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ON VANISHING THEOREMS OF SQUARE-INTEGRABLE $\bar{\partial}$-COHOMOLOGY SPACES ON HOMOGENEOUS KAHLER MANIFOLDS

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1. Introduction

Let G be a connected non-compact semi-simple Lie group. We assume that G has a complex form G^C and a compact Cartan subgroup H. The quotient manifold $D=G/H$ carries a G-invariant complex structure and a G-invariant hermitian metric. Then, corresponding to each character λ of H, one can construct a homogeneous hermitian line bundle $\mathcal{L}_\lambda=G\times_H C$ over D. Let $H^q_2(\mathcal{L}_\lambda)$ be the q-th square-integrable $\bar{\partial}$-cohomology space with coefficients in the bundle $\mathcal{L}_\lambda$, i.e. the Hilbert space of all square-integrable $\mathcal{L}_\lambda$-valued harmonic $(0, q)$-forms on D. P.A. Griffiths and W. Schmid [4] have obtained some vanishing theorems for these cohomology spaces, assuming that the character λ is sufficiently non-singular.

Now the manifold D does not necessarily admit a G-invariant Kähler metric. In fact, P.A. Griffiths and W. Schmid used a non-Kähler hermitian metric on D. The purpose of this paper is to prove certain vanishing theorems for these $\bar{\partial}$-cohomology spaces under the assumption that D has a G-invariant Kähler metric. The main result is Theorem 2 in §7. In some cases, our result is considerably better than the one given in [4]. (cf. §7. Example)

In §2, we recall some facts about Lie algebras and homogeneous vector bundles. In §3 and following sections, we assume further that the Riemannian symmetric space G/K is hermitian symmetric, where K is a maximal compact subgroup of G containing H. Under this assumption, we introduce canonically an invariant complex structure and an invariant Kähler metric on the manifold D. Next, we shall define in §4 the q-th square-integrable $\bar{\partial}$-cohomology space $H^q_2(\mathcal{L}_\lambda)$ on D with coefficients in $\mathcal{L}_\lambda$. Also we shall give explicit formulas for the differential operator $\bar{\partial}$ and the inner product on the space of all compactly supported $\mathcal{L}_\lambda$-valued C^∞-forms on D.

In [1], A. Andreotti and E. Vesentini expressed the Laplace-Beltrami operator $\square$ on a hermitian manifold in terms of the metric connection and showed that this expression of $\square$ becomes simpler if the manifold is Kählerian.

In §5, we construct the metric connection in the bundle $\mathcal{L}_\lambda$ and the Riemannian connection of D, applying Wang's results about invariant connections. Moreover, in §6, we express the operators $\bar{\partial}$, δ and $\square$ in terms of these connections. From the fact that the metric on D is Kählerian, we get a simple explicit formula for the operator $\square$ (cf. §6. Proposition 2). In §7, we shall prove the main vanishing theorem. In this proof, we use the criterion for the vanishing of square-integrable $\bar{\partial}$-cohomology spaces which has been established in [1].

Finally, the author wishes to express her hearty thanks to Prof. S. Murakami who suggested this program to her and gave helpful advices in the preparation of this paper.

2. Preliminaries

Let G be a connected non-compact semi-simple Lie group. We denote by $\mathfrak{g}_0$ the Lie algebra of left invariant vector fields on G and by $\mathfrak{g}$ the complexification of $\mathfrak{g}_0$. Throughout this paper, we assume that G has a compact Cartan subgroup H. Let K be a maximal compact subgroup of G which contains H. Let $\mathfrak{k}_0$ and $\mathfrak{h}_0$ be the subalgebras of $\mathfrak{g}_0$ corresponding to the subgroups K and H, and $\mathfrak{k}$ and $\mathfrak{h}$ the complexifications of $\mathfrak{k}_0$ and $\mathfrak{h}_0$ respectively. For each $x \in \mathfrak{g}$, we denote by $\bar{x}$ the image of x under the conjugation of $\mathfrak{g}$ with respect to the real form $\mathfrak{g}_0$. Let Δ be the set of all non-zero roots of $\mathfrak{g}$ with respect to the Cartan subalgebra $\mathfrak{h}$. Then, the Lie algebra $\mathfrak{g}$ decomposes into the direct sum

$$(2.1) \qquad \mathfrak{g} = \mathfrak{h} \oplus \sum_{\alpha \in \Delta} \mathfrak{g}^\alpha ,$$

where we put

$$\mathfrak{g}^\alpha = \{x \in \mathfrak{g} \mid [h, x] = \langle \alpha, h \rangle x \quad \text{for all} \quad h \in \mathfrak{h}\} .$$

Since H is compact, each root $\alpha \in \Delta$ takes purely imaginary values on $\mathfrak{h}_0$. Thus, we may consider Δ as a subset of the dual space $\mathfrak{h}_R{}^*$ of $\mathfrak{h}_R = \sqrt{-1}\,\mathfrak{h}_0$, and we have

$$\bar{\mathfrak{g}}^\alpha = \mathfrak{g}^{-\alpha} \qquad \text{for all} \quad \alpha \in \Delta .$$

Now, let B be the Killing form of $\mathfrak{g}$. We denote by $(\ , \ )$ the natural inner product on $\mathfrak{h}_R{}^*$ obtained from the restriction of B on $\mathfrak{h}_R$. Put

$$\mathfrak{p} = \{x \in \mathfrak{g} \mid B(x, y) = 0 \qquad \text{for all} \quad y \in \mathfrak{k}\} .$$

Then we have

$$(2.2) \qquad \mathfrak{g} = \mathfrak{k} \oplus \mathfrak{p}, \ [\mathfrak{k}, \mathfrak{p}] \subset \mathfrak{p}, \ [\mathfrak{p}, \mathfrak{p}] \subset \mathfrak{k} .$$

A root α is called compact or non-compact according as $\mathfrak{g}^\alpha \subset \mathfrak{k}$ or $\mathfrak{g}^\alpha \subset \mathfrak{p}$. We denote by $\Delta_{\mathfrak{k}}$(resp. $\Delta_{\mathfrak{p}}$) the set of all compact (resp. non-compact) roots. Then we have

$$\Delta = \Delta_{\mathfrak{k}} \cup \Delta_{\mathfrak{p}} .$$

For each $\alpha \in \Delta$, let h_α be the element of $\mathfrak{h}$ such that

$$B(h, h_\alpha) = \alpha(h) \qquad \text{for all} \quad h \in \mathfrak{h} . \tag{2.3}$$

Then we can choose root vectors $e_\alpha \in \mathfrak{g}^\alpha$ ($\alpha \in \Delta$) satisfying the following conditions:

(2.4) 1) $[e_\alpha, e_{-\alpha}] = h_\alpha$,
2) $[e_\alpha, e_\beta] = 0$ if $\alpha+\beta \neq 0$ and $\alpha+\beta \notin \Delta$,
3) $[e_\alpha, e_\beta] = N_{\alpha,\beta} e_{\alpha+\beta}$ if $\alpha+\beta \in \Delta$,
4) $\bar{e}_\alpha = \varepsilon_\alpha e_{-\alpha}$,

where the $N_{\alpha,\beta}$'s are non-zero real constants, and $\varepsilon_\alpha = -1$ if $\alpha \in \Delta_{\mathfrak{k}}$ and $\varepsilon_\alpha = 1$ if $\alpha \in \Delta_{\mathfrak{p}}$ [5]. Moreover, the $N_{\alpha,\beta}$'s satisfy following equalities:

$$\begin{aligned} &N_{-\alpha,-\beta} = -N_{\alpha,\beta} \\ &N_{-\alpha,-\beta} = N_{-\beta,\alpha+\beta} = N_{\alpha+\beta,-\alpha} . \end{aligned} \tag{2.5}$$

For convenience, we define $N_{\alpha,\beta} = 0$ if $\alpha+\beta \neq 0$ and $\alpha+\beta \notin \Delta$. We denote by $\{\omega^\alpha \mid \alpha \in \Delta\}$ the left-invariant 1-forms on G which are dual to $\{e_\alpha \mid \alpha \in \Delta\}$.

We consider the quotient manifold $D = G/H$. In the decomposition (2.1) of $\mathfrak{g}$, we put

$$\mathfrak{n} = \sum_{\alpha \in \Delta} \mathfrak{g}^\alpha, \quad \mathfrak{n}_0 = \mathfrak{n} \cap \mathfrak{g}_0 .$$

Then, we have

$$\mathfrak{g}_0 = \mathfrak{h}_0 \oplus \mathfrak{n}_0, \quad [\mathfrak{h}_0, \mathfrak{n}_0] \subset \mathfrak{n}_0 , \tag{2.6}$$

and D is a reductive homogeneous space. The tangent space of D at the point $o = eH$ may be identified with the subspace $\mathfrak{n}_0$ of $\mathfrak{g}_0$, where e denotes the identity element of G. Now, let $\pi: H \to GL(E)$ be a representation of H in a complex vector space E. We denote by E_π the homogeneous vector bundle over D associated with the representation π of H. Let $A^p(E_\pi)$ be the space of E_π-valued C^∞ p-forms on D. A form in $A^p(E_\pi)$ can be identified with an E-valued C^∞ p-form φ on G satisfying the conditions

$$\begin{cases} \theta(h)\varphi = -\pi(h)\varphi \\ i(h)\varphi = 0 \end{cases} \qquad \text{for all} \quad h \in \mathfrak{h} , \tag{2.7}$$

where $\theta(h)$ and $i(h)$ denote the operator of Lie derivation and interior product by the vector field h and π is the representation of $\mathfrak{h}$ in E induced by the representation π of H [7]. Let $C^\infty(G)$ be the space of all complex-valued

C^∞-functions on G. Let $\mathfrak{n}^*$ be the dual space of $\mathfrak{n}$ and $\overset{p}{\wedge}\mathfrak{n}^*$ the p-th exterior product of $\mathfrak{n}^*$. For an ordered p-tuple $C=(\lambda_1, \cdots, \lambda_p)$ of roots, we put

$$\omega^C = \omega^{\lambda_1}\wedge\cdots\wedge\omega^{\lambda_p} .$$

Let Λ be a set of ordered p-tuples such that $\{\omega^C \mid C\in\Lambda\}$ forms a basis of $\overset{p}{\wedge}\mathfrak{n}^*$. The vector space $C^\infty(G)\otimes E\otimes\overset{p}{\wedge}\mathfrak{n}^*$ is generated by monomials $F\omega^C$ with $F\in C^\infty(G)\otimes E$ and $C\in\Lambda$. By (2.7), the space $A^p(E_\pi)$ can be identified with the subspace of $C^\infty(G)\otimes E\otimes\overset{p}{\wedge}\mathfrak{n}^*$ consisting of all elements $\varphi=\sum_{C\in\Lambda}F_C\omega^C$ satisfying the condition

$$hF_C = -\pi(h)F_C+\langle |C|, h\rangle F_C . \tag{2.8}$$

for all $C\in\Lambda$ and $h\in\mathfrak{h}$, where $|C|=\lambda_1+\cdots+\lambda_p$ and hF_C denotes the differentiation of the function F_C by the vector field h. In particular, the space $A^0(E_\pi)$ is identified with the subspace of $C^\infty(G)\otimes E$ consisting of all elements $F\in C^\infty(G)\otimes E$ such that

$$hF = -\pi(h)F \qquad \text{for all} \quad h\in\mathfrak{h} .$$

3. Homogeneous Kähler manifolds

Let G be a connected non-compact semi-simple Lie group with a compact Cartan subgroup H. In the following, we assume that there is a complex Lie group G^C with Lie algebra $\mathfrak{g}$ which contains G as a Lie subgroup corresponding to the subalgebra $\mathfrak{g}_0$.

We introduce an ordering for the roots and denote by Δ_+ the set of all positive roots with respect to this ordering. Put

$$\mathfrak{n}_+ = \sum_{\alpha\in\Delta_+}\mathfrak{g}^\alpha, \quad \mathfrak{n}_- = \sum_{\alpha\in\Delta_+}\mathfrak{g}^{-\alpha} .$$

Then we have

$$\begin{aligned} &\mathfrak{g} = \mathfrak{h}\oplus\mathfrak{n}_+\oplus\mathfrak{n}_- \\ &\bar{\mathfrak{n}}_+ = \mathfrak{n}_- \quad \bar{\mathfrak{n}}_- = \mathfrak{n}_+ \\ &[\mathfrak{h}, \mathfrak{n}_+]\subset\mathfrak{n}_+, \quad [\mathfrak{n}_+, \mathfrak{n}_+]\subset\mathfrak{n}_+ . \end{aligned} \tag{3.1}$$

For the quotient manifold $D=G/H$, the complexified tangent space of D at the point o may be identified with the vector space $\mathfrak{n}=\mathfrak{n}_+\oplus\mathfrak{n}_-$, and then, by (3.1), D has a G-invariant complex structure such that the holomorphic tangent space of D at o corresponds to $\mathfrak{n}_+$. This complex structure of D can also be obtained in the following way. Let B be the Borel subgroup of G^C corresponding to the Borel subalgebra $\mathfrak{h}\oplus\mathfrak{n}_-$ of $\mathfrak{g}$. The group G acts on the homogeneous complex manifold G^C/B. Since we have

$$\begin{aligned}\mathfrak{g}_0 \cap (\mathfrak{h} \oplus \mathfrak{n}_-) &= \mathfrak{g}_0 \cap (\mathfrak{h} \oplus \mathfrak{n}_-) \cap (\overline{\mathfrak{h} \oplus \mathfrak{n}_-}) \\ &= \mathfrak{g}_0 \cap \mathfrak{h} = \mathfrak{h}_0 ,\end{aligned}$$

the Lie algebra of the isotropy subgroup $G \cap B$ is $\mathfrak{h}_0$. Therefore, H is the identity component of the subgroup $G \cap B$ and $G \cap B$ normalizes H. The normalizer of H in G is compact and H is a maximal compact subgroup of the Borel subgroup B. Hence we have

$$G \cap B = H ,$$

and $D = G/H$ is identified with the G-orbit of eB in G^C/B. Since

$$\dim G^C/B = \dim \mathfrak{n}_+ = \dim G/H ,$$

D is open in G^C/B and D has a G-invariant complex structure as an open submanifold of G^C/B. Then, it is easily seen that the holomorphic tangent space of D at o corresponds to $\mathfrak{n}_+$ [4].

