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A simple proof of Slater's transformations for bilateral series ${}_r\psi_r$, ${}_{2r}\psi_{2r}$ and ${}_{2r-1}\psi_{2r-1}$

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Abstract

We give a simple proof of Slater's transformations for bilateral series ${}_r\psi_r$, ${}_{2r}\psi_{2r}$, and ${}_{2r-1}\psi_{2r-1}$, using only the residue theorem only, without technical manipulation of the series.

Keywords Slater's transformations · ${}_r\psi_r$ series · Very-well-poised ${}_{2r}\psi_{2r}$ series · Bilateral basic hypergeometric series

Mathematics Subject Classification Primary 33D15; Secondary 05A30 · 33D60

1 Introduction

The bilateral basic hypergeometric series ${}_n\psi_n$ is defined by

$$ {}_n\psi_n\left(\begin{matrix} a_1, \ldots, a_n \\ b_1, \ldots, b_n \end{matrix}; q, x\right) := \sum_{-\infty<k<\infty} \frac{(a_1, \ldots, a_n; q)_k}{(b_1, \ldots, b_n; q)_k} x^k $$

for

$$ \left|b_1 \cdots b_n a_1^{-1} \cdots a_n^{-1}\right| < |x| < 1, \quad |q| < 1, \quad a_i, b_i, x, q \in \mathbb{C}, \text{ and } a_i, qb_i^{-1} \notin q^{\mathbb{Z}_{>0}}, $$

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where

$$(a_1, a_2, \ldots, a_n; q)_k = (a_1; q)_k (a_2; q)_k \cdots (a_n; q)_k,$$
$$(a; q)_k = (a; q)_\infty / (aq^k; q)_\infty,$$

and

$$(a; q)_\infty = \prod_{i \geq 0} (1 - aq^i).$$

In [6, 7], Slater derived transformations for bilateral basic hypergeometric series. She applied the results by Sears [5] in [6], and she considered the basic analogue of the Barnes-type integrals in [7]. Exposition on the work by Slater is provided in Chapter 5 of the book by Gasper and Rahman [1]. On the other hand, Ito and Sanada gave a proof of the transformation for ${}_r\psi_r$ series (Theorem 1 below) and a proof of the transformation for very-well-poised-balanced ${}_{2r}\psi_{2r}$ series (Theorem 2 below) from the viewpoint of the connection problem associated with a Jackson integral [2].

The purpose of the present paper is to give a simple proof of these results by means of the residue theorem only, without technical manipulation of the series, and also to study the integrals which represent several functions in q-analysis. We refer the reader to our previous work [4] for the proof of Ramanujan's ${}_1\psi_1$-sum and Bailey's ${}_6\psi_6$-sum from the same point of view (See also [3]).

We also use the symbols

$$(a_1, a_2, \ldots, a_n; q)_\infty = (a_1; q)_\infty (a_2; q)_\infty \cdots (a_n; q)_\infty,$$
$$\theta(a) = (a, qa^{-1}; q)_\infty$$

and

$$\theta(a_1, \ldots, a_n) = \theta(a_1) \cdots \theta(a_n).$$

In this paper, the base q is fixed to be a real number satisfying $0 < q < 1$ for simplicity.

2 ${}_r\psi_r$ series

Slater's transformation for ${}_r\psi_r$ series is given by the following, which is (5.4.3) of [1] ((4) of [6], (7.2.5) of [7], and (5.1) of [2]):

Theorem 1 *Suppose that*

$$c_i,\ c_i c_j^{-1} \notin q^{\mathbb{Z}},\quad 1 \leq i \neq j \leq r,$$

for $r \geq 1$. Then we have

$$\frac{\theta(Ax)\,\prod_{i=1}^{r}(b_i,\ qa_i^{-1}\,;\,q)_\infty}{\prod_{i=1}^{r}\theta(c_i)}\,{}_r\psi_r\left(\begin{matrix}a_1,\ a_2,\ \ldots,\ a_r\\ b_1,\ b_2,\ \ldots,\ b_r\end{matrix}\,;\ q,\ x\right)$$
$$=\sum_{s=1}^{r}\frac{q}{c_s}\frac{\theta(Axc_sq^{-1})}{\theta(c_s)}\frac{\prod_{i=1}^{r}(qb_ic_s^{-1},\ c_sa_i^{-1}\,;\,q)_\infty}{\prod_{\substack{1\le i\le r\\ i\ne s}}\theta(c_sc_i^{-1})}$$
$$\times\,{}_r\psi_r\left(\begin{matrix}qa_1c_s^{-1},\ qa_2c_s^{-1},\ \ldots,\ qa_rc_s^{-1}\\ qb_1c_s^{-1},\ qb_2c_s^{-1},\ \ldots,\ qb_rc_s^{-1}\end{matrix}\,;\ q,\ x\right),\tag{2.1}$$

where $A=\prod_{i=1}^{r}a_ic_i^{-1}$ *and* $\left|b_1\cdots b_ra_1^{-1}\cdots a_r^{-1}\right|<|x|<1$.

Proof Let $F(t)$ be a function defined by

$$F(t)=\frac{1}{t}\frac{\theta(Axt^{-1})}{\theta(t^{-1})}\prod_{i=1}^{r}\frac{(b_it,\ qa_i^{-1}t^{-1}\,;\,q)_\infty}{\theta(c_i\,t)}.\tag{2.2}$$

First, for $k\in\mathbb{Z}$, we have

$$\mathrm{Res}_{t=q^k}F(t)\,dt=\lim_{t\to q^k}(t-q^k)F(t)$$
$$=\frac{\theta(Ax)\,\prod_{i=1}^{r}\theta(a_i)}{(q\,;\,q)_\infty^2\,\prod_{i=1}^{r}\theta(c_i)}\prod_{i=1}^{r}\frac{(b_iq^k\,;\,q)_\infty}{(a_iq^k\,;\,q)_\infty}x^k,\tag{2.3}$$

and

$$\mathrm{Res}_{t=c_s^{-1}q^{1+k}}F(t)\,dt=\lim_{t\to c_s^{-1}q^{1+k}}(t-c_s^{-1}q^{1+k})F(t)$$
$$=-\frac{q}{c_s}\frac{\theta(Axc_sq^{-1})\,\prod_{i=1}^{r}\theta(c_sa_i^{-1})}{(q\,;\,q)_\infty^2\,\theta(c_s)\,\prod_{\substack{1\le i\le r\\ i\ne s}}\theta(c_sc_i^{-1})}\prod_{i=1}^{r}\frac{(qb_ic_s^{-1}q^k\,;\,q)_\infty}{(qa_ic_s^{-1}q^k\,;\,q)_\infty}x^k.\tag{2.4}$$

