p)p^2 h(-p) - \frac{1}{4\varepsilon^2} p^2 \bar{h}_{ab}\bar{h}^{ab} \right.$$

$$+ \frac{1+2\varepsilon}{2\varepsilon} p_a p_b \bar{h}^{ab}(p)\cdot h(-p) + \frac{2+\varepsilon}{4\varepsilon^2} p_a \bar{h}^a{}_c(p) p_b \bar{h}^{bc}(-p)$$

$$\left. - \frac{1+2\varepsilon}{\varepsilon} \frac{(p_a p_b \bar{h}^{ab})^2}{p^2} \right]$$

(B.24)

which has also $\frac{1}{\varepsilon^2}$ divergences. The sum of the contributions given by (B.22) and (B.24) becomes

$$d1+d2 = +\frac{1}{6\varepsilon}\frac{\alpha'}{16\pi} S_{IJK}S^{IJK}\int\frac{d^n p}{(2\pi)^n}\Big[-\frac{1}{4}\bar{h}^{ab}(p)p^2\bar{h}_{ab}(-p) + \frac{1}{2}\bar{h}^{ab}(p)p_a p_c\bar{h}^c{}_b(-p) + \frac{\varepsilon}{8}h(p)p^2h(-p)\Big]$$

$$+ \frac{\alpha'}{16\pi} S_{IJK}S^{IJK}\int\frac{d^n p}{(2\pi)^n}\Big\{-\frac{5}{12}\frac{(p_a p_b\bar{h}^{ab}(p)-\frac{1}{2}p^2h(p))^2}{p^2} + \frac{1}{6}\bar{h}^{ab}(p)p_a p_b h(-p) - \frac{5}{48}h(p)p^2h(-p)\Big\} \tag{B.25}$$

It is worthwhile to emphasize that $\frac{1}{\varepsilon^2}$ divergent terms in (B.22) and (B.24) cancel out. The recent results[39] of the calculations of the β-functions at the two-loop order showed that the correct prescription is

$$\varepsilon^{ab}\varepsilon^{cd} = (1-\frac{\varepsilon}{2})(\delta^{ac}\delta^{bd}-\delta^{ad}\delta^{bc}). \tag{B.26}$$

Since $\frac{1}{\varepsilon^2}$ divergences appear at the two-loop calculations of β-functions, the $\mathcal{O}(\varepsilon)$ term in (B.26) contributes to the equations for background fields derived from the vanishing conditions of β-functions. However, it is sufficient to use the rule (B.20) in our calculations, because no $\frac{1}{\varepsilon^2}$ divergences arise.

Other graphs can be calculated with ease. The contributions to the effective action are

$$d3 = -c\,\frac{2\pi\alpha'}{3}S_{IJK}S^{IJK}\int\frac{d^n p}{(2\pi)^n}\bar{h}^{ab}(p)\,\varepsilon h(-p)\;\;\varepsilon^{ef}\varepsilon^{gh}$$

$$\times\int\frac{d^n q}{(2\pi)^n}\frac{d^n k}{(2\pi)^n}\frac{q_a(p+q)_b q_e(q-k)_f\{2(p+q)_g+(q-k)_g\}k_h}{q^2(p+q)^2(q-k)^2k^2}$$

$$= -\frac{c}{6}\frac{\alpha'}{16\pi} S_{IJK}S^{IJK}\int\frac{d^n p}{(2\pi)^n}\Big[\bar{h}^{ab}(p)p_a p_b h(-p) - \frac{1}{2}h(p)p^2h(-p)\Big] \tag{B.27}$$

$$d4 = + c^2\frac{\pi\alpha'}{18} S_{IJK}S^{IJK} \int\frac{d^n p}{(2\pi)^n}\varepsilon h(p)\varepsilon h(-p)\ \varepsilon^{ef}\varepsilon^{gh}$$

$$\times \int\frac{d^n q}{(2\pi)^n}\frac{d^n k}{(2\pi)^n}\frac{(q-k-p)_e q_f\{(q-k-p)_g - 2k_g\} q_h}{q^2(q-k-p)^2 k^2}$$

$$= + \frac{c^2}{72}\frac{\alpha'}{16\pi} S_{IJK}S^{IJK}\int\frac{d^n p}{(2\pi)^n}\ h(p)p^2 h(-p)$$

$$d5+d6 = \mathcal{O}(m^2) \tag{B.29}$$

$$d7 = + \frac{\alpha'}{2} S_{IJK}S^{IJK}\int\frac{d^n p}{(2\pi)^n}\bar{h}^{ab}(p)\bar{h}^{cd}(-p)\int\frac{d^n q}{(2\pi)^n}\frac{q_a q_c}{q^2}\frac{(q+p)_b(q+p)_d}{(q+p)^2}$$

$$= -\frac{2}{3\varepsilon}\frac{\alpha'}{16\pi} S_{IJK}S^{IJK}\int\frac{d^n p}{(2\pi)^n}\Big[-\frac{1}{4}\bar{h}^{ab}(p)p^2\bar{h}_{ab}(-p) + \frac{1}{2}\bar{h}^{ab}(p)p_a p_c\bar{h}^c{}_b(-p) + \frac{\varepsilon}{8}h(p)p^2 h(-p)\Big]$$

$$+ \frac{\alpha'}{48\pi} S_{IJK}S^{IJK}\int\frac{d^n p}{(2\pi)^n}\frac{(p_a p_b\bar{h}^{ab}(p) - \frac{1}{2}p^2 h(p))^2}{p^2} \tag{B.30}$$

Note that the results given by (B.27) and (B.28) depend on the unknown parameter c introduced in (B.7). Adding all contributions to the effective action, we eventually arrive at

$$W = -\Big\{ D - \frac{3}{2}\alpha'(R + 4\nabla^\mu\nabla_\mu\Phi - 4\nabla^\mu\Phi\nabla_\mu\Phi - \frac{1}{3}S_{\mu\nu\rho}S^{\mu\nu\rho})\Big\}$$

$$\times \frac{1}{96\pi}\int\frac{d^2 p}{(2\pi)^2}\frac{(p_a p_b\bar{h}^{ab}(p) - \frac{1}{2}p^2 h(p))^2}{p^2}$$

$$-\frac{1}{2\varepsilon}\frac{\alpha'}{16\pi} S_{IJK}S^{IJK}\int\frac{d^n p}{(2\pi)^n}\Big[-\frac{1}{4}\bar{h}^{ab}(p)p^2\bar{h}_{ab}(-p) + \frac{1}{2}\bar{h}^{ab}(p)p_a p_c\bar{h}^c{}_b(-p) + \frac{\varepsilon}{8}h(p)p^2 h(-p)\Big]$$

$$
-\left(\frac{1}{48}-\frac{c^2}{72}\right)\frac{\alpha'}{16\pi}S_{IJK}S^{IJK}\int\frac{d^np}{(2\pi)^n}\,h(p)p^2h(-p)
$$

$$
-\frac{1}{6}(c-1)\frac{\alpha'}{16\pi}S_{IJK}S^{IJK}\int\frac{d^np}{(2\pi)^n}
$$

$$
\times\left[\bar{h}^{ab}(p)p_ap_bh(-p)-\frac{1}{2}h(p)p^2h(-p)\right]. \tag{B.31}
$$

No infrared divergences appear in the final expression. If we look at (B.31), a new counterterm of order $\frac{1}{\varepsilon}$ should be added in the case where the anti-symmetric tensor field exists:

$$
S_{c.t.}^{(2)}=+\frac{1}{2\varepsilon}\frac{\alpha'}{16\pi}S_{IJK}S^{IJK}\int\frac{d^np}{(2\pi)^n}\left[-\frac{1}{4}\bar{h}^{ab}(p)p^2\bar{h}_{ab}(-p)\right.
$$

$$
\left.+\frac{1}{2}\bar{h}^{ab}(p)p_ap_c\bar{h}^c{}_b(-p)+\frac{\varepsilon}{8}h(p)p^2h(-p)\right]. \tag{B.32}
$$

Moreover, a finite local counterterm should be added, so that the renormalized effective action becomes invariant under the reparametrization. The term is

$$
S_{\text{non-cov}}=\left(\frac{1}{48}-\frac{c^2}{72}\right)\frac{\alpha'}{16\pi}S_{IJK}S^{IJK}\int\frac{d^np}{(2\pi)^n}\,h(p)p^2h(-p)
$$

$$
+\frac{1}{6}(c-1)\frac{\alpha'}{16\pi}S_{IJK}S^{IJK}\int\frac{d^np}{(2\pi)^n}
$$

$$
\times\left[\bar{h}^{ab}(p)p_ap_bh(-p)-\frac{1}{2}h(p)p^2h(-p)\right], \tag{B.33}
$$

which is finite local counterterm as it should be. By taking account of (B.32) and (B.33), the two-loop renormalized action is given by

$$S = S_0 + S_{\rm int} + S_{\rm c.t.} + \Delta S + S_{\rm non\text{-}cov} + S_{\rm c.t.}^{(2)}, \tag{B.34}$$

where the counterterms $S_{\rm c.t.}$ and $S_{\rm c.t.}^{(2)}$ are local and invariant.

