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ON RIGIDITY OF HOLOMORPHIC MAPS OF RIEMANN SURFACES

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0. Introduciton

In this paper, we investigate holomorphic maps of Riemann surfaces using homology groups and free homotopty. There is a famous finiteness theorem concering holomorphic maps of compact Riemann surfaces.

Theorem 1 (de Franchis [3]). *Let $\tilde{X}$ be a compact Riemann surface of genus* >1.

(1) *For a fixed compact Riemann surface X of genus >1, the number of nonconstant holomorphic maps $\tilde{X}\to X$ is finite.*

(2) *There are only finitely many compact Riemann surfaces X_i of genus >1 such that there exists a nonconstant holomorphic map $\tilde{X}\to X_i$.*

The second assertion (2) is often attributed to Severi. For algebraic proofs of Theorem 1, see e.g. Kani [9], Martens [11][12], and Howard and Sommese [5]. Imayoshi [6][7] gave analytic proofs of these for Riemann surfaces of finite types.

Here we will study holomorphic maps of compact Riemann surfaces in terms of homology groups. We will show some rigidity theorems which guarantee Theorem 1. Let $\tilde{X}$, X be compact Riemann surfaces of genera $\tilde{g}$, $g(>1)$, and let $\{\tilde{\chi}_1,\cdots,\tilde{\chi}_{2\tilde{g}}\}$, $\{\chi_1,\cdots,\chi_{2g}\}$ be canonical homology bases on $\tilde{X}$, X, respectively. Let $h_i:\tilde{X}\to X$ be a nonconstant holomorphic map, and $M_i\in M\ (2g,2\tilde{g};\boldsymbol{Z})$ be a maxtrix representation of h_i $(i=1,2)$ with respect to $\{\tilde{\chi}_1,\cdots,\tilde{\chi}_{2\tilde{g}}\}$, $\{\chi_1,\cdots,\chi_{2g}\}$. Then, we will show

Theorem 2. *If there is an integer $l>\sqrt{8(\tilde{g}-1)}$ with $M_1\equiv M_2$ (mod. l), then $h_1=h_2$.*

In Theorem 2, the assumption concerns all of the entries of M_1, M_2. If we take l larger, we may assume conditions concerning merely a half number of entries of M_1, M_2 to get the same conclusion. For M_i, write

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$$M_i=\begin{pmatrix} M_{i1} & M_{i2} \\ M_{i3} & M_{i4} \end{pmatrix},$$

where M_{ij} $(j=1,\cdots,4)$ is $g\times\tilde{g}$ sized.

Theorem 3. *Suppose that there is an integer* $l>8(\tilde{g}-1)$ *with* $M_{1j}\equiv M_{2j}$ (mod. l) *for* $j=1,2$ *or for* $j=1,3$. *Then* $h_1=h_2$.

In Theorems 2, 3, the target X is fixed. But the following theorem says that if the number l in Theorem 2 is slightly large, then the matrix representation determines the target. We will show

Theorem 4. *Let* X_1 *and* X_2 *be compact Riemann surfaces of the same genus* $g>1$. *Let* $\{\tilde{\chi}_1,\cdots,\tilde{\chi}_{2\tilde{g}}\}$, $\{\chi_{i,1},\cdots,\chi_{i,2g}\}$ $(i=1,2)$ *be canonical homology bases on* $\tilde{X}$, X_i $(i=1,2)$. *Let* $h_i:\tilde{X}\to X_i$ *be a nonoconstant holomorphic map, and* M_i *be the matrix representation of* h_i *with repsect to* $\{\tilde{\chi}_1,\cdots,\tilde{\chi}_{2\tilde{g}}\}$, $\{\chi_{i,1},\cdots,\chi_{i,2g}\}$ $(i=1,2)$. *Suppose that there is an integer* $l>\sqrt{8}(\tilde{g}-1)$ *with* $M_1\equiv M_2$ (mod. l). *Then* X_1, X_2 *are conformally equivalent and there exists a conformal map* $f:X_1\to X_2$ *with* $f\circ h_1=h_2$.

Also, we will study conditions for an automorphism of a Riemann surface to be the identity in terms of free homotopy. Related to this problem, Marden, Richards, and Rodin [10] derived various results by using certain covering surfaces. Some of their results were improved by Jenkins and Suita [8]. Taniguchi [13] proved several theorems including an interesting theorem below.