Secondly, for real positive numbers R_1 and R_2 satisfying

$$R_j,\ c_iR_j\notin q^{\mathbb{Z}},\quad 1\le i\le r,\quad j=1,\ 2,$$

we have

$$F\left(R_1\,e^{\sqrt{-1}\,\theta}q^{-l}\right)$$
$$=\frac{\left(AxR_1^{-1}\,e^{-\sqrt{-1}\,\theta}q^l,\ qA^{-1}x^{-1}R_1\,e^{\sqrt{-1}\,\theta}\,;\,q\right)_\infty}{R_1\,e^{\sqrt{-1}\,\theta}q^{-l}\left(R_1^{-1}\,e^{-\sqrt{-1}\theta}q^l,\ qR_1\,e^{\sqrt{-1}\theta}\,;\,q\right)_\infty}$$

$$\times \prod_{i=1}^{r} \frac{\left(b_i R_1 e^{\sqrt{-1}\theta},\ q a_i^{-1} R_1^{-1} e^{-\sqrt{-1}\theta} q^l\, ;q\right)_\infty}{\left(c_i R_1 e^{\sqrt{-1}\theta},\ q c_i^{-1} R_1^{-1} e^{-\sqrt{-1}\theta} q^l\, ;q\right)_\infty}$$

$$\times \frac{\left(Ax R_1^{-1} e^{-\sqrt{-1}\,\theta}\, ;q\right)_l}{\left(R_1^{-1} e^{-\sqrt{-1}\theta}\, ;q\right)_l} \prod_{i=1}^{r} \frac{\left(q b_i^{-1} R_1^{-1} e^{-\sqrt{-1}\theta}\, ;q\right)_l}{\left(q c_i^{-1} R_1^{-1} e^{-\sqrt{-1}\theta}\, ;q\right)_l} \left(\frac{b_1\cdots b_r}{a_1\cdots a_r}\frac{1}{x}\right)^l,$$

$$F\left(R_2 e^{\sqrt{-1}\,\theta} q^l\right)$$

$$= \frac{\left(Ax R_2^{-1} e^{-\sqrt{-1}\,\theta},\ q A^{-1} x^{-1} R_2 e^{\sqrt{-1}\,\theta} q^l\, ;q\right)_\infty}{R_2 e^{\sqrt{-1}\,\theta} q^l\, (R_2^{-1} e^{-\sqrt{-1}\theta},\ q R_2 e^{\sqrt{-1}\theta} q^l\, ;q)_\infty}$$

$$\times \prod_{i=1}^{r} \frac{\left(b_i R_2 e^{\sqrt{-1}\theta} q^l,\ q a_i^{-1} R_2^{-1} e^{-\sqrt{-1}\theta}\, ;q\right)_\infty}{\left(c_i R_2 e^{\sqrt{-1}\theta} q^l,\ q c_i^{-1} R_2^{-1} e^{-\sqrt{-1}\theta}\, ;q\right)_\infty}$$

$$\times \frac{\left(q A^{-1} x^{-1} R_2 e^{\sqrt{-1}\,\theta}\, ;q\right)_l}{\left(R_2^{-1} e^{\sqrt{-1}\theta},\ ;q\right)_l} \prod_{i=1}^{r} \frac{\left(a_i R_2 e^{\sqrt{-1}\theta}\, ;q\,\right)_l}{\left(c_i R_2 e^{\sqrt{-1}\theta}\, ;q\right)_l}\, x^l,$$

and thus

$$\left|\int_{C_1^{(l)}} F(t)\,dt\right|$$

$$= \left|\int_0^{2\pi} F(R_1 e^{\sqrt{-1}\theta} q^{-l})\, R_1 \sqrt{-1}\, e^{\sqrt{-1}\theta} q^{-l}\, \mathrm{d}\theta\right| \le M_1 \left|\frac{b_1\cdots b_r}{a_1\cdots a_r}\frac{1}{x}\right|^l, \qquad (2.5)$$

$$\left|\int_{C_2^{(l)}} F(t)\,dt\right| = \left|\int_0^{2\pi} F(R_2 e^{\sqrt{-1}\theta} q^l)\, R_2 \sqrt{-1}\, e^{\sqrt{-1}\theta} q^l\, \mathrm{d}\theta\right| \le M_2\, |\,x\,|^l, \quad (2.6)$$

where $l \in \mathbb{Z}_{\ge 0}$, $C_1^{(l)} = \{\, R_1 e^{\sqrt{-1}\theta} q^{-l} \in \mathbb{C} \mid 0 \le \theta \le 2\pi\}$, $C_2^{(l)} = \{\, R_2 e^{\sqrt{-1}\theta} q^l \in \mathbb{C} \mid 0 \le \theta \le 2\pi\}$, and $M_1,\ M_2$ are positive numbers independent of l.

The residue theorem combined with inequalities (2.5) and (2.6) leads to

$$\sum_{-\infty<k<\infty} \mathrm{Res}_{t=q^k} F(t)\,dt + \sum_{s=1}^{r} \sum_{-\infty<k<\infty} \mathrm{Res}_{t=c_s^{-1}q^{1+k}} F(t)\,dt$$

$$= \lim_{l\to\infty} \frac{1}{2\pi\sqrt{-1}} \left(\int_{C_1^{(l)}} + \int_{C_2^{(l)}}\right) F(t)\,dt = 0, \qquad (2.7)$$

if $\left|b_1\cdots b_r a_1^{-1}\cdots a_r^{-1}\right| < |\,x\,| < 1$, where $C_1^{(l)}$ is in the counterclockwise direction and $C_2^{(l)}$ is in the clockwise direction.