Taking account of $S_{\rm non\text{-}cov}$ and $S_{\rm c.t.}^{(2)}$ and combining the results in (B.31), we finally obtain the two-loop effective action

$$W = -\frac{1}{96\pi}\left\{ D - \frac{3}{2}\alpha'\left(R + 4\nabla^{\mu}\nabla_{\mu}\Phi - 4\nabla^{\mu}\Phi\nabla_{\mu}\Phi - \frac{1}{3}S_{\mu\nu\rho}S^{\mu\nu\rho}\right)\right\}$$
$$\times \int d^2x d^2y \sqrt{g(x)}R(x)G(x,y)\sqrt{g(y)}R(y) + \ldots \tag{B.35}$$

where

$$\frac{1}{\sqrt{g}}\partial_a(\sqrt{g}g^{ab}\partial_b)G(x,y) = -\frac{1}{\sqrt{g}}\delta^2(x-y). \tag{B.36}$$

Here ... means some finite terms proportional to $R^{(2)}$.

Appendix C

Let us calculate the conformal factor dependence of $\langle T_{++}(z)\rangle$ at $\mathcal{O}(\phi)$. The graphs contributing to $\langle T_{++}(z)\rangle$ has been shown in fig. 5. Since each graph can be calculated straightforwardly, we shall list only the results below. They are represented in momentum space, i.e. ,

$$\int d^2z \ \langle T_{++}(z)\rangle \ e^{-ipz} \ . \tag{C.1}$$

The sum of all contributions leads to (4.5) in coordinate space.

(1)flat space-time

$$a1 = \mathcal{O}(\varepsilon) \tag{C.2}$$

$$a2 = -\frac{D}{24\pi} \ p_+p_+\phi(p) \tag{C.3}$$

(2)$G_{\mu\nu}$

$$b1 = \frac{\alpha'}{16\pi}R \ p_+p_+\phi(p) \tag{C.4}$$

$$b2 + b6 = \mathcal{O}(\varepsilon) \tag{C.5}$$

$$b3 + b5 = \mathcal{O}(\varepsilon) \tag{C.6}$$

$$b4 + b7 = \mathcal{O}(\varepsilon) \tag{C.7}$$

(3)Φ

$$c1 = \frac{\alpha'}{8\pi} \ \nabla_M\nabla^M\Phi p_+p_+\phi(p) \tag{C.8}$$

$$c2 = \frac{\alpha'}{8\pi} \ \nabla_M\nabla^M\Phi p_+p_+\phi(p) \tag{C.9}$$

$$c3 + c4 = \mathcal{O}(\varepsilon) \tag{C.10}$$

$$c5 = -\frac{\alpha'}{4\pi} \ \nabla_M\Phi \ \nabla^M\Phi p_+p_+\phi(p) \tag{C.11}$$

(4) $B_{\mu\nu}$

$$d1 = -\frac{3\alpha'}{32\pi} S_{IJK} S^{IJK} \; p_+ p_+ \phi(p) \tag{C.12}$$

$$d2 = -\frac{\alpha'}{32\pi} S_{IJK} S^{IJK} \; p_+ p_+ \phi(p) \tag{C.13}$$

$$d3 = +\, c\, \frac{\alpha'}{24\pi} S_{IJK} S^{IJK} \; p_+ p_+ \phi(p) \tag{C.14}$$

$$d4 = +\frac{\alpha'}{8\pi} S_{IJK} S^{IJK} \; p_+ p_+ \phi(p) \tag{C.15}$$

$$d5 = -\frac{\alpha'}{16\pi} S_{IJK} S^{IJK} \; p_+ p_+ \phi(p) \tag{C.16}$$

$$d6 = -\frac{\alpha'}{16\pi} S_{IJK} S^{IJK} \; p_+ p_+ \phi(p) \tag{C.17}$$

$$d7 = \mathcal{O}(\varepsilon) \tag{C.18}$$

$$d8 = \mathcal{O}(\varepsilon) \tag{C.19}$$

$$d9 = +\frac{\alpha'}{16\pi} S_{IJK} S^{IJK} \; p_+ p_+ \phi(p) \tag{C.20}$$

$$d10 = -\frac{(c-1)\alpha'}{24\pi} S_{IJK} S^{IJK} \; p_+ p_+ \phi(p) \tag{C.21}$$

The expectation value of the energy-momentum tensor is derivable from the variation with respect to the metric tensor of the effective action. Using the effective action W obtained in Appendix B, one finds that

$$\langle T_{ab}(z)\rangle = -\frac{1}{96\pi}\left\{D-\frac{3}{2}\alpha'\left(R+4\nabla^{\mu}\nabla_{\mu}\Phi-4\nabla^{\mu}\Phi\nabla_{\mu}\Phi-\frac{1}{3}S_{\mu\nu\rho}S^{\mu\nu\rho}\right)\right\}$$

$$\times\frac{2}{\sqrt{g(x)}}\int d^2x d^2y\left[2\frac{\delta(\sqrt{g(x)}R(x))}{\delta g^{ab}(z)}G(x,y)\sqrt{g(y)}R(y)\right.$$

$$\left.+\sqrt{g(x)}R(x)\frac{\delta G(x,y)}{\delta g^{ab}}\sqrt{g(y)}R(y)\right] \quad (C.22)$$

with

$$\frac{\delta(\sqrt{g(x)}R(x))}{\delta g^{ab}(z)} = \sqrt{g(x)}\left\{R_{ab}(x)-\frac{1}{2}g_{ab}(x)R(x)\right\}\delta^2(x-z)$$

$$+\sqrt{g(x)}g^{cd}(x)g_{ab}(x)\nabla^x_d\nabla^x_c\delta^2(x-z)$$

$$-\sqrt{g(x)}g^c{}_a(x)g^d{}_b(x)\nabla^x_d\nabla^x_c(x)\delta^2(x-z) \quad (C.23)$$

and

$$\frac{\delta G(x,y)}{\delta g^{ab}(z)} = \int d^2w G(x,w)\partial^w_c\left[\sqrt{g(w)}\left\{-\frac{1}{2}g_{ab}(w)g^{cd}(w)+\delta_a{}^c\delta_b{}^d\right\}\delta^2(w-z)\right.$$

$$\left.\times\ \partial^w_d G(w,y)\right] \quad (C.24)$$

Eq.(C.24) has been derived from (B.36). In the conformal coordinates $g_{ab}=e^{\phi}\delta_{ab}$ one finds that the conformal factor dependence of the energy-momentum tensor becomes

$$\langle T^X_{++}(z)\rangle = \frac{1}{24\pi}\left\{D-\frac{3}{2}\alpha'\left(R+4\nabla^{\mu}\nabla_{\mu}\Phi-4\nabla^{\mu}\Phi\nabla_{\mu}\Phi-\frac{1}{3}S_{\mu\nu\rho}S^{\mu\nu\rho}\right)\right\}$$

$$\times(\partial_z\partial_z\phi-\frac{1}{2}\partial_z\phi\partial_z\phi). \quad (C.25)$$

Appendix D

Let us look for the conformal factor dependence of $\langle \xi_I(1)\xi_J(2)T_{++}(z)\rangle$. The graphs contributing to $\langle \xi_I(1)\xi_J(2)T_{++}(z)\rangle$ have been given in fig. 6. The results listed below are represented in momentum space, i.e.

$$\int d^2 1\, d^2 2\, d^2 z \ \langle \xi_I(1)\xi_J(2)T_{++}(z)\rangle \ e^{-ip1-iq2+i(p+q+k)z} \tag{D.1}$$

The sum of all contributions amounts to (4.22) and (4.23) in coordinate space.

