Kundu, D., Nuccio, F.A.E. and Ramdorai, S.
Osaka J. Math.
61 (2024), 121–146

STRUCTURE OF FINE SELMER GROUPS IN ABELIAN p-ADIC LIE EXTENSIONS

Debanjana KUNDU, Filippo Alberto Edoardo NUCCIO MORTARINO MAJNO DI CAPRIGLIO and Sujatha RAMDORAI

(Received September 21, 2022, revised December 26, 2022)

Abstract

This paper studies fine Selmer groups of elliptic curves in abelian p-adic Lie extensions. A class of elliptic curves are provided where both the Selmer group and the fine Selmer group are trivial in the cyclotomic $\mathbb{Z}_p$-extension. The fine Selmer groups of elliptic curves with complex multiplication are shown to be pseudonull over the trivializing extension in some new cases. Finally, a relationship between the structure of the fine Selmer group for some CM elliptic curves and the Generalized Greenberg's Conjecture is clarified.

1. Introduction

The fine Selmer group (see §2.3) is a module over an Iwasawa algebra that is of interest in the arithmetic of elliptic curves. It plays a key role in the formulation of the main conjecture in Iwasawa theory. Moreover, it enables us to propose analogues of important conjectures in classical Iwasawa theory to elliptic curves over certain p-adic Lie extensions of their field of definition. J. Coates and the third named author initiated a systematic study of the structure of fine Selmer groups and proposed two conjectures (see [11, Conjectures A and B]). While Conjecture A is a generalization of the Iwasawa $\mu = 0$ Conjecture to the context of elliptic curves, Conjecture B is in the spirit of generalizing R. Greenberg's pseudonullity conjecture to elliptic curves. Recently, there has been a renewed interest in studying pseudonull modules over Iwasawa algebras, [5, 36]. It is thus natural to investigate Conjecture B, and this article makes progress in this direction. These conjectures have been generalized to fine Selmer groups of ordinary Galois representations associated to modular forms in [28], and their $\mod p$-versions for supersingular elliptic curves have been studied by the second and third author in [48]. This article restricts attention to the fine Selmer groups of elliptic curves, with good reduction at a prime p, over abelian p-adic Lie extensions of the base field.

We now outline the main results in the paper. Given a number field F and an odd prime number p, let E/F be an elliptic curve, with good reduction at all the primes of F that lie above p. Consider an admissible p-adic Lie extension $\mathcal{L}$ of F (see §2.2 for the precise definition) with Galois group $\mathrm{Gal}(\mathcal{L}/F) =: G_{\mathcal{L}/F}$. The dual fine Selmer group of E at a prime p over $\mathcal{L}$ is a finitely generated module over the associated Iwasawa algebra (see §2.3). While Conjecture A asserts that the dual fine Selmer group over the cyclotomic $\mathbb{Z}_p$-extension F_{cyc}/F is finitely generated as a $\mathbb{Z}_p$-module, Conjecture B is an assertion on the

2020 Mathematics Subject Classification. Primary 11R23; Secondary 11R34.

structure of the dual fine Selmer group over admissible p-adic Lie extensions of dimension at least 2. This conjecture predicts that the dual fine Selmer group over any admissible p-adic Lie extension is pseudonull as a module over the associated Iwasawa algebra. In this article, both conjectures are established in previously unknown cases. Using a result of Greenberg, we prove a general theorem that gives sufficient conditions for the dual fine Selmer group of E over the cyclotomic $\mathbb{Z}_p$-extension $F_{\rm cyc}$ to be trivial. More precisely, we have the following theorem (we refer the reader to Corollary 3.5 for finer estimates):

Theorem 3.1. *Let E/F be the base-change of a rational elliptic curve $\mathsf{E}/\mathbb{Q}$. Suppose that it has rank* 0 *over F and that the Shafarevich–Tate group of E/F is finite. When E has CM by an order of an imaginary quadratic field K, assume further that the Galois closure of F, denoted by F^c, contains K. Then, the Selmer group* $\mathrm{Sel}\left(\mathsf{E}/F_{\rm cyc}\right)$ *is trivial for a set of prime numbers of density at least $\frac{1}{[F^c:\mathbb{Q}]}$. In particular,* Conjecture A *holds for E/F at all such primes.*

Denote by $F(\mathsf{E}_{p^\infty})$ the field obtained by adjoining the coordinates of all p-power torsion points. When p is a prime of good ordinary reduction, using a result of B. Perrin-Riou [52, Lemme 1.1(i) and Lemme 1.3] we prove that Conjecture B holds for special classes of admissible p-adic Lie extensions whenever the dual fine Selmer group over the cyclotomic extension is finite for a CM elliptic curve. We obtain the following result:

Theorem 4.6. *Let E/F be an elliptic curve defined over a number field F. Suppose that F contains the imaginary quadratic field K and that E has CM by $\mathcal{O}_K$. Assume further that $p \geq 3$ is a prime of good ordinary reduction that splits in K and that* $\mathrm{Gal}(F(\mathsf{E}_{p^\infty})/F) \simeq \mathbb{Z}_p^2$. *If the fine Selmer group over the cyclotomic $\mathbb{Z}_p$-extension $F_{\rm cyc}/F$ is finite, then* Conjecture B *holds for* $(\mathsf{E}, F(\mathsf{E}_{p^\infty}))$.

Over the cyclotomic $\mathbb{Z}_p$-extension $F_{\rm cyc}$ of F, there is a connection between the Galois group of the maximal abelian unramified pro-p extension of $F_{\rm cyc}$ and the fine Selmer groups of elliptic curves defined over F, see [11, Theorem 3.4]. This phenomenon can be extended to (both abelian and non-abelian) admissible p-adic Lie extensions of higher dimension. In fact, Conjecture B can be viewed as an elliptic curve analogue of an old conjecture of Greenberg on Galois modules associated with pro-p Hilbert class fields (see §2.4 for the precise statement). This has been explored in [11, p. 827]. It is therefore pertinent to investigate the precise connections between Conjecture B for admissible, abelian p-adic Lie extensions, and Greenberg's conjecture. For CM elliptic curves, the Generalized Greenberg's Conjecture is shown to be equivalent to Conjecture B for certain admissible pro-p, p-adic Lie extensions in Theorem 4.10. This result provides a framework for proving new cases of the Generalized Greenberg's Conjecture. In particular, we prove the following result[1].

Theorem 5.4 and Corollary 5.5. *Let $K/\mathbb{Q}$ be an imaginary quadratic field. If there exists one CM elliptic curve E/K such that the dual fine Selmer group is pseudonull over the trivializing extension $K(\mathsf{E}_{p^\infty})$, then the* Generalized Greenberg's Conjecture *holds for K and $K(\mathsf{E}_p)$.*

[1] The proof of Theorem 5.4 does not require E to be defined over K. This formulation is used in this introduction, for simplicity.

Little is known about Conjecture B and the Generalized Greenberg's Conjecture in full generality. We recall some cases where Conjecture B is proven in the literature. When there is a unique prime above p in the p-adic Lie extension of interest Conjecture B is proven in [49, Theorem 1.3] and [62, § 4]. Also, when the p-adic Lie extension has large dimension there are explicit examples where Conjecture B is known, detailed in [4, Example 23]. Certain analogues of Conjecture B have also been considered in [27, 36]. For evidence towards the Generalized Greenberg's Conjecture (both theoretical and computational) see [63, Remark 1.3], as well as [41, 38, 51, 47, 61, 16]. As per the knowledge of the authors, most results in this latter direction require the crucial hypothesis that p does not divide the class number of the number field. One exception is the result of R. Sharifi and W. McCallum, where the conjecture for $\mathbb{Q}(\mu_p)$ is proven under certain assumptions on a cup-product (see [39, Corollary 10.5]); another is of Sharifi [61, Theorem 1.3], where computational evidence for the Generalized Greenberg's Conjecture is provided when $F = \mathbb{Q}(\mu_p)$ and $p < 1000$ is an irregular prime. Our approach suggests a new line of attack for the Generalized Greenberg's Conjecture even in the case when p divides the class number of the base field.

The paper consists of five sections. Section 2 is preliminary in nature, wherein we recall the precise assertions of Conjecture A, Conjecture B, and the Generalized Greenberg's Conjecture and we introduce the main objects of study. In Section 3, new evidence for Conjecture A is provided by proving the triviality of the fine Selmer group over the cyclotomic extension. Some simple cases of Conjecture B are proven in Section 4. In Section 5 the relation between Conjecture B for CM elliptic curves and the Generalized Greenberg's Conjecture is clarified.

2. Preliminaries

Throughout this article, p denotes an odd prime number. For an abelian group M and a positive integer n, write M_{p^n} for the subgroup of elements of M annihilated by p^n. Put

$$M_{p^\infty} := \bigcup_{n\geq 1} M_{p^n}, \quad T_p(M) := \varprojlim M_{p^n}$$

and, when M is a discrete p-primary (*resp.* compact pro-p) abelian group M, its Pontryagin dual is defined as

$$M^\vee = \mathrm{Hom}_{\mathrm{cont}}(M, \mathbb{Q}_p/\mathbb{Z}_p).$$

Given any p-adic analytic group G, its *Iwasawa algebra* is defined as

$$\Lambda(G) = \varprojlim_{U} \mathbb{Z}_p[G/U]$$

for U running through all open, normal subgroups of G. When G is compact and p-valued in the sense of M. Lazard, $\Lambda(G)$ is a noetherian Auslander regular ring (see [8, Proposition 6.2]). In the special case when G is abelian with no elements of order p, there is an isomorphism

$$\Lambda(G) \simeq \mathbb{Z}_p[\![T_1, \ldots, T_d]\!]$$

where d is the dimension of G as a p-adic analytic manifold. If M is a compact (*resp.* discrete) $\Lambda(G)$-module then its Pontryagin dual is discrete (*resp.* compact). Given a finitely generated $\Lambda(G)$-module M, its Krull dimension is defined as the Krull dimension of $\Lambda(G)/\operatorname{Ann}(M)$ and it is denoted $\dim(M)$.

2.1. Suppose that G is an abelian p-analytic group without elements of order p. A finitely generated $\Lambda(G)$-module M is *torsion* (*resp. pseudonull*) if $\dim(M) \leq \dim(\Lambda(G)) - 1$ (*resp.* $\dim(M) \leq \dim(\Lambda(G)) - 2$). Equivalently (see [64, p. 273]), M is pseudonull if for every prime ideal $\mathfrak{p}$ such that

$$\operatorname{Ann}_{\Lambda(G)}(M) := \{a \in \Lambda(G) : aM = 0\} \subseteq \mathfrak{p}$$

we have $\operatorname{ht}(\mathfrak{p}) \geq 2$ (see [45, Definition 5.1.4]).

Let W (*resp.* M) be a discrete (*resp.* compact) G-module. The profinite cohomology groups (*resp.* homology groups) of W (*resp.* M) are denoted $H^i(G, W)$ (*resp.* $H_i(G, M)$). The subgroup of elements of W fixed by G is denoted W^G, and M_G denotes the largest quotient of M on which G acts trivially.

2.2. For a number field F, denote by F_{cyc} its cyclotomic $\mathbb{Z}_p$-extension. Suppose that $S = S(F)$ is a finite set of primes of F containing the primes above p and the archimedean primes. Let F_S be the maximal extension of F unramified outside S and set $G_S(F) = \operatorname{Gal}(F_S/F)$. For any (finite or infinite) extension $\mathcal{L}/F$ contained in F_S, denote by $G_S(\mathcal{L})$ the Galois group $\operatorname{Gal}(F_S/\mathcal{L})$. Throughout the paper, the focus is on S-admissible p-adic Lie extensions $\mathcal{L}/F$, in the following sense:

DEFINITION 2.1. An *S-admissible* p-adic Lie extension is a Galois extension $\mathcal{L}/F$ satisfying the following conditions:

- the group $\operatorname{Gal}(\mathcal{L}/F)$ is a pro-p, p-adic Lie group with no elements of order p;
- the field $\mathcal{L}$ contains the cyclotomic $\mathbb{Z}_p$-extension F_{cyc};
- the field $\mathcal{L}$ is contained in F_S.