Consequently, combining (2.3), (2.4) with (2.7), we obtain

$$
\begin{aligned}
0 &= \frac{\theta(Ax)\prod_{i=1}^{r}\theta(a_i)}{(q\,;q)_\infty^2\prod_{i=1}^{r}\theta(c_i)}\sum_{-\infty<k<\infty}\prod_{i=1}^{r}\frac{(b_iq^k\,;q)_\infty}{(a_iq^k\,;q)_\infty}x^k\\
&\quad-\sum_{s=1}^{r}\frac{q}{c_s}\frac{\theta(Axc_sq^{-1})\prod_{i=1}^{r}\theta(c_sa_i^{-1})}{(q\,;q)_\infty^2\,\theta(c_s)\prod_{\substack{1\le i\le r\\ i\neq s}}\theta(c_sc_i^{-1})}\sum_{-\infty<k<\infty}\prod_{i=1}^{r}\frac{(qb_ic_s^{-1}q^k\,;q)_\infty}{(qa_ic_s^{-1}q^k\,;q)_\infty}x^k\\
&=\frac{\theta(Ax)\prod_{i=1}^{r}(qa_i^{-1},\,b_i\,;q)_\infty}{(q\,;q)_\infty^2\prod_{i=1}^{r}\theta(c_i)}\sum_{-\infty<k<\infty}\prod_{i=1}^{r}\frac{(a_i\,;q)_k}{(b_i\,;q)_k}x^k\\
&\quad-\sum_{s=1}^{r}\frac{q}{c_s}\frac{\theta(Axc_sq^{-1})\prod_{i=1}^{r}(c_sa_i^{-1},\,qb_ic_s^{-1}\,;q)_\infty}{(q\,;q)_\infty^2\,\theta(c_s)\prod_{\substack{1\le i\le r\\ i\neq s}}\theta(c_sc_i^{-1})}\sum_{-\infty<k<\infty}\prod_{i=1}^{r}\frac{(qa_ic_s^{-1}\,;q)_k}{(qb_ic_s^{-1}\,;q)_k}x^k,
\end{aligned}
$$

if $\left|b_1\cdots b_ra_1^{-1}\cdots a_r^{-1}\right| < |\,x\,| < 1$. This completes the proof of (2.1). □

3 Very-well-poised-balanced ${}_{2r}\psi_{2r}$ series

Slater's transformation for very-well-poised-balanced ${}_{2r}\psi_{2r}$ series is given by the following, which is (5.5.2) of [1] ((1.1) of [2]):

Theorem 2 *Suppose that*

$$
a,\ a_i,\ a_ia^{-1},\ a_ia_j^{-1},\ a_i^2a^{-1},\ a_ia_ja^{-1}\ \notin q^{\mathbb{Z}},\ \ 1\le i\neq j\le r-2,
$$

for $r\ge 3$, *and* $\left|a^{r-1}q^{r-2}\right| < \left|\prod_{i=1}^{2(r-1)}b_i\right|$. *Then we have*

$$
\begin{aligned}
&\frac{\prod_{i=1}^{2(r-1)}(qb_i^{-1},\,qab_i^{-1}\,;q)_\infty}{(qa,\,qa^{-1}\,;q)_\infty\prod_{i=1}^{r-2}\theta(a_ia^{-1},\,a_i)}\\
&\times{}_{2r}\psi_{2r}\left(\begin{matrix}qa^{\frac12},\ -qa^{\frac12},\ b_1,\ b_2,\ \ldots,\ b_{2(r-1)}\\ a^{\frac12},\ -a^{\frac12},\ qab_1^{-1},\ qab_2^{-1},\ \ldots,\ qab_{2(r-1)}^{-1}\end{matrix}\,;q,\ \frac{q^{r-2}a^{r-1}}{\prod_{i=1}^{2(r-1)}b_i}\right)\\
&=\sum_{s=1}^{r-2}\frac{\prod_{i=1}^{2(r-1)}(qaa_s^{-1}b_i^{-1},\,qa_sb_i^{-1}\,;q)_\infty}{(qa_s^2a^{-1},\,qaa_s^{-2}\,;q)_\infty\,\theta(a_sa^{-1},\,a_s)\prod_{\substack{1\le i\le r-2\\ i\neq s}}\theta(a_ia_s^{-1},\,a_sa_ia^{-1})}\\
&\times{}_{2r}\psi_{2r}\left(\begin{matrix}qa_sa^{-\frac12},\ -qa_sa^{-\frac12},\ b_1a_sa^{-1},\ b_2a_sa^{-1},\ \ldots,\ b_{2(r-1)}a_sa^{-1}\\ a_sa^{-\frac12},\ -a_sa^{-\frac12},\ qa_sb_1^{-1},\ qa_sb_2^{-1},\ \ldots,\ qa_sb_{2(r-1)}^{-1}\end{matrix}\,;q,\ \frac{q^{r-2}a^{r-1}}{\prod_{i=1}^{2(r-1)}b_i}\right).
\end{aligned}
\tag{3.1}
$$

Proof Let $F(t)$ be a function defined by

$$F(t) = \frac{\prod_{i=1}^{2(r-1)} (qb_i^{-1}t^{-1},\, qab_i^{-1}t\,;q)_\infty}{\theta(t^{-1},\, at)\,\prod_{i=1}^{r-2}\theta(a_ia^{-1}t^{-1},\, a_it)}$$
$$= \frac{\prod_{i=1}^{2(r-1)} (qb_i^{-1}t^{-1},\, qab_i^{-1}t\,;q)_\infty}{(t^{-1},\, qt,\, at,\, qa^{-1}t^{-1}\,;q)_\infty \prod_{i=1}^{r-2}(a_ia^{-1}t^{-1},\, qa_i^{-1}at,\, a_it,\, qa_i^{-1}t^{-1}\,;q)_\infty}, \tag{3.2}$$

which satisfies $F(t) = F(a^{-1}t^{-1})$.