In the following, we will assume that the Riemannian symmetric space G/K is hermitian symmetric. By (2.2), the complexified tangent space of G/K at eK may be identified with $\mathfrak{p}$. Let $\mathfrak{p}_+$(resp. $\mathfrak{p}_-$) be the subspace of $\mathfrak{p}$ corresponding to the holomorphic (resp. anti-holomorphic) tangent space of G/K at eK under this identification. We know that there exists an element h_0 belonging to the center of $\mathfrak{k}_0$ such that

$$[h_0, x] = \begin{cases} \sqrt{-1}\, x & \text{for } x \in \mathfrak{p}_+ \\ -\sqrt{-1}\, x & \text{for } x \in \mathfrak{p}_- . \end{cases} \tag{3.2}$$

It follows that

$$\begin{aligned}&[\mathfrak{p}_+, \mathfrak{p}_+] = 0, \quad [\mathfrak{p}_-, \mathfrak{p}_-] = 0 \\ &[\mathfrak{k}, \mathfrak{p}_+] \subset \mathfrak{p}_+, \quad [\mathfrak{k}, \mathfrak{p}_-] \subset \mathfrak{p}_-\end{aligned}$$

and in particular

$$[\mathfrak{h}, \mathfrak{p}_+] \subset \mathfrak{p}_+, \quad [\mathfrak{h}, \mathfrak{p}_-] \subset \mathfrak{p}_- .$$

Hence, we see that for some subset Δ_0 of Δ

$$\mathfrak{p}_+ = \sum_{\alpha \in \Delta_0} \mathfrak{g}^{\alpha}, \quad \mathfrak{p}_- = \sum_{\alpha \in \Delta_0} \mathfrak{g}^{-\alpha} .$$

We may choose an ordering for the roots in such a way that the roots belonging to Δ_0 are all positive, i.e.

$$\Delta_0 = \Delta_+ \cap \Delta_{\mathfrak{p}} . \tag{3.3}$$

We choose such an ordering once for all, and introduce an invariant complex structure on D defined by this ordering.

Lemma 1. *There exists an element τ of $\mathfrak{h}_R{}^*$ satisfying following conditions:*

$$(3.4)\qquad \begin{cases} (\alpha, \tau)>0 & \text{for all } \alpha\in\Delta_+\cap\Delta_{\mathfrak{k}} \\ (\alpha, \tau)<0 & \text{for all } \alpha\in\Delta_+\cap\Delta_{\mathfrak{p}}. \end{cases}$$

Proof. Since the element h_0 belongs to the center of $\mathfrak{k}_0$, we have

$$\alpha(\sqrt{-1}\, h_0)=0 \qquad \text{for} \quad \alpha\in\Delta_+\cap\Delta_{\mathfrak{k}}.$$

By (3.2) and (3.3), we have also

$$\alpha(\sqrt{-1}\, h_0)=-1 \qquad \text{for} \quad \alpha\in\Delta_+\cap\Delta_{\mathfrak{p}}.$$

We denote by τ_0 the element of $\mathfrak{h}_R{}^*$ such that

$$B(h, \sqrt{-1}\, h_0)=\tau_0(h) \qquad \text{for all} \quad h\in\mathfrak{h}_R.$$

Then, we obtain

$$(\alpha, \tau_0)=\begin{cases} 0 & \text{for} \quad \alpha\in\Delta_+\cap\Delta_{\mathfrak{k}} \\ -1 & \text{for} \quad \alpha\in\Delta_+\cap\Delta_{\mathfrak{p}}. \end{cases}$$

On the other hand, we know that for the element $\rho=\frac{1}{2}\sum_{\alpha\in\Delta_+}\alpha$ of $\mathfrak{h}_R{}^*$ we have $(\rho, \alpha)>0$ for $\alpha\in\Delta_+$. Therefore, if we put $\tau=\rho+c\tau_0$ with a sufficiently large constant c, we get

$$(\alpha, \tau)=\begin{cases} (\alpha, \rho)>0 & \text{for} \quad \alpha\in\Delta_+\cap\Delta_{\mathfrak{k}} \\ (\alpha, \rho)+c(\alpha, \tau_0)<0 & \text{for} \quad \alpha\in\Delta_+\cap\Delta_{\mathfrak{p}}. \end{cases}$$

q.e.d.

Now, let τ be an element of $\mathfrak{h}_R{}^*$ satisfying the condition (3.4). Using this τ, we shall construct an invariant Kähler metric on D. We define a complex symmetric bilinear form B_τ on $\mathfrak{n}=\sum_{\alpha\in\Delta}\mathfrak{g}^\alpha$ by the following formula:

$$(3.5)\qquad \begin{aligned} &B_\tau(e_\alpha, e_\beta)=B_\tau(e_{-\alpha}, e_{-\beta})=0 \\ &B_\tau(e_\alpha, e_{-\beta})=-\delta_{\alpha,\beta}(\alpha, \tau) \end{aligned}$$

for $\alpha, \beta\in\Delta_+$. Clearly, B_τ is invariant under the adjoint action of H on $\mathfrak{n}$ and the restriction of B_τ on the real subspace $\mathfrak{n}_0=\mathfrak{n}\cap\mathfrak{g}_0$ is a positive-definite symmetric bilinear form on $\mathfrak{n}_0$. Define the endomorphism J on $\mathfrak{n}$ by

$$Jx=\begin{cases} \sqrt{-1}\, x & \text{for} \quad x\in\mathfrak{n}_+ \\ -\sqrt{-1}\, x & \text{for} \quad x\in\mathfrak{n}_-. \end{cases}$$

Then we have also

$$B_\tau(Jx, Jy)=B_\tau(x, y)$$

for all $x, y \in \mathfrak{n}$. Therefore, we can define a G-invariant hermitian metric on D such that the metric on the tangent space of D at o corresponds to $B_\tau | \mathfrak{n}_0$ on $\mathfrak{n}_0$. The Kähler form of this hermitian metric corresponds to the complex 2-form Ω on G given by

$$\Omega = \sum_{\alpha \in \Delta_+} \sqrt{-1}\,(\alpha, \tau)\omega^\alpha \wedge \omega^{-\alpha} .$$

We see easily that

$$d\Omega = 0 .$$

Thus the metric on D induced by B_τ is Kählerian. We denote by g_τ this invariant Kähler metric on D.

REMARK. In the above, we constructed an invariant Kähler metric on D under the assumption that the symmetric space G/K is hermitian and the natural projection $G/H \to G/K$ is holomorphic. Conversely, it is known that if the manifold D has a G-invariant Kähler metric, then G/K is hermitian and the fibering $G/H \to G/K$ is holomorphic [2].

4. Homogeneous line bundles and square-integrable $\bar{\partial}$-cohomology spaces

Let $D = G/H$ be the homogeneous complex manifold of $2n$ real dimension with the G-invariant Kähler metric g_τ. Let λ be a character of H. We consider the homogeneous real line bundle $\mathcal{L}_\lambda = G \times_H C$ over D associated with λ. Let B be the Borel subgroup of G^C such that the quotient manifold G^C/B contains D as an open submanifold (cf § 3). The character λ can be extended to the unique holomorphic character on B [3]. We can consider the homogeneous complex line bundle $G^C \times_B C$ over G^C/B. Then, the bundle $\mathcal{L}_\lambda$ is isomorphic to the restriction of the bundle $G^C \times_B C$ on D as a real line bundle. Therefore, $\mathcal{L}_\lambda$ has a G-invariant complex structure as an open submanifold of $G^C \times_B C$. We get thus a hermitian line bundle $\mathcal{L}_\lambda$ with a natural hermitian metric in the fibers.

Let $A^{p,q}(\mathcal{L}_\lambda)$ be the space of all $\mathcal{L}_\lambda$-valued C^∞-forms of type (p, q) on D, and $A_0^{p,q}(\mathcal{L}_\lambda)$ the subspace of all compactly supported forms in $A^{p,q}(\mathcal{L}_\lambda)$. The hermitian metric on D defines the complex linear operator $*$ of $A^{p,q}(\mathcal{L}_\lambda)$ into $A^{n-q,n-p}(\mathcal{L}_\lambda)$. On the other hand, the hermitian metric on the fibers of $\mathcal{L}_\lambda$ gives rise to a conjugate linear isomorphism

$$\sharp\colon\ A^{p,q}(\mathcal{L}_\lambda) \to A^{q,p}(\mathcal{L}_\lambda{}^*)$$

where $\mathcal{L}_\lambda{}^*$ is the complex dual bundle of $\mathcal{L}_\lambda$. We define an inner product $(\ ,\ )$ on $A_0^{0,q}(\mathcal{L}_\lambda)$ by

$$(\varphi, \psi) = \int_D \varphi \wedge * \sharp \psi$$

for φ, ψ in $A_0^{0,q}(\mathcal{L}_\lambda)$. Let $L_2^{0,q}(\mathcal{L}_\lambda)$ be the completion of $A_0^{0,q}(\mathcal{L}_\lambda)$ with respect to this inner product. The type (0, 1)-component of exterior differentiation defines the differential operator

$$\bar{\partial}\colon A^{0,q}(\mathcal{L}_\lambda) \to A^{0,q+1}(\mathcal{L}_\lambda).$$

Let $\delta\colon A^{0,q}(\mathcal{L}_\lambda) \to A^{0,q-1}(\mathcal{L}_\lambda)$ be the formal adjoint operator of $\bar{\partial}$. We define the Laplace-Beltrami operator $\square$ by

$$\square = \bar{\partial}\delta + \delta\bar{\partial}.$$

Then the space

$$H_2^q(\mathcal{L}_\lambda) = \{\varphi \in L_2^{0,q}(\mathcal{L}_\lambda) \cap A^{0,q}(\mathcal{L}_\lambda) \mid \square\varphi = 0\}$$

is called the q-th square-integrable $\bar{\partial}$-cohomology space of D with coefficients in the bundle $\mathcal{L}_\lambda$. (cf. [1]).

Let $\mathfrak{n}_-^*$ be the dual space of $\mathfrak{n}_-$ and $\overset{q}{\wedge}\mathfrak{n}_-^*$ the q-th exterior product of $\mathfrak{n}_-^*$. We denote by $\{\alpha_1, \cdots, \alpha_n\}$ the set of positive roots Δ_+. For an ordered q-tuple of positive roots $A=(\alpha_{i_1}, \cdots, \alpha_{i_q})$, we put

$$\omega^{-A} = \omega^{-\alpha_{i_1}} \wedge \cdots \wedge \omega^{-\alpha_{i_q}}.$$

Let $\mathfrak{A}$ be the set of all ordered q-tuples $A=(\alpha_{i_1}, \cdots, \alpha_{i_q})$ such that $1 \leqq i_1 < \cdots < i_q \leqq n$. Then the space $C^\infty(G) \otimes \overset{q}{\wedge}\mathfrak{n}_-^*$ is generated by monomials $f\omega^{-A}$ with $f \in C^\infty(G)$ and $A \in \mathfrak{A}$. From the discussion in § 2, the space $A^{0,q}(\mathcal{L}_\lambda)$ is identified with the subspace of $C^\infty(G) \otimes \overset{q}{\wedge}\mathfrak{n}_-^*$. Let $\varphi = \sum_{A \in \mathfrak{A}} f_A \omega^{-A}$ be an element of $C^\infty(G) \otimes \overset{q}{\wedge}\mathfrak{n}_-^*$. Then, according to the condition (2.8), φ belongs to $A^{0,q}(\mathcal{L}_\lambda)$ if and only if the following condition is satisfied:

$$hf_A = \langle -\lambda - |A|, h \rangle f_A \tag{4.1}$$

for all $A \in \mathfrak{A}$ and $h \in \mathfrak{h}$, where λ is the representation of $\mathfrak{h}$ induced by the character λ of H. Under this identification, the space $A_0^{0,q}(\mathcal{L}_\lambda)$ corresponds to a subspace of $C_0^\infty(G) \otimes \overset{q}{\wedge}\mathfrak{n}_-^*$, where $C_0^\infty(G)$ is the space of all compactly supported functions in $C^\infty(G)$.

Now, we give an expression of the inner product on $A_0^{0,q}(\mathcal{L}_\lambda)$. The bilinear form B_τ on $\mathfrak{n}$ induces the following hermitian inner product B_τ^- on $\mathfrak{n}_-$:

$$\begin{aligned} B_\tau^-(e_{-\alpha}, e_{-\beta}) &= B_\tau(e_{-\alpha}, \bar{e}_{-\beta}) \\ &= \delta_{\alpha,\beta}(-\varepsilon_\alpha(\alpha, \tau)). \end{aligned}$$

From B_τ^-, we obtain the hermitian inner product $(\ ,\ )_-$ on $\overset{q}{\wedge}\mathfrak{n}_-^*$ as follows:

$$(4.2)\qquad \begin{cases} (\omega^{-A}, \omega^{-B})_- = 0 & \text{if } (A) \neq (B) \text{ as sets,} \\ (\omega^{-A}, \omega^{-A})_- = \prod_{\alpha \in A} \left(-\dfrac{1}{\varepsilon_\alpha(\alpha, \tau)} \right). \end{cases}$$

Let dg be a G-invariant volume element of G. Then dg defines an inner product $(\ ,\ )_G$ on $C_0^\infty(G)$. These inner products $(\ ,\ )_-$ and $(\ ,\ )_G$ define an inner product on $C_0^\infty(G) \otimes \overset{q}{\wedge} \mathfrak{n}_-^*$ in a canonical way. In fact, for two elements $\varphi = \sum_{A \in \mathfrak{A}} f_A \omega^{-A}$, $\psi = \sum_{A \in \mathfrak{A}} g_A \omega^{-A}$ in $C_0^\infty(G) \otimes \overset{q}{\wedge} \mathfrak{n}_-^*$, this inner product (φ, ψ) is given by

$$\begin{aligned} (\varphi, \psi) &= \sum_{A \in \mathfrak{A}} (f_A, g_A)_G \cdot (\omega^{-A}, \omega^{-A})_- \\ &= \sum_{A \in \mathfrak{A}} \prod_{\alpha \in A} \left(-\frac{1}{\varepsilon_\alpha(\alpha, \tau)} \right) \cdot \int_G f_A \cdot \overline{g_A}\, dg . \end{aligned}$$

The following lemma asserts that if we choose a suitable volume element dg of G, the inner product on $A_0^{0,q}(\mathcal{L}_\lambda)$ is the restriction of this inner product $(\ ,\ )$ on the subspace $A_0^{0,q}(\mathcal{L}_\lambda)$.