Theorem 5 (Taniguchi [13]). *Let* X *be a compact Riemann surface of genus* $g(>1)$, $\{\alpha_i\}_{i=1}^n$ *be an admissible curve system on* X, *and let* $T\in\mathrm{Aut}(X)$ *such that* $T(\alpha_i)\sim\alpha_i$ *for every* i.

If $n\geq g$, *then* T *is the identity.*

Here $\sim$ denotes the equivalence in free homotopy.

We will use certain surfgeries of Riemann surfaces in order to get a relation between fixed points and freely homotopically fixed loops of an automorphism. As a theorem we will give

Theorem 6. *Let* X *be a compact Riemann surface of genus* $g>1$, *and* $id.\neq T\in\mathrm{Aut}(X)$. *Suppose that there is an admissible curve system* $\{\alpha_i\}_{i=1}^n$ *on* X *such that*

$X\backslash\{\alpha_i\}_{i=1}^n$ *is connected and* $T(\alpha_i)\sim\alpha_i$ *for every* $i\in\{1,\cdots,n\}$. *Then we have*

$$k+4n\leq 2g+2,$$

where $k \geq 0$ *is the number of fixed points of* T.

We see immediately that this theorem is an extension of a well-known fact that $T \in \mathrm{Aut}(X)$ which is not the identity has at most $2g+2$ fixed points. We will also give a simple proof of Theorem 5.

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1. The space of Hurwitz relations and the Rosati adjoint

Let $\tilde{X}$, X be compact Riemann surfaces of genera $\tilde{g}$, $g(>1)$. We denote by $H_1(X)$ the first homology group (with integer coefficients) of X. Any basis for $H_1(X)$ (say $\{\chi_1,\cdots,\chi_{2g}\}$), with intersection matrix (that is a matrix whose (k,j)-entry is given by the intersection number $\chi_k \cdot \chi_j$),

$$J=\begin{pmatrix} 0 & E \\ -E & 0 \end{pmatrix}$$

will be called a canonical homology basis, where E is the identity matrix of $g \times g$ sized. Similarly for $\tilde{X}$. Let $\{\tilde{\chi}_1,\cdots,\tilde{\chi}_{2\tilde{g}}\}$ (resp. $\{\chi_1,\cdots,\chi_{2g}\}$) be a canonical homology basis on $\tilde{X}$ (resp. X). Let $\{\tilde{w}^1,\cdots,\tilde{w}^{\tilde{g}}\}$ and $\{w^1,\cdots,w^g\}$ be the dual bases for holomorphic differentials on $\tilde{X}$, X (i.e. $\int_{\chi_j} w^k = \delta_{jk}$ where δ_{jk} is Kronecker's delta), and $\tilde{\Pi}=(\tilde{E},\tilde{Z})$, $\Pi=(E,Z)$ be the associated period matrices. Let $h:\tilde{X} \to X$ be a nonconstant holomorphic map. Then h induces a homomorphism $h_*:H_1(\tilde{X}) \to H_1(X)$. Let $M=(m_{kj}) \in M(2g,2\tilde{g};\boldsymbol{Z})$, where $h_*(\tilde{\chi}_j)=\Sigma_{k=1}^{2g} m_{kj}\chi_k$. (We denote by $M(m,n;K)$ the set of $m \times n$ matrices with K-coefficients.) We will call M the matrix representation of h with respect to $\{\tilde{\chi}_1,\cdots,\tilde{\chi}_{2\tilde{g}}\}$ and $\{\chi_1,\cdots,\chi_{2g}\}$.

The integral $\int_{h_*(\tilde{\chi}_j)} w^i$ is evaluated in two ways; by expressing $h_*(\tilde{\chi}_j)$ in $H_1(X)$ or by expressing the pull back of w^i in terms of the holomorphic differentials on $\tilde{X}$. This leads us to the so-called Hurwitz relation (see [11, p.210])

$$\text{(a)} \qquad A\tilde{\Pi}=\Pi M,$$

where $A \in M(g,\tilde{g};\boldsymbol{C})$. The set of all $M \in M(2g,2\tilde{g};\boldsymbol{Q})$ satisfying (a) for some $A \in M(g,\tilde{g};\boldsymbol{C})$ is called the space of Hurwitz relations. It depends on the choice of the period matrices. We see immediately that it is a $\boldsymbol{Q}$-vector space. We denote it by $S(\tilde{\Pi},\Pi)$.

Lemma 1 ([12, p.534]). *In the space of Hurwitz relations* $S(\tilde{\Pi},\Pi)$, $\langle M,N\rangle = tr(\tilde{J}\,{}^tMJ^{-1}N)$ *defines an inner product* (${}^tM$ *denotes the transposition of* M).

