



Title	EXISTENCE OF INFINITELY MANY SOLUTIONS TO SEMILINEAR ELLIPTIC NEUMANN PROBLEMS WITH CONCAVE-CONVEX TYPE NONLINEARITY
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EXISTENCE OF INFINITELY MANY SOLUTIONS TO SEMILINEAR ELLIPTIC NEUMANN PROBLEMS WITH CONCAVE-CONVEX TYPE NONLINEARITY

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Abstract

In this paper, we consider the semilinear elliptic problem $-\Delta u = a(x)|u|^{p-2}u + \lambda b(x)|u|^{q-2}u$ in a bounded domain Ω with Neumann boundary condition. We show the existence infinitely many solutions by applying critical point theory with a suitable decomposition of the Sobolev space $W^{1,2}(\Omega)$. Also we prove the C^α regularity of the solutions.

1. Introduction

Let $\Omega \subset \mathbb{R}^n$, $n \geq 2$ be an open bounded domain with smooth boundary. We consider the following semilinear elliptic problem

$$(1.1) \quad \begin{aligned} -\Delta u &= \lambda a(x)|u|^{p-2}u + \mu b(x)|u|^{q-2}u \text{ in } \Omega, \\ \frac{\partial u}{\partial \eta} &= 0 \text{ on } \partial\Omega, \end{aligned}$$

where $1 < p < 2 < q < 2^* = 2n/(n-2)$, λ, μ are positive real parameters and $a, b : \Omega \rightarrow \mathbb{R}$ are functions satisfying the following hypotheses:

- (H_1) $a \in L^\infty(\Omega)$ and $\int_\Omega a(x) dx \neq 0$;
- (H'_1) $a \in L^\infty(\Omega)$ and $\alpha := \inf_{x \in \Omega} a(x) > 0$;
- (H_2) $b \in L^\infty(\Omega)$ and $\beta := \inf_{x \in \Omega} b(x) > 0$;
- (H'_2) $b \in L^\infty(\Omega)$ and $\int_\Omega b(x) dx \neq 0$.

Since the work of Ambrosetti et al. [3], semilinear elliptic problems with concave-convex nonlinearities have been investigated widely. We refer to [1, 2, 9, 12, 10] for more concave-convex problems with Dirichlet boundary conditions. In [3], the authors studied the existence of solutions of the following problem

$$(1.2) \quad \begin{aligned} -\Delta u &= |u|^{p-2}u + \lambda |u|^{q-2}u \text{ in } \Omega, \\ u &= 0 \text{ on } \partial\Omega, \end{aligned}$$

and proved that there exists $\lambda^* > 0$ such that for all $\lambda \in (0, \lambda^*)$ there exist sequences of solutions $\{u_n\}, \{v_n\}$ such that $I(u_n) < 0$ and $I(v_n) > 0$. The authors also studied the existence of positive solutions and proved that there exists $\Lambda > 0$ such that the problem (1.2) has at least two positive solutions for $\lambda < \Lambda$, at least one positive solution for $\lambda = \Lambda$ and no positive solution for $\lambda > \Lambda$. In [8], De Figueiredo et al. extended these previous results to a problem

with variable coefficients whose prototype is

$$(1.3) \quad \begin{aligned} -\Delta u &= a(x)|u|^{p-2}u + \lambda b(x)|u|^{q-2}u \text{ in } \Omega, \\ u &= 0 \text{ on } \partial\Omega. \end{aligned}$$

The function $b(x)$ is assumed to be non-negative and $a(x)$ may change sign. Recently in [15], Quoirin and Umezū considered a problem similar to (1.1) and studied the existence of positive solutions. However the authors did not discuss the existence of infinitely many solutions. We refer to [5, 11, 16] where the existence of infinitely many solutions are studied for some major concave-convex problems with nonlinear boundary conditions. In these papers, the left hand side of the problems involve $-\Delta u + u$, which corresponds to the term $(1/2) \int_{\Omega} |\nabla u|^2 + u^2$ in the energy functional. The map $u \mapsto (\int_{\Omega} |\nabla u|^2 + u^2)^{1/2}$, defines a norm in the space $W^{1,2}(\Omega)$. In this paper, we consider (1.1), which does not involve the extra term u . This makes the problem more challenging and the methods in [5, 11, 16] for studying the existence of solutions are not applicable to (1.1). We use some ideas from [10] and consider a suitable decomposition of the space $W^{1,2}(\Omega)$. To the best of our knowledge the existence of infinitely many solutions of (1.1) is not yet addressed. In this paper we address this aspect of (1.1) and prove the following:

Theorem 1.1. *Assume that $(H_1), (H_2)$ hold and $1 < p < 2 < q < 2^*$. Then there exists $\Lambda > 0$ such that for all $\lambda \in (0, \Lambda)$ and $\mu > 0$ there exists an unbounded sequence of solutions (u_n) of (1.1) such that $I(u_n) \rightarrow \infty$ as $n \rightarrow \infty$.*

Theorem 1.2. *Assume that $(H'_1), (H'_2)$ hold and $1 < p < 2 < q < 2^*$. Then for all $\lambda > 0$ and $\mu > 0$ there exists a sequence of solutions (v_n) of (1.1) such that $I(v_n) < 0$ and $I(v_n) \rightarrow 0$ as $n \rightarrow \infty$.*

The paper is organized as follows: In Section 2, we prove some preliminary results required for the proof of the main theorems. In Section 3, we give the proofs of the Theorems 1 and 2. We also prove some regularity results for the solution. More precisely, we prove that the $H^1(\Omega)$ solutions of (1.1) are Hölder continuous. This is discussed in Section 4.

2. Preliminaries

Let $E = W^{1,2}(\Omega)$ be the usual Sobolev space with the norm given by

$$\|u\|_{1,2} = \left(\int_{\Omega} |\nabla u|^2 + u^2 \right)^{\frac{1}{2}}.$$

DEFINITION 2.1. We say a function $u \in E$ a weak solution of (1.1) if for every $\varphi \in E$ we have

$$(2.4) \quad \int_{\Omega} \nabla u \nabla \varphi \, dx - \lambda \int_{\Omega} a(x)|u|^{p-2}u\varphi \, dx - \mu \int_{\Omega} b(x)|u|^{q-2}u\varphi \, dx = 0.$$

For $u \in E$ we define the energy functional for problem (1.1) by

$$I(u) = \frac{1}{2} \int_{\Omega} |\nabla u|^2 \, dx - \frac{\lambda}{p} \int_{\Omega} a(x)|u|^p \, dx - \frac{\mu}{q} \int_{\Omega} b(x)|u|^q \, dx.$$

Since $1 < p < 2 < q < 2^*$ we have that $I \in C^1(E, \mathbb{R})$ and the critical points of I are the weak

solutions of (1.1). Hence the existence of solutions of (1.1) is equivalent to the existence of critical points of I .

We consider the following decomposition of the space E (see [10]) which is crucial in the proof of the main results.

Lemma 2.1. (i) *Let $X = \{u \in W^{1,2}(\Omega) : \int_{\partial\Omega} u \, d\sigma_x = 0\}$. Then the Poincaré's inequality holds in X , that is, there exists a constant $C = C(n, \Omega)$ such that*

$$\|u\|_{L^2(\Omega)} \leq C \|\nabla u\|_{L^2(\Omega)}, \text{ for any } u \in X.$$

(ii) *For any $u \in E$ there exists a unique $t_u \in \mathbb{R}$ and $u^\perp \in X$ such that $u = t_u + u^\perp$, that is, $E = \langle 1 \rangle \oplus X$ and for any $u, v \in E(\Omega)$*

$$\langle u, v \rangle = t_u \cdot t_v + \int_{\Omega} \nabla u \cdot \nabla v$$

defines an inner product in E . Moreover, the corresponding norm is equivalent to the usual norm in E .

Next we prove a technical lemma which we need in the proof of Theorem 1.2.