First, for $k \in \mathbb{Z}$, we have

$$\mathrm{Res}_{t=q^k} F(t)\,dt = \lim_{t\to q^k}(t-q^k)F(t)$$
$$= \frac{q^k \times \prod_{i=1}^{2(r-1)}\theta(b_i)}{(q\,;q)_\infty^2\,\theta(a)\,\prod_{i=1}^{r-2}\theta(a_ia^{-1},\, a_i)_\infty}\prod_{i=1}^{2(r-1)}\frac{(qab_i^{-1}q^k\,;q)_\infty}{(b_iq^k\,;q)_\infty}\left(\frac{(aq)^{r-1}}{\prod_{i=1}^{2(r-1)} b_i}\right)^k, \tag{3.3}$$

and

$$\mathrm{Res}_{t=a_sa^{-1}q^k} F(t)\,dt = \lim_{t\to a_sa^{-1}q^k}(t-a_sa^{-1}q^k)F(t)$$
$$= \frac{a_sa^{-1}q^k \times \prod_{i=1}^{2(r-1)}\theta(qaa_s^{-1}b_i^{-1})}{(q\,;q)_\infty^2\,\theta(aa_s^{-1},\, a_s,\, a_s^2a^{-1})\prod_{\substack{1\le i\le r-2\\ i\ne s}}\theta(a_ia_s^{-1},\, a_sa_ia^{-1})}$$
$$\times \prod_{i=1}^{2(r-1)}\frac{(qa_sb_i^{-1}q^k\,;q)_\infty}{(a_sb_ia^{-1}q^k\,;q)_\infty}\left(\frac{(qa)^{r-1}}{\prod_{i=1}^{2(r-1)} b_i}\right)^k. \tag{3.4}$$

Secondly, for $k \in \mathbb{Z}$, we have

$$\mathrm{Res}_{t=a^{-1}q^{-k}} F(t)\,dt = -a^{-1}\mathrm{Res}_{u=q^k} F(a^{-1}u^{-1})\,u^{-2}\,\mathrm{d}u$$
$$= -a^{-1}\mathrm{Res}_{u=q^k} F(u)\,u^{-2}\,\mathrm{d}u = -a^{-1}q^{-2k}\mathrm{Res}_{u=q^k} F(u)\,\mathrm{d}u \tag{3.5}$$

and

$$\mathrm{Res}_{t=a_s^{-1}q^{-k}} F(t)\,dt = -a^{-1}\mathrm{Res}_{u=a^{-1}a_sq^k} F(a^{-1}u^{-1})\,u^{-2}\,\mathrm{d}u$$
$$= -a^{-1}\mathrm{Res}_{u=a^{-1}a_sq^k} F(u)\,u^{-2}\,\mathrm{d}u = -aa_s^{-2}q^{-2k}\mathrm{Res}_{u=a^{-1}a_sq^k} F(u)\,\mathrm{d}u \tag{3.6}$$

by the change of integration variable from t to u by $t = a^{-1}u^{-1}$ with the equality $F(t) = F(a^{-1}t^{-1})$.

Thirdly, for real positive numbers R_1 and R_2 satisfying

$$R_j,\ aR_j,\ a_iR_j,\ aa_i^{-1}R_j \notin q^{\mathbb{Z}},\quad 1\le i\le r-2,\quad j=1,\ 2,$$

we have

$$\begin{aligned}
&F\left(R_1\,e^{\sqrt{-1}\,\theta}q^{-l}\right)\\
&=\frac{\prod_{i=1}^{2(r-1)}(qb_i^{-1}R_1^{-1}\,e^{-\sqrt{-1}\theta}q^l,\ qab_i^{-1}R_1\,e^{\sqrt{-1}\theta}\,;q)_\infty}{(R_1^{-1}\,e^{-\sqrt{-1}\theta}q^l,\ qR_1\,e^{\sqrt{-1}\theta},\ aR_1\,e^{\sqrt{-1}\theta},\ qa^{-1}R_1^{-1}\,e^{-\sqrt{-1}\theta}q^l\,;q)_\infty}\\
&\quad\times\frac{1}{\prod_{i=1}^{r-2}(a_ia^{-1}R_1^{-1}\,e^{-\sqrt{-1}\theta}q^l,\ qaa_i^{-1}R_1\,e^{\sqrt{-1}\theta},\ a_iR_1\,e^{\sqrt{-1}\theta},\ qa_i^{-1}R_1^{-1}\,e^{-\sqrt{-1}\theta}q^l\,;q)_\infty}\\
&\quad\times\frac{\prod_{i=1}^{2(r-1)}(b_ia^{-1}R_1^{-1}\,e^{-\sqrt{-1}\theta}\,;q)_l}{(R_1^{-1}\,e^{-\sqrt{-1}\theta}q^l,\ a^{-1}R_1^{-1}\,e^{-\sqrt{-1}\theta}\,;q)_l\prod_{i=1}^{r-2}(a_ia^{-1}R_1^{-1}\,e^{-\sqrt{-1}\theta},\ a_i^{-1}R_1^{-1}\,e^{-\sqrt{-1}\theta}\,;q)_l}\\
&\quad\times\left(\frac{(qa)^{r-1}}{\prod_{i=1}^{2(r-1)}b_i}\right)^l,\\
&F(R_2\,e^{\sqrt{-1}\,\theta}q^l)\\
&=\frac{\prod_{i=1}^{2(r-1)}(qb_i^{-1}R_2^{-1}\,e^{-\sqrt{-1}\theta},\ qab_i^{-1}R_2\,e^{\sqrt{-1}\theta}q^l\,;q)_\infty}{(R_2^{-1}\,e^{-\sqrt{-1}\theta},\ qR_2\,e^{\sqrt{-1}\theta}q^l,\ aR_2\,e^{\sqrt{-1}\theta}q^l,\ qa^{-1}R_2^{-1}\,e^{-\sqrt{-1}\theta}\,;q)_\infty}\\
&\quad\times\frac{1}{\prod_{i=1}^{r-2}(a_ia^{-1}R_2^{-1}\,e^{-\sqrt{-1}\theta},\ qaa_i^{-1}R_2\,e^{\sqrt{-1}\theta}q^l,\ a_iR_2\,e^{\sqrt{-1}\theta}q^l,\ qa_i^{-1}R_2^{-1}\,e^{-\sqrt{-1}\theta}\,;q)_\infty}\\
&\quad\times\frac{\prod_{i=1}^{2(r-1)}(b_iR_2\,e^{\sqrt{-1}\theta}\,;q)_l}{(qR_2\,e^{\sqrt{-1}\theta},\ aR_2\,e^{\sqrt{-1}\theta}\,;q)_l\prod_{i=1}^{r-2}(qaa_i^{-1}R_2\,e^{\sqrt{-1}\theta},\ a_iR_2\,e^{\sqrt{-1}\theta},\ ;q)_l}\left(\frac{(qa)^{r-1}}{\prod_{i=1}^{2(r-1)}b_i}\right)^l,
\end{aligned}$$