Lemma 2. *If we choose a suitable G-invariant volume element dg on G, the inner product of $A_0^{0,q}(\mathcal{L}_\lambda)$ is given by the following formula:*

$$(\varphi, \psi) = \sum_{A \in \mathfrak{A}} \prod_{\alpha \in A} \left(-\frac{1}{\varepsilon_\alpha(\alpha, \tau)} \right) \cdot \int_G f_A \cdot \overline{g_A}\, dg$$

where $\varphi = \sum_{A \in \mathfrak{A}} f_A \omega^{-A}$ and $\psi = \sum_{A \in \mathfrak{A}} g_A \omega^{-A}$ are forms in $A_0^{0,q}(\mathcal{L}_\lambda)$.

Proof. We apply the methods used in the proof of Proposition 5.1 in [7]. Let dv_D be the G-invariant volume element on D determined by the metric g_τ. Then, we can choose invariant volume elements dg on G and dh on H such that

$$\int_G f(g) dg = \int_D \left(\int_H f(gh) dh \right) dv_D$$

for all $f \in C_0^\infty(G)$ ([5], p. 369, Theorem 1.7). Let p: $G \to D$ be the natural projection. Then we have

$$(4.3)\qquad \int_D f' dv_D = \frac{1}{v_H} \int_G f' \circ p\, dg$$

for every compactly supported C^∞-function f' on D, where v_H is the volume of H with respect to dh.

For a root $\alpha_i \in \Delta_+$, we put

$$x_{\alpha_i} = \left(-\frac{1}{\varepsilon_{\alpha_i}(\alpha_i, \tau)} \right)^{1/2} e_{\alpha_i} .$$

Then, $\{x_{\alpha_1}, \cdots, x_{\alpha_n}, \overline{x_{\alpha_1}}, \cdots, \overline{x_{\alpha_n}}\}$ is a basis of $\mathfrak{n}$ and we have

$$B_\tau(x_{\alpha_i}, \overline{x_{\alpha_j}}) = \delta_{ij}\,. \tag{4.4}$$

We take a point $p(g)(g\in G)$ in D. By (4.4), $\{p_*(x_{\alpha_1})_g, \cdots, p_*(x_{\alpha_n})_g, p_*(\overline{x_{\alpha_1}})_g, \cdots, p_*(\overline{x_{\alpha_n}})_g\}$ is a basis of the complexified tangent space of D at $p(g)$ such that

$$g_\tau(p_*(x_{\alpha_i})_g, p_*(\overline{x_{\alpha_j}})_g) = \delta_{ij}\,,$$

where p_* is the differential of p. For a sufficiently small neighbourhood U of $p(g)$ in D, we can find (1, 0)-forms $\theta^1, \cdots, \theta^n$ and (0, 1)-forms $\bar{\theta}^1, \cdots, \bar{\theta}^n$ such that

$$\theta^i_{p(g)}(p_*(x_{\alpha_j})_g) = \bar{\theta}^i_{p(g)}(p_*(\overline{x_{\alpha_j}})_g) = \delta_{ij}$$
$$\theta^i_{p(g)}(p_*(\overline{x_{\alpha_j}})_g) = \bar{\theta}^i_{p(g)}(p_*(x_{\alpha_j})_g) = 0\,.$$

Let φ and ψ be forms in $A_0^{0,q}(\mathcal{L}_\lambda)$. We denote by $\sum_{A\in\mathfrak{A}} f_A\omega^{-A}$(resp. $\sum_{A\in\mathfrak{A}} g_A\omega^{-A}$) an element of $C_0^\infty(G)\otimes\stackrel{q}{\wedge}\mathfrak{n}_-^*$ which corresponds to φ (resp. ψ) under the identification of $A_0^{0,q}(\mathcal{L}_\lambda)$ with the subspace of $C_0^\infty(G)\otimes\stackrel{q}{\wedge}\mathfrak{n}_-^*$. The forms φ and ψ are written on U in the form

$$\varphi = \sum_{i_1<\cdots<i_q} u_{i_1\cdots i_q}\bar{\theta}^{i_1}\wedge\cdots\wedge\bar{\theta}^{i_q}$$
$$\psi = \sum_{i_1<\cdots<i_q} v_{i_1\cdots i_q}\bar{\theta}^{i_1}\wedge\cdots\wedge\bar{\theta}^{i_q}\,.$$

Then, we have

$$\begin{aligned} u_{i_1\cdots i_q}(p(g)) &= \varphi_{p(g)}(p_*(\overline{x_{\alpha_{i_1}}})_g, \cdots, p_*(\overline{x_{\alpha_{i_q}}})_g) \\ &= \nu(g, (\textstyle\sum f_A\omega^{-A})_g((\overline{x_{\alpha_{i_1}}})_g, \cdots, (\overline{x_{\alpha_{i_q}}})_g)) \\ &= \prod_{j=1}^q \left(-\frac{1}{\varepsilon_{\alpha_{i_j}}(\alpha_{i_j}, \tau)}\right)^{1/2}\cdot\nu(g, f_{(\alpha_{i_1}, \cdots, \alpha_{i_q})}(g)) \end{aligned} \tag{4.5}$$

where ν is the projection of $G\times C$ onto $\mathcal{L}_\lambda = G\times_H C$. Similarly, we have also

$$v_{i_1\cdots i_q}(p(g)) = \prod_{j=1}^q \left(-\frac{1}{\varepsilon_{\alpha_{i_j}}(\alpha_{i_j}, \tau)}\right)^{1/2}\nu(g, g_{(\alpha_{i_1}, \cdots, \alpha_{i_q})}(g))\,. \tag{4.6}$$

On the other hand, by the definition of the inner product (,) on $A_0^{0,q}(\mathcal{L}_\lambda)$, we have

$$\begin{aligned} (\varphi, \psi) &= \int_D \varphi\wedge\#*\psi \\ &= \int_D \sum_{i_1<\cdots<i_q} (u_{i_1\cdots i_q}, v_{i_1\cdots i_q})_{\mathcal{L}_\lambda} dv_D \end{aligned}$$

where $(\ ,\ )_{\mathcal{L}_\lambda}$ is the inner product on the fibers of $\mathcal{L}_\lambda$. By (4.5) and (4.6), we get

$$(u_{i_1\cdots i_q}, v_{i_1\cdots i_q})_{\mathcal{L}_\lambda}(p(g)) = \prod_{j=1}^{q}\left(-\frac{1}{\varepsilon_{\alpha_{i_j}}(\alpha_{i_j}, \tau)}\right) f_{(\alpha_{i_1}\cdots\alpha_{i_q})}(g)\,\overline{g_{(\alpha_{i_1}\cdots\alpha_{i_q})}(g)}\,.$$

Therefore, by (4.3), we obtain

$$\begin{aligned}(\varphi, \psi) &= \sum_{A\in\mathfrak{A}} \prod_{\alpha\in A}\left(-\frac{1}{\varepsilon_\alpha(\alpha, \tau)}\right)\cdot\int_D f_A\cdot\overline{g_A}\,dv_D \\ &= \sum_{A\in\mathfrak{A}} \prod_{\alpha\in A}\left(-\frac{1}{\varepsilon_\alpha(\alpha, \tau)}\right)\cdot\int_G f_A\cdot\overline{g_A}\frac{1}{v_H}dg\,.\end{aligned}$$

Thus, taking $\frac{1}{v_H}dg$ as a G-invariant volume element of G, we obtain the lemma. q.e.d.

For later use, we give an expression of the operator $\bar{\partial}$ due to [4]. First, we define some operators. Since we have

$$[\mathfrak{n}_-, \mathfrak{n}_-]\subset\mathfrak{n}_-\,,$$

the vector space $\mathfrak{n}_-$ is an $\mathfrak{n}_-$-module under the adjoint representation, and so $\mathfrak{n}_-^*$ is also an $\mathfrak{n}_-$-module. Then the action of $e_{-\alpha}(\alpha\in\Delta_+)$ on $\mathfrak{n}_-^*$ is given by

$$e_{-\alpha}\omega^{-\beta} = \begin{cases} N_{-\alpha,\beta}\,\omega^{\alpha-\beta} & \text{if } \beta-\alpha\in\Delta_+ \\ 0 & \text{otherwise.}\end{cases} \tag{4.7}$$

This action of $\mathfrak{n}_-$ on $\mathfrak{n}_-^*$ is extended to $\overset{q}{\wedge}\mathfrak{n}_-^*$. On the other hand, for $x\in\mathfrak{n}_+$ and $y\in\mathfrak{n}_-$, we put

$$x\cdot y = [x, y]_{\mathfrak{n}_-}$$

where $[x, y]_{\mathfrak{n}_-}$ is the $\mathfrak{n}_-$-component of $[x, y]$. Then, since $\mathfrak{h}\oplus\mathfrak{n}_+$ is an $\mathfrak{n}_+$-module, $\mathfrak{n}_-$ becomes an $\mathfrak{n}_+$-module, and so $\mathfrak{n}_-^*$ is an $\mathfrak{n}_+$-module. The action of $e_\alpha(\alpha\in\Delta_+)$ on $\mathfrak{n}_-^*$ is given by

$$e_\alpha\,\omega^{-\beta} = N_{\alpha,\beta}\,\omega^{-\alpha-\beta}\,. \tag{4.8}$$

This action of $\mathfrak{n}_+$ on $\mathfrak{n}_-^*$ is also extended to $\overset{q}{\wedge}\mathfrak{n}_-^*$. In the case $q=0$, we define $e_\alpha c=0$ and $e_{-\alpha}c=0$ for $c\in C=\overset{0}{\wedge}\mathfrak{n}_-^*$. Moreover, we define the operators

$$\begin{aligned} e(\omega^{-\alpha});\ \overset{q}{\wedge}\mathfrak{n}_-^* &\to \overset{q+1}{\wedge}\mathfrak{n}_-^* \\ i(\omega^{-\alpha});\ \overset{q}{\wedge}\mathfrak{n}_-^* &\to \overset{q-1}{\wedge}\mathfrak{n}_-^* \end{aligned}$$

by the following formulas:

$$e(\omega^{-\alpha})\omega^{-A} = \omega^{-\alpha}\wedge\omega^{-A} \tag{4.9}$$

$$(4.10)\qquad \begin{cases} i(\omega^{-\alpha})=0 & \text{on } C=\overset{0}{\wedge}\mathfrak{n}_-^*\,, \\ i(\omega^{-\alpha})\omega^{-A}=0 & \text{if } \alpha\notin A\,, \\ i(\omega^{-\alpha})(\omega^{-\alpha}\wedge\omega^{-A})=\omega^{-A} & \text{if } \alpha\notin A\,. \end{cases}$$

Now, returning to the holomorphic line bundle $\mathcal{L}_\lambda$, from the definition of the complex structure of $\mathcal{L}_\lambda$, we obtain the formula

$$(4.11)\qquad \begin{aligned} \bar\partial(f\omega^{-A}) &= \sum_{\alpha\in\Delta_+}(e_{-\alpha}f)\omega^{-\alpha}\wedge\omega^{-A}+\frac{1}{2}\sum_{\alpha\in\Delta_+}f\omega^{-\alpha}\wedge e_{-\alpha}\omega^{-A} \\ &= \sum_{\alpha\in\Delta+}(1\otimes e(\omega^{-\alpha}))((e_{-\alpha}f)\omega^{-A}+\frac{1}{2}fe_{-\alpha}\omega^{-A}) \end{aligned}$$

for each monomial $f\omega^{-A}\in A^{0,q}(\mathcal{L}_\lambda)$, where 1 denotes the identity operator in $C^\infty(G)$.

5. Connections

Let $D=G/H$ be the homogeneous complex manifold with the G-invariant Kähler metric g_τ induced by B_τ, and let $\mathcal{L}_\lambda\to D$ be the homogeneous hermitian line bundle defined by the character λ of H. In this section, we will discuss the metric connection in the bundle $\mathcal{L}_\lambda$ and the Riemannian connection of D.

We consider the bundle $\mathcal{L}_\lambda\to D$. By the reductive decomposition (2.6) of the Lie algebra $\mathfrak{g}$, we can define a canonical G-invariant connection in the principal bundle $G\to D=G/H$. This connection in $G\to D$ induces a connection in the associated line bundle $\mathcal{L}_\lambda\to D$. We denote by ∇_λ: $A^0(\mathcal{L}_\lambda)\to A^1(\mathcal{L}_\lambda)$ the covariant differentiation with respect to this connection. It is easy to see that for a C^∞-section f: $G\to C$ of $\mathcal{L}_\lambda$, $\nabla_\lambda f$ is given by

$$(5.1)\qquad \nabla_\lambda f=\sum_{\alpha\in\Delta}e_\alpha f\otimes\omega^\alpha\,.$$

REMARK. The connection ∇_λ in $\mathcal{L}_\lambda$ is the metric connection in the hermitian vector bundle $\mathcal{L}_\lambda$ i.e. the connection of type (1, 0) such that for C^∞ sections f, f' of $\mathcal{L}_\lambda$ we have

$$d(f,f')_{\mathcal{L}_\lambda}=(\nabla_\lambda f,f')_{\mathcal{L}_\lambda}+(f,\nabla_\lambda f')_{\mathcal{L}_\lambda}$$

where d is the exterior differential operator and $(\ ,\ )_{\mathcal{L}_\lambda}$ is the hermitian inner product on the fibers of $\mathcal{L}_\lambda$ [4].