Lemma 2.2. *Assume that (H'_1) holds and $p > 1$. Then there exists $\gamma > 0$ such that for any $u = t_u + u^\perp \in \langle 1 \rangle \oplus X$ with $\|\nabla u\|_{L^2(\Omega)} \leq \gamma |t_u|$, we have*

$$\int_{\Omega} a(x) |t_u + u^\perp|^p \, dx \geq \frac{|t_u|^p}{2} \int_{\Omega} a(x) \, dx.$$

Proof. Arguing by contradiction, we suppose that there exists a sequence $(u_n) \subset W^{1,2}(\Omega)$ such that $\|\nabla u_n\|_{L^2(\Omega)} \leq \frac{|t_{u_n}|}{n}$ and

$$(2.5) \quad \int_{\Omega} a(x) |t_{u_n} + u_n^\perp|^q \, d\sigma_x < \frac{|t_{u_n}|^q}{2} \int_{\Omega} a(x) \, dx.$$

Now set $v_n := u_n^\perp / t_{u_n}$, then $v_n \in X$ and by our assumption $\|\nabla v_n\|_{L^2(\Omega)} \rightarrow 0$ as $n \rightarrow \infty$. Since, the Poincaré's inequality holds in X , $v_n \rightarrow 0$ in X . Also X is continuously embedded in $L^q(\Omega)$, hence $v_n \rightarrow 0$ in $L^q(\Omega)$. Consequently, we have $v_n \rightarrow 0$ a.e. on Ω . Now dividing both sides of (2.5) by $|t_{u_n}|$ and using Lebesgue theorem we obtain

$$\int_{\Omega} a(x) \, dx < 0,$$

which contradicts that $a(x) > 0$ on Ω . □

3. Proofs of the main results

The following abstract results on existence of critical points will be used.

Theorem 3.1. *Let E be an infinite dimensional Banach space and, $I \in C^1(E, \mathbb{R})$ be even and $I(0) = 0$. If $E = V \oplus X$, where V finite dimensional, and I satisfies*

(PS) any sequence $\{u_n\} \subset E$ for which $I(u_n) \rightarrow c$ and $I'(u_n) \rightarrow 0$ possesses a convergent subsequence,

(I₁) there are constants $r, \alpha > 0$ such that $I|_{\partial B_r \cap X} \geq \alpha$, and

(I₂) for each finite dimensional subspace $\bar{E} \subset E$, there is an $R = R(\bar{E})$ such that $I \leq 0$ on $E \setminus B_{R(\bar{E})}$,

then I possesses an unbounded sequence of critical values.

Theorem 3.2. Let X be a Banach space and $I \in C^1(X, \mathbb{R})$ satisfies

- (I₁) there exists an admissible representation V of G , a compact lie group, such that $X = \oplus_{j \in A} X(j)$ with $A = \mathbb{N}$ or $A = \mathbb{Z}$ and $X(j) \cong V$ for every $j \in A$. The space X is then a Banach space with isometric linear G -action. The functional $I : X \rightarrow \mathbb{R}$ is invariant under this action: $I(gu) = I(u)$ for $g \in G$ and $u \in X$,
- (I₂) for every $k \geq k_0$ there exists $R_k > 0$ such that $I(u) \geq 0$ for every $u \in X_k := \oplus_{j \geq k} X(j)$ with $\|u\| = R_k$,
- (I₃) $b_k := \inf_{u \in B_k} I(u) \rightarrow 0$ as $k \rightarrow \infty$ where $B_k := \{u \in X_k : \|u\| \leq R_k\}$,
- (I₄) for every $k \geq 1$ there exists $r_k \in (0, R_k)$ and $d_k < 0$ such that $I(u) \leq d_k$ for every $u \in X^k := \oplus_{j \leq k} X(j)$ with $\|u\| = r_k$, and
- (I₅) every sequence $u_n \in X_{-n}^n := \oplus_{j=-n}^n X(j)$ with $I(u_n) < 0$ bounded and $(I|_{X_{-n}^n})'(u_n) \rightarrow 0$ as $n \rightarrow \infty$ has a subsequence which converges to a critical point of I .

Then for each $k \geq k_0$, I has a critical value $c_k \in [b_k, d_k]$, hence $c_k \rightarrow 0$ as $k \rightarrow \infty$.

For the proof of Theorem 3.1 we refer to [14, Theorem 9.12] and for the proof of Theorem 3.2 we refer to [6, Theorem 2].

3.1. Proof of Theorem 1.1. Let $E = \langle 1 \rangle \oplus X$ where X is given by Lemma 2.1. We show that I satisfies hypotheses of Theorem 3.1. Clearly $I(0) = 0$ and I is even.