and thus

$$\left|\int_{C_1^{(l)}}F(t)\,dt\right|=\left|\int_0^{2\pi}F(R_1\,e^{\sqrt{-1}\theta}q^{-l})\,R_1\,\sqrt{-1}\,e^{\sqrt{-1}\theta}q^{-l}\,\mathrm{d}\theta\right|\le M_1\left|\frac{a^{r-1}q^{r-2}}{\prod_{i=1}^{2(r-1)}b_i}\right|^l, \tag{3.7}$$

$$\left|\int_{C_2^{(l)}}F(t)\,dt\right|=\left|\int_0^{2\pi}F(R_2\,e^{\sqrt{-1}\theta}q^{l})\,R_2\,\sqrt{-1}\,e^{\sqrt{-1}\theta}q^{l}\,\mathrm{d}\theta\right|\le M_2\left|\frac{a^{r-1}q^{r}}{\prod_{i=1}^{2(r-1)}b_i}\right|^l, \tag{3.8}$$

where $l\in\mathbb{Z}_{\ge 0}$, $C_1^{(l)}=\{\,R_1\,e^{\sqrt{-1}\theta}q^{-l}\in\mathbb{C}\mid 0\le\theta\le 2\pi\}$, $C_2^{(l)}=\{\,R_2\,e^{\sqrt{-1}\theta}q^{l}\in\mathbb{C}\mid 0\le\theta\le 2\pi\}$, and $M_1,\ M_2$ are positive numbers independent of l.

The residue theorem combined with inequalities (3.7) and (3.8) leads to

$$\begin{aligned}
&\sum_{-\infty<k<\infty}\Big\{\mathrm{Res}_{t=q^k}F(t)\,dt+\mathrm{Res}_{t=a^{-1}q^{-k}}F(t)\,dt\Big\}\\
&+\sum_{s=1}^{r-2}\sum_{-\infty<k<\infty}\Big\{\mathrm{Res}_{t=a_sa^{-1}q^k}F(t)\,dt+\mathrm{Res}_{t=a_s^{-1}q^{-k}}F(t)\,dt\Big\}\\
&=\lim_{l\to\infty}\frac{1}{2\pi\sqrt{-1}}\left(\int_{C_1^{(l)}}+\int_{C_2^{(l)}}\right)F(t)\,dt=0,
\end{aligned}\tag{3.9}$$

if $\left|a^{r-1}q^{r-2}\right|<\left|\prod_{i=1}^{2(r-1)}b_i\right|$, where $C_1^{(l)}$ is in the counterclockwise direction and $C_2^{(l)}$ is in the clockwise direction.

On the other hand, for $k\in\mathbb{Z}$, we have

$$\begin{aligned}
&\mathrm{Res}_{t=q^k}F(t)\,dt+\mathrm{Res}_{t=a^{-1}q^{-k}}F(t)\,dt\\
&=\left(q^k-a^{-1}q^{-k}\right)\times\frac{\prod_{i=1}^{2(r-1)}\theta(b_i)}{(q\,;q)_\infty^2\,\theta(a)\,\prod_{i=1}^{r-2}\theta(a_ia^{-1},\,a_i)}\\
&\quad\times\prod_{i=1}^{2(r-1)}\frac{(qab_i^{-1}q^k\,;\,q)_\infty}{(b_i\,q^k\,;\,q)_\infty}\left(\frac{(aq)^{r-1}}{\prod_{i=1}^{2(r-1)}b_i}\right)^k\\
&=-a^{-1}\times\frac{\prod_{i=1}^{2(r-1)}\theta(b_i)}{(q\,;q)_\infty^2\,(qa,\,qa^{-1}\,;\,q)_\infty\,\prod_{i=1}^{r-2}\theta(a_ia^{-1},\,a_i)}\\
&\quad\times\frac{1-aq^{2k}}{1-a}\prod_{i=1}^{2(r-1)}\frac{(qab_i^{-1}q^k\,;\,q)_\infty}{(b_i\,q^k\,;\,q)_\infty}\left(\frac{q^{r-2}a^{r-1}}{\prod_{i=1}^{2(r-1)}b_i}\right)^k
\end{aligned}\tag{3.10}$$

from (3.3) and (3.5), and

$$\begin{aligned}
&\mathrm{Res}_{t=a_sa^{-1}q^k}F(t)\,dt+\mathrm{Res}_{t=a_s^{-1}q^{-k}}F(t)\,dt\\
&=\left(a_sa^{-1}q^k-a_s^{-1}q^{-k}\right)\times\frac{\prod_{i=1}^{2(r-1)}\theta(a_sb_ia^{-1})}{(q\,;q)_\infty^2\,\theta(aa_s^{-1},\,a_s,\,a_s^2a^{-1})\,\prod_{\substack{1\le i\le r-2\\ i\neq s}}\theta(a_ia_s^{-1},\,a_ia_sa^{-1})}\\
&\quad\times\prod_{i=1}^{2(r-1)}\frac{(qa_sb_i^{-1}q^k\,;\,q)_\infty}{(a_sb_ia^{-1}\,q^k\,;\,q)_\infty}\left(\frac{(qa)^{r-1}}{\prod_{i=1}^{2(r-1)}b_i}\right)^k\\
&=-a_s^{-1}\times\frac{\prod_{i=1}^{2(r-1)}\theta(a_sb_ia^{-1})}{(q\,;q)_\infty^2\,(qa_s^2a^{-1},\,qaa_s^{-2}\,;\,q)_\infty\,\theta(aa_s^{-1},\,a_s)\,\prod_{\substack{1\le i\le r-2\\ i\neq s}}\theta(a_ia_s^{-1},\,a_ia_sa^{-1})}\\
&\quad\times\frac{1-a_s^2a^{-1}q^{2k}}{1-a_s^2a^{-1}}\prod_{i=1}^{2(r-1)}\frac{(qa_sb_i^{-1}\,q^k\,;\,q)_\infty}{(a_sb_ia^{-1}\,q^k\,;\,q)_\infty}\left(\frac{q^{r-2}a^{r-1}}{\prod_{i=1}^{2(r-1)}b_i}\right)^k
\end{aligned}\tag{3.11}$$

from (3.4) and (3.6).