Now, we consider the tangent bundle $T(D)$ of D with the Kähler metric g_τ on the fibers. The bundle $T(D)$ may be identified with the homogeneous vector bundle $G\times_H\mathfrak{n}_0$ over D associated with the adjoint representation of H in $\mathfrak{n}_0$. We denote by ν the canonical projection of $G\times\mathfrak{n}_0$ onto $T(D)=G\times_H\mathfrak{n}_0$. Let $L_{g_0}(g_0\in G)$ be the action of g_0 on D, then L_{g_0} induces the transformation $(L_{g_0})_*$ on the bundle $T(D)$ and we have

$$(L_{g_0})_*(\nu(g, x)) = \nu(g_0 g, x)$$

for $(g, x) \in G \times \mathfrak{n}_0$. Let $P(D)$ be the frame bundle of D. We fix a basis of $\mathfrak{n}_0$ and identify the set of all frames of $\mathfrak{n}_0$ with $GL(\mathfrak{n}_0)$. Then the bundle $P(D)$ can be identified with the homogeneous principal bundle $G \times_H GL(\mathfrak{n}_0)$ defined as follows: The group H acts on $G \times GL(\mathfrak{n}_0)$ by

$$(g, M)h = (gh, Ad(h^{-1}) \cdot M)$$

for $(g, M) \in G \times GL(\mathfrak{n}_0)$ and $h \in H$. The space $G \times_H GL(\mathfrak{n}_0)$ is the quotient space $(G \times GL(\mathfrak{n}_0))/H$. We denote by μ the natural projection of $G \times GL(\mathfrak{n}_0)$ onto $P(D) = G \times_H GL(\mathfrak{n}_0)$. The transformation $(L_{g_0})_* (g_0 \in G)$ on $P(D)$ induced by L_{g_0} on D is given by

$$(L_{g_0})_*(\mu(g, M)) = \mu(g_0 g, M)$$

for $(g, M) \in G \times GL(\mathfrak{n}_0)$. We fix a frame $u_0 = \mu(e, 1)$ at the point $o \in D$. We will now apply the following lemma due to Wang ([6], II, p. 191, Theorem 2.1).

Lemma 3. *There is a one-to-one correspondence between the set of G-invariant connections in the bundle $P(D)$ and the set of linear mappings $\Lambda_{\mathfrak{n}_0}$; $\mathfrak{n}_0 \to \mathfrak{gl}(\mathfrak{n}_0)$ such that*

$$\Lambda_{\mathfrak{n}_0}(Ad(h)x) = Ad(h)\Lambda_{\mathfrak{n}_0}(x)Ad(h)^{-1} \tag{5.2}$$

for $h \in H$ and $x \in \mathfrak{n}_0$, where Ad is the adjoint representation of H on $\mathfrak{n}_0$. A linear mapping $\Lambda_{\mathfrak{n}_0}$ satisfying (5.2) *corresponds to the invariant connection whose connection form ω is given by*

$$\omega_{u_0}(\tilde{x}) = \begin{cases} ad(x) & \text{if } x \in \mathfrak{h}_0 \\ \Lambda_{\mathfrak{n}_0}(x) & \text{if } x \in \mathfrak{n}_0 \end{cases} \tag{5.3}$$

where $\tilde{x}$ is the vector field on $P(D)$ defined by the 1-parameter group of transformations $(L_{\exp tx})_*$.

Let $\Lambda_{\mathfrak{n}_0}$ be a linear mapping satisfying (5.2). The connection in $P(D)$ corresponding to $\Lambda_{\mathfrak{n}_0}$ induces the connection in the bundle $T(D)$. We denote by $\nabla_{\Lambda_{\mathfrak{n}_0}}$; $A^0(T(D)) \to A^1(T(D))$ the operator of the covariant differentiation with respect to this connection. The operator $\nabla_{\Lambda_{\mathfrak{n}_0}}$ is complex-linearly extended to the operator of $A^0(T(D)^C)$ into $A^1(T(D)^C)$. The complexified tangent bundle $T(D)^C$ may be identified with the homogeneous vector bundle $G \times_H \mathfrak{n}$ associated with the adjoint representation of H in $\mathfrak{n}$. Therefore the space $A^p(T(D)^C)$ is identified with a subspace of $C^\infty(G) \otimes \mathfrak{n} \otimes \overset{p}{\wedge} \mathfrak{n}^*$.

Lemma 4. *Let F; $G \to \mathfrak{n}$ be a section of $T(D)^C$. Then $\nabla_{\Lambda_{\mathfrak{n}_0}} F \in A^1(T(D)^C)$ is given by*

$$\nabla_{\Lambda_{\mathfrak{n}_0}} F = \sum_{\alpha \in \Delta} (e_\alpha F + \Lambda_{\mathfrak{n}_0}(e_\alpha)F) \otimes \omega^\alpha \tag{5.4}$$

where $\Lambda_{\mathfrak{n}_0}$ *is extended to the complex linear mapping of* $\mathfrak{n}$ *into* $\mathfrak{gl}(\mathfrak{n})$.

Proof. Let x be a vector field in $\mathfrak{n}_0$, and $\tilde{x}$ (resp. x_D) be a vector field on $P(D)$ (resp. D) defined by the 1-parameter group of transformations $(L_{\exp tx})_*$ (resp. $L_{\exp tx}$). The curve $(L_{\exp tx})_*(u_0)$ on $P(D)$ gives rise to the vector $\tilde{x}_{u_0}$ and the curve $(L_{\exp tx})(o)$ on D gives rise to the vector $(x_D)_0$. By Proposition 11.2 [6] II. p. 104, the horizontal lift v_t of the curve $L_{\exp tx}(o)$ such that $v_0 = u_0$ is given by

$$v_t = (L_{\exp tx})_*(u_0) \cdot a_t^{-1}$$

where a_t is the 1-parameter subgroup of $GL(\mathfrak{n}_0)$ generated by $\omega_{u_0}(\tilde{x})$. By (5.3) in Lemma 3, we have

$$\begin{aligned} v_t &= (L_{\exp tx})_*(u_0) \cdot (\exp t\Lambda_{\mathfrak{n}_0}(x))^{-1} \\ &= \mu(\exp tx, (\exp t\Lambda_{\mathfrak{n}_0}(x))^{-1}) . \end{aligned}$$

Thus, when we denote by τ_0^t the parallel displacement of the tangent space $T_{\exp tx \cdot 0}(D)$ along the curve $L_{\exp tx}(o)$ from $\exp tx \cdot o$ to o, we have

$$\tau_0^t(\nu(\exp tx, F(\exp tx))) = \nu(o, \exp t\Lambda_{\mathfrak{n}_0}(x) \cdot F(\exp tx))$$

Since we have $p_*(x_e) = (x_D)_0$, by the definition of the covariant differentiation, we get

$$\begin{aligned} (\nabla_{\Lambda_{\mathfrak{n}_0}} F)(x_e) &= \frac{d}{dt}(\exp t\Lambda_{\mathfrak{n}_0}(x) \cdot F(\exp tx))_{t=0} \\ &= (xF)(e) + \Lambda_{\mathfrak{n}_0}(x) \cdot F(e) . \end{aligned}$$

Since the connection is G-invariant, we have

$$\nabla_{\Lambda_{\mathfrak{n}_0}}(F)(x) = xF + \Lambda_{\mathfrak{n}_0}(x) \cdot F .$$

If we extend complex linearly the operator $\nabla_{\Lambda_{\mathfrak{n}_0}}$ to the operator of $A^0(T(D)^C)$ into $A^1(T(D)^C)$, we obtain the formula (5.4).

Lemma 5. *Under the correspondence of Lemma* 3, *the Riemannian connection in* $T(D)$ *is given by the following mapping*

$$\Lambda_{\mathfrak{n}_0}(x)y = \frac{1}{2}[x, y]_{\mathfrak{n}_0} + U(x, y) , \tag{5.5}$$

where $U(x, y)$ *is the symmetric bilinear mapping of* $\mathfrak{n}_0 \times \mathfrak{n}_0$ *into* $\mathfrak{n}_0$ *defined by*

$$2B_\tau(U(x, y), z) = B_\tau(x, [z, y]_{\mathfrak{n}_0}) + B_\tau([z, x]_{\mathfrak{n}_0}, y) \tag{5.6}$$

for all $x, y, z \in \mathfrak{n}_0$. *Here,* $[x, y]_{\mathfrak{n}_0}$ *is the* $\mathfrak{n}_0$*-component of* $[x, y]$ *with respect to the decomposition* (2.6) *of* $\mathfrak{g}_0$.

For the proof, see [6] II, p. 201, Theorem 3.3.

We denote by Λ_τ the linear mapping of $\mathfrak{n}_0$ into $\mathfrak{gl}(\mathfrak{n}_0)$ which gives the Riemannian connection of $T(D)$ in Lemma 5, and by the same letter Λ_τ its extention to the complex linear mapping of $\mathfrak{n}$ into $\mathfrak{gl}(\mathfrak{n})$. Then, by (3.5) we can calculate the mapping Λ_τ and we get

$$\text{(5.7)} \qquad \begin{cases} \Lambda_\tau(e_\alpha)e_\beta = \dfrac{(\beta, \tau)}{(\alpha+\beta, \tau)}[e_\alpha, e_\beta] \\ \Lambda_\tau(e_\alpha)e_{-\beta} = [e_\alpha, e_{-\beta}]_{\mathfrak{n}_-} \end{cases} \qquad \begin{cases} \Lambda_\tau(e_{-\alpha})e_\beta = [e_{-\alpha}, e_\beta]_{\mathfrak{n}_+} \\ \Lambda_\tau(e_{-\alpha})e_{-\beta} = \dfrac{(\beta, \tau)}{(\alpha+\beta, \tau)}[e_{-\alpha}, e_{-\beta}] \end{cases}$$

where α and β are positive roots and $[x, y]_{\mathfrak{n}_+}$ (resp. $[x, y]_{\mathfrak{n}_-}$) is the $\mathfrak{n}_+$(resp. $\mathfrak{n}_-$) component of $[x, y]$. By (5.5) and (5.6), we verify easily the following property of Λ_τ:

$$\text{(5.8)} \qquad B_\tau(\Lambda_\tau(x)y, z)+B_\tau(y, \Lambda_\tau(x)z) = 0$$

where $x, y, z\in\mathfrak{n}$. We denote by $\nabla_{T(D)^C}; A^0(T(D)^C)\to A^1(T(D)^C)$ the covariant differentiation with respect to this Riemannian connection. By Lemma 4, the operator $\nabla_{T(D)^C}$ is given by

$$\nabla_{T(D)^C}F = \sum_{\alpha\in\Delta}(e_\alpha F+\Lambda_\tau(e_\alpha)F)\otimes\omega^\alpha$$

for a section $F; G\to\mathfrak{n}$ of $T(D)^C$.

Let $\Theta(D)$ be the holomorphic tangent bundle of D. The bundle $T(D)^C$ decomposes into the Whitney sum

$$\begin{aligned} T(D)^C &= \Theta(D)\oplus\overline{\Theta}(D) \\ &= (G\times_H\mathfrak{n}_+)\oplus(G\times_H\mathfrak{n}_-) \end{aligned}$$

where $\overline{\Theta}(D)$ is the conjugate bundle of $\Theta(D)$. Since

$$\Lambda_\tau(x)(\mathfrak{n}_+)\subset\mathfrak{n}_+, \quad \Lambda_\tau(x)(\mathfrak{n}_-)\subset\mathfrak{n}_-$$

for all $x\in\mathfrak{n}$, we have

$$\begin{aligned} &\nabla_{T(D)^C}(A^0(\Theta(D)))\subset A^1(\Theta(D)) \\ &\nabla_{T(D)^C}(A^0(\overline{\Theta}(D)))\subset A^1(\overline{\Theta}(D)) . \end{aligned}$$

Therefore, the restriction of $\nabla_{T(D)^C}$ on $A^0(\Theta(D))$ (resp. $A^0(\overline{\Theta}(D))$) defines a connection in $\Theta(D)$ (resp. $\overline{\Theta}(D)$) which we denote by ∇_Θ (resp. $\nabla_{\overline{\Theta}}$). The connection $\nabla_{\bar{\Theta}}$ induces a connection $\nabla_{\bar{\Theta}^*}$ in the dual bundle $\bar{\Theta}^*(D)$ of $\bar{\Theta}(D)$ [1]. It is easy to see that, for a section F^*; $G\to \mathfrak{n}_-^*$ of $\bar{\Theta}^*(D)$, $\nabla_{\bar{\Theta}^*}F^*$ is given by

$$\nabla_{\bar{\Theta}^*}F^* = \sum_{\alpha\in\Delta}(e_\alpha F^* - {}^t\Lambda_\tau(e_\alpha)F^*)\otimes\omega^\alpha, \tag{5.9}$$

where the linear mapping ${}^t\Lambda_\tau(e_\alpha)$; $\mathfrak{n}_-^*\to\mathfrak{n}_-^*$ is the transposed mapping of $\Lambda_\tau(e_\alpha)|_{\mathfrak{n}_-}$; $\mathfrak{n}_-\to\mathfrak{n}_-$ and given by

$$\begin{cases} {}^t\Lambda_\tau(e_\alpha)\omega^{-\beta} = -e_\alpha\omega^{-\beta} \\ {}^t\Lambda_\tau(e_{-\alpha})\omega^{-\beta} = \dfrac{(\alpha-\beta,\tau)}{(\beta,\tau)}e_{-\alpha}\omega^{-\beta} \end{cases} \tag{5.10}$$

for α, $\beta\in\Delta_+$.