Let $u \in E$ then there exists t_u and $u^\perp \in X$ such that $u = t_u + u^\perp$. By using the embeddings $L^p \hookrightarrow X, L^q \hookrightarrow X$ and since Poincares inequality holds in X we have

$$\begin{aligned} I(u^\perp) &= \frac{1}{2} \int_{\Omega} |\nabla u^\perp|^2 \, dx - \frac{\lambda}{p} \int_{\Omega} a(x) |u^\perp|^p \, dx - \frac{\mu}{q} \int_{\Omega} b(x) |u^\perp|^q \, dx \\ &\geq \frac{1}{2} \|u^\perp\|^2 - \lambda c_1 \|u^\perp\|^p - \mu c_2 \|u^\perp\|^q \\ &= \frac{1}{2} \rho^2 - \lambda c_1 \rho^p - \mu c_2 \rho^q \\ &= \rho^2 \left(\frac{1}{2} - \lambda c_1 \rho^{p-2} - \mu c_2 \rho^{q-2} \right). \end{aligned}$$

Let $\rho = (1/(8\mu c_2))^{1/(q-2)}$. There exists $\Lambda > 0$ such that for $\lambda \in (0, \Lambda)$ we have $\lambda c_1 \rho^{p-2} \leq 1/8$ and

$$I(u^\perp) \geq \frac{\rho^2}{4} := \alpha,$$

for all $u \in X$ with $\|u\| = \rho$. This shows that I satisfies (I₁).

Let $\bar{E} \subset E$ be any finite dimensional subspace of E and let $u \in \bar{E} \setminus \{0\}$. Let $t > 0$. Since, $\bar{E}$ is finite dimensional and all norms in finite dimensional spaces are equivalent, we have

$$\begin{aligned} I(tu) &= \frac{t^2}{2} \int_{\Omega} |\nabla u|^2 - \frac{\lambda t^p}{p} \int_{\Omega} a(x) |u|^p - \frac{\mu t^q}{q} \int_{\Omega} b(x) |u|^q \\ &\leq \frac{t^2}{2} \|u\|^2 - \lambda c_1 t^p \|u\|^p - \mu c_2 t^q \|u\|^q. \end{aligned}$$

Since $q > 2$ and $b > 0$, we get $I(tu) \rightarrow -\infty$ as $t \rightarrow \infty$. Hence (I_2) holds.

Next, we show that I satisfies the Palais-smale condition (PS). Let $(u_n) \subset E$ be a sequence and $I(u_n) \rightarrow c$, $I'(u_n) \rightarrow 0$ as $n \rightarrow \infty$. Since the operators $A, B : E \rightarrow \mathbb{R}$ defined by

$$A(u) = \int_{\Omega} a(x)|u|^p dx \text{ and } B(u) = \int_{\Omega} b(x)|u|^q dx$$

are weakly continuous and their derivatives A', B' are compact it is sufficient to show that (u_n) is bounded in E . Suppose by contradiction, there exists a subsequence denoted by (u_n) with $\|u_n\| \rightarrow \infty$ as $n \rightarrow \infty$. Set $v_n := u_n/\|u_n\|$. Then $\|v_n\| = 1$ for all n . Hence up to subsequence $v_n \rightharpoonup v$ weakly in E and $v_n \rightarrow v$ in $L^p(\Omega)$ and $L^q(\Omega)$. By simple calculations we get

$$I(u_n) - \frac{1}{2}I'(u_n)u_n = \lambda\left(\frac{1}{2} - \frac{1}{p}\right) \int_{\Omega} a(x)|u_n|^p + \mu\left(\frac{1}{2} - \frac{1}{q}\right) \int_{\Omega} b(x)|u_n|^q.$$

Thus we can write

$$\int_{\Omega} b(x)|u|^q \leq c + c_1\|u_n\|^p.$$

Also since $I(u_n) \rightarrow c$ we have

$$\frac{1}{2} \int_{\Omega} |\nabla u_n|^2 = \frac{\lambda}{p} \int_{\Omega} a(x)|u_n|^p + \frac{\mu}{q} \int_{\Omega} b(x)|u_n|^q + c + o_n(1).$$