DEFINITION. We define a norm in $S(\tilde{\Pi},\Pi)$ by

$$\|M\| = \langle M,M\rangle^{\frac{1}{2}}.$$

Lemma 2. *If M is a matrix representation of a holomorphic map $h:\tilde{X}\to X$, then*

$$\|M\|^2=2dg\leq 4(\tilde{g}-1),$$

where d is the degree of the holomorphic map h.

Although the equality $\|M\|^2=2dg$ is already known (see e.g. [12, p.534]), we will give a new proof using harmonic differentials here.

Proof. We will show that $M\tilde{J}{}^tM=dJ$ which implies $\|M\|^2=trM\tilde{J}{}^tMJ^{-1}=2dg$.

Let $\{\tilde{\alpha}^1,\cdots,\tilde{\alpha}^{2\tilde{g}}\}$ (resp. $\{\alpha^1,\cdots,\alpha^{2g}\}$) be the dual basis for harmonic differentials (i.e. $\int_{\tilde{\chi}j}\tilde{\alpha}^k=\delta_{jk}$) on $\tilde{X}$ (resp. X). We denote by $\alpha^k\circ h$ the pull-back of α^k. Then, denoting by h_* the induced homomorphism between homology groups, we may write

$$\alpha^k\circ h=\sum_{j=1}^{2\tilde{g}}\alpha_{kj}\tilde{\alpha}^j \qquad (a_{kj}\in \boldsymbol{C}),$$

and

$$m_{kj}=\int_{h_*(\tilde{\chi}j)}\alpha^k=\int_{\tilde{\chi}j}\alpha^k\circ h=\int_{\tilde{\chi}j}\sum_{j=1}^{2\tilde{g}}a_{kj}\tilde{\alpha}^j=a_{kj}.$$

Thus

$$\alpha^k\circ h=\sum_{j=1}^{2\tilde{g}}m_{kj}\tilde{\alpha}^j.$$

Then we have

$$\iint_{h_*(\tilde{X})}\alpha^k\wedge\alpha^j=\iint_{\tilde{X}}\alpha^k\circ h\wedge\alpha^j\circ h=\iint_{\tilde{X}}\sum_{i=1}^{2\tilde{g}}m_{ki}\tilde{\alpha}^i\wedge\sum_{i=1}^{2\tilde{g}}m_{ji}\tilde{\alpha}^i.$$

The left-hand side is equal to the (k,j)-entry of dJ, and the right-hand side is equal to the (k,j)-entry of $M\tilde{J}{}^tM$, since a matrix whose (k,j)-entry is $\iint_X\alpha^k\wedge\alpha^j$ has the form

$$J=\begin{pmatrix}0 & E\\ -E & 0\end{pmatrix}.$$

Now we see that $M\tilde{J}{}^tM=dJ$.

To see the inequality holds, recall the Riemann-Hurwitz relation

$$2(\tilde{g}-1)=2d(g-1)+B,$$

where B is the total branch number of h. By the assumption $g>1$, we have

$$d=\frac{\tilde{g}-1}{g-1}-\frac{B}{2g-2}\leq\tilde{g}-1,$$

and

$$dg=\tilde{g}-1-\frac{B}{2}+d\leq 2(\tilde{g}-1).$$

□

We denote the Jacobian variety of X by $J(X)=\boldsymbol{C}^g/\Gamma$, where Γ is the lattice (over $\boldsymbol{Z}$) generated by 2g-columns of Π. Similarly for $J(\tilde{X})$. Let $\tilde{\kappa}:\tilde{X}\to J(\tilde{X})$, $\kappa:X\to J(X)$ be canonical injections. The following lemma is known (see e.g. [2, p.137]).

Lemma 3. *Let $\Phi:X\to T$ be a holomorphic map of a compact Riemann surface X (of genus >0) into a complex torus T. Then there exists a unique holomorphic map $\Psi:J(X)\to T$ such that $\Phi=\Psi\circ\kappa$ holds.*