(1)flat space-time

$$\text{a1} = \mathcal{O}(\varepsilon) \tag{D.2}$$

(2)$G_{\mu\nu}$

$$\text{b1} = +\frac{\alpha'}{6} R_{IJ} \frac{1}{p^2}\frac{1}{q^2}\left[-\frac{1}{2}(p+q)_+ k_+ + p_+ q_+\right]\phi(k) \tag{D.3}$$

$$\text{b2} + \text{b3} = \mathcal{O}(\varepsilon) \tag{D.4}$$

$$\text{b4} = \mathcal{O}(\varepsilon) \tag{D.5}$$

$$\text{b5} = -\frac{\alpha'}{12} R_{IJ} \frac{k_\pm}{p^2}\frac{(p+q)_\pm}{q^2}\phi(k) \tag{D.6}$$

$$\text{b6} = \mathcal{O}(\varepsilon) \tag{D.7}$$

$$\text{b7} + \text{b8} = \mathcal{O}(\varepsilon) \tag{D.8}$$

$$\text{b9} + \text{b10} = \mathcal{O}(\varepsilon) \tag{D.9}$$

$$\text{b11} + \text{b12} = \mathcal{O}(\varepsilon) \tag{D.10}$$

$$\text{b13} + \text{b14} = \mathcal{O}(\varepsilon) \tag{D.11}$$

$$b15 = -\frac{\alpha'}{3} R_{IJ} \frac{q_\pm}{p^2} \frac{1}{q^2} \frac{(p+k)_\pm}{(p+k)^2} \left[-k^2 + p\cdot(p+k)\right]\phi(k) \tag{D.12}$$

(3) Φ

$$c1 = \alpha' \nabla_I \nabla_J \Phi \frac{(p+k)_\pm}{p^2} \frac{q_\pm}{q^2} \frac{k^2}{(p+k)^2} \phi(k) \tag{D.13}$$

$$c2 = \frac{\alpha'}{2} \nabla_I \nabla_J \Phi \frac{p_\pm}{p^2} \frac{k_\pm}{q^2} \phi(k) \tag{D.14}$$

$$c3 = \mathcal{O}(\varepsilon) \tag{D.15}$$

(4) $B_{\mu\nu}$

$$d1 + d11 = \mathcal{O}(\varepsilon) \tag{D.16}$$

$$d2 = c\alpha' S_{IKL} S_J{}^{KL} \frac{q_\pm}{p^2} \frac{(p+k/3)_\pm}{q^2} \phi(k) \tag{D.17}$$

$$d3 = -\alpha' S_{IKL} S_J{}^{KL} \frac{p_\pm}{p^2} \frac{q_\pm}{q^2} \phi(k) \tag{D.18}$$

$$d4 = -\frac{\alpha'}{2} S_{IKL} S_J{}^{KL} \frac{p_\pm}{p^2} \frac{q_\pm}{q^2} \phi(k) \tag{D.19}$$

$$d5 + d12 = \mathcal{O}(\varepsilon) \tag{D.20}$$

$$d6 = -c\alpha' S_{IKL} S_J{}^{KL} \frac{(p+k)_\pm}{p^2} \frac{q_\pm}{q^2} \frac{(p+k)\cdot(p+k/3)}{(p+k)^2} \phi(k) \tag{D.21}$$

$$d7 = +2\alpha' S_{IKL} S_J{}^{KL} \frac{(p+k)_\pm}{p^2} \frac{q_\pm}{q^2} \frac{p\cdot(p+k)}{(p+k)^2} \phi(k) \tag{D.22}$$

$$d8 = -c\alpha' S_{IKL} S_J{}^{KL} \frac{(p+k)_\pm}{p^2} \frac{q_\pm}{q^2} \frac{p\cdot(p+2k/3)}{(p+k)^2} \phi(k) \tag{D.23}$$

$$d9 + d14 = \mathcal{O}(\varepsilon) \tag{D.24}$$

$$d10 + d15 = \mathcal{O}(\varepsilon) \tag{D.25}$$

$$d13 = \alpha' S_{IKL} S_J{}^{KL} \frac{(p+k)_\pm}{p^2} \frac{q_\pm}{q^2} \frac{p\cdot(p+k)}{(p+k)^2} \phi(k) \tag{D.26}$$

$$d16 = \mathcal{O}(\varepsilon) \tag{D.27}$$

$$d17 = \mathcal{O}(\varepsilon) \tag{D.28}$$

$$e1 + e9 = \mathcal{O}(\varepsilon) \tag{D.29}$$

$$e2 + e10 = \mathcal{O}(\varepsilon) \tag{D.30}$$

$$e3 = -\frac{\alpha'}{2} \nabla^K S_{KIJ} \left[\frac{k_\pm}{p^2} \frac{q_\pm}{q^2} - \frac{k_\pm}{p^2} \frac{(p+q)_\pm}{q^2} \frac{k^2}{(p+k)^2} + \frac{p_\pm}{p^2} \frac{q_\pm}{q^2} + \frac{q_\pm}{q^2} \frac{(p+k)_\pm}{(p+k)^2} \right] \phi(k) \tag{D.33}$$

$$e4 + e11 = \mathcal{O}(\varepsilon) \tag{D.34}$$

$$e5 + e12 = \mathcal{O}(\varepsilon) \tag{D.35}$$

$$e6 = \mathcal{O}(\varepsilon) \tag{D.36}$$

$$e7 = \mathcal{O}(\varepsilon) \tag{D.37}$$

$$e8 = \mathcal{O}(\varepsilon) \tag{D.38}$$

(5)Φ and $B_{\mu\nu}$

$$f1 = \mathcal{O}(\varepsilon) \tag{D.39}$$

$$f2 = \mathcal{O}(\varepsilon) \tag{D.40}$$

$$f3 = \mathcal{O}(\varepsilon) \tag{D.41}$$

$$f4 = \alpha' \nabla^K \Phi S_{KIJ} \left[\frac{k_\pm}{p^2} \frac{q_\pm}{q^2} - \frac{k_\pm}{p^2} \frac{(p+q)_\pm}{q^2} \frac{k^2}{(p+k)^2} + \frac{p_\pm}{p^2} \frac{q_\pm}{q^2} + \frac{q_\pm}{q^2} \frac{(p+k)_\pm}{(p+k)^2} \right] \phi(k) \tag{D.42}$$

$$\mathrm{f5} = \frac{\alpha'}{2} \nabla^K \Phi S_{KIJ} \left[\frac{p_\pm}{p^2} \frac{k_\pm}{q^2} + \frac{(p+k)_\pm}{p^2} \frac{k_\pm}{(p+q)^2}\right] \phi(k) \tag{D.43}$$

$$\mathrm{f6} = -\frac{\alpha'}{2} \nabla^K \Phi S_{KIJ} \left[\frac{p_\pm}{p^2} \frac{k_\pm}{q^2} + \frac{(p+k)_\pm}{p^2} \frac{k_\pm}{(p+q)^2}\right] \phi(k) \tag{D.44}$$

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Figure Captions

Fig. 1 Graphs contributing to the effective action. The solid line represents the classical background X_{B}. The wavy line stands for the weak fields of two-dimensional metric tensor.

Fig. 2 A half of the world sheet diagram for three closed-string vertex. The other half, a mirror image of the figure with respect to the bottom line, is implicit.

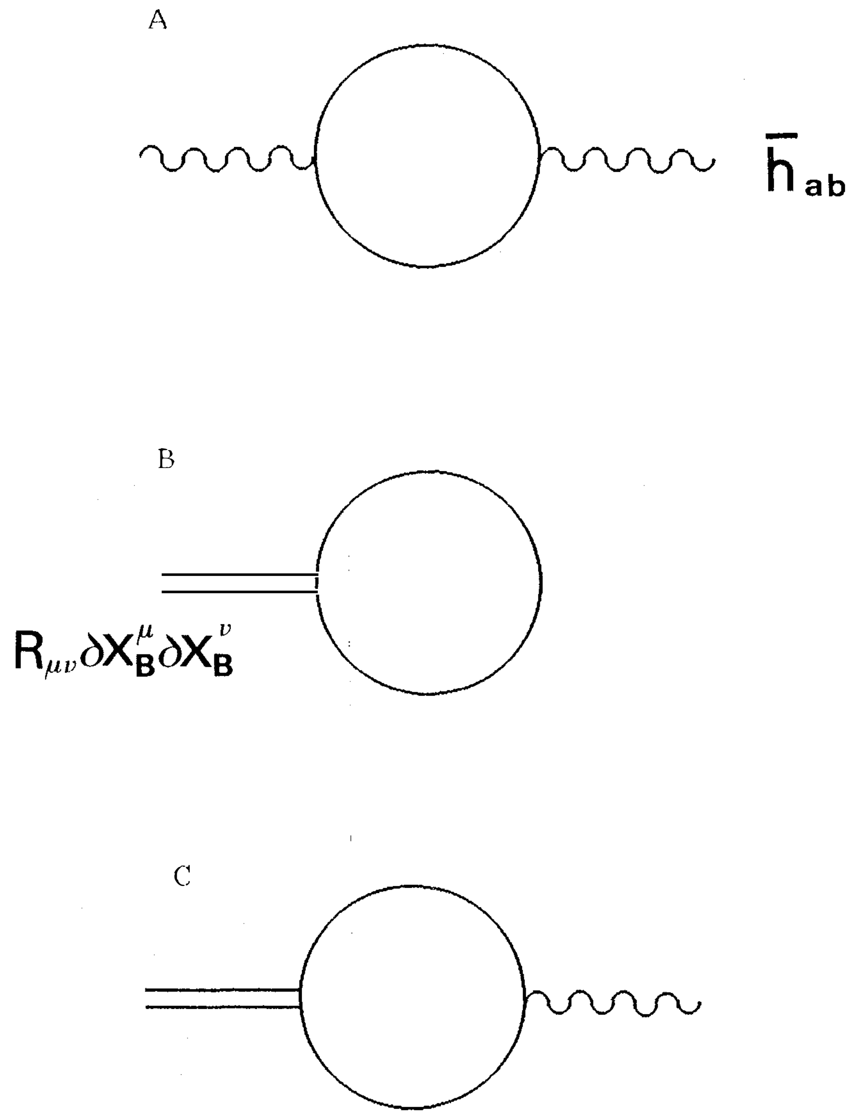
Fig. 3 Image of the three string vertex on the complex z-plane.