Thus $\int_{\Omega} |\nabla v_n|^2 \leq o_n(1) + c_3\|u_n\|^{p-2} \rightarrow 0$ as $n \rightarrow \infty$. If $v_n = t_{v_n} + v_n^\perp$, then $v_n^\perp \rightarrow 0$ in X . Also up to a subsequence $t_{v_n} \rightarrow t_v$ in $\mathbb{R}$ and

$$|t_v| = \|t_v\| = \|v\| = \lim_{n \rightarrow \infty} \|v_n\| = 1.$$

Hence $v_n \rightarrow 1$ strongly in E . Now from $\|u_n\|^{1-q}I'(u_n)t_{v_n} = o_n(1)$ we get

$$\left| \int_{\Omega} b(x)|v_n|^{q-2}v_n t_{v_n} \right| \leq \frac{\lambda}{\mu\|u_n\|^{q-p}} \int_{\Omega} a(x)|v_n|^{p-1}|t_{v_n}| + o_n(1).$$

Since $q > p$ and $v_n \rightarrow 1$ strongly in E the right hand side tends to 0. Hence by Lebesgue theorem we obtain

$$\int_{\Omega} b(x) = 0,$$

which is a contradiction. Thus (u_n) is bounded in E . That is I satisfies (PS) condition. Hence by Theorem 3.1 problem (1.1) has an unbounded sequence of solutions such that $I(u_n) \rightarrow \infty$ as $n \rightarrow \infty$.

3.2. Proof of Theorem 1.2. We show that I satisfies the hypotheses of Theorem 3.2. Consider the Neumann eigenvalue problem

$$\begin{aligned} -\Delta u &= \lambda u \text{ in } \Omega, \\ \frac{\partial u}{\partial \eta} &= 0 \text{ on } \partial\Omega. \end{aligned}$$

Then applying the theory of compact self-adjoint operators, one gets that there exists a sequence of eigenvalues $\lambda_j \rightarrow \infty$ and the corresponding eigenfunctions $\{e_j\}$ form an orthonor-

mal basis of X (see [4, 7]). Moreover, the eigenvalues has the following characterization

$$(3.6) \quad \lambda_j = \max_{u \in \text{span}\{e_1, \dots, e_j\}} \frac{\int_{\Omega} |\nabla u|^2 dx}{\int_{\Omega} u^2 dx}.$$

Let $E = \langle 1 \rangle \oplus \oplus_{j \geq 1} X(j)$, where $X(j) = \text{span}\{e_j\}$. Since I is even, hypothesis (I_1) is satisfied taking $G = \mathbb{Z}/2$ and $V = \mathbb{R}$. For (I_2) we set

$$\delta_k := \sup_{X_k \setminus \{0\}} \frac{\|u\|_{L^p(\Omega)}}{\|u\|}.$$

Then by Rellich's embedding theorem, $\delta_k \rightarrow 0$ as $k \rightarrow \infty$. Hence for $u \in X_k$ using the embedding $L^p \hookrightarrow X$, we obtain

$$I(u) \geq \frac{1}{2} \int_{\Omega} |\nabla u|^2 - \frac{\lambda c_1}{p} (\delta_k)^p \|u\|^p - \frac{\mu c_2}{q} \|u\|^q = \frac{1}{2} \|u\|^2 - \frac{\lambda c_1}{p} (\delta_k)^p \|u\|^p - \frac{\mu c_2}{q} \|u\|^q.$$

Since $q > 2$ choosing $R > 0$ small enough we have $\mu c_2 \|u\|^q / q \leq \|u\|^2 / 4$ if $\|u\| \leq R$. Thus

$$I(u) \geq \frac{1}{4} \|u\|^2 - \frac{\lambda c_1}{p} (\delta_k)^p \|u\|^p.$$

Let $R_k = (4\lambda c_1 (\delta_k)^p / p)^{1/(2-p)}$. Then $R_k \rightarrow 0$. Hence one can find k_0 such that $R_k \leq R$ for $k \geq k_0$. Now for $u \in X_k$ with $\|u\| = R_k$ we obtain

$$I(u) \geq \frac{1}{4} \|u\|^2 - \frac{\lambda c_1}{p} (\delta_k)^p \|u\|^p = 0.$$

That is (I_2) holds. Since B_k is weakly compact and I is weakly lower semicontinuous (I_3) follows from $R_k \rightarrow 0$. Next we show (I_4) in two cases.

Case 1: $\|\nabla u^\perp\| \leq \gamma |t_u|$.