In the above, we have constructed the connections ∇_λ in $\mathcal{L}_\lambda$ and $\nabla_{\bar{\Theta}^*}$ in $\bar{\Theta}^*(D)$. These connections give rise to a connection in the bundle $\mathcal{L}_\lambda\otimes\overset{q}{\wedge}\bar{\Theta}^*(D)$, where $\overset{q}{\wedge}\bar{\Theta}^*(D)$ is the q-th exterior product of the bundle $\bar{\Theta}^*(D)$ [1]. We shall denote this connection by

$$\nabla;\ A^0(\mathcal{L}_\lambda\otimes\overset{q}{\wedge}\bar{\Theta}^*(D))\to A^1(\mathcal{L}_\lambda\otimes\overset{q}{\wedge}\bar{\Theta}^*(D)).$$

Then, for an element $f\omega^{-A}$ of $A^{0,q}(\mathcal{L}_\lambda)=A^0(\mathcal{L}_\lambda\otimes\overset{q}{\wedge}\bar{\Theta}^*(D))$, we get

$$\nabla(f\omega^{-A}) = \sum_{\alpha\in\Delta}(e_\alpha f\omega^{-A} - f({}^t\Lambda_\tau(e_\alpha)\omega^{-A}))\otimes\omega^\alpha, \tag{5.11}$$

where the mapping ${}^t\Lambda_\tau(e_\alpha)$; $\overset{q}{\wedge}\mathfrak{n}_-^*\to\overset{q}{\wedge}\mathfrak{n}_-^*$ is the natural extension of the endomorphism ${}^t\Lambda_\tau(e_\alpha)$; $\mathfrak{n}_-^*\to\mathfrak{n}_-^*$. In the following sections, we shall use this connection ∇.

6. Computation of the Laplace-Beltrami operator

We retain the notation introduced in the preceding sections. In this section, we will give an expression of the Laplace-Beltrami operator $\square=\bar{\partial}\delta+\delta\bar{\partial}$. To begin with, we give expressions of the operators $\bar{\partial}$ and δ in terms of the connection ∇ in §5. For each $e_\alpha\in\mathfrak{g}$, we define a linear mapping

$$\nabla_{e_\alpha};\ C^\infty(G)\otimes\overset{q}{\wedge}\mathfrak{n}_-^*\to C^\infty(G)\otimes\overset{q}{\wedge}\mathfrak{n}_-^*$$

by the following formula:

$$\nabla_{e_\alpha}(f\omega^{-A}) = (e_\alpha f)\omega^{-A} - f({}^t\Lambda_\tau(e_\alpha)\omega^{-A}). \tag{6.1}$$

Proposition 1. *Let $f\omega^{-A}$ be a form in $A^{0,q}(\mathcal{L}_\lambda)$. Then we have*

$$\bar{\partial}(f\omega^{-A}) = \sum_{\alpha\in\Delta_+}(1\otimes e(\omega^{-\alpha}))\nabla_{e_{-\alpha}}(f\omega^{-A}) \tag{6.2}$$

$$\delta(f\omega^{-A}) = \sum_{\alpha\in\Delta_+}\frac{1}{(\alpha,\tau)}(1\otimes i(\omega^{-\alpha}))\nabla_{e_\alpha}(f\omega^{-A}). \tag{6.3}$$

Proof. By (5.10) we have

$$\sum_{\alpha\in\Delta_+} e(\omega^{-\alpha})(-{}^t\Lambda_\tau(e_{-\alpha})\omega^{-\beta}) = \sum_{\alpha<\beta} \frac{(\beta-\alpha,\tau)}{(\beta,\tau)}\omega^{-\alpha}\wedge e_{-\alpha}\omega^{-\beta}$$

for $\alpha, \beta\in\Delta_+$. If we replace $\beta-\alpha$ by α in the right side, by (2.5) and (4.7), we get also

$$\begin{aligned}\sum_{\alpha\in\Delta_+} e(\omega^{-\alpha})(-{}^t\Lambda_\tau(e_{-\alpha})\omega^{-\beta}) &= \sum_{\alpha<\beta}\frac{(\alpha,\tau)}{(\beta,\tau)}\omega^{\alpha-\beta}\wedge e_{\alpha-\beta}\omega^{-\beta}\\ &= \sum_{\alpha<\beta}\frac{(\alpha,\tau)}{(\beta,\tau)}N_{\alpha-\beta,\beta}\,\omega^{\alpha-\beta}\wedge\omega^{-\alpha}\\ &= \sum_{\alpha<\beta}\frac{(\alpha,\tau)}{(\beta,\tau)}\omega^{-\alpha}\wedge N_{-\alpha,\beta}\,\omega^{\alpha-\beta}\\ &= \sum_{\alpha<\beta}\frac{(\alpha,\tau)}{(\beta,\tau)}\omega^{-\alpha}\wedge e_{-\alpha}\omega^{-\beta}.\end{aligned}$$

Hence, we get

$$2\sum_{\alpha\in\Delta_+} e(\omega^{-\alpha})(-{}^t\Lambda_\tau(e_{-\alpha})\omega^{-\beta}) = \sum_{\alpha\in\Delta_+}\omega^{-\alpha}\wedge e_{-\alpha}\omega^{-\beta}.$$

This formula can be extended to the formula for $\omega^{-A}\in\overset{q}{\wedge}\mathfrak{n}_-^*$ and we have

$$\sum_{\alpha\in\Delta_+} e(\omega^{-\alpha})(-{}^t\Lambda_\tau(e_{-\alpha})\omega^{-A}) = \frac{1}{2}\sum_{\alpha\in\Delta_+}\omega^{-\alpha}\wedge e_{-\alpha}\omega^{-A}.$$

Then, by (4.11) and (6.1), we obtain

$$\begin{aligned}\bar{\partial}(f\omega^{-A}) &= \sum_{\alpha\in\Delta_+}(e_{-\alpha}f)\omega^{-\alpha}\wedge\omega^{-A} + \frac{1}{2}f(\sum_{\alpha\in\Delta_+}\omega^{-\alpha}\wedge e_{-\alpha}\omega^{-A})\\ &= \sum_{\alpha\in\Delta_+}(e_{-\alpha}f)e(\omega^{-\alpha})\omega^{-A} + f\sum_{\alpha\in\Delta_+}e(\omega^{-\alpha})(-{}^t\Lambda_\tau(e_{-\alpha})\omega^{-A})\\ &= \sum_{\alpha\in\Delta_+}(1\otimes e(\omega^{-\alpha}))((e_{-\alpha}f)\omega^{-A} - f({}^t\Lambda_\tau(e_{-\alpha})\omega^{-A}))\\ &= \sum_{\alpha\in\Delta_+}(1\otimes e(\omega^{-\alpha}))\nabla_{e_{-\alpha}}(f\omega^{-A}).\end{aligned}$$

This proves (6.2).

In order to obtain the expression (6.3) of δ, we construct the adjoint operators of the operator $e_{-\alpha}$ on $C_0^\infty(G)$ and the operator $e(\omega^{-\alpha})$ and ${}^t\Lambda_\tau(e_{-\alpha})$ on $\overset{q}{\wedge}\mathfrak{n}_-^*$. For two functions $f, g\in C_0^\infty(G)$, we have

$$\int_G e_{-\alpha}(f\cdot\bar{g})dg = 0$$

([7], Lemma 5.1). Thus, we see that

$$(6.4)\qquad (e_{-\alpha}f, g)_G = (f, -\varepsilon_\alpha e_\alpha g)_G$$

where $(\ , \ )_G$ is the inner product on $C_0^\infty(G)$ defined in §4. On the other hand, by easy computations, we get

$$(6.5)\qquad (e(\omega^{-\alpha})\omega^{-A}, \omega^{-B})_- = \left(\omega^{-A}, -\frac{1}{\varepsilon_\alpha(\alpha, \tau)} i(\omega^{-\alpha})\omega^{-B}\right)_-$$

where $(\ , \ )_-$ is the inner product on $\overset{q}{\wedge}\mathfrak{n}_-^*$ introduced in §4. Also, by the definition (5.7) of $\Lambda_\tau(e_\alpha)$, we have

$$\overline{\Lambda_\tau(e_\alpha)y} = \varepsilon_\alpha \Lambda_\tau(e_{-\alpha})\bar{y}$$

for each $y \in \mathfrak{n}$ and $\alpha \in \Delta_+$. Since the operator $\Lambda_\tau(e_\alpha)$ satisfies the formula (5.8), we obtain

$$\begin{aligned} B_\tau^-(\Lambda_\tau(e_{-\alpha})x, y) &= B_\tau(\Lambda_\tau(e_{-\alpha})x, \bar{y}) \\ &= B_\tau(x, -\Lambda_\tau(e_{-\alpha})\bar{y}) \\ &= B_\tau(x, -\overline{\varepsilon_\alpha \Lambda_\tau(e_\alpha)y}) \\ &= B_\tau^-(x, -\varepsilon_\alpha \Lambda_\tau(e_\alpha)y)\,. \end{aligned}$$

for $x, y \in \mathfrak{n}_-$. It follows that we have

$$(6.6)\qquad ({}^t\Lambda_\tau(e_{-\alpha})\omega^{-A}, \omega^{-B})_- = (\omega^{-A}, -\varepsilon_\alpha {}^t\Lambda_\tau(e_\alpha)\omega^{-B})_-\,.$$

By the formulas (6.1), (6.2), (6.4)—(6.6) and Lemma 2, the formal adjoint operator δ of $\bar{\partial}$ is given by

$$\begin{aligned} \delta(f\omega^{-A}) &= \sum_{\alpha\in\Delta_+} \frac{1}{(\alpha, \tau)}(1\otimes i(\omega^{-\alpha}))((e_\alpha f)\omega^{-A} - f({}^t\Lambda_\tau(e_\alpha)\omega^{-A})) \\ &= \sum_{\alpha\in\Delta_+} \frac{1}{(\alpha, \tau)}(1\otimes i(\omega^{-\alpha}))\nabla_{e_\alpha}(f\omega^{-A})\,. \end{aligned}$$

q.e.d.

Now, if $\alpha, \beta \in \Delta_+$, we have the following relations among operators on $\overset{q}{\wedge}\mathfrak{n}_\pm^*$:

$$(6.7)\qquad e(\omega^{-\alpha})e(\omega^{-\beta}) = -e(\omega^{-\beta})e(\omega^{-\alpha})$$

$$(6.8)\qquad e(\omega^{-\alpha})i(\omega^{-\beta}) + i(\omega^{-\beta})e(\omega^{-\alpha}) = \delta_{\alpha,\beta}$$

$$(6.9)\qquad [e_\alpha, e(\omega^{-\beta})] = e(e_\alpha \omega^{-\beta})$$

$$(6.10)\qquad [e_\alpha, i(\omega^{-\beta})] = -i(e_{-\alpha}\omega^{-\beta})$$

$$(6.11)\qquad e_\alpha \omega^{-A} = \sum_{\beta\in\Delta_+} e(e_\alpha \omega^{-\beta}) i(\omega^{-\beta})\omega^{-A}\,.$$

All these are easily proved [4], and the equalities (6.9)–(6.11) hold also when we replace α by $-\alpha$.

Lemma 6. *For roots α, $\beta\in\Delta_+$, we have the following relations:*

$$[{}^t\Lambda_\tau(e_\alpha),\ e(\omega^{-\beta})] = e({}^t\Lambda_\tau(e_\alpha)\omega^{-\beta}) \tag{6.12}$$

$$[{}^t\Lambda_\tau(e_\alpha),\ i(\omega^{-\beta})] = i(e_{-\alpha}\,\omega^{-\beta}) \tag{6.13}$$

$$[{}^t\Lambda_\tau(e_{-\alpha}),\ i(\omega^{-\beta})] = \frac{(\beta,\ \tau)}{(\alpha+\beta,\ \tau)}\, i(e_\alpha\,\omega^{-\beta}) \tag{6.14}$$

$${}^t\Lambda_\tau(e_\alpha)\omega^{-A} = \sum_{\beta\in\Delta_+} e({}^t\Lambda_\tau(e_\alpha)\omega^{-\beta})\, i(\omega^{-\beta})\omega^{-A}\,. \tag{6.15}$$

The relations (6.12) *and* (6.15) *hold also when we replace α by $-\alpha$.*

Proof. The equalities (6.12) and (6.15) are easily proved and (6.13) follows from (5.10) and (6.10). We will prove the relation (6.14) on $\overset{q}{\wedge}\mathfrak{n}_-^*$ by the induction on q. For a 1-form $\omega^{-\gamma}$, we have

$$ {}^t\Lambda_\tau(e_{-\alpha})i(\omega^{-\beta})\omega^{-\gamma} = 0$$
$$ e_{-\alpha}\, i(\omega^{-\beta})\omega^{-\gamma} = 0\,.$$

Hence, by (5.10) and (6.10), we get

$$\begin{aligned}[{}^t\Lambda_\tau(e_{-\alpha}),\ i(\omega^{-\beta})]\,\omega^{-\gamma} &= -i(\omega^{-\beta})\,{}^t\Lambda_\tau(e_{-\alpha})\omega^{-\gamma}\\ &= -\frac{(\alpha-\gamma,\ \tau)}{(\gamma,\ \tau)}\, i(\omega^{-\beta})e_{-\alpha}\omega^{-\gamma}\\ &= -\frac{(\alpha-\gamma,\ \tau)}{(\gamma,\ \tau)}\, i(e_\alpha\,\omega^{-\beta})\omega^{-\gamma}\,.\end{aligned}$$