Here, by a complex torus T, we mean the quotient space, $T=\boldsymbol{C}^n/G$, where G is a group of translations generated by $2n$ $\boldsymbol{R}$-linearly independent vectors in $\boldsymbol{C}^n$ (see [2, p.133]). For any holomorphic map $h:\tilde{X}\to X$, there exists a homomorphism $H:J(\tilde{X})\to J(X)$ with $\kappa\circ h=H\circ\tilde{\kappa}$, because of Lemma 3. By an underlying real structure for $J(X)$, we mean the real torus $\boldsymbol{R}^{2g}/\boldsymbol{Z}^{2g}$ together with a map $\boldsymbol{R}^{2g}/\boldsymbol{Z}^{2g}\to J(X)$ induced by a linear map $\boldsymbol{R}^{2g}\ni x\mapsto\Pi x\in\boldsymbol{C}^g$. It is known that for any homomorphism $H:J(\tilde{X})\to J(X)$, there are $A\in M(g,\tilde{g};\boldsymbol{C})$ and $M\in M(2g,2\tilde{g};\boldsymbol{Z})$ such that the following diagram is commutative (see e.g. [2]).

$$\begin{array}{ccccc}
\boldsymbol{R}^{2\tilde{g}} & \xrightarrow{\tilde{\Pi}} & \boldsymbol{C}^{\tilde{g}} & \rightarrow & J(\tilde{X}) \\
\downarrow{\scriptstyle M} & & \downarrow{\scriptstyle A} & & \downarrow{\scriptstyle H} \\
\boldsymbol{R}^{2g} & \xrightarrow{\Pi} & \boldsymbol{C}^{g} & \rightarrow & J(X)
\end{array}$$

In particular, if H is induced by a holomorphic map $h:\tilde{X}\to X$, then $M\in M(2g,2\tilde{g};\boldsymbol{Z})$ is the matrix representation of h.

DEFINITION. For a nonconstant holomorphic map $h:\tilde{X}\to X$, we denote by

$$h^*(Q)=\tilde{Q}_1^{n_1}\cdots\tilde{Q}_k^{n_k},\ (\tilde{Q}_1,\cdots,\tilde{Q}_k\in\tilde{X})$$

a divisor of the preimages of $Q\in X$ with multiplicities. Defining $\tilde{\kappa}(h^*(Q))$ by linearity (i.e. $X\ni Q\mapsto\Sigma_{j=1}^k n_j\tilde{\kappa}(\tilde{Q}_j)\in J(\tilde{X})$), we get a holomorphic map $X\to J(\tilde{X})$. By Lemma 3, it can be extended to a homomorphism

$$H^*: J(X) \to J(\tilde{X}).$$

The homomorphism H^* is called the *Rosati adjoint* of H (see [12, p.535]).

The Rosati adjoint H^* is induced by the matrix $M^* = \tilde{J}\,{}^t\!MJ^{-1}$ acting on the underlying real tori (see [12, p.535]).

The following theorem will play a significant role in Sectiion 2.

Theorem 7 (Martens [11]). *Let X, X_1, X_2 be closed Riemann surfaces of genera >0. Let $h_i: X \to X_i$ be a nonconstant holomorphic map and $h_{i*}: H_1(X) \to H_1(X_i)$ be a homomorphism induced by h_i $(i=1,2)$. Assume that there is a homomorphism $\mathfrak{f}: H_1(X_1) \to H_1(X_2)$ with $\mathfrak{f} \circ h_{1*} = h_{2*}$. Then there is a unique holomorphic map $f: X_1 \to X_2$ with $f \circ h_1 = h_2$ and $f_* = \mathfrak{f}$.*

2. Rigidity theorems in terms of homology groups.

Now, we will give proofs of the rigidity theorems.

Proof of Theorem 2. Let $D = M_2 - M_1$. Then, by the assumption, $D \equiv 0$ (mod. l). Since M_1, M_2 are matrix representations of holomorphic maps, we may use Lemma 2 and have

$$\|D\| = \|M_2 - M_1\| \le \|M_2\| + \|M_1\| = (2d_2 g)^{\frac{1}{2}} + (2d_1 g)^{\frac{1}{2}},$$

where d_i is the digree of h_i $(i=1,2)$. By the inequality in Lemma 2,

$$\|D\|^2 \le 16(\tilde{g}-1). \tag{b}$$

Write

$$D = \begin{pmatrix} D_1 & D_2 \\ D_3 & D_4 \end{pmatrix},$$

where D_j $(j=1,\cdots,4)$ is $g \times \tilde{g}$ sized. Then

$$\tilde{J}\,{}^t\!DJ^{-1}D = \begin{pmatrix} -{}^t\!D_2 D_3 + {}^t\!D_4 D_1 & * \\ * & {}^t\!D_1 D_4 - {}^t\!D_3 D_2 \end{pmatrix}, \tag{c}$$

and we see that $\|D\|^2$ is a multiple of $2l^2$. Hence we may write $\|D\|^2 = 2al^2$, where a is a non-negative integer. But combining the inequality (b) and the assumption $l > \sqrt{8(\tilde{g}-1)}$, we see that a must be 0. Therefore $D=0$, that is $M_2 = M_1$. Using Theorem 7, we obtain $h_1 = h_2$. □