Next, we record some conjectures pertaining to the modules associated with maximal abelian unramified pro-p extension of admissible p-adic Lie extensions. The first conjecture we mention was formulated by K. Iwasawa in [26, pp. 1–2] for the cyclotomic $\mathbb{Z}_p$-extension.

Iwasawa $\mu = 0$ Conjecture. *Let $L(F_{\text{cyc}})$ denote the maximal abelian unramified pro-p extension of F_{cyc} and set*

$$X_{\text{nr}}^{F_{\text{cyc}}} = \operatorname{Gal}(L(F_{\text{cyc}})/F_{\text{cyc}}).$$

Then, the μ-invariant associated with $X_{\text{nr}}^{F_{\text{cyc}}}$ is trivial.

In [25, Theorem 5], Iwasawa proved that $X_{\text{nr}}^{F_{\text{cyc}}}$ is a torsion $\Lambda(\Gamma)$-module; in view of this result, the Iwasawa $\mu = 0$ Conjecture is equivalent to saying that $X_{\text{nr}}^{F_{\text{cyc}}}$ is finitely generated over $\mathbb{Z}_p$. When $F/\mathbb{Q}$ is an abelian extension, the Iwasawa $\mu = 0$ Conjecture is known to be true by the work [14] by B. Ferrero and L. Washington.

Next, we mention a conjecture of Greenberg (see [20, Conjecture 3.5]) which is formulated for certain abelian p-adic Lie extensions.

Generalized Greenberg's Conjecture. *Let $\widetilde{F}$ denote the compositum of all $\mathbb{Z}_p$-extensions of F and let $L(\widetilde{F})$ denote the maximal abelian unramified pro-p extension of $\widetilde{F}$. Then* $\operatorname{Gal}\left(L(\widetilde{F})/\widetilde{F}\right)$ *is a pseudonull module over the Iwasawa algebra* $\Lambda(\operatorname{Gal}(\widetilde{F}/F)) =$

$\mathbb{Z}_p[\![\mathrm{Gal}(\widetilde{F}/F)]\!]$.

2.3. Fix a number field F and an admissible extension $\mathcal{L}/F$. Write $G_{\mathcal{L}/F}$ for the compact, pro-p, p-adic Lie group $\mathrm{Gal}(\mathcal{L}/F)$ and $\Lambda(G_{\mathcal{L}/F})$ for the associated Iwasawa algebra. The main objects of study will be modules over $\Lambda(G_{\mathcal{L}/F})$ that arise in Iwasawa theory, such as the Selmer group and the fine Selmer group. Let E be an elliptic curve defined over F. Choose a set $S = S(F)$ containing the primes above p, the primes of bad reduction of E/F, and the archimedean primes. Write $S \supseteq S_p \cup S_{\mathrm{bad}} \cup S_\infty$, where the notation S_p, S_{bad}, and S_∞ are self-explanatory. For a finite extension L/F and a prime v of F, define

$$J_v(L) = \bigoplus_{w|v} H^1(L_w, \mathsf{E})(p), \qquad \text{and} \qquad K_v(L) = \bigoplus_{w|v} H^1\left(L_w, \mathsf{E}_{p^\infty}\right), \tag{1}$$

where the direct sum is taken over all primes w of L lying above v. Taking direct limits, define

$$J_v(\mathcal{L}) = \varinjlim_L J_v(L), \qquad \text{and} \qquad K_v(\mathcal{L}) = \varinjlim_L K_v(L),$$

where L varies over finite sub-extensions of $\mathcal{L}/F$. Given any finite extension L/F contained in $\mathcal{L}$, the *p-primary Selmer group* $\mathrm{Sel}(\mathsf{E}/L)$ and the *p-primary fine Selmer group* $R(\mathsf{E}/L)$ are defined by the exactness of the following sequences:

$$0 \longrightarrow \mathrm{Sel}(\mathsf{E}/L) \longrightarrow H^1\left(G_S(F), \mathsf{E}_{p^\infty}\right) \longrightarrow \bigoplus_{v\in S(L)} J_v(L),$$
$$0 \longrightarrow R(\mathsf{E}/L) \longrightarrow H^1\left(G_S(F), \mathsf{E}_{p^\infty}\right) \longrightarrow \bigoplus_{v\in S(L)} K_v(L).$$

Moreover, by [11, Equation (58)] we can relate these groups as follows

$$0 \longrightarrow R(\mathsf{E}/L) \longrightarrow \mathrm{Sel}(\mathsf{E}/L) \longrightarrow \bigoplus_{w\in S_p(L)} \left(\mathsf{E}(L_w) \otimes \mathbb{Q}_p/\mathbb{Z}_p\right). \tag{2}$$

Define $\mathrm{Sel}(\mathsf{E}/\mathcal{L}) = \varinjlim_L \mathrm{Sel}(\mathsf{E}/L)$ and $R(\mathsf{E}/\mathcal{L}) = \varinjlim_L R(\mathsf{E}/L)$. It can then be shown (see [9, pp. 14–15] and [11, Equation (46)]) that

$$\mathrm{Sel}(\mathsf{E}/\mathcal{L}) \cong \ker\left(H^1\left(G_S(\mathcal{L}), \mathsf{E}_{p^\infty}\right) \longrightarrow \bigoplus_{v\in S} J_v(\mathcal{L})\right)$$

and

$$R(\mathsf{E}/\mathcal{L}) \cong \ker\left(H^1\left(G_S(\mathcal{L}), \mathsf{E}_{p^\infty}\right) \longrightarrow \bigoplus_{v\in S} K_v(\mathcal{L})\right).$$

Taking direct limits of (2), we obtain that

$$0 \longrightarrow R(\mathsf{E}/\mathcal{L}) \longrightarrow \mathrm{Sel}(\mathsf{E}/\mathcal{L}) \longrightarrow \varinjlim_L \bigoplus_{w\in S_p(L)} \left(\mathsf{E}(L_w) \otimes \mathbb{Q}_p/\mathbb{Z}_p\right).$$

Finally, we set a notation for the Pontryagin dual of these groups:

$$\mathfrak{X}(\mathsf{E}/\mathcal{L}) := \mathrm{Sel}(\mathsf{E}/\mathcal{L})^\vee \quad \text{and} \quad \mathfrak{Y}(\mathsf{E}/\mathcal{L}) := R(\mathsf{E}/\mathcal{L})^\vee. \tag{3}$$

These are compact $\Lambda(G_{\mathcal{L}/F})$-modules and it follows from (2) that $\mathfrak{Y}(\mathsf{E}/\mathcal{L})$ is a quotient of $\mathfrak{X}(\mathsf{E}/\mathcal{L})$.

In this paper, we are interested in a certain class of S-admissible p-adic Lie extensions generated by the p-primary torsion points of an elliptic curve. When the elliptic curve E/F is clear from the context, we write

$$F_\infty := \bigcup_{n\geq 1} F(\mathsf{E}_{p^n}).$$

It follows from the Weil pairing that F_∞ contains F_{cyc} and the choice of S ensures that F_∞ is contained in F_S. The Galois group $\text{Gal}(F_\infty/F)$ has no p-torsion if $p \geq 5$ (see, for example, [22, Lemma 4.7]) and contains an open, normal, pro-p subgroup (see [13, Corollary 8.34]). In fact, the extension $F_\infty/F(\mathsf{E}_p)$ is always pro-p and hence S-admissible. If E is an elliptic curve with CM, and F contains the field of complex multiplication, then $\text{Gal}(F_\infty/F)$ contains an open subgroup which is abelian and isomorphic to $\mathbb{Z}_p^2$.

2.4. Fix a number field F. In this section, we record the two conjectures formulated by Coates and the third named author in [11] which will be studied in this paper.

Conjecture A ([11, Section 3]). *Let E be an elliptic curve defined over F. Then $\mathfrak{Y}(\mathsf{E}/F_{\text{cyc}})$ is a finitely generated $\mathbb{Z}_p$-module.*

This conjecture is closely related to the Iwasawa $\mu = 0$ Conjecture. Their connection can be made precise:

Theorem 2.2 ([11, Theorem 3.4]). *Let E/F be an elliptic curve and suppose that* $\text{Gal}(F_\infty/F)$ *is pro-p. Then* Conjecture A *for E/F is equivalent to the* Iwasawa $\mu = 0$ Conjecture *for F.*

In [1, Theorems A,C], there are more examples for which Conjecture A holds.

The dimension theory for finitely generated modules over Iwasawa algebras allows framing an analogue of the Generalized Greenberg's Conjecture in a more general setting. This is Conjecture B and concerns the dual fine Selmer group over admissible p-adic Lie extensions (not necessarily abelian) of dimension ≥ 2. It asserts that this module is smaller than intuitively expected.

Conjecture B ([11, Section 4]). *Let E/F be an elliptic curve and let $\mathcal{L}/F$ be an S-admissible p-adic Lie extension such that $G_{\mathcal{L}/F}$ =* $\text{Gal}(\mathcal{L}/F)$ *has dimension strictly greater than 1. Then Conjecture A holds for E/F and $\mathfrak{Y}(\mathsf{E}/\mathcal{L})$ is a pseudonull $\Lambda(G_{\mathcal{L}/F})$-module.*

2.5. Fix a number field F and let T denote a finitely generated $\mathbb{Z}_p$-module, endowed with a continuous action of $G_S(F)$, where S contains the primes above p, the archimedean primes, and the primes v such that the inertia group of v does not act trivially on T. Note that if T is the Tate module $T_p\mathsf{E}$ of an elliptic curve E/F, then the inertia group of v acts trivially on T for every prime v of good reduction. Fix an S-admissible extension $\mathcal{L}/F$. Define the i-th *Iwasawa cohomology group* as the inverse limit

$$\mathcal{Z}_S^i(T/\mathcal{L}) = \varprojlim_L H^i(G_S(L), T), \text{ for } i = 0, 1, 2, \tag{4}$$

where L ranges over all finite extensions of F contained in $\mathcal{L}$ and the limit is taken with respect to the corestriction maps. It is well-known that $\mathcal{Z}_S^0(T/\mathcal{L})$ vanishes (see, for example, [11, Proposition 2.1]). In this article, we consider $T = \mathbb{Z}_p(1) = \varprojlim \mu_{p^n}$ or $T = T_p(\mathsf{E}) = \varprojlim_n \mathsf{E}_{p^n}$. Here $\mathbb{Z}_p(1)$ denotes the Tate twist of $\mathbb{Z}_p$. We remark that the dual fine Selmer group $\mathbb{Z}_p(1)$ has also been studied under various guises in [57, 46]. The weak Leopoldt conjecture is known to be true for the cyclotomic $\mathbb{Z}_p$-extension, see [45, Theorem 10.3.25]. In other words,

$$H^2\left(G_S(F_{\text{cyc}}), \mathbb{Q}_p/\mathbb{Z}_p\right) = 0.$$

Hence $H^2\left(G_S(\mathcal{L}), \mathbb{Q}_p/\mathbb{Z}_p\right)$ vanishes (see [11, p. 815 (20)]). An argument identical to [11, Lemma 3.1] but for the module $\mathbb{Q}_p/\mathbb{Z}_p$, shows that this vanishing is equivalent to the fact that $\mathcal{Z}_S^2\left(\mathbb{Z}_p(1)/\mathcal{L}\right)$ is $\Lambda(G_{\mathcal{L}/F})$-torsion. Analogously, [11, Lemma 3.1] shows that $H^2\left(G_S(\mathcal{L}), \mathsf{E}_{p^\infty}\right) = 0$ if and only if $\mathcal{Z}_S^2\left(T_p(\mathsf{E})/\mathcal{L}\right)$ is $\Lambda(G_{\mathcal{L}/F})$-torsion, but the equivalent of the weak Leopoldt conjecture is not known in the case of elliptic curves. When $G_S(\mathcal{L})$ acts trivially on E_{p^∞}, then $H^2\left(G_S(\mathcal{L}), \mathsf{E}_{p^\infty}\right) = 0$ (see for example [11, Lemma 2.4]).