Fig. 4 Graphs contributing to the effective action.

Fig. 5 Graphs contributing to $\langle T_{++}(z)\rangle$. The cross stands for the coupling of energy-momentum tensor.

Fig. 6 Graphs contributing to $\langle \xi_I(1)\xi_J(2)T_{++}(z)\rangle$.

Fig. 1



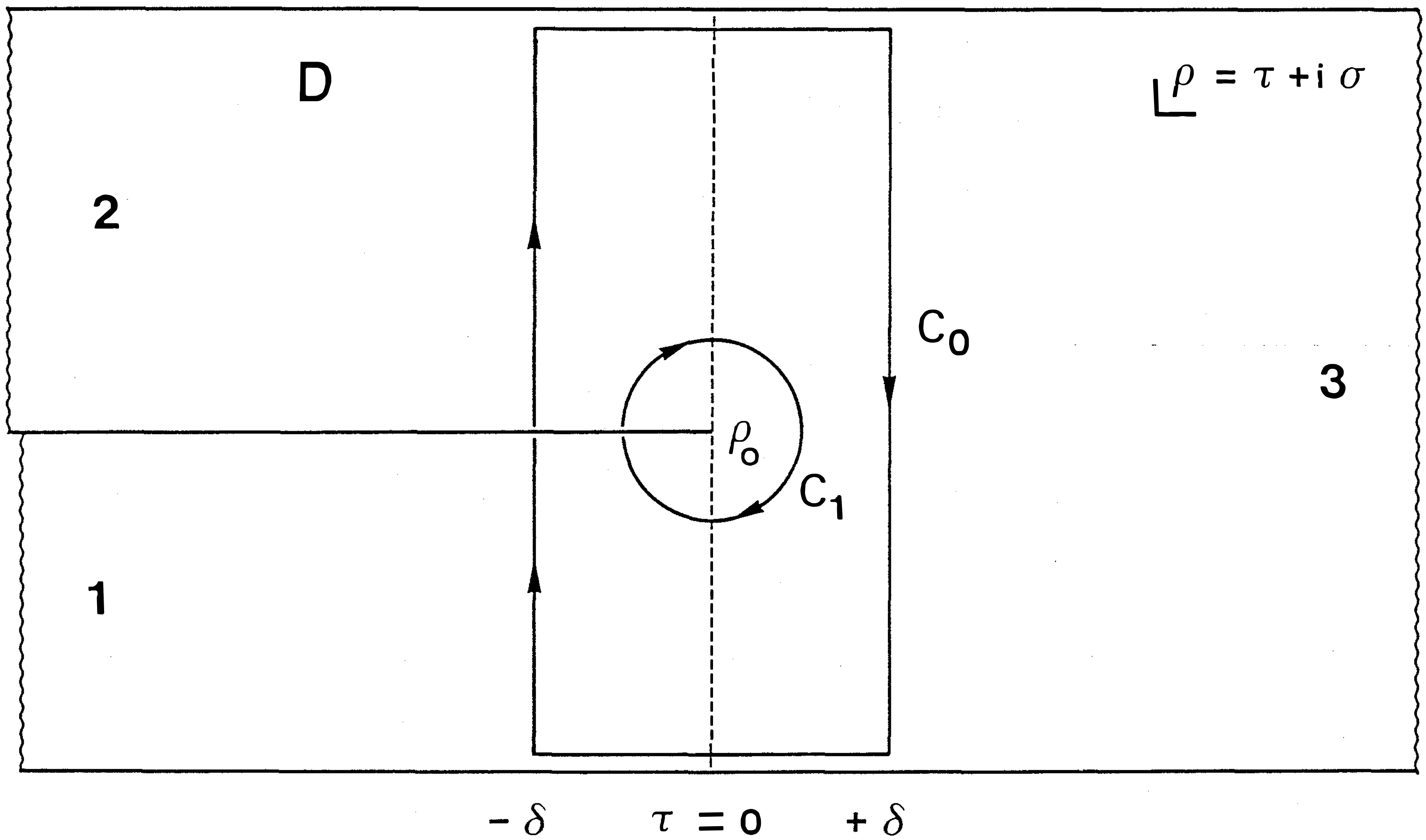


Fig. 2

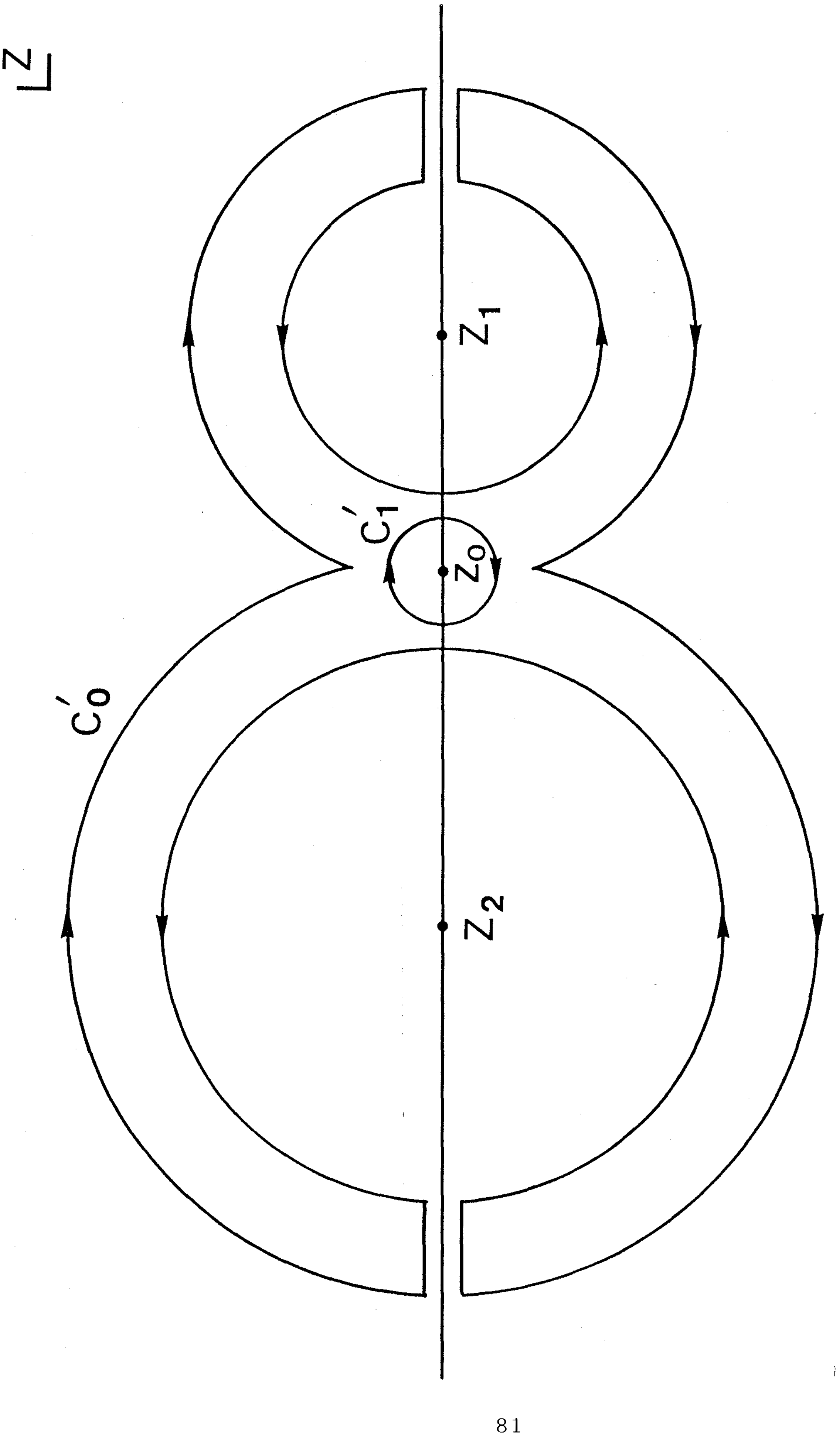


Fig. 3

Fig. 4

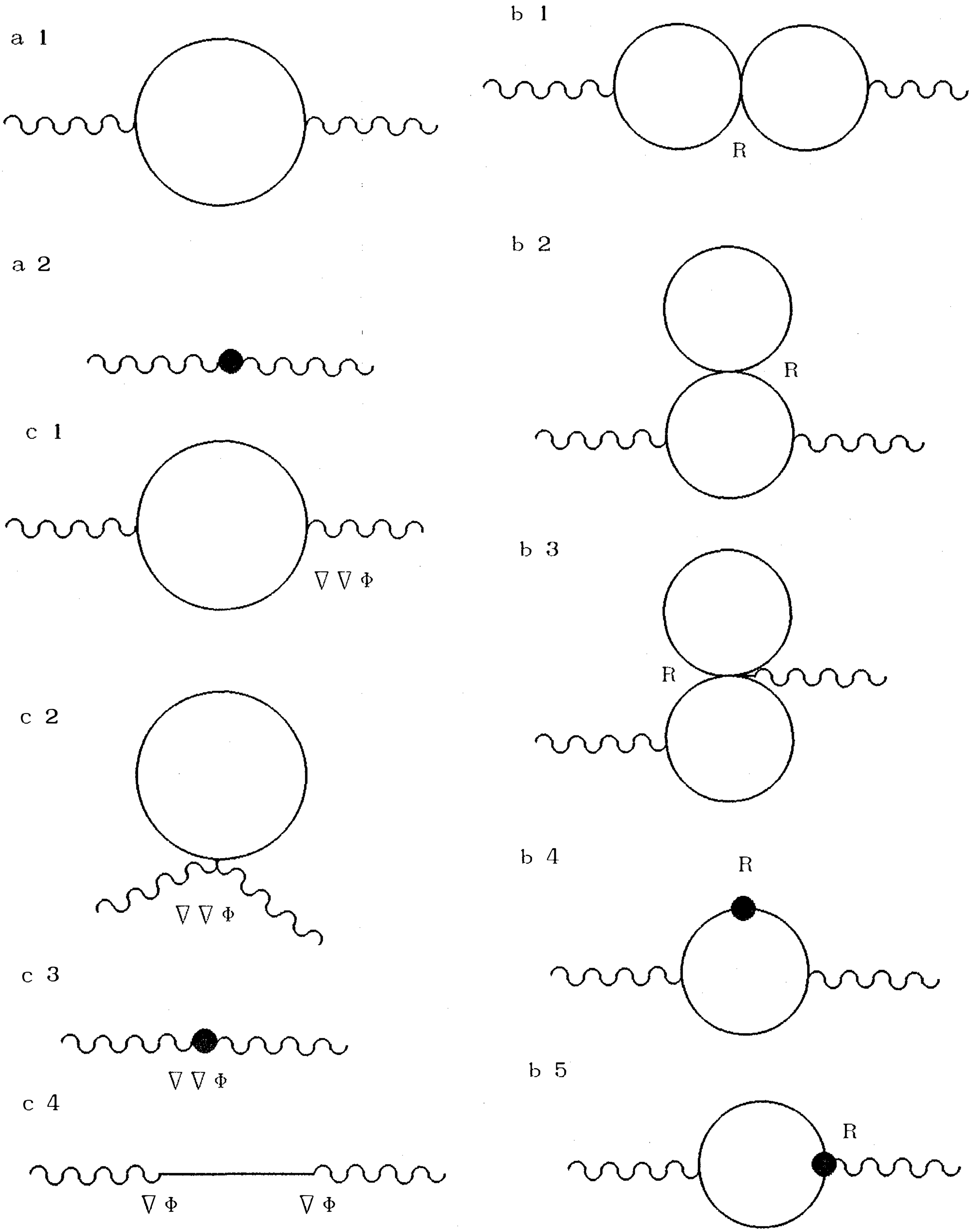


Fig. 4 (Continued)

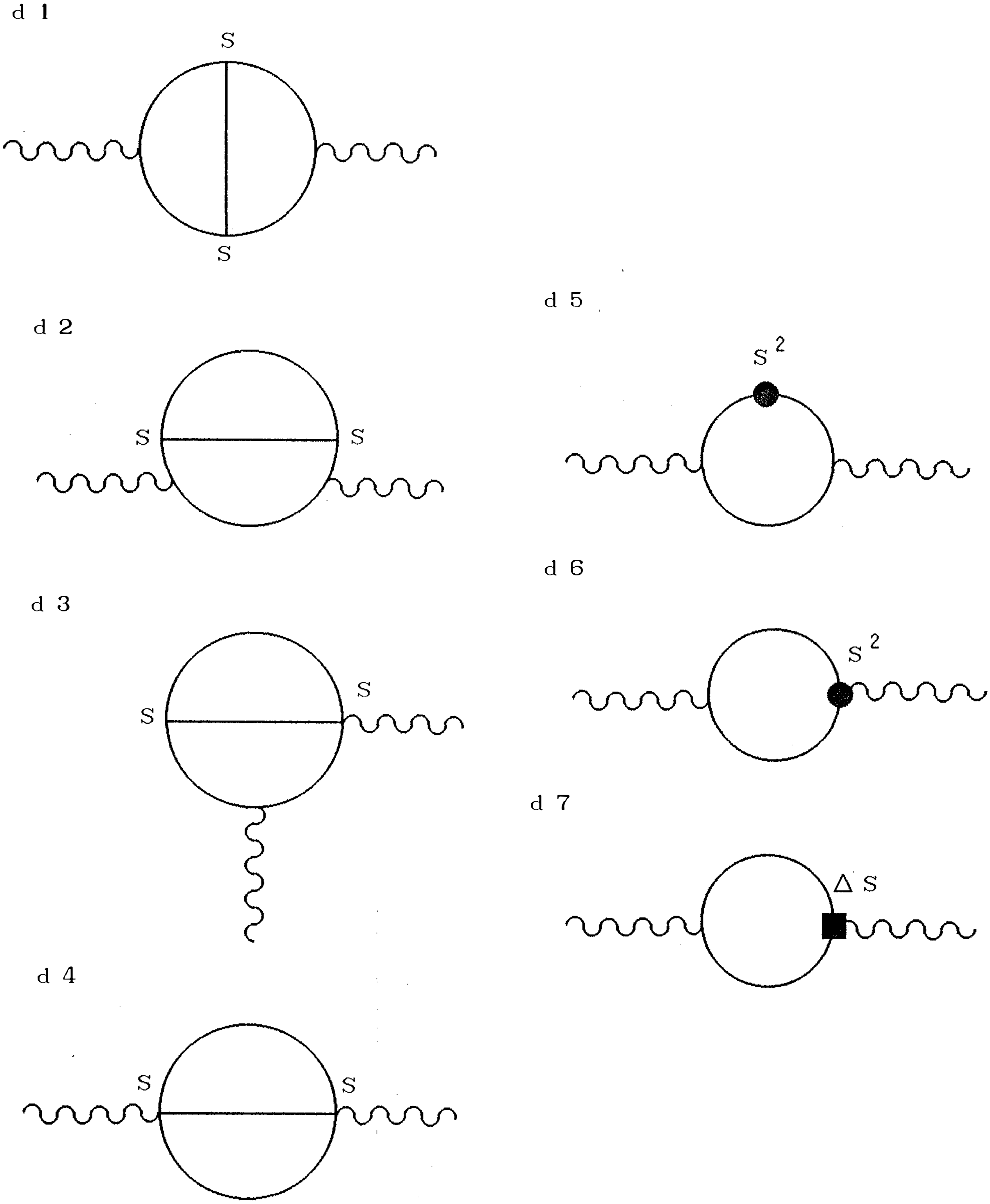
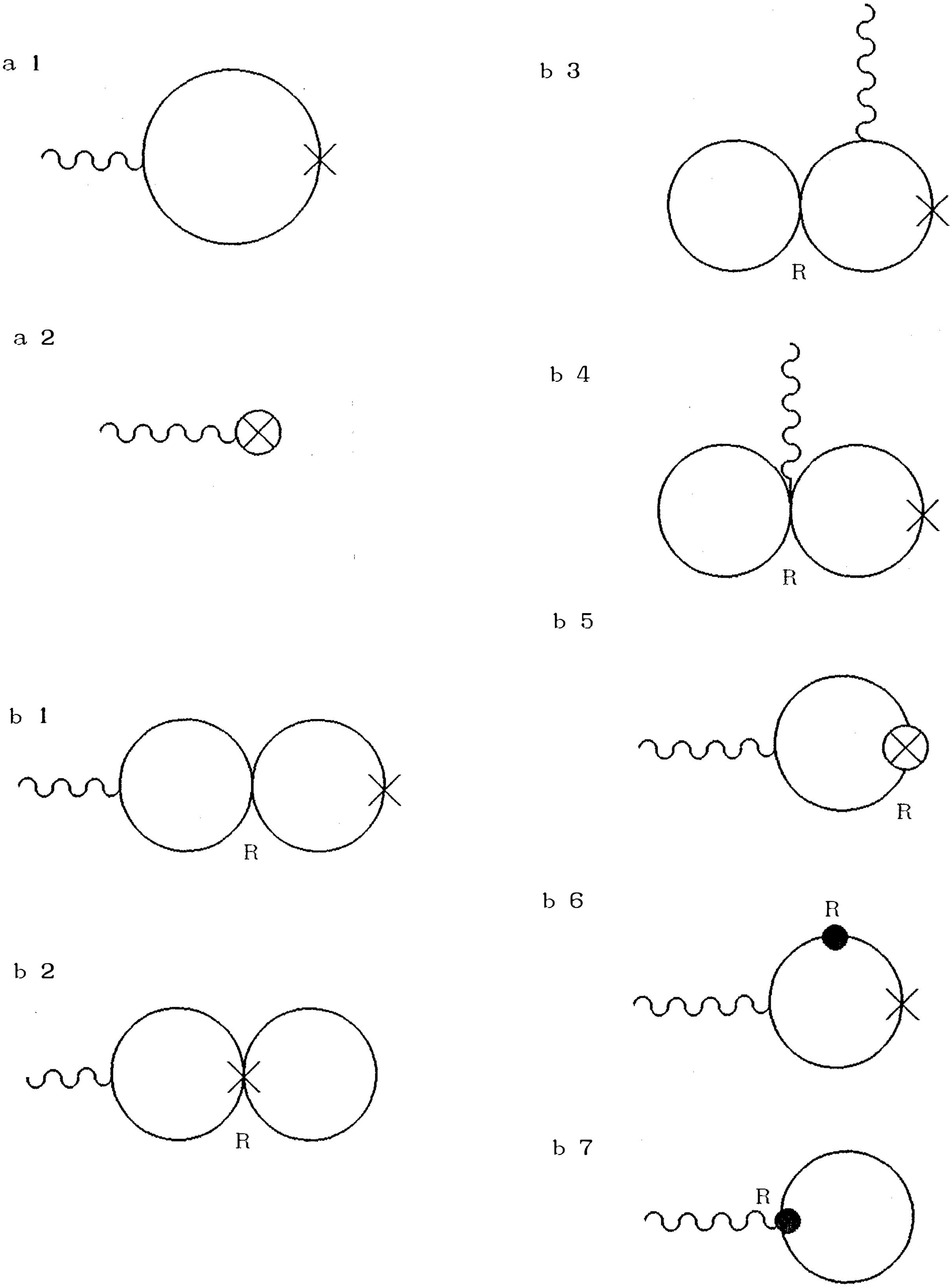