Consequently, combining (3.10), (3.11) with (3.9), we obtain

$$0 = -a^{-1} \times \frac{\prod_{i=1}^{2(r-1)} \theta(b_i)}{(q\,;q)_\infty^2\,(qa,\,qa^{-1}\,;q)_\infty \prod_{i=1}^{r-2} \theta(a_i a^{-1},\,a_i)}$$
$$\times \sum_{-\infty<k<\infty} \frac{1-aq^{2k}}{1-a} \prod_{i=1}^{2(r-1)} \frac{(qab_i^{-1}q^k\,;q)_\infty}{(b_i\,q^k\,;q)_\infty} \left(\frac{q^{r-2}a^{r-1}}{\prod_{i=1}^{2(r-1)} b_i}\right)^k$$
$$-\sum_{s=1}^{r-2} a_s^{-1} \times \frac{\prod_{i=1}^{2(r-1)} \theta(a_s b_i a^{-1})}{(q\,;\,q)_\infty^2\,(qa_s^2a^{-1},\,qaa_s^{-2}\,;\,q)_\infty\,\theta(aa_s^{-1},\,a_s) \prod_{\substack{1\le i\le r-2\\ i\neq s}} \theta(a_i a_s^{-1},\,a_i a_s a^{-1})}$$
$$\times \sum_{-\infty<k<\infty} \frac{1-a_s^2a^{-1}q^{2k}}{1-a_s^2a^{-1}} \prod_{i=1}^{2(r-1)} \frac{(qa_s b_i^{-1}\,q^k\,;\,q)_\infty}{(a_s b_i a^{-1}\,q^k\,;\,q)_\infty} \left(\frac{q^{r-2}a^{r-1}}{\prod_{i=1}^{2(r-1)} b_i}\right)^k, \tag{3.12}$$

which is equivalent to

$$\frac{\prod_{i=1}^{2(r-1)} (qb_i^{-1},\,qab_i^{-1}\,;\,q)_\infty}{(qa,\,qa^{-1}\,;\,q)_\infty \prod_{i=1}^{r-2} \theta(a_i a^{-1},\,a_i)}$$
$$\times \sum_{-\infty<k<\infty} \frac{1-aq^{2k}}{1-a} \prod_{i=1}^{2(r-1)} \frac{(b_i\,;\,q)_k}{(qab_i^{-1}\,;\,q)_k} \left(\frac{q^{r-2}a^{r-1}}{\prod_{i=1}^{2(r-1)} b_i}\right)^k$$
$$= \sum_{s=1}^{r-2} \frac{\prod_{i=1}^{2(r-1)} (qaa_s^{-1}b_i^{-1},\,qa_s b_i^{-1}\,;\,q)_\infty}{(qa_s^2a^{-1},\,qaa_s^{-2}\,;\,q)_\infty\,\theta(a_s a^{-1},\,a_s) \prod_{\substack{1\le i\le r-2\\ i\neq s}} \theta(a_i a_s^{-1},\,a_i a_s a^{-1})}$$
$$\times \sum_{-\infty<k<\infty} \frac{1-a_s^2a^{-1}q^{2k}}{1-a_s^2a^{-1}} \prod_{i=1}^{2(r-1)} \frac{(a_s b_i a^{-1}\,;\,q)_k}{(qa_s b_i^{-1}\,;\,q)_k} \left(\frac{q^{r-2}a^{r-1}}{\prod_{i=1}^{2(r-1)} b_i}\right)^k. \tag{3.13}$$

This completes the proof of (3.1). □

Remark 1 If we change r to $r+2$ and substitute $b_{2r+1} = a^{\frac{1}{2}},\ b_{2r+2} = -a^{\frac{1}{2}}$ in (3.1), we obtain (5.5.1) of [1] ((7) of [6], and (7.2.1.1) of [7]), since

$$\frac{(a^{\frac{1}{2}},\,-a^{\frac{1}{2}};q)_\infty}{\theta(a)} = \frac{(qa^{\frac{1}{2}},\,-qa^{\frac{1}{2}};q)_\infty}{(qa,\,qa^{-1};q)_\infty},\quad a_s\,\frac{(a^{\frac{1}{2}}a_s^{-1},\,-a^{\frac{1}{2}}a_s^{-1};q)_\infty}{\theta(aa_s^{-1},\,a_s^2a^{-1})}$$
$$= \frac{(qa^{\frac{1}{2}}a_s^{-1},\,-qa^{\frac{1}{2}}a_s^{-1};q)_\infty}{\theta(a_s a^{-1})\,(qa_s^2a^{-1},\,qaa_s^{-2};q)_\infty}.$$

Remark 2 To obtain (5.5.1) of [1] by the same manner as in the proof of Theorem 2, it is enough to consider the function

$$F(t)=\frac{\prod_{i=1}^{2r}\left(qb_i^{-1}t^{-1},\, qab_i^{-1}t\,;q\right)_\infty}{\left(t^{-1},\, qt,\, at,\, qa^{-1}t^{-1}\,;q\right)_\infty \prod_{i=1}^{r}\left(a_ia^{-1}t^{-1},\, qa_i^{-1}at,\, a_it,\, qa_i^{-1}t^{-1}\,;q\right)_\infty}.$$