REMARK. Theorem 2 guarantees assertion (1) of Theorem 1 because it implies that the number of nonconstant holomorphic maps is less than $\# M(2g, 2\tilde{g}; \boldsymbol{Z}/(l)) = l^{4\tilde{g}g}$

for $l>\sqrt{8(\tilde{g}-1)}$.

Proof of Theorem 3. By the equality (c) in the proof of Theorem 2, we see that $\langle D,D\rangle=2al$, where $D=M_2-M_1$ and a is a non-negative integer. By the same consideration as in the proof of Theorem 2, we have $h_1=h_2$. □

REMARK. Theorem 3 also guarantees assertion (1) of Theorem 1.

Proof of Theorem 4. We use the Rosati adjoint. Let H_i^* be the Rosati adjoin of h_i, and let $G_i=M_i^*M_i=\tilde{J}\,{}^tM_iJ^{-1}M_i$ $(i=1,2)$. Then we have endomorphisms of $J(\tilde{X})$ with the matrices G_1, G_2 acting on the underlying real tori, that is, for G_i, there exists an $A\in M(\tilde{g},\tilde{g};C)$ with $A_i\tilde{\Pi}=\tilde{\Pi}G_i$ $(i=1,2)$. Martens ([12, p.535]) showed that the restricted map $H_i^*|_X$ is conformal. Also, he pointed out that if $G_1=G_2$, the targets X_1, X_2 are conformally equivalent and there exists a conformal map $f\colon X_1\to X_2$ with $f\circ h_1=h_2$. Indeed, we have the inverse map of $H_2^*|_X$, and if $G_1=G_2$, we compose $H_1^*|_X$ and the inverse map to obtain $f\colon X_1\to X_2$ with $f\circ h_1=h_2$. Thus, it suffices to show that $G_1=G_2$. Recall that $M_i\tilde{J}\,{}^tM_iJ^{-1}=d_iE$, where d_i is the degree of h_i. Thus we have

$$\tilde{J}^tG_i\tilde{J}^{-1}G_i=\tilde{J}({}^tM_iJM_i\tilde{J}^{-1})\tilde{J}^{-1}(\tilde{J}\,{}^tM_iJ^{-1}M_i)=d_i\tilde{J}\,{}^tM_iJ^{-1}M_i,$$

and

$$\|G_i\|^2=2d_i^2g,$$

where $\|\cdot\|$ is the norm in $S(\tilde{\Pi},\tilde{\Pi})$. By the assumption, we may write $M_2=M_1+D$, where $D\equiv 0$ (mod. l). Then,

$$G_2=\tilde{J}^t(M_1+D)J^{-1}(M_1+D)=G_1+\tilde{D},$$

where $\tilde{D}\equiv 0$ (mod. l). Using the triangle inequality, we have

$$\|\tilde{D}\|\le\|G_1\|+\|G_2\|=(2d_1^2g)^{1/2}+(2d_2^2g)^{1/2}.$$

By the inequality of Lemma 2 and

$$d=\frac{\tilde{g}-1}{g-1}-\frac{B}{2g-2}\le\tilde{g}-1,$$

we have

$$2d_i^2g\le 4(\tilde{g}-1)^2,$$

and we obtain $\|\tilde{D}\|^2\le 16(\tilde{g}-1)^2$. By the same consideration as in the proof of Theorem 2, we see $\tilde{D}=0$ or $G_1=G_2$. □

For a matrix $M\in M(m,n;\boldsymbol{Z})$, we denote by $M_l\in M(m,n;\boldsymbol{Z}/(l))$ the natural projection of M_i (i.e. $m_{ij} \mapsto [m_{ij}] \in \boldsymbol{Z}/(l)$, where $(m_{ij}) = M$).