The following notions will be useful in the reformulation of Conjecture B in Section 4.2. For $i \geq 0$ and $T = \mathbb{Z}_p(1)$, choose S to be a finite set of places of F containing the primes above p and the archimedean primes. For a finite extension L/F, let $\mathcal{O}_L[1/S]$ be the subring of L consisting of elements that are integral at every finite place of L not lying over S, and let $H^i_{\text{ét}}$ denote étale cohomology. An equivalent definition of the i-th Iwasawa cohomology group is the following (see [29, § 2.2 p. 552])

$$\mathcal{Z}_S^i\left(\mathbb{Z}_p(1)/\mathcal{L}\right) = \varprojlim_L H^i_{\text{ét}}\left(\mathcal{O}_L[1/S], \mathbb{Z}_p(1)\right) \tag{5}$$

where L ranges over all finite extensions of F contained in $\mathcal{L}$ and the limit is taken with respect to the corestriction maps. The dual fine Selmer group of $\mathbb{Z}_p(1)$ was introduced in [10] and is defined as $\mathrm{Gal}(M(\mathcal{L})/\mathcal{L})$ where $M(\mathcal{L})$ is the maximal abelian, pro-p unramified extension of $\mathcal{L}$ such that all primes above p split completely. An equivalent definition has been given in [29, §2.4, p. 554]. In particular,

$$\mathfrak{Y}(\mathbb{Z}_p(1)/\mathcal{L}) = \varprojlim_L \mathrm{Pic}\left(\mathcal{O}_L[1/S]\right)_{p^\infty}. \tag{6}$$

Moreover, there is an exact sequence (see for example, [10, p. 330 (2.6)])

$$0 \to \mathfrak{Y}\left(\mathbb{Z}_p(1)/\mathcal{L}\right) \to \mathcal{Z}_S^2\left(\mathbb{Z}_p(1)/\mathcal{L}\right) \to \bigoplus_{v \in S(\mathcal{L})} \mathbb{Z}_p \to \mathbb{Z}_p \to 0$$

and an isomorphism (see [56, §I.6.1])

$$R\left(\mathbb{Q}_p/\mathbb{Z}_p/\mathcal{L}\right) \simeq \mathrm{Hom}\left(\mathrm{Gal}\left(M\left(\mathcal{L}\right)/\mathcal{L}\right), \mathbb{Q}_p/\mathbb{Z}_p\right). \tag{7}$$

Since the dual fine Selmer group is independent of the choice of S, it is not included in the notation.

3. Fine Selmer Groups in the Cyclotomic Extension

The results in this section provide evidence for Conjecture A. First, we prove that for a set of ordinary primes of positive density, the Selmer group is trivial over the cyclotomic $\mathbb{Z}_p$-extension for rank 0 elliptic curves. Next, we provide evidence for Conjecture A for a class of elliptic curves defined over p-rational number fields.

3.1. **Trivial Fine Selmer Groups in the Cyclotomic Tower.** Throughout this section, assume that $\mathsf{E}/\mathbb{Q}$ is a rational elliptic curve. Fix a number field F and consider the base-change E/F of the curve to F. Given a prime number p, by slight abuse of notation, we denote by F_{cyc}/F the cyclotomic $\mathbb{Z}_p$-extension and by $\Gamma = \mathrm{Gal}\left(F_{\mathrm{cyc}}/F\right) \simeq \mathbb{Z}_p$ its Galois group, without mention of the prime p, as it can be inferred by the context.

At a prime v in F, the reduction of E modulo v is denoted $\widetilde{\mathsf{E}}_v$; it is a curve over the residue field κ_v. Following [37, Section 1(b)], a prime $v \mid p$ is called *anomalous* if p divides $\left|\widetilde{\mathsf{E}}_v(\kappa_v)\right|$.

In the remaining part of this section, we extend results of Greenberg [18, Proposition 5.1] and C. Wuthrich [68, Section 9] to base fields other than $\mathbb{Q}$. In Theorem 3.1 we provide evidence for Conjecture A for elliptic curves over a general number field. We stress that the prime p is not fixed in the remainder of this section and will vary over primes of good reduction.

In the statement of the next theorem we denote by F^c the Galois closure of $F/\mathbb{Q}$.

Theorem 3.1. *Let E/F be the base-change of a rational elliptic curve $\mathsf{E}/\mathbb{Q}$. Suppose that it has rank 0 over F and that the Shafarevich–Tate group of E/F is finite. When E has CM by an order in an imaginary quadratic field K, assume further that F^c contains K. Then the Selmer group* $\mathrm{Sel}\left(\mathsf{E}/F_{\mathrm{cyc}}\right)$ *is trivial for a set of prime numbers of density at least $\frac{1}{[F^c:\mathbb{Q}]}$. In particular,* Conjecture A *holds for E/F at all such primes.*

Proof. By assumption, the Selmer group over F is finite since both the Mordell–Weil and the Shafarevich–Tate groups are finite. If we further know that p is a prime of good ordinary reduction for E, it follows from Mazur's Control Theorem that the cyclotomic p-primary Selmer group $\mathrm{Sel}\left(\mathsf{E}/F_{\mathrm{cyc}}\right)$ is $\Lambda(\Gamma)$-cotorsion (see [19, Corollary 4.9]). In this setting, let $f_{\mathsf{E}}(T)$ be a power series generating the characteristic ideal of $\mathfrak{X}\left(\mathsf{E}/F_{\mathrm{cyc}}\right)$. Since $\mathrm{Sel}(\mathsf{E}/F)$ is finite, $f_{\mathsf{E}}(0) \neq 0$. Denote by c_v the local Tamagawa number at a prime v and by $c_v^{(p)}$ the highest power of p dividing it. Then, [18, Theorem 4.1] asserts that

$$f_{\mathsf{E}}(0) \sim \left(\prod_{v\,\mathrm{bad}} c_v^{(p)}\right)\left(\prod_{v|p}\left|\widetilde{\mathsf{E}}_v(\kappa_v)_p\right|^2\right)\left|\mathrm{Sel}(\mathsf{E}/F)\right| \Big/ \left|\mathsf{E}(F)_p\right|^2 \tag{8}$$

where $a \sim b$ for $a, b \in \mathbb{Q}_p^\times$ indicates that a, b have the same p-adic valuation.

For a prime number p, consider the following five properties:

(i) p is a prime of good ordinary reduction for E;
(ii) E has no non-trivial p-torsion points defined over F;
(iii) E/F has good ordinary reduction at all primes $v \mid p$ and all these primes are non-anomalous;
(iv) the p-primary part $Ш(\mathsf{E}/F)_{p^\infty}$ of the Shafarevich–Tate group is trivial;
(v) p does not divide the local Tamagawa number, i.e, $c_v^{(p)} = 1$ for every prime v of bad reduction.

Since E/F is assumed to have rank 0, the condition $\mathsf{E}(F)_p = 0$ implies that $\mathrm{Sel}(\mathsf{E}/F) = Ш(\mathsf{E}/F)_{p^\infty}$. It follows from (8) that for a prime number satisfying (i)–(v) above, $f_{\mathsf{E}}(0)$ is a unit.

When $f_{\mathsf{E}}(0)$ is a unit, elementary properties of characteristic power series show that $\mathfrak{X}\left(\mathsf{E}/F_{\mathrm{cyc}}\right)$ (and hence $\mathfrak{Y}\left(\mathsf{E}/F_{\mathrm{cyc}}\right)$) is finite, (see notation introduced in (3)). Equivalently, both $\mathrm{Sel}\left(\mathsf{E}/F_{\mathrm{cyc}}\right)$ and $R\left(\mathsf{E}/F_{\mathrm{cyc}}\right)$ are finite. When $\mathsf{E}(F)_p = 0$, [18, Proposition 4.14] implies that $\mathfrak{X}\left(\mathsf{E}/F_{\mathrm{cyc}}\right)$ has no non-trivial finite $\Lambda(\Gamma)$-submodules. In other words, $\mathfrak{X}\left(\mathsf{E}/F_{\mathrm{cyc}}\right)$ is trivial, whenever it is finite. Thus, $\mathfrak{Y}\left(\mathsf{E}/F_{\mathrm{cyc}}\right)$ is also trivial. Hence, Conjecture A holds for E/F when E/F is an elliptic curve satisfying (i)–(v).

To complete the proof, we show that for E/F satisfying the assumptions of the theorem, properties (i)–(v) hold for a set of prime numbers of density at least $\frac{1}{[F^c:\mathbb{Q}]}$.

When $\mathsf{E}/\mathbb{Q}$ is an elliptic curve without CM, we know by [59, Théorème 20] that all primes in $\mathbb{Q}$ outside a set of density 0 have good ordinary reduction. When E/F is an elliptic curve with CM by an order in K, Deuring's Criterion (see, for instance, [34, Chapter 13, §4, Theorem 12]) asserts that the primes of ordinary reduction are those lying above rational primes that split in $K/\mathbb{Q}$ and the density of such prime numbers equals $1/2$ by the Chebotarev density theorem. Next, it follows from the celebrated result [40, Théorème] of L. Merel that for all but finitely many prime numbers, we have $\mathsf{E}(F)_p = 0$. Assuming the finiteness of the Shafarevich–Tate group, condition (iv) holds for all but finitely many prime numbers, and the same is true for (v) since the local Tamagawa number c_v is equal to 1 at the primes of good reduction.

The analysis of (iii) requires more care. By definition, a prime $v \mid p$ is anomalous when $a_v = 1+|\kappa_v|-\left|\widetilde{\mathsf{E}}_v(\kappa_v)\right|$ is congruent to $1 \pmod{p}$. Observe that by the Hasse bound, $|a_v| \leq 2\sqrt{|\kappa_v|}$. Therefore, if $v \mid p$ is a prime in F that splits completely, so that $\kappa_v = \mathbb{F}_p$, then $a_v \equiv 1 \pmod{p}$ implies that $a_v = 1$ for $p > 5$. By the Chebotarev density theorem, the density of rational primes that split completely in F^c is $\frac{1}{[F^c:\mathbb{Q}]}$. Therefore, at least $\frac{1}{[F^c:\mathbb{Q}]}$ of the primes in $\mathbb{Q}$ split in F, as well. By the previous discussion, the density of rational primes which split completely in F and whose divisors are primes of good ordinary reduction for E/F is at least $\frac{1}{[F^c:\mathbb{Q}]}$. Finally, since E is defined over $\mathbb{Q}$, the Modularity Theorem guarantees that E is associated with an eigencuspform of weight 2. This allows us to appeal to the work of V. K. Murty [43]. We conclude from [43, pp. 288–289 or Theorem 5.1 and Remark 5.2] that for $\mathsf{E}/\mathbb{Q}$, the set of prime numbers with the property that $a_p = 1$ has density 0. Since for all prime numbers p that split completely and for all $v \mid p$, we have $a_v(\mathsf{E}/F) = a_p(\mathsf{E}/\mathbb{Q})$, we deduce that the set of prime numbers p such that $a_v = 1$ for at least one $v \mid p$ is a set of density 0. This completes the proof of the theorem. □

REMARK 3.2.

(1) It should be clear from the proof that one can insist that at all primes dividing the prime numbers in the set of positive density whose existence is stated in the theorem, the reduction type is good and ordinary.
(2) The key difficulty in extending this result to elliptic curves defined over F is that we rely on [43] to show that anomalous primes have density 0. Since these results are proven for normalized weight 2 eigencuspforms, we need to invoke the Modularity Theorem.

An analogous statement can be proven in the supersingular case as well.

Theorem 3.3. *Let* $\mathsf{E}/\mathbb{Q}$ *be an elliptic curve, and suppose that* $\mathrm{Sel}(\mathsf{E}/F)$ *is finite. Then* Conjecture A *holds for* E/F *for all but finitely many primes of supersingular reduction.*

Proof. For an elliptic curve E/F it is known that the Selmer group is not $\Lambda(\Gamma)$-cotorsion at a prime p of supersingular reduction, see [9, p. 19]. However, there is a notion of ±-Selmer groups[2] when $p > 3$, denoted by $\mathrm{Sel}^{\pm}\left(\mathsf{E}/F_{\mathrm{cyc}}\right)$. In the setting of the theorem, and under the additional hypothesis that $p > 3$ is an unramified prime in F, it is known that $\mathrm{Sel}^{\pm}\left(\mathsf{E}/F_{\mathrm{cyc}}\right)$ are $\Lambda(\Gamma)$-cotorsion, see [30, first line of the proof of Corollary 3.15]. Therefore, in this case, we can define a pair of signed characteristic power series $f^{\pm}_{\mathsf{E}}(T)$ for the Pontryagin duals $\mathfrak{X}(\mathsf{E}/F_{\mathrm{cyc}})^{\pm}$ of $\mathrm{Sel}^{\pm}\left(\mathsf{E}/F_{\mathrm{cyc}}\right)$. It follows from the definitions that the fine Selmer group is a subgroup of the signed Selmer groups. To prove the theorem it thus suffices to show that either of the signed Selmer groups is finite for all but finitely many primes of good supersingular reduction as this will ensure that the fine Selmer group is also finite and its corresponding μ and λ invariants vanish.