Fig. 5



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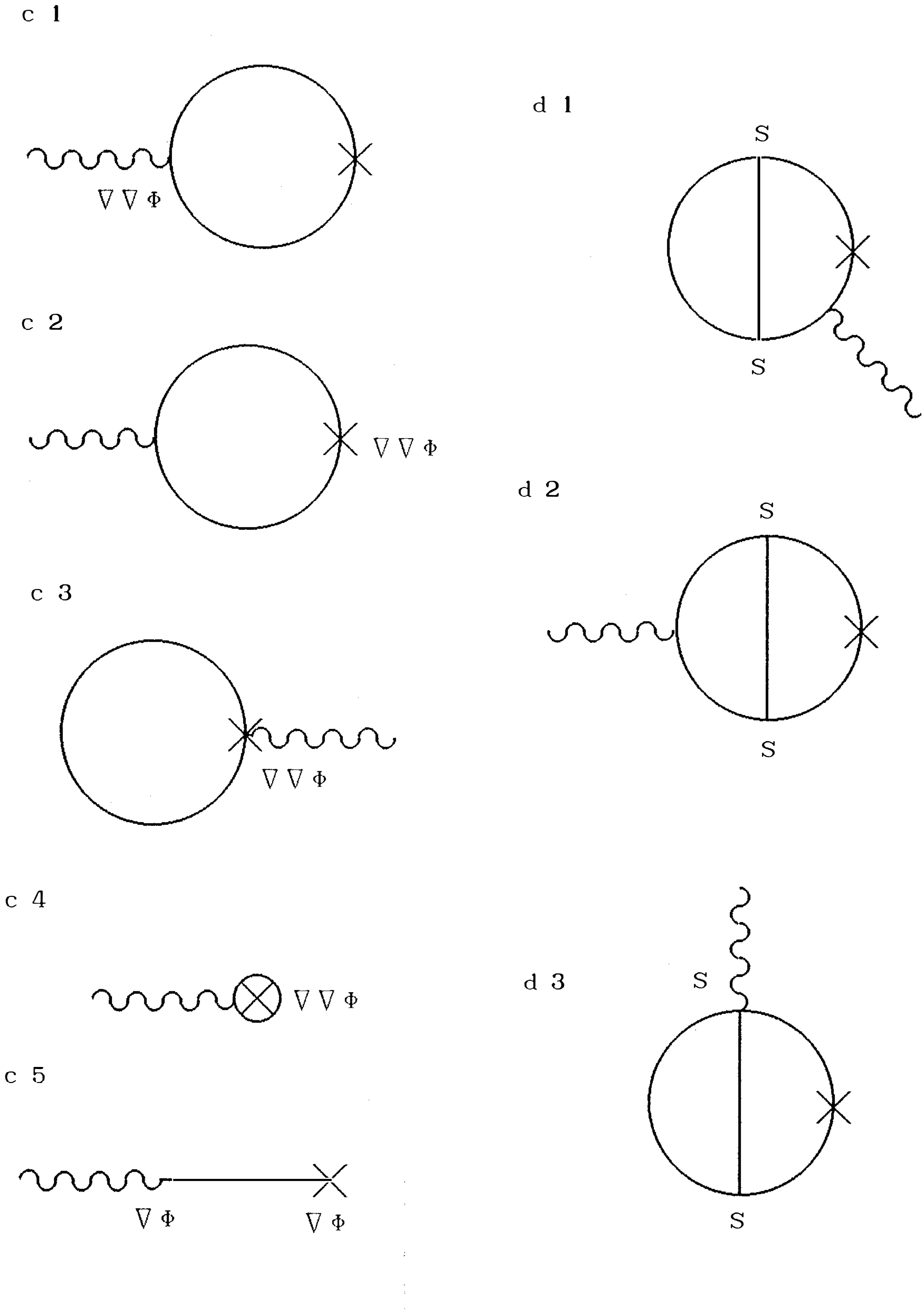