Corollary. *Let X_1 and X_2 be compact Riemann surfaces of the same genus $g > 1$. Let $\{\tilde{\chi}_1, \cdots, \tilde{\chi}_{2\tilde{g}}\}$, $\{\chi_{i,1}, \cdots, \chi_{i,2g}\}$ $(i = 1,2)$ be cannonical homology bases on $\tilde{X}$, X_i $(i = 1,2)$. Let $h_i : \tilde{X} \to X_i$ be a nonconstant holomorphic map, and M_i be the matrix representation of h_i with respect to $\{\tilde{\chi}_1, \cdots, \tilde{\chi}_{2\tilde{g}}\}$, $\{\chi_{i,1}, \cdots, \chi_{i,2g}\}$ $(i = 1,2)$. Suppose that there is a prime number $l > \sqrt{8}(\tilde{g} - 1)$ and $S_l \in Sp(2g; \boldsymbol{Z}/(l))$ with $S_l M_{1l} = M_{2l}$, where $Sp(2g; \boldsymbol{Z}/(l))$ denotes the symplectic group. Then X_1, X_2 are conformally equvalent and there exists a conformal map $f : X_1 \to X_2$ with $f \circ h_1 = h_2$.*

Proof. Recall that the group $Sp(2g; \boldsymbol{Z}/(l))$ is generated by the symplectic transvections (see [1, Chapter 3.5]). A symplectic transvection clearly has preimages in $Sp(2g; \boldsymbol{Z})$ and the natural projection $Sp(2g; \boldsymbol{Z}) \to Sp(2g; \boldsymbol{Z}/(l))$ is homomorphic. Thus, for any given $S_l \in Sp(2g; \boldsymbol{Z}/(l))$, there is a $S \in Sp(2g; \boldsymbol{Z})$ whose natural projection is S_l. We denote by S_*^{-1} the automorphism of $H_1(X_1)$ induced by $S^{-1} \in Sp(2g; \boldsymbol{Z})$. Then, SM_1 is the matrix representation of h_1 with respect to $\{\tilde{\chi}_1, \cdots, \tilde{\chi}_{2\tilde{g}}\}$, $\{S_*^{-1}(\chi_{1,1}), \cdots, S_*^{-1}(\chi_{1,2g})\}$, while M_1 is the matrix representation of h_1 with respect to $\{\tilde{\chi}_1, \cdots, \tilde{\chi}_{2\tilde{g}}\}$, $\{\chi_{1,1}, \cdots, \chi_{1,2g}\}$. $SM_1 \equiv M_2$ (mod. l) and Theorem 4 complete the proof. □

REMARK. Theorem 4 implies that the number of isomorphism classes of maps is less than $\sharp M(2g, 2\tilde{g}; \boldsymbol{Z}/(l)) = l^{4\tilde{g}g}$ for $l > \sqrt{8}(\tilde{g} - 1)$, and we know that the number of automorphisms is finite. Thus, it implies Theorem 1.

Kani ([9]) showed that the number of isomorphism classes of maps is

$$\leq (\tilde{g} - 1)2^{2\tilde{g}^2 - 2}(2^{2\tilde{g}^2 - 1} - 1).$$

To the best knowledge of the author, it is the smallest bound depending only on $\tilde{g}$.

3. Automorphisms and free homotopy

In this seciton, we will deal with another problem. We will give conditions for an automorphism of a Riemann surface to be the identity in terms of free homotopy.

In this section, we treat Riemann surfaces whose universal covering surfaces are the upper half-plane. By an automorphism, we mean a holomorphic bijective map on a Riemann surface. We will denote by Aut(X) the set of all automorphisms on a Riemann surface X. We will denote by $[\alpha]$ the free homotopy class of a closed curve α. We will use the symbol $\sim$ to denote the equivalence in free homotopy. Now let us recall the definition of an admissible curve system.

DEFINITION (see e.g. [4, p.192]). A system of closed curves $\{\alpha_i\}_{i=1}^n$ on a compact Riemann surface X is called *admissible* if

(a) each α_i is simple and no α_i intersects any α_j for $i \neq j$;
(b) no α_i is homotopic to any α_j for $i \neq j$; and
(c) no α_i is homotopically trivial.

In above definition, we identify a curve α with $-\alpha$ homotopically. The following lemma will be useful.