When $\mathrm{Sel}\,(\mathsf{E}/F)$ is finite and $p > 3$ is an unramified prime in F, we know from [30, Theorem 1.2] that

$$f^{\pm}_{\mathsf{E}}(0) \sim \left|\mathrm{Sel}(\mathsf{E}/F)\right| \prod_{v\,\mathrm{bad}} c_v^{(p)}. \tag{9}$$

If $f^{\pm}_{\mathsf{E}}(0) \sim 1$, then it follows from the Structure Theorem that $\mathrm{Sel}^{\pm}\left(\mathsf{E}/F_{\mathrm{cyc}}\right)$ are finite. To complete the proof we show that $f^{\pm}_{\mathsf{E}}(0) \sim 1$ for all but finitely many primes of good supersingular reduction.

(i) Since F is fixed, there are only finitely many primes which can ramify in F. In other words, (9) holds for all but finitely many primes.

(ii) By assumption, $\mathrm{Sel}(\mathsf{E}/F)$ is finite. There are only finitely many primes which can divide its order.

(iii) The local Tamagawa number c_v is equal to 1 at the primes of good reduction. Therefore, there are only finitely many primes which can divide $\prod_{v\,\mathrm{bad}} c_v$.

Therefore, as p varies over all supersingular primes of E, both signed Selmer groups $\mathrm{Sel}^{\pm}\left(\mathsf{E}/F_{\mathrm{cyc}}\right)$ are finite for all but finitely many such primes. Hence, $R\left(\mathsf{E}/F_{\mathrm{cyc}}\right)$ is also finite for such p. □

REMARK 3.4. In fact, more is true. [30, Theorem 1.1 (or Theorem 3.14)] applies in the setting of Theorem 3.3 and ensures that the $\mathfrak{X}^{-}(\mathsf{E}/F_{\mathrm{cyc}})$ does not contain any non-trivial finite index submodules. Therefore, if $\mathrm{Sel}^{-}\left(\mathsf{E}/F_{\mathrm{cyc}}\right)$ is finite, it must be trivial. Since $R\left(\mathsf{E}/F_{\mathrm{cyc}}\right)$ is a subgroup of $\mathrm{Sel}^{-}\left(\mathsf{E}/F_{\mathrm{cyc}}\right)$, it must be trivial as well. For the assertion that $\mathfrak{X}(\mathsf{E}/F_{\mathrm{cyc}})^{+}$ has no non-trivial finite index submodules, the additional hypothesis that p is completely split in F is required.

Combining Theorems 3.1 and 3.3, the next result is immediate.

Corollary 3.5. *Let E be CM rational elliptic curve and let E/F be its base-change to F. Suppose that E/F has rank 0, that the Shafarevich–Tate group of E/F is finite, and that the Galois closure F^c of F contains K. Then* Conjecture A *holds for E/F for a set of prime numbers of density $\frac{1}{2} + \frac{1}{[F^c:\mathbb{Q}]}$.*

[2] We avoid giving the precise definition of these Selmer groups because their definition is intricate and also not relevant for the remainder of this paper. For a precise definition, we refer the reader to [32] or [30].

Proof. By Deuring's Criterion we know that $1/2$ of the primes are supersingular and Theorem 3.3 asserts that there is a contribution of density $1/2$. But, there is also a contribution from the primes of good ordinary reduction by Theorem 3.1. The corollary follows. □

Let us now turn to a special class of number fields, called p-rational number fields.

3.2. **Conjecture A over p-Rational Number Fields.** For the number field F and a fixed prime p, choose S to be a finite set of primes of F containing the primes above p and the archimedean primes. The weak Leopoldt conjecture for $\mathcal{L}/F$ is the following assertion (see for example [45, Theorem 10.3.22])

$$H^2\left(\mathrm{Gal}\,(F_S/\mathcal{L})\,, \mathbb{Q}_p/\mathbb{Z}_p\right) = 0. \tag{10}$$

It is known to hold for the cyclotomic $\mathbb{Z}_p$-extension F_{cyc}/F (see [45, Theorem 10.3.25]). If (10) holds for a finite set S as above, it also holds for the set $\Sigma = S_p \cup S_\infty$ (see [45, Theorem 11.3.2]). Therefore, the weak Leopoldt Conjecture is independent of the choice of S, when S contains Σ. Henceforth, fix $S = \Sigma$. An equivalent formulation of the Iwasawa $\mu = 0$ Conjecture for F is the assertion that $\mathcal{G}_\Sigma\left(F_{\mathrm{cyc}}\right) = \mathrm{Gal}\left(F_\Sigma(p)/F_{\mathrm{cyc}}\right)$ is a free pro-p group (see [45, Theorem 11.3.7]). Moreover, a pro-p group G is free if and only if its p-cohomological dimension $\mathrm{cd}_p(G)$ is less or equal to 1 (see [45, Corollary 3.5.17]). Combining these results with [60, Chapter I, Section 4, Proposition 21], one obtains the following equivalent formulation:

$$\text{the Iwasawa } \mu = 0 \text{ Conjecture for } F \text{ is true} \iff H^2\left(\mathcal{G}_\Sigma\left(F_{\mathrm{cyc}}\right), \mathbb{Z}/p\mathbb{Z}\right) = 0. \tag{11}$$

To state the results in this section, we recall the notion of a special class of number fields, called *p-rational*, which were introduced in [42]. We refer the reader to [17, Theorem IV.3.5 and Definition IV.3.4.4] for a detailed discussion.

DEFINITION 3.6. Denote by F_{S_p} the maximal extension of F unramified outside S_p and let $F_{S_p}(p)/F$ be its maximal pro-p sub-extension. Set $\mathcal{G}_{S_p}(F) = \mathrm{Gal}\left(F_{S_p}(p)/F\right)$. If $\mathcal{G}_{S_p}(F)$ is free pro-p, then F is called p-rational.

Some examples of p-rational fields include:

(i) the field $\mathbb{Q}$ of rational numbers;
(ii) imaginary quadratic fields such that p does not divide the class number (see [21, Proposition 4.1.1]);
(iii) cyclotomic fields $\mathbb{Q}(\mu_{p^n})$, where p is a regular prime and $n \geq 1$ (combine [17, Example II.7.8.1.1] with [67, Proposition 13.22]);
(iv) more generally, number fields F containing μ_p with the property that $\#S_p(F) = 1$ and such that p does not divide the class number of F (see [17, Theorem 3.5-(iii)]).

p-rational number fields have been studied by Greenberg in [21], where he explains heuristic reasons to believe that a number field F should be p-rational for all primes outside a set of density 0 (see [21, §7.4.4]). In [3, Table 4.1], R. Barbulescu and J. Ray provide examples of non-abelian p-rational number fields.

The following result is easily deduced from the aforementioned results in Galois cohomology. A proof is included for the sake of completeness.

Theorem 3.7. *Let F be a p-rational number field. Then the following assertions hold.*

(i) *The* Iwasawa $\mu = 0$ Conjecture *holds for F.*

(ii) *Suppose that F contains μ_p and that E/F is an elliptic curve such that $\mathsf{E}(F)_p \neq 0$. Then* Conjecture A *holds for E/F.*

Proof.

(i) Since $p \neq 2$, we can replace S_p by Σ in the definition of p-rational fields. This is because the archimedean primes are unramified in $F_{S_p}(p)/F$ when p is odd. By definition, if F is p-rational, $\mathcal{G}_\Sigma(F) = \mathrm{Gal}\,(F_\Sigma(p)/F)$ has p-cohomological dimension at most 1. Hence

$$H^2\,(\mathcal{G}_\Sigma(F), \mathbb{Z}/p\mathbb{Z}) = 0.$$

Since $\mathcal{G}_\Sigma(F_{\mathrm{cyc}}) = \mathrm{Gal}(F_\Sigma(p)/F_{\mathrm{cyc}})$ is a closed normal subgroup of $\mathcal{G}_\Sigma(F)$, it follows from [45, Proposition 3.3.5] that

$$\mathrm{cd}_p\left(\mathcal{G}_\Sigma(F_{\mathrm{cyc}})\right) \leq \mathrm{cd}_p\,(\mathcal{G}_\Sigma(F)) \leq 1.$$

Thus $H^2(\mathcal{G}_\Sigma(F_{\mathrm{cyc}}), \mathbb{Z}/p\mathbb{Z}) = 0$, and the result follows from (11).

(ii) Since $F \supseteq \mu_p$ and $\mathsf{E}(F)_p \neq 0$ by assumption, the Weil pairing ensures that $F(\mathsf{E}_p)/F$ is either trivial or of degree p. Thus, $F(\mathsf{E}_{p^\infty})/F$ is pro-p. The theorem follows from the first point together with Theorem 2.2. □

4. Conjecture B for Elliptic Curves with CM: Special Cases

In this section, we provide evidence for Conjecture B. First, in Section 4.1 we provide sufficient conditions for Conjecture B to hold when p is a prime of good ordinary reduction, see Theorem 4.6. In Section 4.2 we give a different formulation of Conjecture B for CM elliptic curves and prove cases of the conjecture when p is a prime of good supersingular reduction. We start with a lemma about good reduction of CM elliptic curves that can be found extracted from [55, proof of Theorem 5.7-(i)].

Lemma 4.1. *Let F be a number field and let E/F be an elliptic curve with CM by an order inside the ring of integers $\mathcal{O}_K$ of an imaginary quadratic field K. Let p be an odd prime number and suppose that the following hypotheses hold:*

(i) *E has good reduction at all primes above p.*

(ii) *The Galois group $G = \mathrm{Gal}(F_\infty/F)$ is isomorphic to $\mathbb{Z}_p^2$, where F_∞ denotes $F(\mathsf{E}_{p^\infty})$.*

Then E has good reduction everywhere over F.

Proof. It follows from the theory of complex multiplication that F contains the Hilbert class field K' of K. Since the extension F_∞/F is a p-extension and $[F(\mathsf{E}_p) : F]$ is prime-to-p, it follows that $F = F(\mathsf{E}_p)$. Therefore, $K'(\mathsf{E}_p) \subseteq F$.

Since all primes above p are of good reduction, we only need to check that at primes away from p, the curve E has good reduction. This follows from the criterion of Néron–Ogg–Shafarevich, because every such prime is unramified in the $\mathbb{Z}_p^2$-extension F_∞/F. □

4.1. Fix a number field F. We will work in the following setting.

Ass 1

(i) $p \neq 2, 3$ is a fixed prime which splits in an imaginary quadratic field K;
(ii) E is an elliptic curve defined over F with CM by $\mathcal{O}_K$, and K is contained in F;
(iii) E has good reduction at primes above p;
(iv) the Galois group $G = \mathrm{Gal}(F_\infty/F)$ is isomorphic to $\mathbb{Z}_p^2$, where F_∞ denotes $F(\mathsf{E}_{p^\infty})$.

In the setting of **Ass 1**, write $H = \mathrm{Gal}(F_\infty/F_{\mathrm{cyc}})$, and fix a finite set S containing $S_p \cup S_\infty$.

Note that **Ass 1** ensures that E has good ordinary reduction at p, see [34, Chapter 13 Theorem 12 (Deuring's Criterion)]. Observe that given any p-adic Lie group $\mathcal{G}$ and a finitely generated $\Lambda(\mathcal{G})$-module M, the group $M_{\mathcal{G}} := H_0(\mathcal{G}, M)$ is finitely generated as a $\mathbb{Z}_p$-module.