On the other hand, by (4.8) we have

$$\begin{aligned} i(e_\alpha\,\omega^{-\beta})\omega^{-\gamma} &= N_{\alpha,\beta}\, i(\omega^{-\alpha-\beta})\omega^{-\gamma}\\ &= \begin{cases} 0 & \text{if } \alpha+\beta\neq\gamma\,,\\ N_{\alpha,\beta} & \text{if } \alpha+\beta=\gamma\,.\end{cases}\end{aligned}$$

Therefore, we obtain the equality

$$[{}^t\Lambda_\tau(e_{-\alpha}),\ i(\omega^{-\beta})]\,\omega^{-\gamma} = \frac{(\beta,\ \tau)}{(\alpha+\beta,\ \tau)}\, i(e_\alpha\omega^{-\beta})\omega^{-\gamma}\,.$$

Now, assume that the equality (6.14) holds on the space $\overset{q-1}{\wedge}\mathfrak{n}_-^*$. By (6.8) and (6.12) we have

$$\begin{aligned} &[{}^t\Lambda_\tau(e_{-\alpha}),\ i(\omega^{-\beta})]e(\omega^{-\gamma})\\ &= {}^t\Lambda_\tau(e_{-\alpha})i(\omega^{-\beta})e(\omega^{-\gamma}) - i(\omega^{-\beta})\,{}^t\Lambda_\tau(e_{-\alpha})e(\omega^{-\gamma})\\ &= {}^t\Lambda_\tau(e_{-\alpha})\delta_{\beta,\gamma} - {}^t\Lambda_\tau(e_{-\alpha})e(\omega^{-\gamma})i(\omega^{-\beta}) - i(\omega^{-\beta})e(\omega^{-\gamma})\,{}^t\Lambda_\tau(e_{-\alpha}) - i(\omega^{-\beta})e({}^t\Lambda_\tau(e_{-\alpha})\omega^{-\gamma})\end{aligned}$$

$$= -e(\omega^{-\gamma})^t\Lambda_\tau(e_{-\alpha})i(\omega^{-\beta})-e(^t\Lambda_\tau(e_{-\alpha})\omega^{-\gamma})i(\omega^{-\beta})$$
$$+e(\omega^{-\gamma})i(\omega^{-\beta})^t\Lambda_\tau(e_{-\alpha})-i(\omega^{-\beta})e(^t\Lambda_\tau(e_{-\alpha})\omega^{-\gamma})$$
$$= -e(\omega^{-\gamma})[^t\Lambda_\tau(e_{-\alpha}),\ i(\omega^{-\beta})]-\frac{(\alpha-\gamma,\ \tau)}{(\gamma,\ \tau)}N_{-\alpha,\gamma}\{e(\omega^{\alpha-\gamma})i(\omega^{-\beta})+i(\omega^{-\beta})e(\omega^{\alpha-\gamma})\}$$
$$= -e(\omega^{-\gamma})[^t\Lambda_\tau(e_{-\alpha}),\ i(\omega^{-\beta})]-\frac{(\alpha-\gamma,\ \tau)}{(\gamma,\ \tau)}N_{-\alpha,\gamma}\delta_{\gamma-\alpha,\beta}\,.$$

Since

$$N_{-\alpha,\alpha+\beta}=N_{\alpha,\beta}\,, \tag{6.16}$$

we get

$$[^t\Lambda_\tau(e_{-\alpha}),\ i(\omega^{-\beta})]e(\omega^{-\gamma}) = -e(\omega^{-\gamma})[^t\Lambda_\tau(e_{-\alpha}),\ i(\omega^{-\beta})]+\frac{(\beta,\ \tau)}{(\alpha+\beta,\ \tau)}N_{\alpha,\beta}\delta_{\alpha+\beta,\gamma}\,.$$

By the assumption and (6.8), for a q-form $\omega^{-\gamma}\wedge\omega^{-A}$ with $\omega^{-A}\in\overset{q-1}{\wedge}\mathfrak{n}_-^*$, we obtain

$$[^t\Lambda_\tau(e_{-\alpha}),\ i(\omega^{-\beta})](\omega^{-\gamma}\wedge\omega^{-A})$$
$$= -e(\omega^{-\gamma})[^t\Lambda_\tau(e_{-\alpha}),\ i(\omega^{-\beta})]\omega^{-A}+\frac{(\beta,\ \tau)}{(\alpha+\beta,\ \tau)}N_{\alpha,\beta}\,\delta_{\alpha+\beta,\gamma}\omega^{-A}$$
$$= \frac{(\beta,\ \tau)}{(\alpha+\beta,\ \tau)}\{-e(\omega^{-\gamma})i(e_\alpha\omega^{-\beta})\omega^{-A}+N_{\alpha,\beta}\,\delta_{\alpha+\beta,\gamma}\,\omega^{-A}\}$$
$$= \frac{(\beta,\ \tau)}{(\alpha+\beta,\ \tau)}i(e_\alpha\,\omega^{-\beta})e(\omega^{-\gamma})\omega^{-A}$$
$$= \frac{(\beta,\ \tau)}{(\alpha+\beta,\ \tau)}i(e_\alpha\,\omega^{-\beta})(\omega^{-\gamma}\wedge\omega^{-A})\,.$$

This proves (6.14) and the lemma is proved. q.e.d.

We recall also following equalities proved in [4]:

$$\sum_{\beta<\alpha}N_{\beta,\alpha-\beta}N_{-\beta,\alpha}=(2\rho-\alpha,\ \alpha) \tag{6.17}$$

where $\rho=\frac{1}{2}\sum_{\alpha\in\Delta_+}\alpha$ ([4], p. 266, Lemma 3.1), and

$$(e_{-\alpha}e_\gamma-e_\gamma e_{-\alpha})\omega^{-\beta}-[e_{-\alpha},\ e_\gamma]\omega^{-\beta}=\begin{cases}0 & \text{if }\ \alpha<\beta\\ (\gamma,\ \beta)\omega^{-\gamma} & \text{if }\ \alpha=\beta\\ N_{-\alpha,\beta}N_{\gamma,\beta-\alpha}\,\omega^{\alpha-\beta-\gamma} & \text{if }\ \alpha>\beta\end{cases} \tag{6.18}$$

for all α, β, $\gamma\in\Delta_+$ ([4], p. 281).

Proposition 2. *Let $f\omega^{-A}$ be a form in $A^{0,q}(\mathcal{L}_\lambda)$. Then*

$$\square(f\omega^{-A})=\sum_{\alpha\in\Delta_+}\frac{1}{(\alpha,\tau)}\nabla_{e_\alpha}\nabla_{e_{-\alpha}}(f\omega^{-A})+\Big(\sum_{\alpha\in A}\frac{(\alpha,\lambda+2\rho)}{(\alpha,\tau)}\Big)f\omega^{-A}.$$

Proof. By (4.7) we have $e_{-\alpha}\omega^{-\alpha}=0$ and thus

$$^t\Lambda_\tau(e_{-\alpha})\omega^{-\alpha}=0.$$

Hence, by (6.12) and (6.13) in Lemma 6, $e(\omega^{-\alpha})$ commutes with $^t\Lambda_\tau(e_{-\alpha})$ and $i(\omega^{-\beta})$ with $^t\Lambda_\tau(e_\beta)$. In particular, $1\otimes i(\omega^{-\beta})$ commutes with ∇_{e_β}. Using Proposition 1 and (6.8) we have

$$\begin{aligned}\square(f\omega^{-A}) &= (\bar{\partial}\delta+\delta\bar{\partial})(f\omega^{-A})\\
&=\sum_{\alpha,\beta\in\Delta_+}\frac{1}{(\beta,\tau)}(1\otimes e(\omega^{-\alpha}))\nabla_{e_{-\alpha}}\nabla_{e_\beta}(1\otimes i(\omega^{-\beta}))(f\omega^{-A})\\
&\quad+\sum_{\alpha,\beta\in\Delta_+}\frac{1}{(\beta,\tau)}\nabla_{e_\beta}(1\otimes i(\omega^{-\beta}))(1\otimes e(\omega^{-\alpha}))\nabla_{e_{-\alpha}}(f\omega^{-A})\\
&=\sum_{\alpha,\beta\in\Delta_+}\frac{1}{(\beta,\tau)}(1\otimes e(\omega^{-\alpha}))\nabla_{e_{-\alpha}}\nabla_{e_\beta}(1\otimes i(\omega^{-\beta}))(f\omega^{-A})\\
&\quad+\sum_{\alpha\in\Delta_+}\frac{1}{(\alpha,\tau)}\nabla_{e_\alpha}\nabla_{e_{-\alpha}}(f\omega^{-A})\\
&\quad-\sum_{\alpha,\beta\in\Delta_+}\frac{1}{(\beta,\tau)}\nabla_{e_\beta}(1\otimes e(\omega^{-\alpha}))(1\otimes i(\omega^{-\beta}))\nabla_{e_{-\alpha}}(f\omega^{-A}).\end{aligned}$$

By the definition (6.1) of $\nabla_{e_{-\alpha}}$ and ∇_{e_β}, we get

$$\begin{aligned}\square(f\omega^{-A}) &=\sum_{\alpha\in\Delta_+}\frac{1}{(\alpha,\tau)}\nabla_{e_\alpha}\nabla_{e_{-\alpha}}(f\omega^{-A})\\
&\quad+\sum_{\alpha,\beta\in\Delta_+}\frac{1}{(\beta,\tau)}(e_{-\alpha}e_\beta f-e_\beta e_{-\alpha}f)e(\omega^{-\alpha})i(\omega^{-\beta})\omega^{-A}\\
&\quad-\sum_{\alpha,\beta\in\Delta_+}\frac{1}{(\beta,\tau)}e_\beta fe(\omega^{-\alpha})({}^t\Lambda_\tau(e_{-\alpha})i(\omega^{-\beta})-i(\omega^{-\beta}){}^t\Lambda_\tau(e_{-\alpha}))\omega^{-A}\\
&\quad-\sum_{\alpha,\beta\in\Delta_+}\frac{1}{(\beta,\tau)}e_{-\alpha}f(e(\omega^{-\alpha}){}^t\Lambda_\tau(e_\beta)-{}^t\Lambda_\tau(e_\beta)e(\omega^{-\alpha}))i(\omega^{-\beta})\omega^{-A}\\
&\quad+\sum_{\alpha,\beta\in\Delta_+}\frac{1}{(\beta,\tau)}f(e(\omega^{-\alpha}){}^t\Lambda_\tau(e_{-\alpha}){}^t\Lambda_\tau(e_\beta)i(\omega^{-\beta})\\
&\quad-{}^t\Lambda_\tau(e_\beta)e(\omega^{-\alpha})i(\omega^{-\beta}){}^t\Lambda_\tau(e_{-\alpha}))\omega^{-A}.\end{aligned}$$

By (6.14), the third term can be written as follows:

$$-\sum_{\alpha,\beta\in\Delta_+}\frac{1}{(\alpha+\beta,\tau)}e_\beta fe(\omega^{-\alpha})i(e_\alpha\omega^{-\beta})\omega^{-A},$$

and by (5.10), (6.12) the fourth term as follows:

$$-\sum_{\alpha,\beta\in\Delta_+}\frac{1}{(\beta,\tau)}e_{-\alpha}fe(e_\beta\omega^{-\alpha})i(\omega^{-\beta})\omega^{-A}.$$

On the other hand, from the formula

$$[e_\alpha, e_{-\alpha}] = h_\alpha, \quad [e_{-\alpha}, e_\beta] = N_{-\alpha,\beta} e_{\beta-\alpha},$$

we have

$$\begin{aligned}
&\sum_{\alpha,\beta\in\Delta_+} \frac{1}{(\beta, \tau)} (e_{-\alpha}e_\beta f - e_\beta e_{-\alpha} f) e(\omega^{-\alpha}) i(\omega^{-\beta})\omega^{-A} \\
&= - \sum_{\alpha\in\Delta_+} \frac{1}{(\alpha, \tau)} h_\alpha f e(\omega^{-\alpha}) i(\omega^{-\alpha})\omega^{-A} \\
&\quad + \sum_{0<\alpha<\beta} \frac{1}{(\beta, \tau)} e_{\beta-\alpha} f e(\omega^{-\alpha}) i(N_{-\alpha,\beta}\,\omega^{-\beta})\omega^{-A} \\
&\quad + \sum_{\alpha>\beta>0} \frac{1}{(\beta, \tau)} e_{\beta-\alpha} f e(N_{-\alpha,\beta}\,\omega^{-\alpha}) i(\omega^{-\beta})\omega^{-A}.
\end{aligned}$$

Here, we replace $\beta-\alpha$ by β in the second term and $\beta-\alpha$ by $-\alpha$ in the third term. Then, by (2.5) and (4.8), we have

$$\begin{aligned}
&\sum_{\alpha,\beta\in\Delta_+} \frac{1}{(\beta, \tau)} (e_{-\alpha}e_\beta f - e_\beta e_{-\alpha} f) e(\omega^{-\alpha}) i(\omega^{-\beta})\omega^{-A}. \\
&= - \sum_{\alpha\in\Delta_+} \frac{1}{(\alpha, \tau)} h_\alpha f e(\omega^{-\alpha}) i(\omega^{-\alpha})\omega^{-A} \\
&\quad + \sum_{\alpha,\beta\in\Delta_+} \frac{1}{(\alpha+\beta, \tau)} e_\beta f e(\omega^{-\alpha}) i(e_\alpha\omega^{-\beta})\omega^{-A} \\
&\quad + \sum_{\alpha,\beta\in\Delta_+} \frac{1}{(\beta, \tau)} e_{-\alpha} f e(e_\beta\omega^{-\alpha}) i(\omega^{-\beta})\omega^{-A}.
\end{aligned}$$