Lemma 4 ([4, p.210], [14, Lemma 4.9]). 1) *Let p be the minimal number of self intersections of a closed curve in the free fomotopy class $[\alpha_1]$. The Poincaré geodesic in $[\alpha_1]$ has p intersections.*

2) *Let q be the minimal number of intersections between curves chosen from the free homotopy classes $[\alpha_1]$, $[\alpha_2]$. The corresponding Poincaré geodesics intersect q times to each other.*

Before proving Theorem 6, we will introduce certain surgeries of Riemann surfaces. Let α be a homotopically non-trivial closed curve on a Riemann surface X, and let $T \in \mathrm{Aut}(X)$ with $T(\alpha) \sim \alpha$. We know that each homotopy class has a unique representative which is a Poincaré geodesic in X. We consider α as the geodesic. Then, $T(\alpha) = \alpha$ set-theoretically. Let l denote the length of α for the Poincaré metric. Then α with the Poincaré metric and $S = \{z \in \boldsymbol{C}; |z| = l/2\pi\}$ with the Euclidian metric are isometric. We denote by T_S the action of T on S. Then we may write

$$T_S: z \mapsto e^{i\theta} z \qquad (0 \leq \theta < 2\pi),$$

since $T \in \mathrm{Aut}(X)$ is isometric with respect to the Poincaré metric. The map T_S can be extended to the closed disk $D = \{z \in \boldsymbol{C}; |z| \leq l/2\pi\}$. We denote by $T_D: z \mapsto e^{i\theta} z$ $(z \in D)$ the extended map. Of course, $0 \in D$ is a fixed point of T_D. Next we cut X along the geodesic α and paste two copies of D by identifying α with S to get a Riemann surface (or a union of Riemann surfaces) $\tilde{X}$. We denote by $\tilde{T}: \tilde{X} \to \tilde{X}$ the extension of T whose restricted map $\tilde{T}|_D = T_D$. Then $\tilde{T}$ has a fixed point on each of the pasted disks. We will call the above surgery the *α-surgery*.

We sum up the above as a propositon.

Proposition. *Let X be a Riemann surface whose universal covering surface is the upper half-plane, and α be a simple closed curve on X. Let $\tilde{X}$ be a Riemann surface (or a union of Riemann surfaces) obtained by using the α-surgery. Let $T \in \mathrm{Aut}(X)$. Suppose that $T(\alpha) \sim \alpha$. Then T can be extended to a conformal map $\tilde{T}: \tilde{X} \to \tilde{X}$ naturally, and $\tilde{T}$ has a fixed point on each of the pasted disks.*

REMARK. By Proposition, we see at once the following.

Let X be a Riemann surface whose universal covering surface is the upper *half-plane and let* $T\in\mathrm{Aut}(X)$. *Suppose that there is a curve syetem* $\{\alpha_i\}_{i=1}^n$ ($n=2$ or 3) *such that one of components of* $X\backslash\{\alpha_i\}_{i=1}^n$ *is a pair of pants whose boundaries are* $\{\alpha_i\}_{i=1}^n$ *and* $T(\alpha_i)\sim\alpha_i$ *for every i. Then* $T=id$.

Proof of Theorem 6. By Lemma 4, we may assume that each element of the admissible curve system $\{\alpha_i\}_{i=1}^n$ is the geodesic in the free homotopy class of each. For each α_i, we apply Proposition and obtain a compact Riemann surface $\tilde{X}$ since $X\backslash\{\alpha_i\}_{i=1}^n$ is connected. Then, the extended map $\tilde{T}$ has $k+2n$ fixed points on $\tilde{X}$ (recall that k is the number of fixed points of T).

Calculating the Euler characteristic, we see that the genus of $\tilde{X}$ is $g-n$. Since $\tilde{T}\neq id.$, $\tilde{T}$ has at most $2(g-n)+2$ fixed points. Now we see $k+2n\leq 2(g-n)+2$ and $k+4n\leq 2g+2$. □

Now let us recall the difinition of γ-hyperellipticity. A compact Riemann surface X is called γ-hyperelliptic ($\gamma\in\boldsymbol{N}$) provided there is a compact Riemann surface X_γ of genus γ and a holomorphic mapping of degree 2, $\pi: X\to X_\gamma$. On every γ-hyperelliptic Riemann surface X, there is a γ-hyperelliptic involution ; that is a $T\in\mathrm{Aut}(X)$ such that there are $2g+2-4\gamma$ fixed points of T and ord $T=2$, where g is the genus of X. "0-hyperelliptic" is corresponding to the usual notion of "hyperelliptic". The γ-hyperelliptic involution on a surface of genus g is unique (if it exists) provided $g>4\gamma+1$ (see e.g. [2, Chapter 5.1.9]).

Corollary. *Let X be a* γ*-hyperelliptic Riemann surface and let* $T\in\mathrm{Aut}(X)$ *be a* γ*-hyperelliptic involution. If there is an admissible curve system* $\{\alpha_i\}_{i=1}^n$ *on X such that* $X\backslash\{\alpha_i\}_{i=1}^n$ *is connected and* $T(\alpha_i)\sim\alpha_i$ *for every* $i\in\{1,\cdots,n\}$, *then* $n\leq\gamma$.