Lemma 4.2. *Suppose that* **Ass 1** *holds. Then, the following map of* $\Lambda(H)$*-modules is a pseudo-isomorphism,* i. e. *it has a finite kernel and cokernel,*

$$\mathfrak{Y}(\mathsf{E}/F_\infty)_H \to \mathfrak{Y}\left(\mathsf{E}/F_{\mathrm{cyc}}\right).$$

Proof. Let L be a finite extension of F contained in F_S. For each $v \in S$, write $W_v(L) = \bigoplus_{w|v} \mathsf{E}(L_w) \otimes \mathbb{Q}_p/\mathbb{Z}_p$. We have the maps

$$r_{\mathrm{cyc}}\colon\ \mathrm{Sel}(\mathsf{E}/F_{\mathrm{cyc}}) \longrightarrow \bigoplus_{v|p} W_v(F_{\mathrm{cyc}})$$

and

$$r_\infty\colon\ \mathrm{Sel}(\mathsf{E}/F_\infty) \longrightarrow \bigoplus_{v|p} W_v(F_\infty)$$

where $W_v(F_{\mathrm{cyc}})$ (*resp.* $W_v(F_\infty)$) is the direct limit of $W_v(L)$ with respect to the restriction map as L ranges over all finite extensions of F contained in F_{cyc} (*resp.* F_∞). Write $C(F_{\mathrm{cyc}})$ (*resp.* $C(F_\infty)$) for the image of r_{cyc} (*resp.* r_∞). Consider the following diagram

$$\begin{array}{ccccccccc} 0 & \longrightarrow & R(\mathsf{E}/F_{\mathrm{cyc}})_p & \longrightarrow & \mathrm{Sel}(\mathsf{E}/F_{\mathrm{cyc}})_p & \longrightarrow & C(F_{\mathrm{cyc}})_p & \longrightarrow & 0 \\ & & \big\downarrow{\scriptstyle\alpha} & & \big\downarrow{\scriptstyle\beta} & & \big\downarrow{\scriptstyle\gamma} & & \\ 0 & \longrightarrow & R(\mathsf{E}/F_\infty)_p^H & \longrightarrow & \mathrm{Sel}(\mathsf{E}/F_\infty)_p^H & \longrightarrow & C(F_\infty)_p^H\ . & & \end{array}$$

Note that β is an isomorphism (see [52, Lemme 1.1(i) and Lemme 1.3]). Therefore $\ker(\beta)$ and $\mathrm{coker}(\beta)$ are trivial; hence $\ker(\alpha) = 0$. Further, observe that there is an inclusion

$$\ker\gamma \subseteq \ker\left(\bigoplus_{v|p} K_v\left(F_{\mathrm{cyc}}\right) \xrightarrow{\delta_v} K_v\left(F_\infty\right)^H\right).$$

Now, observe that

$$\bigoplus_{v|p} \ker(\delta_v) = \bigoplus_{v|p} H^1\left(H_v, \mathsf{E}(F_{\infty,v})_{p^\infty}\right).$$

This latter object is known to be finite by using an argument identical to [11, proof of

Lemma 4.2]. Therefore, by the snake lemma, coker(α) must be finite. □

REMARK 4.3. Another way to prove this lemma was pointed out to us by the referee. Consider the fundamental diagram

$$\begin{array}{ccccccccc} 0 & \longrightarrow & R(\mathsf{E}/F_{\mathrm{cyc}})_p & \longrightarrow & H^1\left(G_S(F_{\mathrm{cyc}}), \mathsf{E}_{p^\infty}\right) & \longrightarrow & \bigoplus_{v\in S} K_v\left(F_{\mathrm{cyc}}\right) & \longrightarrow & 0 \\ & & \downarrow{\scriptstyle\alpha} & & \downarrow{\scriptstyle\beta} & & \downarrow{\scriptstyle\gamma} & & \\ 0 & \longrightarrow & R(\mathsf{E}/F_\infty)_p^H & \longrightarrow & H^1\left(G_S(F_\infty), \mathsf{E}_{p^\infty}\right)^H & \longrightarrow & \bigoplus_{v\in S} K_v\left(F_\infty\right)^H . & & \end{array}$$

The map β is the restriction map; it is surjective and $\ker(\beta) = H^1\left(H, \mathsf{E}(F_\infty)_{p^\infty}\right)$. Similarly, $\ker(\gamma) = \bigoplus_{v\in S_p} H^1\left(H_v, \mathsf{E}(F_{\infty,v})_{p^\infty}\right)$. To show that $\ker(\gamma)$ is finite, we use an argument similar to [11, proof of Lemma 4.2]. First, recall a result of H. Imai [24, Theorem] which asserts that $\mathsf{E}(F_{\mathrm{cyc},v})_{p^\infty}$ is finite. Note that $H_v \simeq \mathbb{Z}_p$ and $\mathsf{E}(F_{\infty,v})_{p^\infty}^\vee$ is a torsion $\Lambda(H_v)$-module (since it is in fact finitely generated over $\mathbb{Z}_p$). It follows that $H^1\left(H_v, \mathsf{E}(F_{\infty,v})_{p^\infty}\right)$ is also finite. In fact, $H^1\left(H_v, \mathsf{E}(F_{\infty,v})_{p^\infty}\right) = 0$ which can be proven in the same way as [12, Lemma 5.4] using the fact that H_v has p-cohomological dimension 1. Furthermore, since $\mathsf{E}(F_{\mathrm{cyc}})_{p^\infty}$ is finite by a result of K. Ribet [54, Theorem 1], the global version of the above argument ensures that $\ker(\beta)$ is also finite, see also [11, pp. 834–835]. Applying the snake lemma, the lemma follows.

Since E is an elliptic curve with CM, both G and H are abelian. Under the assumption that $G \simeq \mathbb{Z}_p^2$, we further know that $\Lambda(H) \simeq \mathbb{Z}_p[\![T]\!]$. We now state an equivalent condition for a $\Lambda(G)$-module to be pseudonull.

Proposition 4.4. *Let M be a finitely generated $\Lambda(G)$-module which is also finitely generated as a $\Lambda(H)$-module. Then the module M is $\Lambda(G)$-torsion. Further, M is $\Lambda(H)$-torsion if and only if it is $\Lambda(G)$-pseudonull.*

Proof. Note that $G \simeq H \times \Gamma$ where $\Gamma \simeq \mathbb{Z}_p$. The first assertion follows from the fact that $\Lambda(G)$ is not finitely generated over $\Lambda(H)$. The second assertion is a special case of [65, Proposition 5.4]. □

Lemma 4.5. *Let M be a finitely generated $\Lambda(G)$-module which is also finitely generated over $\Lambda(H)$. If M_H is finite, then M is a pseudonull $\Lambda(G)$-module.*

Proof. We are grateful to the referee for suggesting the following proof, which is simpler than the one we had in a first version of our manuscript. Since $H \cong \mathbb{Z}_p$, it follows from the structure theory of $\Lambda(H)$-modules that whenever M_H is finite, then M is torsion over $\Lambda(H)$. The conclusion of the lemma is now immediate from Proposition 4.4. □

The main theorem of this section is the following.

Theorem 4.6. *Suppose that* **Ass 1** *holds. If $\mathfrak{Y}(\mathsf{E}/F_{\mathrm{cyc}})$ is finite, then* Conjecture B *holds for* (E, F_∞).

Proof. By Lemma 4.2, if $\mathfrak{Y}(\mathsf{E}/F_{\mathrm{cyc}})$ is finite, then so is $\mathfrak{Y}\left(\mathsf{E}/F_\infty\right)_H$. The theorem follows from Lemma 4.5. □

REMARK 4.7. We point out that for a given prime p, we cannot conclude that Conjecture B holds for (E, F_∞) for a rank 0 elliptic curve E/F with CM by combining Theorems 3.1 and 4.6. This is because, in the proof of Theorem 3.1 it was required that the elliptic curve does not admit any non-trivial p-torsion point over F. However, in proving Theorem 4.6, we assume that F_∞/F is a pro-p extension; hence F must contain non-trivial p-torsion points.

Another case where we can show Conjecture B is the following.

Proposition 4.8. *Suppose that* **Ass 1** *holds. Further assume that* $\mathfrak{X}(\mathsf{E}/F_{\mathrm{cyc}})$ *is a finitely generated* $\mathbb{Z}_p$*-module of* $\mathbb{Z}_p$*-rank* 2 *and that* $\mathsf{E}(F_\infty)$ *has a point of infinite order. Then* Conjecture B *holds for* (E, F_∞).

Proof. By **Ass 1**, we know that E/F has good reduction everywhere. Next, it follows from [23, Theorem 2.8] that

$$\operatorname{rank}_{\Lambda(H)} \mathfrak{X}(\mathsf{E}/F_\infty) = \operatorname{rank}_{\mathbb{Z}_p} \mathfrak{X}(\mathsf{E}/F_{\mathrm{cyc}}).$$

We explain this briefly. To apply [23, Theorem 2.8] one must assume that Conjecture 2.5 *ibid.* holds. As mentioned on p. 649 *ibid.*, this conjecture is equivalent to Conjecture 2.6 *ibid.* when all primes above p have good ordinary reduction. This conjecture predicts that $\mathfrak{X}(\mathsf{E}/F_{\mathrm{cyc}})$ is $\Lambda(\Gamma)$-torsion and our hypothesis that $\mathfrak{X}(\mathsf{E}/F_{\mathrm{cyc}})$ is a finitely generated $\mathbb{Z}_p$-module accounts for it. For the final assumption in Theorem 2.8 *ibid,* the inclusion $\mu_p \subseteq F$ is ensured by the Weil pairing.

It is shown in [11, Theorem 4.5-(ii)] that

$$\operatorname{rank}_{\Lambda(H)} \mathfrak{Y}(\mathsf{E}/F_\infty) \leq \operatorname{rank}_{\Lambda(H)} \mathfrak{X}(\mathsf{E}/F_\infty) - 2,$$

and our assumption, combined with (4.1), implies that $\operatorname{rank}_{\Lambda(H)} \mathfrak{Y}(\mathsf{E}/F_\infty) = 0$, showing that $\mathfrak{Y}(\mathsf{E}/F_\infty)$ is $\Lambda(G_{F_\infty/F})$-pseudonull. □

REMARK 4.9. We are grateful to the referee for the following observation. In [11, Theorem 4.5-(i)] it is shown that if $\operatorname{rank}_{\Lambda(H)} \mathfrak{X}(\mathsf{E}/F_\infty)$ is odd, then

$$\operatorname{rank}_{\Lambda(H)} \mathfrak{Y}(\mathsf{E}/F_\infty) \leq \operatorname{rank}_{\Lambda(H)} \mathfrak{X}(\mathsf{E}/F_\infty) - 1.$$

Since $\mathfrak{X}(\mathsf{E}/F_{\mathrm{cyc}})$ is a finitely generated $\mathbb{Z}_p$-module, if $\operatorname{rank}_{\mathbb{Z}_p} \mathfrak{X}(\mathsf{E}/F_{\mathrm{cyc}}) = 1$, it would follow that $\mathfrak{Y}(\mathsf{E}/F_\infty)$ is $\Lambda(G_{F_\infty/F})$-pseudonull. Unfortunately, though, one can show that under our assumptions $\operatorname{rank}_{\Lambda(H)} \mathfrak{X}(\mathsf{E}/F_\infty)$ is always even, and therefore the above argument cannot be used to show Conjecture B in more cases. The argument forcing $\operatorname{rank}_{\Lambda(H)} \mathfrak{X}(\mathsf{E}/F_\infty)$ to be even comes from CM theory: indeed, by functoriality, the $\Lambda(H)$-module $\mathfrak{X}(\mathsf{E}/F_\infty)$ comes endowed with a structure of an $\mathcal{O}_K$-module, and therefore it is ultimately a $\Lambda(H) \otimes \mathcal{O}_K$-module. For brevity, denote $\Lambda(H) \otimes \mathcal{O}_K$ by $\widetilde{\Lambda(H)}$, and set $Q(H) = \operatorname{Frac}(\Lambda(H)), \widetilde{Q(H)} = \operatorname{Frac}(\widetilde{\Lambda(H)})$. Clearly, $\widetilde{\Lambda(H)}$ is a finite extension of $\Lambda(H)$ of rank 2: in particular, it is integral so that $\widetilde{Q(H)} = \widetilde{\Lambda(H)} \otimes_{\Lambda(H)} Q(H)$ and $\dim_{Q(H)} \widetilde{Q(H)} = 2$. Now, by definition,

$$\begin{aligned} 2 \cdot \operatorname{rank}_{\widetilde{\Lambda(H)}} \mathfrak{X}(\mathsf{E}/F_\infty) &= 2 \cdot \dim_{\widetilde{Q(H)}} \Big(\mathfrak{X}(\mathsf{E}/F_\infty) \otimes_{\widetilde{\Lambda(H)}} \widetilde{Q(H)} \Big) \\ &= 2 \cdot \dim_{\widetilde{Q(H)}} \Big(\mathfrak{X}(\mathsf{E}/F_\infty) \otimes_{\widetilde{\Lambda(H)}} \widetilde{\Lambda(H)} \otimes_{\Lambda(H)} Q(H) \Big) \end{aligned}$$

$$= 2 \cdot \dim_{\widetilde{Q(H)}}\Big(\mathfrak{X}(\mathsf{E}/F_\infty) \otimes_{\Lambda(H)} Q(H)\Big)$$
$$= \dim_{Q(H)}\Big(\mathfrak{X}(\mathsf{E}/F_\infty) \otimes_{\Lambda(H)} Q(H)\Big)$$
$$= \operatorname{rank}_{\Lambda(H)} \mathfrak{X}(\mathsf{E}/F_\infty),$$

showing that $\operatorname{rank}_{\Lambda(H)} \mathfrak{X}(\mathsf{E}/F_\infty)$ is indeed even.