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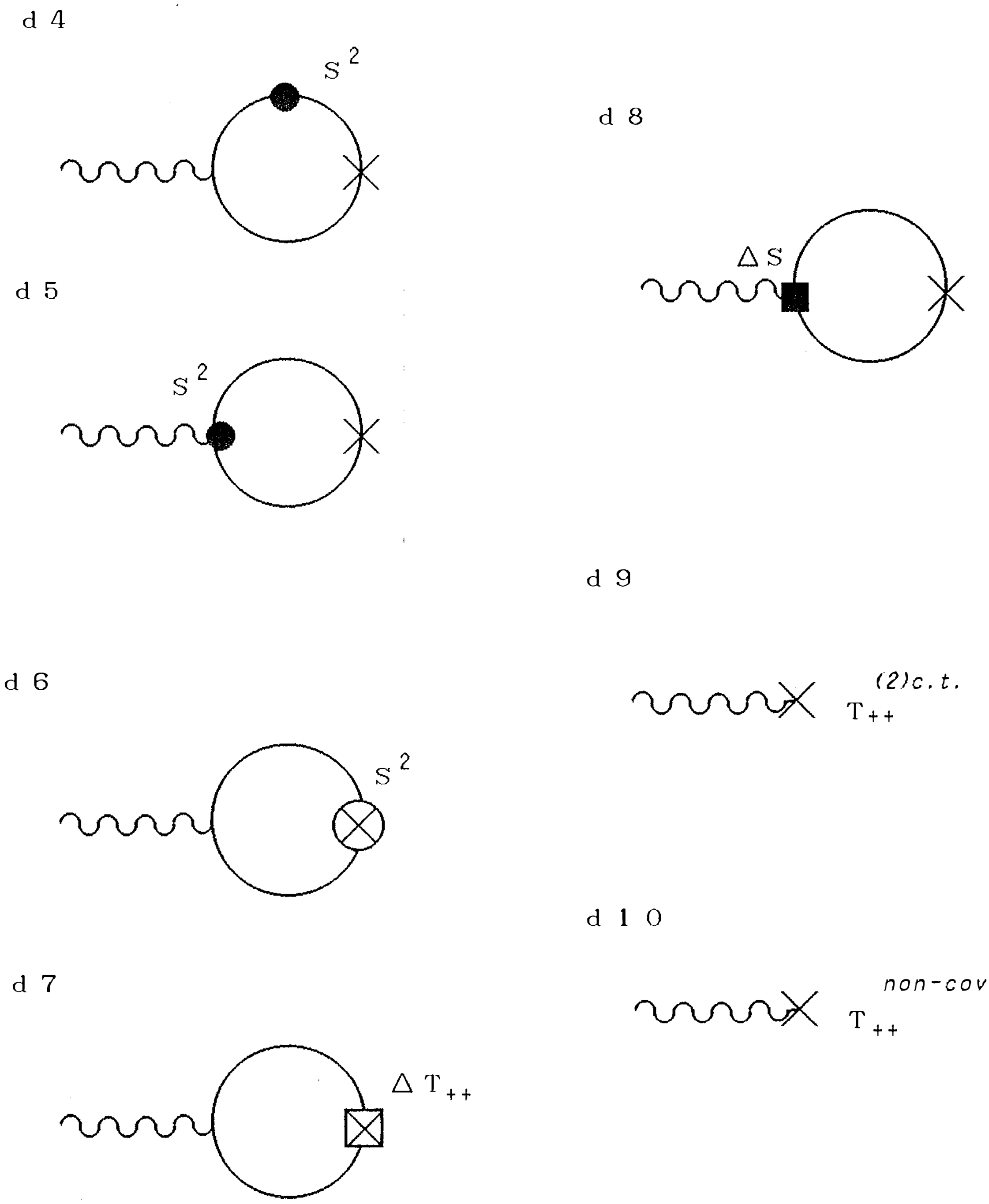
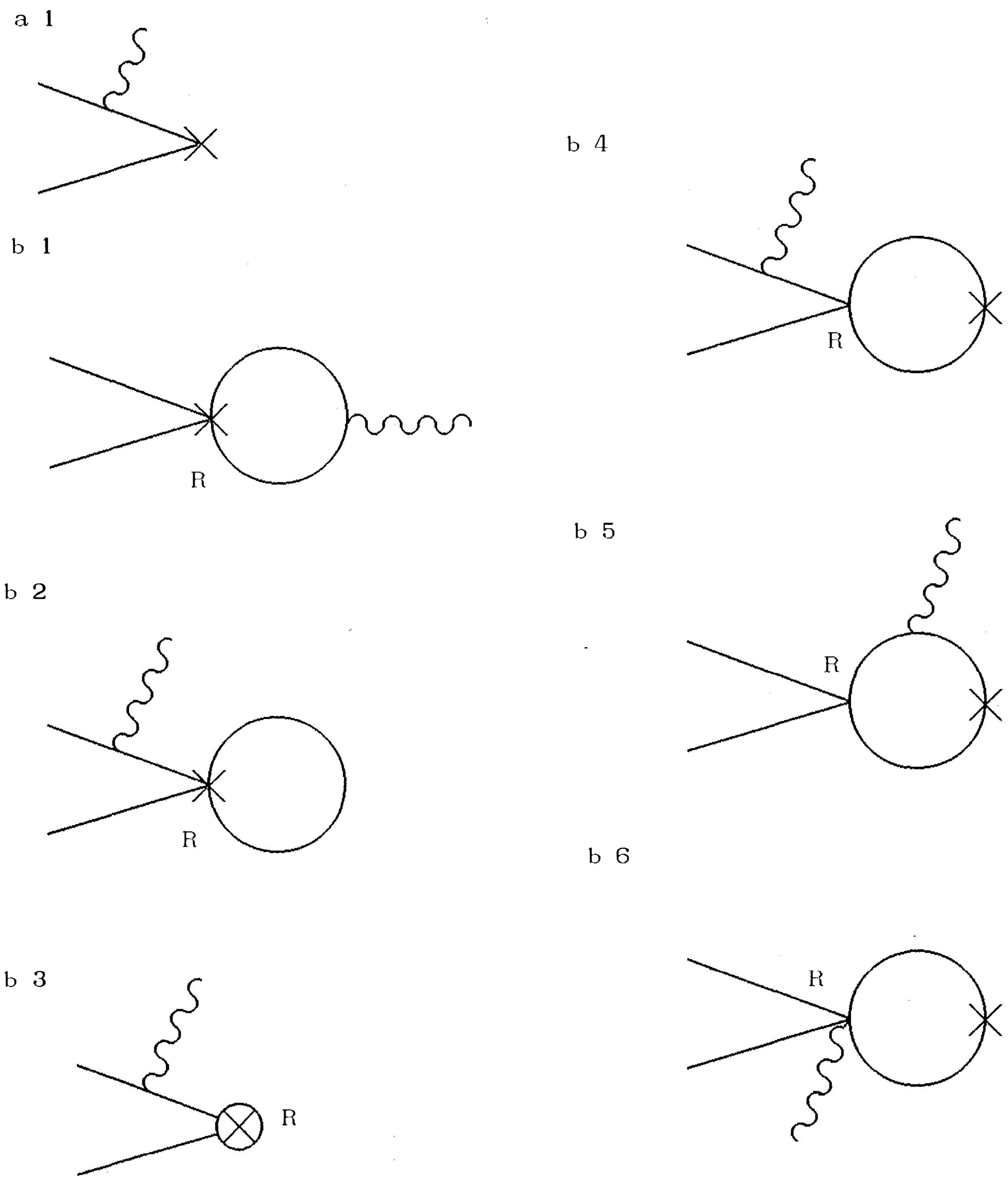


Fig. 6



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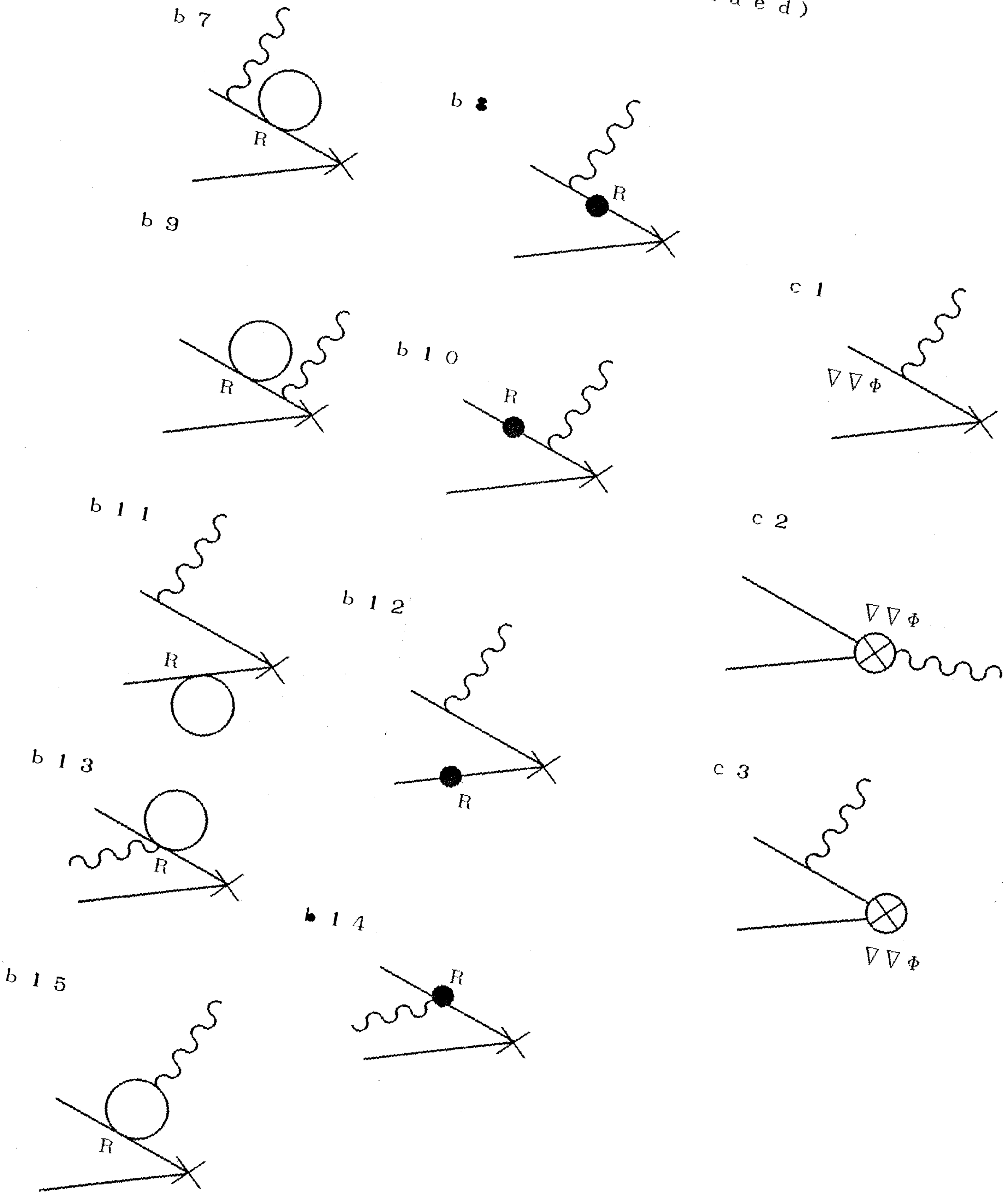