Let $\|u\| = r$. Then $\|u\|^2 = t_u^2 + \|\nabla u^\perp\|_2^2 \implies t_u \geq \frac{r}{\sqrt{1+\gamma^2}}$. Now using Lemma 2.2 we get

$$I(u) \leq \frac{1}{2} \|u\|^2 + c_3 \|u\|^q - \frac{|t_u|^p}{2} \int_{\Omega} a(x) dx \leq r^2 \left(\frac{1}{2} + c_3 r^{q-2} - \frac{1}{2} c_4 (1 + \gamma^2)^{-p/2} r^{p-2} \right).$$

Since $1 < p < 2$, $r^{p-2} \rightarrow \infty$ as $r \rightarrow 0$. Thus choosing r_k small enough we get $(1/2) + c_3 r^{q-2} - (c_4/2)(1 + \gamma^2)^{-p/2} r^{p-2} \leq -1/2$. Hence

$$I(u) \leq -\frac{r_k^2}{2} := d_k < 0.$$

Case 2: $\|\nabla u^\perp\| > \gamma |t_u|$.

In this case for $\|u\| = r$ we have $r^2 = t_u^2 + \|\nabla u^\perp\|_2^2 \implies \|\nabla u^\perp\|_2 > \beta^{-1} r$, where $\beta = \sqrt{1 + \gamma^2}$. Now since X^k is finite dimensional, we have $\|u\|_{L^2(\Omega)} \leq c \|u\|_{L^p(\Omega)}$. Hence using (3.6) we obtain

$$\int_{\Omega} |u|^p \geq c_3 \left(\int_{\Omega} |u|^2 \right)^{\frac{p}{2}} \geq c_k \left(\int_{\Omega} |\nabla u^\perp|^2 \right)^{\frac{p}{2}} \geq c_k \beta^{-p} r^p,$$

for some $c_k > 0$ depending on k . Thus we have

$$I(u) \leq \frac{1}{2} \|u\|^2 + c_3 \|u\|^q - \frac{\alpha}{p} \int_{\Omega} |u|^p \leq \frac{r^2}{2} + c_3 r^q - c_k \beta^{-p} r^p.$$

Again similar to Case 1 we can find $r_k > 0$ small enough so that

$$I(u) \leq -\frac{r_k^2}{2} = d_k < 0.$$

Hence (I_4) holds. Finally the Palais-smale condition I_5 can be shown as in the proof of part (a).

4. C^α regularity of solutions

In this section we show that the solutions of (1.1) are in $C^\alpha(\Omega)$. We follow the method used in [11, Lemma 4.2] to prove our result. We prove the following.

Proposition 4.1. *Let $1 < p < 2 < q < 2^*$. Then any $H^1(\Omega)$ solution u of (1.1) is in $C^\alpha(\Omega)$ and satisfies*

$$\|u\|_{C^\alpha(\Omega)} \leq C \left(\|u\|_{L^2(\Omega)} + \|u\|_{L^\infty(\Omega)}^{p-1} + \|u\|_{L^\infty(\Omega)}^{q-1} \right).$$

Proof. Let $u \in H^1(\Omega)$ be any solution of (1.1). We first show that $u \in L^\infty(\Omega)$. We use a bootstrap argument. Let $\nu > 1$ be a parameter to be chosen later. Multiplying both sides of (1.1) by $|u|^{\nu-1}u$, and integrating we obtain

$$\nu \int_{\Omega} |\nabla u|^2 |u|^{\nu-1} = \lambda \int_{\Omega} a(x) |u|^{p+\nu-1} + \mu \int_{\Omega} b(x) |u|^{q+\nu-1}.$$

Since, $a, b \in L^\infty(\Omega)$ we have

$$(4.7) \quad \nu \int_{\Omega} |\nabla u|^2 |u|^{\nu-1} \leq c \left(\|u\|_{\nu+p-1}^{\nu+p-1} + \|u\|_{\nu+q-1}^{\nu+q-1} \right).$$

Using the identity

$$|\nabla(|u|^{\frac{\nu+1}{2}})|^2 = \frac{(\nu+1)^2}{4} |\nabla u|^2 |u|^{\nu-1},$$

and putting $w = |u|^{\frac{\nu+1}{2}}$ we obtain

$$(4.8) \quad \frac{4\nu}{(\nu+1)^2} \int_{\Omega} |\nabla w|^2 \leq c \left(\|u\|_{\nu+p-1}^{\nu+p-1} + \|u\|_{\nu+q-1}^{\nu+q-1} \right).$$

Without loss of generality we can assume

$$(4.9) \quad q - p = \frac{2}{n-2}.$$

Indeed, set q_1 such that $p + 2/(n-2) < q_1 < 2n/(n-2)$ and $q < q_1$. Let $p_1 = q_1 - 2/(n-2)$. Then $p < p_1 < (n-1)/(n-2)$. Then (4.7) and (4.8) are satisfied with p_1, q_1 .