4 Very-well-poised ${}_{2r}\psi_{2r}$ and ${}_{2r-1}\psi_{2r-1}$ series

If we change r to $r+1$ and substitute $b_{2r-1}=q^{\frac{1}{2}}a^{\frac{1}{2}},\ b_{2r}=-q^{\frac{1}{2}}a^{\frac{1}{2}}$ in (3.1), we obtain the following transformation for very-well-poised ${}_{2r}\psi_{2r}$ series:

Theorem 3 *Suppose that*

$$a,\, a_i,\, a_ia^{-1},\, a_ia_j^{-1},\, a_i^2a^{-1},\, a_ia_ja^{-1}\notin q^{\mathbb{Z}},\ 1\le i\ne j\le r-1,$$

for $r\ge 2$, and $\left|a^{r-2}q^{r-1}\right|<\left|\prod_{i=1}^{2(r-1)}b_i\right|$. Then we have

$$\begin{aligned}&\frac{\prod_{i=1}^{2(r-1)}(qb_i^{-1},\, qab_i^{-1}\,;q)_\infty}{(q^2a,\, q^2a^{-1}\,;q^2)_\infty\prod_{i=1}^{r-1}\theta(a_ia^{-1},\, a_i)}\\ &\times{}_{2r}\psi_{2r}\left(\begin{matrix}qa^{\frac{1}{2}},\ -qa^{\frac{1}{2}},\ b_1,\, b_2,\,\dots,\, b_{2(r-1)}\\ a^{\frac{1}{2}},\ -a^{\frac{1}{2}},\ qab_1^{-1},\, qab_2^{-1},\,\dots,\, qab_{2(r-1)}^{-1}\end{matrix};q,\ -\frac{q^{r-2}a^{r-1}}{\prod_{i=1}^{2(r-1)}b_i}\right)\\ &=\sum_{s=1}^{r-1}\frac{\prod_{i=1}^{2(r-1)}(qaa_s^{-1}b_i^{-1},\, qa_sb_i^{-1}\,;q)_\infty}{(q^2a_s^2a^{-1},\, q^2aa_s^{-2}\,;q^2)_\infty\,\theta(a_sa^{-1},\, a_s)\prod_{\substack{1\le i\le r-1\\ i\ne s}}\theta(a_ia_s^{-1},\, a_sa_ia^{-1})}\\ &\times{}_{2r}\psi_{2r}\left(\begin{matrix}qa_sa^{-\frac{1}{2}},\ -qa_sa^{-\frac{1}{2}},\ b_1a_sa^{-1},\, b_2a_sa^{-1},\,\dots,\, b_{2(r-1)}a_sa^{-1}\\ a_sa^{-\frac{1}{2}},\ -a_sa^{-\frac{1}{2}},\ qa_sb_1^{-1},\, qa_sb_2^{-1},\,\dots,\, qa_sb_{2(r-1)}^{-1}\end{matrix};q,\ -\frac{q^{r-2}a^{r-1}}{\prod_{i=1}^{2(r-1)}b_i}\right).\end{aligned}\tag{4.1}$$

Remark 3 To obtain (4.1) by the same manner as in the proof of Theorem 2, it is enough to consider the function

$$F(t)=\frac{(q^{\frac{1}{2}}a^{-\frac{1}{2}}t^{-1},\, q^{\frac{1}{2}}a^{\frac{1}{2}}t,\, -q^{\frac{1}{2}}a^{-\frac{1}{2}}t^{-1},\, -q^{\frac{1}{2}}a^{\frac{1}{2}}t;q)_\infty\prod_{i=1}^{2(r-1)}(qb_i^{-1}t^{-1},\, qab_i^{-1}t;q)_\infty}{(t^{-1},\, qt,\, at,\, qa^{-1}t^{-1};q)_\infty\prod_{i=1}^{r-1}(a_ia^{-1}t^{-1},\, qa_i^{-1}at,\, a_it,\, qa_i^{-1}t^{-1};q)_\infty}.$$

Remark 4 If we change r to $r+2$ and substitute $b_{2r+1}=a^{\frac{1}{2}},\ b_{2r+2}=-a^{\frac{1}{2}}$ in (4.1), we obtain (5.5.4) of [1] ((8) of [6], and (7.2.1.3) of [7]), since

$$\begin{aligned}&(qa^{-\frac{1}{2}},\, qa^{\frac{1}{2}},\, -qa^{-\frac{1}{2}},\, -qa^{\frac{1}{2}}\,;q)_\infty(q^{\frac{1}{2}}a^{-\frac{1}{2}},\, q^{\frac{1}{2}}a^{\frac{1}{2}},\, -q^{\frac{1}{2}}a^{-\frac{1}{2}},\, -q^{\frac{1}{2}}a^{\frac{1}{2}}\,;q)_\infty\\ &=(qa,\, qa^{-1}\,;q)_\infty,\end{aligned}$$

and

$$\begin{aligned}&(qa^{\frac{1}{2}}a_s^{-1},\, qa_sa^{-\frac{1}{2}},\, -qa^{\frac{1}{2}}a_s^{-1},\, -qa_sa^{-\frac{1}{2}}\,;q)_\infty\\ &\times(q^{\frac{1}{2}}a^{\frac{1}{2}}a_s^{-1},\, q^{\frac{1}{2}}a^{-\frac{1}{2}}a_s,\, -q^{\frac{1}{2}}a^{\frac{1}{2}}a_s^{-1},\, -q^{\frac{1}{2}}a^{-\frac{1}{2}}a_s\,;q)_\infty=(qa_s^2a^{-1},\, qaa_s^{-2}\,;q)_\infty.\end{aligned}$$