In particular, if T is a hyperelliptic involution, then there is no non-dividing simple closed curve α (*i.e.* $X\backslash\{\alpha\}$ *is connected*) *with* $T(\alpha)\sim\alpha$.

Proof. In the ineguality of Theorem 6, $k+4n=2g+2-4\gamma+4n\leq 2g+2$ since T has $2g+2-4\gamma$ fixed points. Thus $n\leq\gamma$. □

Here we will give another proof of Theorem 5.

Proof of Theorem 5. We may assume that each element of the admissible curve system $\{\alpha_i\}_{i=1}^n$ is the geodesic in the free homotopy class of each, by Lemma 4. Apply Proposition to each α_i to have compact Riemann surfaces $\{\tilde{X}_i\}_{i=1}^m$. Let $\chi(\tilde{X}_i)$ be the Euler characteristic of $\tilde{X}_i$. Then,

$$\sum_{i=1}^m \chi(\tilde{X}_i)=2-2g+2n\geq 2-2g+2g=2.$$

If the genus of a compact Riemann surface is >0, then its Euler characteristic is

≤ 0. Thus, at least one of $\{\tilde{X}_i\}_{i=1}^m$, say $\tilde{X}_1$, must be of genus 0. On $\tilde{X}_1$, the conformal map induced naturally by T has at least three fixed points by the construction of $\tilde{X}_1$. Thus T is the identity. □

Here we exhibit examples showing that the inequality in Theorem 6 is sharp. It is well known that a hyperelliptic involution has $2g+2$ fixed points. Thus the inequality is sharp for $n=0$. We shall consider automorphisms with $n \neq 0$.

EXAMPLE 1. Let $D(a,r)=\{z \in \boldsymbol{C}; |z-a|<r\}$, and $i=\sqrt{-1}$. From the Riemann sphere $\hat{\boldsymbol{C}}$, we remove $2n$ $(n>1)$ disks $D(i, 1/3)$, $D(2i, 1/3)$, $\cdots$, $D(ni, 1/3)$, $D(-i, 1/3)$, $D(-2i, 1/3)$, $\cdots$, $D(-ni, 1/3)$ to get a compact bordered Riemann surface M. Consider now two copies M and M' of M, and construct a compact Riemann surface $N=M \cup M'$ known as the double of M. Here N is of genus $2n-1$. Let $p \in M$ and we denote by p' the point which is on M' and corresponding to p. Let $j: N \rightarrow N$ denote the reflection (the anti-conformal involution). Let z be the usual coordinate on M, and consider an anti-conformal map $a: z \mapsto \bar{z}$, $z \in M$. It is clear that the map a can be extended to an anti-conformal map on N. We also denote it by $a: N \rightarrow N$. Composed map $a \circ j: N \rightarrow N$ is an automorphism. Let $\alpha_k = \{z \in M; |\mathrm{Im} z|=k,\ 1/3 \leq \mathrm{Re} z \leq k\} \cup \{\mathrm{Re} z=k,\ -k \leq \mathrm{Im} z \leq k\} \cup \{z' \in M'; |\mathrm{Im} z'|=k',\ (1/3)' \leq \mathrm{Re} z' \leq k'\} \cup \{\mathrm{Re} z'=k',\ -k' \leq \mathrm{Im} z' \leq k'\}$, $(k=1,\cdots,n)$. It is easy to see that $\{\alpha_1,\cdots,\alpha_n\}$ is an admissible surve system, $M \backslash \{\alpha_1,\cdots,\alpha_n\}$ is connected, and $a \circ j(\alpha_i) \sim \alpha_i$, $(i=1,\cdots,n)$. In the ineguality of the theorem $k+4n=4n$ since $a \circ j$ has no fixed points, and $2g+2=2(2n-1)+2=4n$. We see that the equality holds.

In example 1, the automorphism has no fixed points. Next, we will give an example that the automorphism has fixed points (i.e. in the inequality $n,k \neq 0$).

EXAMPLE 2. We do the α_1-surgery on the surface N (constructed in example 1) to get a compact Riemann surface $\tilde{N}$ of genus $2n-2$. Let $b: \tilde{N} \rightarrow \tilde{N}$ denote the automorphism induced by $a \circ j$. Then b has two fixed points. It is easy to see that equality holds.

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