The Sobolev embedding shows that

$$(4.10) \quad \|w\|_{\frac{2n}{n-2}}^2 \leq c \|w\|_{H^1(\Omega)}^2.$$

Combining (4.8) and (4.10) and using $w = |u|^{\frac{\nu+1}{2}}$ we get

$$\|u\|_{\frac{(\nu+1)n}{n-2}}^{\nu+1} \leq \frac{(\nu+1)^2}{4\nu} c \left(\|u\|_{\nu+p-1}^{\nu+p-1} + \|u\|_{\nu+q-1}^{\nu+q-1} \right) + \int_{\Omega} |u|^{\nu+1} \leq \nu c \left(\|u\|_{\nu+p-1}^{\nu+p-1} + \|u\|_{\nu+q-1}^{\nu+q-1} + \int_{\Omega} |u|^{\nu+1} \right).$$

Now using the Holder's inequality

$$\begin{aligned} \|u\|_{\nu+p-1}^{\nu+p-1} &\leq |\Omega|^{\frac{q-p}{\nu+q-1}} \|u\|_{\nu+q-1}^{\nu+p-1} \leq c \|u\|_{\nu+q-1}^{\nu+p-1}, \\ \int_{\Omega} |u|^{\nu+1} &\leq |\Omega|^{\frac{q-2}{\nu+q-1}} \|u\|_{\nu+q-1}^{\nu+1} \leq c \|u\|_{\nu+q-1}^{\nu+1}, \end{aligned}$$

for some constant $c > 0$. Thus we can write

$$(4.11) \quad \|u\|_{\frac{(\nu+1)n}{n-2}}^{\nu+1} \leq \nu c \left(\|u\|_{\nu+q-1}^{\nu+p-1} + \|u\|_{\nu+q-1}^{\nu+q-1} + \|u\|_{\nu+q-1}^{\nu+1} \right).$$

Now define the sequences a_k, b_k by

$$(4.12) \quad b_1 := 2r, \quad b_k := (b_{k-1} - p + 2)r, \quad r := \frac{n-1}{n-2},$$

$$(4.13) \quad a_k := (b_{k-1} - p + 2) \frac{n}{n-2} = \frac{n}{n-1} b_k.$$

Then

$$(4.14) \quad b_k = r^k + (2-p)r \frac{r^k - 1}{r-1}, \quad \forall k \in \mathbb{N}.$$

There exist constants $c, C > 0$ such that

$$(4.15) \quad cr^k \leq b_k \leq Cr^k.$$

Since $r > 1$, $a_k \rightarrow \infty$ and $b_k \rightarrow \infty$ as $k \rightarrow \infty$. Putting $\nu = b_k - p + 1$ in (4.11) we get

$$(4.16) \quad \|u\|_{a_{k+1}}^{b_k-p+2} \leq cb_k \left(\|u\|_{b_k+\frac{2}{n-2}}^{b_k} + \|u\|_{b_k+\frac{2}{n-2}}^{b_k+\frac{2}{n-2}} + \|u\|_{b_k+\frac{2}{n-2}}^{b_k-p+2} \right),$$

where we have used (4.9). Since b_k is increasing it holds that $b_k \geq 2(n-1)/(n-2)$. This implies

$$b_k + \frac{2}{n-2} \leq a_k.$$

Let δ_k be such that $1/\delta_k = 1 - ((b_k + 2/(n-2))/a_k)$. Then $1/\delta_k \rightarrow 1/n$ as $k \rightarrow \infty$. Using this and the Holder inequality implies

$$\|u\|_{b_k+\frac{2}{n-2}}^{b_k+\frac{2}{n-2}} \leq |\Omega|^{\frac{1}{\delta_k}} \|u\|_{a_k}^{b_k+\frac{2}{n-2}} \leq c \|u\|_{a_k}^{b_k+\frac{2}{n-2}}.$$