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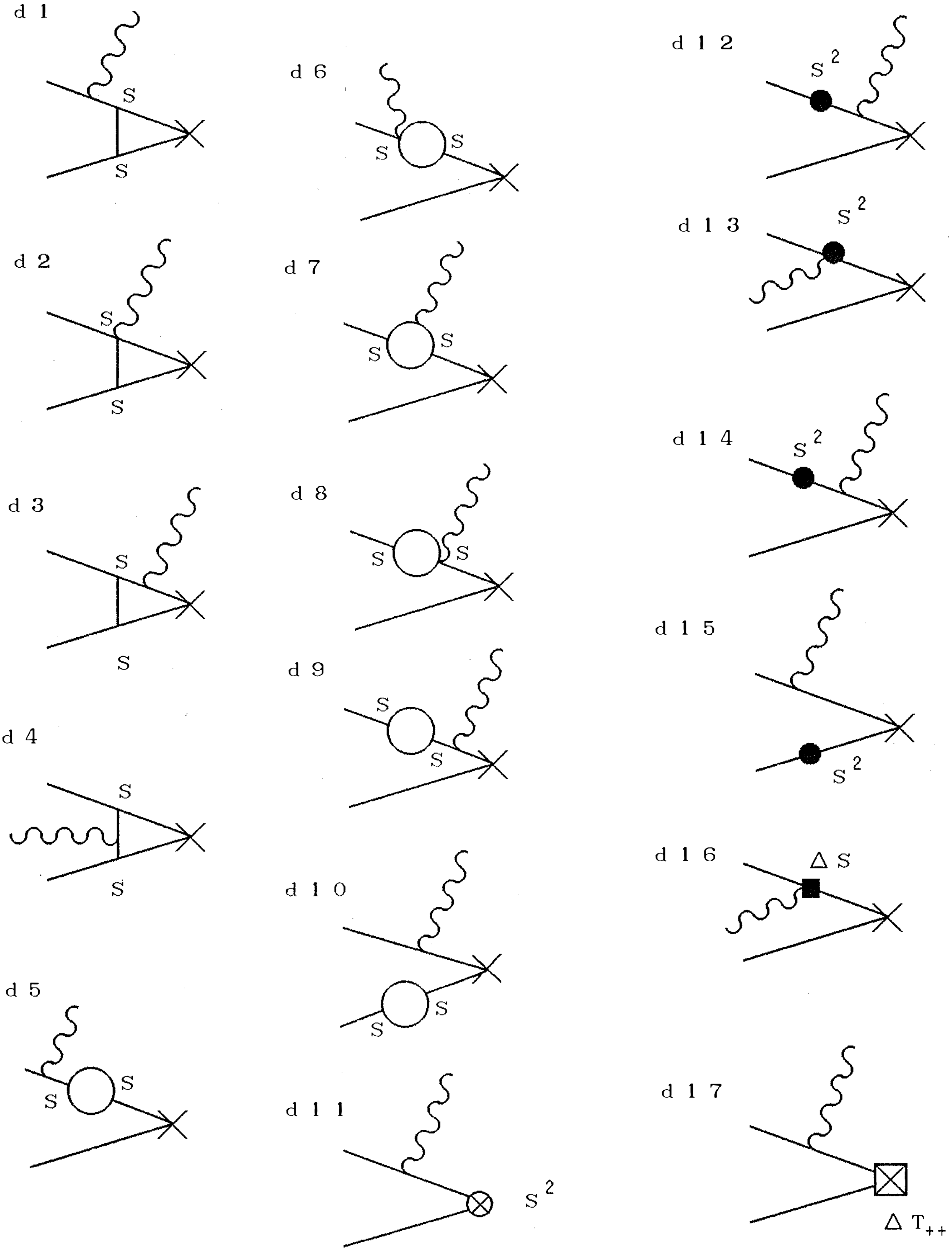

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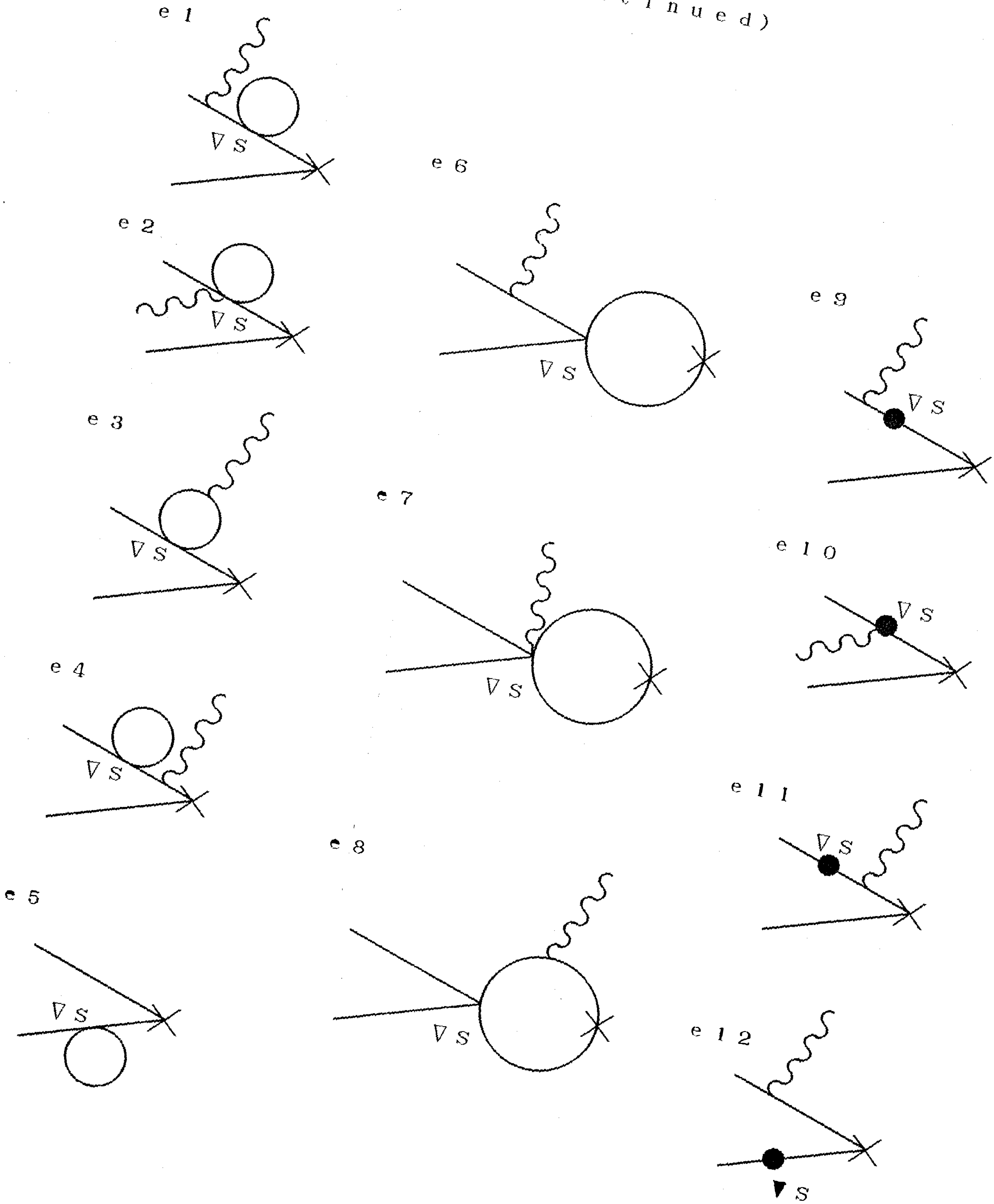

Fig. 6 (continued)