4.2. **Reformulation of Conjecture B.** Let E/F be an elliptic curve, and let $\mathcal{L}$ be an S-admissible p-adic Lie extension containing the trivializing extension F_∞. Throughout this section we suppose that Conjecture A holds for E/F. Since $G_S(\mathcal{L})$ acts trivially on E_{p^∞}, Conjecture B for $(\mathsf{E}, \mathcal{L})$ has an equivalent formulation in terms of the pseudonullity of the Galois group $\mathrm{Gal}(M(\mathcal{L})/\mathcal{L})$, where $M(\mathcal{L})$ is the maximal unramified abelian pro-p extension of $\mathcal{L}$ such that all primes above p in $\mathcal{L}$ split completely.

Reformulation (see [11, p. 827])**.** *Let E/F be an elliptic curve, and let $\mathcal{L}$ be an S-admissible, p-adic Lie extension over F such that $G_S(\mathcal{L})$ acts trivially on E_{p^∞}. Then $\mathfrak{Y}\left(\mathbb{Z}_p(1)/\mathcal{L}\right)$ is $\Lambda(G_{\mathcal{L}/F})$-pseudonull.*

The next result asserts that for an S-admissible p-adic Lie extension $\mathcal{L}/F$ containing F_∞, the $\Lambda(G_{\mathcal{L}/F})$-pseudonullity of the Iwasawa module $X_{\mathrm{nr}}^{\mathcal{L}}$ is equivalent to the pseudonullity of a certain quotient module. (The notation $X_{\mathrm{nr}}^{\mathcal{L}}$ was introduced at the beginning of this section). This result is well-known to experts and follows from results available in the literature, see for example [66, Theorem 4.9]. For the convenience of the reader, a proof is provided here purely relying on techniques that are more germane to our paper.

Theorem 4.10. *Let E/F be an elliptic curve with CM by an order in an imaginary quadratic field K such that $K \subseteq F$ and suppose that* $\mathrm{Gal}(F_\infty/F) \simeq \mathbb{Z}_p^2$. *Let $\mathcal{L}/F$ be an abelian S-admissible p-adic Lie extension containing F_∞. Then, the following statements are equivalent*

(a) *The* Iwasawa $\mu = 0$ Conjecture *is true for F and $X_{\mathrm{nr}}^{\mathcal{L}}$ is $\Lambda(G_{\mathcal{L}/F})$-pseudonull.*
(b) Conjecture B *holds for* $(\mathsf{E}, \mathcal{L})$.
(c) *The* Iwasawa $\mu = 0$ Conjecture *is true for F and $\mathfrak{Y}(\mathbb{Z}_p(1)/\mathcal{L})$ is $\Lambda(G_{\mathcal{L}/F})$-pseudonull.*

Proof. Since E/F has CM by the imaginary quadratic field K contained in F and F_∞/F is a $\mathbb{Z}_p^2$-extension, it follows that F contains $K'(\mathsf{E}_p)$ where K' is the Hilbert class field of K (see Lemma 4.1). Moreover, since $\mathcal{L}/F$ is an abelian extension containing F_∞ and, by definition of being admissible, it contains no element of order p, it must be a $\mathbb{Z}_p^d$-extension for some $d \geq 2$. It follows that the only primes that can ramify in this extension are the primes above p and therefore we can assume that $S = S_p \cup S_\infty$.

Equivalence of (a) *and of* (c): We need to show that

$$\mathfrak{Y}\left(\mathbb{Z}_p(1)/\mathcal{L}\right) \text{ is } \Lambda(G_{\mathcal{L}/F})\text{-pseudonull} \iff X_{\mathrm{nr}}^{\mathcal{L}} \text{ is } \Lambda(G_{\mathcal{L}/F})\text{-pseudonull.}$$

Write $X_{\mathrm{cs}}^{\mathcal{L}}$ to denote the Galois group $\mathrm{Gal}\left(M(\mathcal{L})/\mathcal{L}\right)$. It is known by the work of U. Jannsen (see for example [66, Proposition 4.7-(ii)]) that there is an exact sequence

$$\bigoplus_{v \in S_{\mathrm{cs}} \cup S_{\mathrm{ram}}} \mathrm{Ind}_{G_{\mathcal{L}/F}}^{G_{\mathcal{L}/F,v}} \left(\mathbb{Z}_p\right) \longrightarrow X_{\mathrm{nr}}^{\mathcal{L}} \longrightarrow X_{\mathrm{cs}}^{\mathcal{L}} \longrightarrow 0.$$

Here, S_{cs} denotes the set of non-archimedean primes in S which are completely split in $\mathcal{L}/F$ and S_{ram} denotes the set of non-archimedean primes in S which are ramified in $\mathcal{L}/F$. Note that in our setting $S_{\text{cs}} = \emptyset$ because every prime above p is finitely decomposed in F_{cyc}/F, and $S_{\text{ram}} = S_p$. Since the base field F contains μ_p it follows from (7) that

$$X_{\text{cs}}^{\mathcal{L}} = \text{Gal}\left(M(\mathcal{L})/\mathcal{L}\right) \simeq \mathfrak{Y}\left(\mathbb{Z}_p(1)/\mathcal{L}\right).$$

Therefore, to complete the proof of the equivalence it is enough to show that $X_{\text{nr}}^{\mathcal{L}}$ and $X_{\text{cs}}^{\mathcal{L}}$ are pseudo-isomorphic. In other words, it suffices to prove that

$$\bigoplus_{v \in S_p} \mathbb{Z}_p[\![G_{\mathcal{L}/F}]\!] \otimes_{\mathbb{Z}_p[\![G_{\mathcal{L}/F,v}]\!]} \mathbb{Z}_p = \bigoplus_{v \in S_p} \text{Ind}_{G_{\mathcal{L}/F}}^{G_{\mathcal{L}/F,v}} \left(\mathbb{Z}_p\right)$$

is a $\Lambda(G_{\mathcal{L}/F})$-pseudonull module. We know from [35, Théorème 3.2] (observe that since F contains $K'(\mathsf{E}_p)$, condition (i) *ibid.* is satisfied, by the Weil pairing) that for all $v \mid p$, the decomposition group at v inside $G_{\mathcal{L}/F}$ has dimension at least 2. It follows that $\bigoplus_{v \in S_p} \text{Ind}_{G_{\mathcal{L}/F}}^{G_{\mathcal{L}/F,v}} \left(\mathbb{Z}_p\right)$ is $\Lambda(G_{\mathcal{L}/F,v})$-pseudonull. This completes the proof of the equivalence.

Equivalence of (b) *and* (c): It follows from the discussion in [11, p. 825] that

$$\mathfrak{Y}\left(\mathsf{E}/\mathcal{L}\right) \simeq \left(\mathfrak{Y}(\mathbb{Z}_p(1)/\mathcal{L}) \otimes \mathsf{E}_{p^\infty}^\vee\right). \tag{12}$$

Here $G_{\mathcal{L}/F}$ acts diagonally on the tensor product and $\mathsf{E}_{p^\infty}^\vee$ is a $\mathbb{Z}_p$-module with a $G_{\mathcal{L}/F}$-action induced by the $G_S(F)$-action. This latter action makes sense because F_∞ is the trivializing extension of E_{p^∞}. In this setting, Conjecture A for E/F is equivalent to the Iwasawa $\mu = 0$ Conjecture for F (see Theorem 2.2). Therefore, using [66, Proposition 2.12] and [50, Proposition 3.4], the isomorphism in (12) yields that $\mathfrak{Y}\left(\mathsf{E}/\mathcal{L}\right)$ is $\Lambda(G_{\mathcal{L}/F})$-pseudonull if and only if $\mathfrak{Y}\left(\mathbb{Z}_p(1)/\mathcal{L}\right)$ is $\Lambda(G_{\mathcal{L}/F})$-pseudonull. □

REMARK 4.11. In the above proof, the assumption that E has CM is not really used. We detail in §4.3 the proof in the general case.

We now prove a special case of Conjecture B in the supersingular reduction setting and provide applications pertaining to universal norms. For the remainder of this section, we work in the following setting:

Ass 2

(i) K is an imaginary quadratic field of class number 1;
(ii) E is an elliptic curve defined over K, and with CM by $\mathcal{O}_K$;
(iii) p is an odd prime of good supersingular reduction for E;
(iv) p does not divide the order of the S_p-class group of $F = K(\mathsf{E}_p)$;

REMARK 4.12. It follows from **Ass 2**-(ii) that the Galois group $G = \text{Gal}(F_\infty/F)$ is isomorphic to $\mathbb{Z}_p^2$ and we write $H = \text{Gal}(F_\infty/F_{\text{cyc}})$.

Recall that the i-th Iwasawa cohomology group over F_∞ is defined, when $S = S_p$, as

$$\mathcal{Z}_{S_p}^i\left(\mathbb{Z}_p(1)/F_\infty\right) = \varprojlim_L H_{\text{ét}}^i\left(\mathcal{O}_L[1/p], \mathbb{Z}_p(1)\right),$$

where L ranges over all finite extensions of F contained in F_∞.

Proposition 4.13. *Suppose that* **Ass 2** *holds. Then* $\mathcal{Z}^2_{S_p}\left(\mathbb{Z}_p(1)/F_\infty\right) = 0$*. In particular,* Conjecture B *holds for* (E, F_∞) *and* $\mu_{p^\infty}(F)$ *is a universal norm from* F_∞*.*

Proof. We are grateful to the referee for suggesting the following proof, which is simpler than the one we had in a first version of our manuscript. Using the Poitou–Tate sequence over F as in [29, p. 553 §2.4-(1)], we have that

$$0 \longrightarrow \mathrm{Cl}_{S_p}(F)_{p^\infty} \longrightarrow \mathcal{Z}^2_{S_p}\left(\mathbb{Z}_p(1)/F\right) \longrightarrow \bigoplus_{v\in S_p(F)} \mathbb{Z}_p \longrightarrow \mathbb{Z}_p \longrightarrow 0. \tag{13}$$

Since p is a prime of supersingular reduction for E/K, we know that there exists a unique prime above p in K. Moreover, p is totally ramified in the extension $\mathrm{Gal}(F_\infty/K)$, see for example [53, Section 1]. In particular, there is a unique prime above p in F, *i. e.* $\left|S_p\right| = 1$. Combining this with the assumption that the p-Sylow subgroup of the S_p-class group is trivial yields, through (13), that $\mathcal{Z}^2_{S_p}\left(\mathbb{Z}_p(1)/F\right) = 0$. By Nekovar's spectral sequence (see [44, Corollary 8.4.8.4-(ii)]), we obtain that

$$\mathcal{Z}^2_{S_p}\left(\mathbb{Z}_p(1)/F_\infty\right)_G \simeq \mathcal{Z}^2_{S_p}\left(\mathbb{Z}_p(1)/F\right) = 0.$$

Now, employing Nakayama's Lemma we conclude that $\mathcal{Z}^2_{S_p}\left(\mathbb{Z}_p(1)/F_\infty\right) = 0$.