By (4.1) in §4, the function f; $G\to C$ satisfies the following property:

$$h_\alpha f = -(\lambda+|A|, \alpha) f \qquad \text{for all} \quad \alpha\in\Delta_+.$$

Therefore, if we put

$$(6.19) \qquad \begin{aligned}
\mathfrak{R}_\tau\omega^{-A} = \sum_{\alpha,\beta\in\Delta_+} \frac{1}{(\beta, \tau)} &(e(\omega^{-\alpha})^t\Lambda_\tau(e_{-\alpha})^t\Lambda_\tau(e_\beta) i(\omega^{-\beta}) \\
&- {}^t\Lambda_\tau(e_\beta) e(\omega^{-\alpha}) i(\omega^{-\beta})^t\Lambda_\tau(e_{-\alpha}))\omega^{-A},
\end{aligned}$$

we have

$$(6.20) \qquad \begin{aligned}
\square(f\omega^{-A}) = \sum_{\alpha\in\Delta_+} &\frac{1}{(\alpha, \tau)} \nabla_{e_\alpha}\nabla_{e_{-\alpha}}(f\omega^{-A}) \\
&+ \Big(\sum_{\alpha\in A} \frac{(\alpha, \lambda+|A|)}{(\alpha, \tau)}\Big) f\omega^{-A} + f\cdot\mathfrak{R}_\tau\omega^{-A}.
\end{aligned}$$

It remains to compute the operator $\mathfrak{R}_\tau$: $\overset{q}{\wedge}\mathfrak{n}_-^* \to \overset{q}{\wedge}\mathfrak{n}_-^*$. We begin with the operator $\mathfrak{R}_\tau \cdot e(\omega^{-\gamma})$ for $\gamma \in \Delta_+$. Using (6.7)–(6.15), we will exchange the operator $e(\omega^{-\gamma})$ with the operators $e(\omega^{-\alpha})$, $i(\omega^{-\alpha})$, ${}^t\Lambda_\tau(e_\alpha)$ and ${}^t\Lambda_\tau(e_{-\alpha})$ $(\alpha \in \Delta_+)$ one by one. By (6.8) and (6.12) we have

$$\begin{aligned}\mathfrak{R}_\tau \cdot e(\omega^{-\gamma}) &= \sum_{\alpha,\beta\in\Delta_+} \frac{1}{(\beta, \tau)}(e(\omega^{-\alpha}){}^t\Lambda_\tau(e_{-\alpha}){}^t\Lambda_\tau(e_\beta)i(\omega^{-\beta}) \\ &\quad - {}^t\Lambda_\tau(e_\beta)e(\omega^{-\alpha})i(\omega^{-\beta}){}^t\Lambda_\tau(e_{-\alpha}))\cdot e(\omega^{-\gamma}) \\ &= -\sum_{\alpha,\beta\in\Delta_+} \frac{1}{(\beta, \tau)} e(\omega^{-\alpha}){}^t\Lambda_\tau(e_{-\alpha}){}^t\Lambda_\tau(e_\beta)e(\omega^{-\gamma})i(\omega^{-\beta}) \\ &\quad + \sum_{\alpha\in\Delta_+} \frac{1}{(\gamma, \tau)} e(\omega^{-\alpha}){}^t\Lambda_\tau(e_{-\alpha}){}^t\Lambda_\tau(e_\gamma) \\ &\quad - \sum_{\alpha,\beta\in\Delta_+} \frac{1}{(\beta, \tau)} {}^t\Lambda_\tau(e_\beta)e(\omega^{-\alpha})i(\omega^{-\beta})\, e({}^t\Lambda_\tau(e_{-\alpha})\omega^{-\gamma}) \\ &\quad - \sum_{\alpha,\beta\in\Delta_+} \frac{1}{(\beta, \tau)} {}^t\Lambda_\tau(e_\beta)e(\omega^{-\alpha})i(\omega^{-\beta})e(\omega^{-\gamma}){}^t\Lambda_\tau(e_{-\alpha})\,.\end{aligned}$$

By (5.10) and (6.8), we get

$$\begin{aligned} i(\omega^{-\beta})\, e({}^t\Lambda_\tau(e_{-\alpha})\omega^{-\gamma}) &= \frac{(\alpha-\gamma, \tau)}{(\gamma, \tau)} N_{-\alpha,\gamma} i(\omega^{-\beta})e(\omega^{\alpha-\gamma}) \\ &= \frac{(\alpha-\gamma, \tau)}{(\gamma, \tau)} N_{-\alpha,\gamma}(-e(\omega^{\alpha-\gamma})\, i(\omega^{-\beta}) + \delta_{\gamma-\alpha,\beta}) \\ &= -e({}^t\Lambda_\tau(e_{-\alpha})\omega^{-\gamma})i(\omega^{-\beta}) - \frac{(\beta, \tau)}{(\gamma, \tau)} N_{-\alpha,\gamma}\delta_{\gamma,\alpha+\beta}\end{aligned}$$

provided that $\alpha < \gamma$. By (4.8) and (5.10), we have also

$${}^t\Lambda_\tau(e_\gamma)\omega^{-\beta} = -e_\gamma\omega^{-\beta} = e_\beta\omega^{-\gamma} = -{}^t\Lambda_\tau(e_\beta)\omega^{-\gamma}\,,$$

and thus by (6.15) we have

$$\begin{aligned}{}^t\Lambda_\tau(e_\gamma) &= \sum_{\beta\in\Delta_+} e({}^t\Lambda_\tau(e_\gamma)\omega^{-\beta})i(\omega^{-\beta}) \\ &= -\sum_{\beta\in\Delta_+} e({}^t\Lambda_\tau(e_\beta)\omega^{-\gamma})i(\omega^{-\beta})\,.\end{aligned}$$

Therefore using again (4.8) and (6.12) we get

$$\begin{aligned}\mathfrak{R}_\tau e(\omega^{-\gamma}) &= -\sum_{\alpha,\beta\in\Delta_+} \frac{1}{(\beta, \tau)} e(\omega^{-\alpha}){}^t\Lambda_\tau(e_{-\alpha})e(\omega^{-\gamma}){}^t\Lambda_\tau(e_\beta)i(\omega^{-\beta}) \\ &\quad - \sum_{\alpha,\beta\in\Delta_+} \frac{1}{(\beta, \tau)} e(\omega^{-\alpha}){}^t\Lambda_\tau(e_{-\alpha})e({}^t\Lambda_\tau(e_\beta)\omega^{-\gamma})i(\omega^{-\beta}) \\ &\quad - \sum_{\alpha,\beta\in\Delta_+} \frac{1}{(\gamma, \tau)} e(\omega^{-\alpha}){}^t\Lambda_\tau(e_{-\alpha})e({}^t\Lambda_\tau(e_\beta)\omega^{-\gamma})i(\omega^{-\beta})\end{aligned}$$

$$+\sum_{\alpha,\beta\in\Delta_+}\frac{1}{(\beta,\tau)}{}^t\Lambda_\tau(e_\beta)e(\omega^{-\alpha})e({}^t\Lambda_\tau(e_{-\alpha})\omega^{-\gamma})i(\omega^{-\beta})$$
$$+\sum_{0<\alpha<\gamma}\frac{1}{(\gamma,\tau)}N_{-\alpha,\gamma}{}^t\Lambda_\tau(e_{\gamma-\alpha})e(\omega^{-\alpha})$$
$$-\sum_{\alpha,\beta\in\Delta_+}\frac{1}{(\beta,\tau)}{}^t\Lambda_\tau(e_\beta)e(\omega^{-\gamma})e(\omega^{-\alpha})i(\omega^{-\beta}){}^t\Lambda_\tau(e_{-\alpha})$$
$$-\sum_{\alpha\in\Delta_+}\frac{1}{(\gamma,\tau)}{}^t\Lambda_\tau(e_\gamma)e(\omega^{-\alpha}){}^t\Lambda_\tau(e_{-\alpha})$$
$$=\sum_{\alpha,\beta\in\Delta_+}\frac{1}{(\beta,\tau)}e(\omega^{-\gamma})e(\omega^{-\alpha}){}^t\Lambda_\tau(e_{-\alpha}){}^t\Lambda_\tau(e_\beta)i(\omega^{-\beta})$$
$$-\sum_{\alpha,\beta\in\Delta_+}\frac{1}{(\beta,\tau)}e(\omega^{-\alpha})e({}^t\Lambda_\tau(e_{-\alpha})\omega^{-\gamma}){}^t\Lambda_\tau(e_\beta)i(\omega^{-\beta})$$
$$-\sum_{\alpha,\beta\in\Delta_+}\left(\frac{1}{(\beta,\tau)}+\frac{1}{(\gamma,\tau)}\right)e(\omega^{-\alpha})e({}^t\Lambda_\tau(e_{-\alpha}){}^t\Lambda_\tau(e_\beta)\omega^{-\gamma})i(\omega^{-\beta})$$
$$-\sum_{\alpha,\beta\in\Delta_+}\frac{1}{(\beta,\tau)}e(\omega^{-\alpha})e({}^t\Lambda_\tau(e_\beta)\omega^{-\gamma}){}^t\Lambda_\tau(e_{-\alpha})i(\omega^{-\beta})$$
$$-\sum_{\alpha,\beta\in\Delta_+}\frac{1}{(\gamma,\tau)}e(\omega^{-\alpha})e({}^t\Lambda_\tau(e_\beta)\omega^{-\gamma}){}^t\Lambda_\tau(e_{-\alpha})i(\omega^{-\beta})$$
$$+\sum_{\alpha,\beta\in\Delta_+}\frac{1}{(\beta,\tau)}e(\omega^{-\alpha}){}^t\Lambda_\tau(e_\beta)e({}^t\Lambda_\tau(e_{-\alpha})\omega^{-\gamma})i(\omega^{-\beta})$$
$$+\sum_{\alpha,\beta\in\Delta_+}e({}^t\Lambda_\tau(e_\beta)\omega^{-\alpha})e({}^t\Lambda_\tau(e_{-\alpha})\omega^{-\gamma})i(\omega^{-\beta})$$
$$+\sum_{\substack{0<\alpha<\gamma\\ \beta\in\Delta_+}}\frac{1}{(\gamma,\tau)}N_{-\alpha,\gamma}e(\omega^{-\alpha})e({}^t\Lambda_\tau(e_{\gamma-\alpha})\omega^{-\beta})i(\omega^{-\beta})$$
$$+\sum_{0<\alpha<\gamma}\frac{1}{(\gamma,\tau)}N_{-\alpha,\gamma}e({}^t\Lambda_\tau(e_{\gamma-\alpha})\omega^{-\alpha})$$
$$-\sum_{\alpha,\beta\in\Delta_+}\frac{1}{(\beta,\tau)}e(\omega^{-\gamma}){}^t\Lambda_\tau(e_\beta)e(\omega^{-\alpha})i(\omega^{-\beta}){}^t\Lambda_\tau(e_{-\alpha})$$
$$+\sum_{\alpha,\beta\in\Delta_+}\frac{1}{(\beta,\tau)}e(\omega^{-\alpha})e({}^t\Lambda_\tau(e_\beta)\omega^{-\gamma})i(\omega^{-\beta}){}^t\Lambda_\tau(e_{-\alpha})$$
$$+\sum_{\alpha,\beta\in\Delta_+}\frac{1}{(\gamma,\tau)}e(\omega^{-\alpha})e({}^t\Lambda_\tau(e_\beta)\omega^{-\gamma})i(\omega^{-\beta}){}^t\Lambda_\tau(e_{-\alpha})$$
$$-\sum_{\alpha,\beta\in\Delta_+}\frac{1}{(\gamma,\tau)}e({}^t\Lambda_\tau(e_\gamma)\omega^{-\alpha})e({}^t\Lambda_\tau(e_{-\alpha})\omega^{-\beta})i(\omega^{-\beta})\,.$$

Changing suitably the order of the terms in the above equality, by (6.19), we get

$$\mathfrak{R}_\tau\cdot e(\omega^{-\gamma})=e(\omega^{-\gamma})\cdot\mathfrak{R}_\tau+\sum_{0<\alpha<\gamma}\frac{1}{(\gamma,\tau)}N_{-\alpha,\gamma}e({}^t\Lambda_\tau(e_{\gamma-\alpha})\omega^{-\alpha})$$

$$-\sum_{\alpha,\beta\in\Delta_+}\left(\frac{1}{(\beta,\tau)}+\frac{1}{(\gamma,\tau)}\right)e(\omega^{-\alpha})e({}^t\Lambda_\tau(e_{-\alpha}){}^t\Lambda_\tau(e_\beta)\omega^{-\gamma})i(\omega^{-\beta})$$
$$-\sum_{\alpha,\beta\in\Delta_+}\left(\frac{1}{(\beta,\tau)}+\frac{1}{(\gamma,\tau)}\right)e(\omega^{-\alpha})e({}^t\Lambda_\tau(e_\beta)\omega^{-\gamma})({}^t\Lambda_\tau(e_{-\alpha})i(\omega^{-\beta})-i(\omega^{-\beta}){}^t\Lambda_\tau(e_{-\alpha}))$$
$$+\sum_{\alpha,\beta\in\Delta_+}\frac{1}{(\beta,\tau)}e({}^t\Lambda_\tau(e_\beta)\omega^{-\alpha})e({}^t\Lambda_\tau(e_{-\alpha})\omega^{-\gamma})i(\omega^{-\beta})$$
$$+\sum_{\alpha,\beta\in\Delta_+}\frac{1}{(\beta,\tau)}e(\omega^{-\alpha})({}^t\Lambda_\tau(e_\beta)e({}^t\Lambda_\tau(e_{-\alpha})\omega^{-\gamma})-e({}^t\Lambda_\tau(e_{-\alpha})\omega^{-\gamma}){}^t\Lambda_\tau(e_\beta))i(\omega^{-\beta})$$
$$+\sum_{\substack{0<\alpha<\gamma\\ \beta\in\Delta_+}}\frac{1}{(\gamma,\tau)}e(\omega^{-\alpha})e(N_{-\alpha,\gamma}{}^t\Lambda_\tau(e_{\gamma-\alpha})\omega^{-\beta})\,i(\omega^{-\beta})$$
$$-\sum_{\alpha\in\Delta_+}\frac{1}{(\gamma,\tau)}e({}^t\Lambda_\tau(e_\gamma)\omega^{-\alpha})e({}^t\Lambda_\tau(e_{-\alpha})\omega^{-\beta})i(\omega^{-\beta})\,.$$