Remark 5 To obtain (5.5.4) of [1] by the same manner as in the proof of Theorem 2, it is enough to consider the function

$$F(t)=\frac{(q^{\frac12}a^{-\frac12}t^{-1},\ q^{\frac12}a^{\frac12}t,\ -q^{\frac12}a^{-\frac12}t^{-1},\ -q^{\frac12}a^{\frac12}t;q)_\infty\,(qa^{-\frac12}t^{-1},\ qa^{\frac12}t,\ -qa^{-\frac12}t^{-1},\ -qa^{\frac12}t;q)_\infty}{(t^{-1},\ qt,\ at,\ qa^{-1}t^{-1};q)_\infty}\times\frac{\prod_{i=1}^{2r}(qb_i^{-1}t^{-1},\ qab_i^{-1}t;q)_\infty}{\prod_{i=1}^{r+1}(a_ia^{-1}t^{-1},\ qa_i^{-1}at,\ a_it,\ qa_i^{-1}t^{-1};q)_\infty}.$$

The substitution $b_{2(r-1)}=\pm q^{\frac12}a^{\frac12}$ in (4.1) implies the following transformation for very-well-poised ${}_{2r-1}\psi_{2r-1}$ series:

Theorem 4 *Suppose that*

$$a,\ a_i,\ a_ia^{-1},\ a_ia_j^{-1},\ a_i^2a^{-1},\ a_ia_ja^{-1}\notin q^{\mathbb{Z}},\ 1\le i\ne j\le r-2,$$

for $r\ge3$, *and* $\left|a^{r-\frac52}q^{r-\frac32}\right|<\left|\prod_{i=1}^{2r-3}b_i\right|$. *Then we have*

$$\begin{aligned}&\frac{(\pm q^{\frac12}a^{-\frac12},\ \pm q^{\frac12}a^{\frac12};q)_\infty\prod_{i=1}^{2r-3}(qb_i^{-1},\ qab_i^{-1};q)_\infty}{(qa,\ qa^{-1};q)_\infty\prod_{i=1}^{r-2}\theta(a_ia^{-1},\ a_i)}\\&\times{}_{2r-1}\psi_{2r-1}\left(\begin{matrix}qa^{\frac12},\ -qa^{\frac12},\ b_1,\ b_2,\ \ldots,\ b_{2r-3}\\ a^{\frac12},\ -a^{\frac12},\ qab_1^{-1},\ qab_2^{-1},\ \ldots,\ qab_{2r-3}^{-1}\end{matrix};q,\ \pm\frac{q^{r-\frac52}a^{r-\frac32}}{\prod_{i=1}^{2r-3}b_i}\right)\\&=\sum_{s=1}^{r-2}\frac{(\pm q^{\frac12}a^{\frac12}a_s^{-1},\ \pm q^{\frac12}a^{-\frac12}a_s;q)_\infty\prod_{i=1}^{2r-3}(qaa_s^{-1}b_i^{-1},\ qa_sb_i^{-1};q)_\infty}{(qa_s^2a^{-1},\ qaa_s^{-2};q)_\infty\,\theta(a_sa^{-1},\ a_s)\prod_{\substack{1\le i\le r-2\\ i\ne s}}\theta(a_ia_s^{-1},\ a_sa_ia^{-1})}\\&\times{}_{2r-1}\psi_{2r-1}\left(\begin{matrix}qa_sa^{-\frac12},\ -qa_sa^{-\frac12},\ b_1a_sa^{-1},\ b_2a_sa^{-1},\ \ldots,\ b_{2r-3}a_sa^{-1}\\ a_sa^{-\frac12},\ -a_sa^{-\frac12},\ qa_sb_1^{-1},\ qa_sb_2^{-1},\ \ldots,\ qa_sb_{2r-3}^{-1}\end{matrix};q,\ \pm\frac{q^{r-\frac52}a^{r-\frac32}}{\prod_{i=1}^{2r-3}b_i}\right).\end{aligned}\tag{4.2}$$

Here either all the upper or all the lower signs are taken throughout.

Remark 6 To obtain (4.2) by the same manner as in the proof of Theorem 2, it is enough to consider the function

$$F(t)=\frac{(\pm q^{\frac12}a^{-\frac12}t^{-1},\ \pm q^{\frac12}a^{\frac12}t;q)_\infty\prod_{i=1}^{2r-3}(qb_i^{-1}t^{-1},\ qab_i^{-1}t;q)_\infty}{(t^{-1},\ qt,\ at,\ qa^{-1}t^{-1};q)_\infty\prod_{i=1}^{r-2}(a_ia^{-1}t^{-1},\ qa_i^{-1}at,\ a_it,\ qa_i^{-1}t^{-1};q)_\infty}.$$

Remark 7 If we change r with $r+2$ and substitute $b_{2r}=a^{\frac12},\ b_{2r+1}=-a^{\frac12}$ in (4.2), we obtain (5.5.5) of [1] ((9) of [6], and (7.2.1.4) of [7]), since

$$\begin{aligned}&(a_sa^{-\frac12},\ -a_sa^{-\frac12},\\&qa^{\frac12}a_s^{-1},\ -qa^{\frac12}a_s^{-1};q)_\infty=-a_s^2a^{-1}(qa_sa^{-\frac12},\ -qa_sa^{-\frac12},\ a^{\frac12}a_s^{-1},\ -a^{\frac12}a_s^{-1};q)_\infty,\end{aligned}$$

and

$$\theta(a_s a^{-1}) = -a_s a^{-1}\theta(a a_s^{-1}).$$

Remark 8 To obtain (5.5.5) of [1] by the same manner as in the proof of Theorem 2, it is enough to consider the function

$$F(t) = \frac{\left(\pm q^{\frac{1}{2}} a^{-\frac{1}{2}} t^{-1},\, \pm q^{\frac{1}{2}} a^{\frac{1}{2}} t\,;q\right)_\infty \left(q a^{-\frac{1}{2}} t^{-1},\, q a^{\frac{1}{2}} t,\, -q a^{-\frac{1}{2}} t^{-1},\, -q a^{\frac{1}{2}} t\,;q\right)_\infty}{(t^{-1},\, qt,\, at,\, q a^{-1} t^{-1}\,;q)_\infty} \times \frac{\prod_{i=1}^{2r-1} (q b_i^{-1} t^{-1},\, q a b_i^{-1} t\,;q)_\infty}{\prod_{i=1}^{r} (a_i a^{-1} t^{-1},\, q a_i^{-1} a t,\, a_i t,\, q a_i^{-1} t^{-1}\,;q)_\infty}.$$

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Declarations

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