Next, consider the exact sequence (see, for example, [10, p. 330 (2.6)])

$$0 \longrightarrow \mathfrak{Y}\left(\mathbb{Z}_p(1)/F_\infty\right) \longrightarrow \mathcal{Z}^2_{S_p}\left(\mathbb{Z}_p(1)/F_\infty\right) \longrightarrow \bigoplus_{v\in S_p(F_\infty)} \mathbb{Z}_p \longrightarrow \mathbb{Z}_p \longrightarrow 0.$$

It follows from the first part of the proof that $\mathfrak{Y}\left(\mathbb{Z}_p(1)/F_\infty\right) = 0$. Moreover, the same descent argument as above using Nekovar's spectral sequence shows that $\mathfrak{Y}\left(\mathbb{Z}_p(1)/F_{\mathrm{cyc}}\right) = X_{\mathrm{cs}}^{F_{\mathrm{cyc}}} = 0$: in particular, $\mu(X_{\mathrm{cs}}^{F_{\mathrm{cyc}}}) = 0$. By [45, Corollary 11.3.16], we know that $\mu(X_{\mathrm{cs}}^{F_{\mathrm{cyc}}}) = \mu(X_{\mathrm{nr}}^{F_{\mathrm{cyc}}}) = 0$. Therefore, we obtain that the Iwasawa $\mu = 0$ Conjecture holds for F, and we can apply Theorem 4.10, showing that Conjecture B holds.

To prove the final assertion, consider the exact sequence (see [10, p. 335 (3.26)])

$$\begin{aligned} 0 \longrightarrow H_2\Big(G, \mathcal{Z}^2_{S_p}\left(\mathbb{Z}_p(1)/F_\infty\right)\Big) \longrightarrow \mathcal{Z}^1_{S_p}\left(\mathbb{Z}_p(1)/F_\infty\right)_G &\xrightarrow{\tau_{F_\infty/F}} \mathcal{Z}^1_{S_p}\left(\mathbb{Z}_p(1)/F\right) \\ &\longrightarrow H_1\Big(G, \mathcal{Z}^2_{S_p}\left(\mathbb{Z}_p(1)/F_\infty\right)\Big) \longrightarrow 0. \end{aligned}$$

We have shown above that $\mathcal{Z}^2_{S_p}\left(\mathbb{Z}_p(1)/F_\infty\right) = 0$; hence, $\tau_{F_\infty/F}$ is an isomorphism. It follows that $\mu_{p^\infty}(F)$ is a universal norm from F_∞ (see [10, Corollary 3.27] for details). □

The following corollary provides asymptotics for the growth of the p-primary torsion of the fine Selmer group at each layer of the $\mathbb{Z}_p^2$-extension.

Corollary 4.14. *Suppose that* **Ass 1** *holds and that either*

(i) $\mathfrak{Y}(\mathsf{E}/F_{\mathrm{cyc}})$ *is finite; or*

(ii) $\mathfrak{X}(\mathsf{E}/F_{\mathrm{cyc}})$ *is a finitely generated* $\mathbb{Z}_p$*-module of* $\mathbb{Z}_p$*-rank equal to* 2 *and* $\mathsf{E}(F_\infty)$ *has a point of infinite order.*

Then

$$\mathrm{ord}_p\Big(R\big(\mathsf{E}/F(\mathsf{E}_{p^n})\big)^\vee[p^\infty]\Big) = O\left(p^n\right).$$

If, moreover, **Ass 2** *holds, then* $\mathrm{ord}_p\left(R\left(\mathsf{E}/F(\mathsf{E}_{p^n})\right)^\vee[p^\infty]\right) = 0.$

Proof. Conjecture A holds by assumption in each case and Conjecture B holds by Theorem 4.6 in case (i) and by Proposition 4.8 in case (ii) The first claim follows from [33, Corollary 6.14].

When **Ass 2** holds, a better estimate can be obtained, and we thank the referee for this observation. Since F_∞ is the trivializing extension, we have

$$\mathcal{Z}^2_{S_p}(T_p\mathsf{E}/F_\infty) \simeq \mathcal{Z}^2_{S_p}(\mathbb{Z}_p(1)/F_\infty)\otimes T_p\mathsf{E}$$

and thus Proposition 4.13 implies that $\mathcal{Z}^2_{S_p}(T_p\mathsf{E}/F_\infty) = 0$. By Nekovar's spectral sequence, we also know that

$$\mathcal{Z}^2_{S_p}\left(T_p\mathsf{E}/F(\mathsf{E}_{p^n})\right) \simeq \mathcal{Z}^2_{S_p}(T_p\mathsf{E}/F_\infty)_{G_n} = 0.$$

Since $\mathfrak{Y}\left(\mathsf{E}/F(\mathsf{E}_{p^n})\right)$ is contained in $\mathcal{Z}^2_{S_p}\left(T_p\mathsf{E}/F(\mathsf{E}_{p^n})\right)$, the result follows. □

4.3. **The noncommutative setting.** Even though this paper largely treats the commutative case, an analogue of Theorem 4.10 is valid even in the noncommutative setting. We would like to thank the referee for pointing this out, and for insisting that the general case be included. Let S be a finite set of primes of F containing the primes above p , the archimedean primes and the primes of bad reduction for E.

Theorem 4.15. *Let* E/F *be an elliptic curve without complex multiplication and* $p \geq 5$ *be a rational prime. Assume that* F *contains the* p*-torsion points of* E. *Suppose that the elliptic curve has either potential ordinary or potential multiplicative reduction at all the primes* $v \mid p$*, that* F *contains the* p*-torsion points of* E*, and let* $F_\infty = F(\mathsf{E}_{p^\infty})/F$ *be the trivializing extension. Write* $S = S_p \cup S_{\mathrm{bad}} \cup S_\infty$ *and set* $G = \mathrm{Gal}(F_\infty/F), H = \mathrm{Gal}(F_\infty/F_{\mathrm{cyc}})$*. Then the following assertions are equivalent:*

(a) *The* Iwasawa $\mu = 0$ Conjecture *is true for* F *and* $X_{\mathrm{nr}}^{F_\infty}$ *is* $\Lambda(G)$*-pseudonull.*
(b) Conjecture B *holds for* (E, F_∞).
(c) *The* Iwasawa $\mu = 0$ Conjecture *is true for* F *and* $\mathfrak{Y}(\mathbb{Z}_p(1)/F_\infty)$ *is* $\Lambda(G)$*-pseudonull.*

Proof. As in the proof of Theorem 4.10, the equivalence between (a) and of (c) will follow once we prove that $X_{\mathrm{nr}}^{F_\infty}$ and $X_{\mathrm{cs}}^{F_\infty}$ are pseudo-isomorphic. We know from [7, Lemma 2.8] that $G_{F_\infty/F,v}$ has dimension at least 2 at primes $v \in S$; here we have used the fact that when v does not divide p, it is not possible in our setting that E has bad but potential good reduction (see for example, [50, the paragraph after Theorem 5.2]). It follows that $\bigoplus_{v\in S}\mathrm{Ind}_{G_{F_\infty/F}}^{G_{F_\infty/F,v}}\left(\mathbb{Z}_p\right)$ is $\Lambda(G_{F_\infty/F,v})$-pseudonull.

We next prove the implication (c) $\Rightarrow$ (b): assume that assertion (c) holds. Then, by [11, Lemma 3.8, p. 825], there is an isomorphism

$$\mathfrak{Y}(\mathsf{E}/F_\infty) \simeq \mathfrak{Y}\left(\mathbb{Z}_p(1)/F_\infty\right)\otimes \mathsf{E}^\vee_{p^\infty},$$

where the tensor product is over $\mathbb{Z}_p$, and the action of $\mathrm{Gal}(F(\mathsf{E}_{p^\infty})/F)$ on the right hand-side is the diagonal action. The hypothesis that the Iwasawa $\mu = 0$ Conjecture holds for F assures us of the finite generation of $\mathfrak{Y}\left(\mathbb{Z}_p(1)/F_\infty\right)$ as a $\Lambda(H)$-module. Hence, to show that assertion (b) holds, it suffices to prove that $\mathfrak{Y}(\mathsf{E}/F_\infty)$ is a torsion $\Lambda(H)$-module.

The pseudonullity of $\mathfrak{Y}(\mathsf{E}/F_\infty)$ as a $\Lambda(G)$-module is equivalent to $\mathfrak{Y}(\mathsf{E}/F_\infty)$ being torsion as a $\Lambda(H)$-module. By our hypothesis that $\mathfrak{Y}\left(\mathbb{Z}_p(1)/F_\infty\right)$ is pseudonull, it follows that it is also finitely generated and torsion as a $\Lambda(H)$-module. Thus there exists a finite, free $\Lambda(H)$-resolution of $\mathfrak{Y}\left(\mathbb{Z}_p(1)/F_\infty\right)$ such that the alternating sum of the $\Lambda(H)$-ranks of the free modules in the resolution is 0. Tensoring such a resolution over $\mathbb{Z}_p$ with $\mathsf{E}_{p^\infty}^\vee$ preserves the exactness and gives a finite, free $\Lambda(H)$-resolution of $\mathfrak{Y}\left(\mathbb{Z}_p(1)/F_\infty\right) \otimes \mathsf{E}_{p^\infty}^\vee$. Further, the alternating sum of the $\Lambda(H)$-ranks of the free modules is still 0, whence $\mathfrak{Y}\left(\mathbb{Z}_p(1)/F_\infty\right) \otimes \mathsf{E}_{p^\infty}^\vee$ is $\Lambda(H)$-torsion. This proves assertion (b). Note that this argument also proves the equality of the $\Lambda(H)$-ranks of $\mathfrak{Y}(\mathbb{Z}_p(1)/F_\infty)$ and $\mathfrak{Y}(\mathsf{E}/F_\infty)$.

It remains to prove the implication (b) $\Rightarrow$ (c). Suppose that Conjecture B is true for (E, F_∞). Then Conjecture A is true and the dual fine Selmer group $\mathfrak{Y}(\mathsf{E}/F_{\mathrm{cyc}})$ of E over the cyclotomic $\mathbb{Z}_p$-extension is a finitely generated $\mathbb{Z}_p$-module. The vanishing of the Iwasawa μ-invariant for F_{cyc} is a consequence of [11, Theorem 3.4]. Now suppose that $\mathfrak{Y}(\mathbb{Z}_p(1)/F_\infty)$ is not $\Lambda(H)$-torsion, and hence has positive rank as a $\Lambda(H)$-module. By the remark above on the equality of $\Lambda(H)$-ranks, this implies that $\mathfrak{Y}(\mathsf{E}/F_\infty)$ also has positive $\Lambda(H)$-rank, contradicting the hypothesis. This completes the proof of the equivalence. □

5. Conjecture B and the Generalized Greenberg's Conjecture

The aim of this section is to clarify the connection between the Generalized Greenberg's Conjecture and Conjecture B for CM elliptic curves. For the sake of brevity, we henceforth refer to the Generalized Greenberg's Conjecture as GGC.

Both conjectures pertain to the pseudonullity of certain Iwasawa modules. Even though Conjecture B was proposed as a generalization of GGC, the precise formulation of this connection is rather intricate. Using Theorem 4.10, we make precise in which sense Conjecture B for CM elliptic curves is a generalization of GGC (see Theorem 5.4).

Fix an imaginary quadratic field K and denote its Hilbert class field by K'. Given an elliptic curve E/K' with CM by an order in K, set

$$F = K'(\mathsf{E}_p), \quad F_\infty = K'(\mathsf{E}_{p^\infty}) = F(\mathsf{E}_{p^\infty}),$$
$$G = \mathrm{Gal}(F_\infty/F), \quad \mathcal{G}_\infty = \mathrm{Gal}(F_\infty/K), \quad \mathcal{G}'_\infty = \mathrm{Gal}(F_\infty/K').$$

Note that $G \simeq \mathbb{Z}_p^2$. Set $\widetilde{K}$ (*resp.* $\widetilde{K'}$, $\widetilde{F}$) to be the compositum of all $\mathbb{Z}_p$-extensions of K (*resp.* of K', of F). Since the Leopoldt conjecture is true for imaginary quadratic fields, $\widetilde{K}$ is the unique $\mathbb{Z}_p^2$-extension of K. For the rest of this section, we make the following assumption.