By a similar calculation we also have

$$\begin{aligned} \|u\|_{b_k+\frac{2}{n-2}}^{b_k} &\leq c \|u\|_{a_k}^{b_k}, \\ \|u\|_{b_k+\frac{2}{n-2}}^{b_k-p+2} &\leq c \|u\|_{a_k}^{b_k-p+2}. \end{aligned}$$

Substituting this in (4.16) we have

$$(4.17) \quad \|u\|_{a_{k+1}}^{b_k-p+2} \leq cb_k \left(\|u\|_{a_k}^{b_k+\frac{2}{n-2}} + \|u\|_{a_k}^{b_k} + \|u\|_{a_k}^{b_k-p+2} \right).$$

Define $\theta_k := \max\{\|u\|_{a_k}, 1\}$. Then (4.17) implies

$$(4.18) \quad \theta_{k+1}^{b_k-p+2} \leq cb_k \theta_k^{b_k+\frac{2}{n-2}}.$$

Set

$$\alpha_k = (cb_k)^{\frac{1}{b_k-p+2}}, \quad \beta_k = \frac{b_k + \frac{2}{n-2}}{b_k - p + 2}.$$

Then (4.18) can be written as

$$(4.19) \quad \theta_{k+1} \leq \alpha_k \theta_k^{\beta_k}.$$

Now repeated use of (4.19) produces

$$(4.20) \quad \theta_k \leq \alpha_{k-1} \alpha_{k-2}^{\beta_{k-1}} \alpha_{k-3}^{\beta_{k-1}\beta_{k-2}} \cdots \alpha_1^{\beta_{k-1}\cdots\beta_2} \theta_1^{\beta_{k-1}\cdots\beta_2}.$$

We wish to show that $\theta_k \leq c$ for all $k \in \mathbb{N}$. For this we first show that

$$(4.21) \quad 0 < \prod_{k=1}^{\infty} \beta_k < \infty.$$

By (4.14) there exists a constant $c > 0$ such that $b_k - p + 2 \geq cr^k$ for all $k \in \mathbb{N}$. Since

$$q - p = 2/(n - 2), \beta_k = 1 + \frac{q - 2}{b_k - p + 2},$$

(4.21) follows from the fact that

$$\sum_{k=1}^{\infty} \frac{q - 2}{b_k - p + 2} < \infty.$$

Put $\sigma = \prod_{k=1}^{\infty} \beta_k$. Then $1 < \sigma < \infty$. Since $\beta_k > 1$, it follows that

$$\beta_{k-1} \cdots \beta_i \leq \sigma \text{ for } i \leq k - 1.$$

Since $\alpha_k > 1$, we can write

$$\alpha_{k-1} \alpha_{k-2}^{\beta_{k-1}} \alpha_{k-3}^{\beta_{k-1}\beta_{k-2}} \cdots \alpha_1^{\beta_{k-1}\cdots\beta_2} \leq (\alpha_1 \cdots \alpha_{k-1})^{\sigma}.$$

Next, we show that

$$(4.22) \quad 0 < \prod_{k=1}^{\infty} \alpha_k < \infty.$$

Using (4.15) and the definition of α_k , we get

$$\log \left(\prod_{k=1}^m \alpha_k \right) = \sum_{k=1}^m \log \alpha_k \leq \sum_{k=1}^{\infty} \log \alpha_k < \infty.$$

Hence (4.22) holds. Thus there exists a constant $c > 0$ such that

$$\theta_k \leq c \theta_1^{\sigma}, \forall k \in \mathbb{N}.$$

Taking $k \rightarrow \infty$ we obtain

$$\|u\|_{L^{\infty}(\Omega)} \leq c \theta_1^{\sigma} < \infty.$$

Thus $u \in L^{\infty}(\Omega)$.

Now using [13, Proposition 3.6] we obtain $u \in C^{\alpha}(\Omega)$ and u satisfies

$$\|u\|_{C^{\alpha}(\Omega)} \leq C \left(\|u\|_{L^2} + \|u\|_{L^{\infty}(\Omega)}^{p-1} + \|u\|_{L^{\infty}(\Omega)}^{q-1} \right).$$

□

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