Ass 3
(i) p is an odd prime that is unramified in K;
(ii) the prime p is such that $K' \cap \widetilde{K} = K$.

By the theory of complex multiplication, $\mathcal{G}_\infty = G \times \Delta$ and $\mathcal{G}'_\infty = G \times \Delta'$ where $\Delta \simeq \mathrm{Gal}(F/K)$ (*resp.* $\Delta' \simeq \mathrm{Gal}(F/K')$) is a finite abelian group. Recall from [58, Remark on p. IV-13] that Δ' is a Cartan subgroup of $\mathrm{GL}_2(\mathbb{F}_p)$ and hence it either has order $p^2 - 1$ or $(p-1)^2$: in any case, $p \nmid \left|\Delta'\right|$.

Remark 5.1.

(1) In fact, it is forced by **Ass 3**-(i) that $\mu_p \not\subset K$. This can be seen as follows: the

only pair (p, K) for which $\mu_p \subset K$ is when $p = 3$ and $K = \mathbb{Q}(\sqrt{-3})$; but this contradicts **Ass 3**-(i).

(2) We now discuss **Ass 3**-(ii) in a little more detail. This assumption is trivially satisfied when p does not divide the class number of K. But observe that, in general, $K' \cap \widetilde{K}$ is contained in the anti-cyclotomic $\mathbb{Z}_p$-extension of K, denoted by K_{ac}. For a proof of this fact, see [15, Lemma 2.2]. Therefore, **Ass 3**-(ii) is equivalent to the following condition:

(ii′) The prime p is such that $K' \cap K_{\mathrm{ac}} = K$.

To know more about non-trivial examples where this condition is satisfied, we refer the reader to [6]. For a specific example, see Example 4 *ibid*. Moreover, **Ass 3**-(ii) is closely related to the notion of p-rationality (see [6, p. 2133]) but we will not discuss this point any further.

Set the notation K'_∞ to denote the composite of the fields K' and $\widetilde{K}$. The theory of complex multiplication guarantees that $F_\infty = F\widetilde{K} = FK'_\infty$. Recall that F_∞ is the trivializing extension for the Galois representation associated to $T_p\mathsf{E}$ and it is an S-admissible p-adic Lie extension. We note that $F_\infty \subseteq \widetilde{F}$.

Denote by $L(\widetilde{F})$ (*resp.* $L(F_\infty)$) the maximal abelian unramified pro-p-extension of $\widetilde{F}$ (*resp.* of F_∞). Denote by $\mathscr{F}_S$ the maximal abelian pro-p extension of $\widetilde{F}$ unramified outside S. Set the notation

$$X_{\mathrm{nr}}^{\widetilde{F}} = \mathrm{Gal}\left(L(\widetilde{F})/\widetilde{F}\right), \quad X_{\mathrm{nr}}^{F_\infty} = \mathrm{Gal}\left(L(F_\infty)/F_\infty\right), \quad X_S^{\widetilde{F}} = \mathrm{Gal}\left(\mathscr{F}_S/\widetilde{F}\right). \tag{14}$$

As in the previous sections, given any extension $\mathcal{L}/F$, we denote by $M(\mathcal{L})$ the maximal unramified abelian p-extension of $\mathcal{L}$ where all primes above p in $\mathcal{L}$ split completely; this group is related to the fine Selmer group (see (7)). For most of the discussion, $\mathcal{L}$ will either be F_∞ or $\widetilde{F}$. For convenience, the diagram of fields is drawn in Figure 1.

Recall the statement of GGC for F (the statement for K is analogous, by replacing $F, \widetilde{F}, \Lambda(G_{\widetilde{F}/F})$ by $K, \widetilde{K}, \Lambda(G_{\widetilde{K}/K})$, respectively).

GGC. *With notation as above, $X_{\mathrm{nr}}^{\widetilde{F}}$ is a pseudonull $\Lambda(G_{\widetilde{F}/F})$-module.*

The following results are required to relate GGC to the pseudonullity of the fine Selmer group. The first lemma assures pseudonullity over a larger tower, once it holds for a proper subextension.

Lemma 5.2 (Pseudonullity Lifting Lemma)**.** *Let $n \geq 3$, let $\mathcal{F}/\mathbb{Q}$ be a finite Galois extension containing μ_p, and denote by $\widetilde{\mathcal{F}}$ the compositum of all $\mathbb{Z}_p$-extensions of $\mathcal{F}$. Suppose that* $\mathrm{Gal}(\widetilde{\mathcal{F}}/\mathcal{F}) \simeq \mathbb{Z}_p^n$ *and let $\mathcal{F}^{(d)} \subsetneq \widetilde{\mathcal{F}}$ be such that* $\mathrm{Gal}(\mathcal{F}^{(d)}/\mathcal{F}) \simeq \mathbb{Z}_p^d$ *for some $2 \leq d < n$. If $X_{\mathrm{nr}}^{\mathcal{F}^{(d)}}$ is $\Lambda(G_{\mathcal{F}^{(d)}/\mathcal{F}})$-pseudonull then* GGC *holds for $\widetilde{\mathcal{F}}/\mathcal{F}$.*

Proof. This lemma is a special case of [2, Theorem 12]. Since $\mathcal{F}$ contains μ_p, the technical conditions in the mentioned theorem are satisfied by [35, Theorem 3.2] or [2, Remark 15]. □

The next result studies pseudonullity of Galois modules under base change.

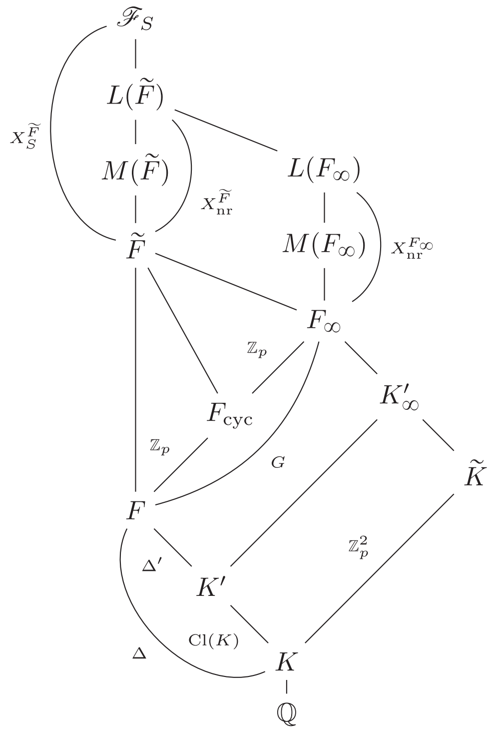

Fig. 1. The diagram of fields occurring in Theorem 5.4

Lemma 5.3 (Pseudonullity Shifting Down Lemma)**.** *Let $\mathcal{F}$ be a number field and let $\mathcal{F}^{(d)}/\mathcal{F}$ be a $\mathbb{Z}_p^d$-extension. Suppose that $\mathcal{F}_1/\mathcal{F}$ is a finite extension and set $\mathcal{K} = \mathcal{F}_1 \cdot \mathcal{F}^{(d)}$. If $X_{\mathrm{nr}}^{\mathcal{K}}$ is a $\Lambda(G_{\mathcal{K}/\mathcal{F}_1})$-pseudonull module, then $X_{\mathrm{nr}}^{\mathcal{F}^{(d)}}$ is a $\Lambda(G_{\mathcal{F}^{(d)}/\mathcal{F}})$-pseudonull module.*

Proof. For a proof, see [31, Theorem 3.1-(i)]. □

The purpose of the next result is to show that Conjecture B is indeed a generalization of GGC. We resume the notation introduced at the beginning of this section.

Theorem 5.4. *In the setting of* **Ass 3***, suppose that there exists an elliptic curve E/K' with CM by an order in K such that* Conjecture B *holds for* (E, F_∞)*. Then* GGC *holds for K.*

Proof. Let E/K' be an elliptic curve with CM by an order in K such that Conjecture B holds for (E, F_∞). Regarding it as being defined over $F = K'(\mathsf{E}_p)$, Theorem 4.10 shows that $X_{\mathrm{nr}}^{F_\infty}$ is $\Lambda(G_{F_\infty/F})$-pseudonull.

Applying Lemma 5.3 with $\mathcal{F} = K'$, $\mathcal{F}_1 = F$, $\mathcal{F}^{(2)} = K'_\infty$, and $\mathcal{K} = F_\infty = FK'_\infty$, the $\Lambda(G_{F_\infty/F})$-pseudonullity of $X_{\mathrm{nr}}^{F_\infty}$ can be shifted down to $\Lambda(G_{K'_\infty/K'})$-pseudonullity of $X_{\mathrm{nr}}^{K'_\infty}$. Therefore, we have shown that

$$\text{Conjecture B for } (\mathsf{E}, F_\infty) \Longrightarrow X_{\mathrm{nr}}^{K'_\infty} \text{ is } \Lambda(G_{K'_\infty/K'})\text{-pseudonull.}$$

Another application of Lemma 5.3 with $\mathcal{F} = K$, $\mathcal{F}_1 = K'$, $\mathcal{F}^{(2)} = \widetilde{K}$, and $\mathcal{K} = K'_\infty = K'\widetilde{K}$, shows that $\Lambda(G_{K'_\infty/K'})$-pseudonullity of $X_{\mathrm{nr}}^{K'_\infty}$ can be shifted down to $\Lambda(G_{\widetilde{K}/K})$-pseudonullity of $X_{\mathrm{nr}}^{\widetilde{K}}$. This is GGC for K. □

Corollary 5.5. *With the same hypotheses of* Theorem 5.4*,* GGC *holds also for any number field L such that $K'(\mu_p) \subseteq L \subseteq F$.*

Proof. Applying Lemma 5.3 with $\mathcal{F} = L$, $\mathcal{F}_1 = F$, $\mathcal{F}^{(2)} = LK'_\infty$, and $\mathcal{K} = F_\infty = F\mathcal{F}^{(2)}$, the $\Lambda(G_{F_\infty/F})$-pseudonullity of $X_{\mathrm{nr}}^{F_\infty}$ obtained in Theorem 4.10 can be shifted down to $\Lambda(G_{\mathcal{F}^{(2)}/\mathcal{F}})$-pseudonullity of $X_{\mathrm{nr}}^{\mathcal{F}^{(2)}}$. Therefore, we have shown that

$$\text{Conjecture B for } (\mathsf{E}, F_\infty) \Longrightarrow X_{\mathrm{nr}}^{\mathcal{F}^{(2)}} \text{ is } \Lambda(G_{\mathcal{F}^{(2)}/L})\text{-pseudonull.}$$

As discussed in Remark 5.1-(1), $\mu_p \not\subset K$, hence $L \neq K$ and L admits at least two complex embeddings. Letting $\widetilde{L}$ denote the compositum of all $\mathbb{Z}_p$-extensions of L, [67, Theorem 13.4] implies that $\mathrm{Gal}(\widetilde{L}/L) \cong \mathbb{Z}_p^n$ for some $n \geq 3$. Using Lemma 5.2 with $\mathcal{F} = L$, pseudonullity of $X_{\mathrm{nr}}^{\mathcal{F}^{(2)}}$ as a $\Lambda(G_{\mathcal{F}^{(2)}/L})$-module implies GGC for L. □

Acknowledgements. D. K. is supported by a PIMS Postdoctoral fellowship. S. R. is supported by the NSERC Discovery Grant 2019-03987. We thank the referee for their timely and thorough reading of the manuscript which helped strengthen some of the results and also contributed to a clearer exposition.

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Debanjana Kundu
Department of Mathematical and Statistical Sciences
University of Texas Rio Grande Valley
Edinburg, TX 78539
USA
e-mail: dkundu@math.toronto.edu

Filippo Alberto Edoardo Nuccio Mortarino Majno di Capriglio
Université Jean Monnet Saint-Étienne, CNRS
Institut Camille Jordan UMR 5208
F–42023, Saint-Étienne
France
e-mail: filippo.nuccio@univ-st-etienne.fr

Sujatha Ramdorai
Mathematics Department
1984, Mathematics Road
University of British Columbia
Vancouver BC, V6T 1Z2
Canada
e-mail: sujatha@math.ubc.ca