Now, by (5.10) and (6.17)

$$(6.21)\qquad \sum_{0<\alpha<\gamma}\frac{1}{(\gamma,\tau)}N_{-\alpha,\gamma}e({}^t\Lambda_\tau(e_{\gamma-\alpha})\omega^{-\alpha})=\frac{1}{(\gamma,\tau)}\Big(\sum_{0<\alpha<\gamma}N_{-\alpha,\gamma}N_{\gamma-\alpha,\alpha}\Big)e(\omega^{-\gamma})$$
$$=\frac{(2\rho-\gamma,\gamma)}{(\gamma,\tau)}e(\omega^{-\gamma})\,.$$

By (5.10), we have

$$\begin{aligned}{}^t\Lambda_\tau(e_{-\alpha}){}^t\Lambda_\tau(e_\beta)\omega^{-\gamma}&=-N_{\beta,\gamma}{}^t\Lambda_\tau(e_{-\alpha})\omega^{-\beta-\gamma}\\&=\begin{cases}\dfrac{(\alpha-\beta-\gamma,\tau)}{(\beta+\gamma,\tau)}N_{\gamma,\beta}N_{-\alpha,\beta+\gamma}\omega^{\alpha-\beta-\gamma} & \text{if}\quad \alpha<\beta+\gamma\\ 0 & \text{otherwise.}\end{cases}\end{aligned}$$

Hence, as for the third term in the above equality, we obtain

$$(6.22)\qquad -\sum_{\alpha,\beta\in\Delta_+}\left(\frac{1}{(\beta,\tau)}+\frac{1}{(\gamma,\tau)}\right)e(\omega^{-\alpha})e({}^t\Lambda_\tau(e_{-\alpha}){}^t\Lambda_\tau(e_\beta)\omega^{-\gamma})i(\omega^{-\beta})$$
$$=-\sum_{\substack{0<\alpha<\beta+\gamma\\ \beta\in\Delta_+}}\frac{(\alpha-\beta-\gamma,\tau)}{(\beta,\tau)(\gamma,\tau)}e(\omega^{-\alpha})e(N_{\gamma,\beta}N_{-\alpha,\beta+\gamma}\omega^{\alpha-\beta-\gamma})i(\omega^{-\beta})$$
$$=-\sum_{\substack{0<\alpha<\beta\\ \alpha\neq\gamma\\ \beta\in\Delta_+}}\frac{(\alpha-\beta-\gamma,\tau)}{(\beta,\tau)(\gamma,\tau)}e(\omega^{-\alpha})e(N_{\gamma,\beta}N_{-\alpha,\beta+\gamma}\omega^{\alpha-\beta-\gamma})i(\omega^{-\beta})$$
$$-\sum_{\substack{\beta<\alpha<\beta+\gamma\\ \alpha\neq\gamma\\ \beta\in\Delta_+}}\frac{(\alpha-\beta-\gamma,\tau)}{(\beta,\tau)(\gamma,\tau)}e(\omega^{-\alpha})e(N_{\gamma,\beta}N_{-\alpha,\beta+\gamma}\omega^{\alpha-\beta-\gamma})i(\omega^{-\beta})$$

$$+\sum_{\beta\in\Delta_+}\frac{1}{(\beta,\tau)}e(\omega^{-\beta})e(N_{\gamma,\beta}N_{-\beta,\beta+\gamma}\omega^{-\gamma})i(\omega^{-\beta})$$
$$+\sum_{\beta\in\Delta_+}\frac{1}{(\gamma,\tau)}e(\omega^{-\gamma})e(N_{\gamma,\beta}N_{-\gamma,\beta+\gamma}\omega^{-\beta})i(\omega^{-\beta})\,.$$

If we use (5.10) and (6.14) and replace $\alpha+\beta$ by β in the fourth term, we have

$$\sum_{\alpha,\beta\in\Delta_+}\left(\frac{1}{(\beta,\tau)}+\frac{1}{(\gamma,\tau)}\right)e(\omega^{-\alpha})e({}^t\Lambda_\tau(e_\beta)\omega^{-\gamma})({}^t\Lambda_\tau(e_{-\alpha})i(\omega^{-\beta})-i(\omega^{-\beta}){}^t\Lambda_\tau(e_{-\alpha}))$$
$$=\sum_{\alpha,\beta\in\Delta_+}\frac{(\beta+\gamma,\tau)}{(\gamma,\tau)(\alpha+\beta,\tau)}e(\omega^{-\alpha})e({}^t\Lambda_\tau(e_\beta)\omega^{-\gamma})i(e_\alpha\omega^{-\beta})$$
$$=-\sum_{\alpha,\beta\in\Delta_+}\frac{(\beta+\gamma,\tau)}{(\gamma,\tau)(\alpha+\beta,\tau)}e(\omega^{-\alpha})e(N_{\beta,\gamma}N_{\alpha,\beta}\omega^{-\beta-\gamma})i(\omega^{-\alpha-\beta})$$
$$=\sum_{\substack{0<\alpha<\beta\\ \beta\in\Delta_+}}\frac{(\alpha-\beta-\gamma,\tau)}{(\gamma,\tau)(\beta,\tau)}e(\omega^{-\alpha})e(N_{\beta-\alpha,\gamma}N_{\alpha,\beta-\alpha}\omega^{\alpha-\beta-\gamma})i(\omega^{-\beta})\,.$$

Hence, from (2.5), the fourth term is equal to

$$\sum_{\substack{0<\alpha<\beta\\ \beta\in\Delta_+}}\frac{(\alpha-\beta-\gamma,\tau)}{(\gamma,\tau)(\beta,\tau)}e(\omega^{-\alpha})e(N_{-\alpha,\beta}N_{\gamma,\beta-\alpha}\omega^{\alpha-\beta-\gamma})i(\omega^{-\beta})\,.$$

As for the fifth term, we use (5.10) and replace $\alpha+\beta$ by α. Then we get

$$\sum_{\alpha,\beta\in\Delta_+}\frac{1}{(\beta,\tau)}e({}^t\Lambda_\tau(e_\beta)\omega^{-\alpha})e({}^t\Lambda_\tau(e_{-\alpha})\omega^{-\gamma})i(\omega^{-\beta})$$
$$=\sum_{\substack{0<\alpha<\gamma\\ \beta\in\Delta_+}}\frac{(\alpha-\gamma,\tau)}{(\beta,\tau)(\gamma,\tau)}e(\omega^{-\alpha-\beta})e(-N_{\beta,\alpha}N_{-\alpha,\gamma}\omega^{\alpha-\gamma})i(\omega^{-\beta})$$
$$=\sum_{\substack{\beta<\alpha<\beta+\gamma\\ \beta\in\Delta_+}}\frac{(\alpha-\beta-\gamma,\tau)}{(\beta,\tau)(\gamma,\tau)}e(\omega^{-\alpha})e(-N_{\beta,\alpha-\beta}N_{\beta-\alpha,\gamma}\omega^{\alpha-\beta-\gamma})i(\omega^{-\beta})\,.$$

Hence, from (2.5), the fifth term is equal to

$$\sum_{\substack{\beta<\alpha<\beta+\gamma\\ \beta\in\Delta_+}}\frac{(\alpha-\beta-\gamma,\tau)}{(\beta,\tau)(\gamma,\tau)}e(\omega^{-\alpha})e(N_{-\alpha,\beta}N_{\gamma,\beta-\alpha}\omega^{\alpha-\beta-\gamma})i(\omega^{-\beta})\,.$$

Therefore, the sum of the fourth and fifth term is equal to

$$(6.23)\qquad \sum_{\substack{0<\alpha<\beta\\ \alpha\neq\gamma,\beta\in\Delta_+}}\frac{(\alpha-\beta-\gamma,\tau)}{(\gamma,\tau)(\beta,\tau)}e(\omega^{-\alpha})e(N_{-\alpha,\beta}N_{\gamma,\beta-\alpha}\omega^{\alpha-\beta-\gamma})i(\omega^{-\beta})$$
$$+\sum_{\substack{\beta<\alpha<\beta+\gamma\\ \alpha\neq\gamma,\beta\in\Delta_+}}\frac{(\alpha-\beta-\gamma,\tau)}{(\gamma,\tau)(\beta,\tau)}e(\omega^{-\alpha})e(N_{-\alpha,\beta}N_{\gamma,\beta-\alpha}\omega^{\alpha-\beta-\gamma})i(\omega^{-\beta})$$

$$+\sum_{\beta\in\Delta_+}\frac{1}{(\gamma,\tau)}e(\omega^{-\gamma})e(N_{-\gamma,\beta}N_{\beta-\gamma,\gamma}\omega^{-\beta})i(\omega^{-\beta}).$$

By (5.10) and (6.12), we have

$$\begin{aligned}&e({}^t\Lambda_\tau(e_{-\alpha})\omega^{-\gamma}){}^t\Lambda_\tau(e_\beta)-{}^t\Lambda_\tau(e_\beta)e({}^t\Lambda_\tau(e_{-\alpha})\omega^{-\gamma})\\&=e({}^t\Lambda_\tau(e_\beta){}^t\Lambda_\tau(e_{-\alpha})\omega^{-\gamma})\\&=\begin{cases}\dfrac{(\alpha-\gamma,\tau)}{(\gamma,\tau)}e(N_{-\alpha,\gamma}N_{\beta,\gamma-\alpha}\omega^{\alpha-\beta-\gamma}) & \text{if } \alpha<\gamma\\ 0 & \text{otherwise.}\end{cases}\end{aligned}$$

By (5.10), we have also

$$e(N_{-\alpha,\gamma}{}^t\Lambda_\tau(e_{\gamma-\alpha})\omega^{-\beta})=e(N_{-\alpha,\gamma}N_{\beta,\gamma-\alpha}\omega^{\alpha-\beta-\gamma})$$

provided that $\alpha<\gamma$. Thus, the sum of the sixth and seventh term is equal to

$$-\sum_{\substack{0<\alpha<\gamma\\ \beta\in\Delta_+}}\frac{(\alpha-\beta-\gamma,\tau)}{(\beta,\tau)(\gamma,\tau)}e(\omega^{-\alpha})e(N_{-\alpha,\gamma}N_{\beta,\gamma-\alpha}\omega^{\alpha-\beta-\gamma})i(\omega^{-\beta}).$$

In the eighth term, we use (5.10) and replace $\alpha+\gamma$ by α. Then by (2.5) we get

$$\begin{aligned}&-\sum_{\alpha\in\Delta_+}\frac{1}{(\gamma,\tau)}e({}^t\Lambda_\tau(e_\gamma)\omega^{-\alpha})e({}^t\Lambda_\tau(e_{-\alpha})\omega^{-\beta})i(\omega^{-\beta})\\&=-\sum_{\substack{0<\alpha<\beta\\ \beta\in\Delta_+}}\frac{(\alpha-\beta,\tau)}{(\gamma,\tau)(\beta,\tau)}e(-N_{\gamma,\alpha}\omega^{-\alpha-\gamma})e(N_{-\alpha,\beta}\omega^{\alpha-\beta})i(\omega^{-\beta})\\&=-\sum_{\substack{\gamma<\alpha<\beta+\gamma\\ \beta\in\Delta_+}}\frac{(\alpha-\beta-\gamma,\tau)}{(\gamma,\tau)(\beta,\tau)}e(\omega^{-\alpha})e(N_{-\alpha,\gamma}N_{\beta,\gamma-\alpha}\omega^{\alpha-\beta-\gamma})i(\omega^{-\beta}).\end{aligned}$$

Therefore, the sum of the sixth, seventh and eighth term is equal to

$$\begin{aligned}(6.24)\qquad &-\sum_{\substack{0<\alpha<\beta\\ \alpha\neq\gamma,\beta\in\Delta_+}}\frac{(\alpha-\beta-\gamma,\tau)}{(\beta,\tau)(\gamma,\tau)}e(\omega^{-\alpha})e(N_{-\alpha,\gamma}N_{\beta,\gamma-\alpha}\omega^{\alpha-\beta-\gamma})i(\omega^{-\beta})\\&-\sum_{\substack{\beta<\alpha<\beta+\gamma\\ \alpha\neq\gamma,\beta\in\Delta_+}}\frac{(\alpha-\beta-\gamma,\tau)}{(\beta,\tau)(\gamma,\tau)}e(\omega^{-\alpha})e(N_{-\alpha,\gamma}N_{\beta,\gamma-\alpha}\omega^{\alpha-\beta-\gamma})i(\omega^{-\beta})\\&+\sum_{\beta\in\Delta_+}\frac{1}{(\beta,\tau)}e(\omega^{-\beta})e(N_{-\beta,\gamma}N_{\beta,\gamma-\beta}\omega^{-\gamma})i(\omega^{-\beta}).\end{aligned}$$

Now, we use the equality (6.18). In the case $\alpha<\beta$ we have

$$(N_{\gamma,\beta}N_{-\alpha,\beta+\gamma}-N_{-\alpha,\beta}N_{\gamma,\beta-\alpha})\omega^{\alpha-\beta-\gamma}-N_{-\alpha,\gamma}N_{\gamma-\alpha,\beta}\omega^{\alpha-\beta-\gamma}=0.$$

Since $e_{-\alpha}\omega^{-\beta}=0$ for $\alpha>\beta$, in the case $\alpha>\beta$ we have also

$$(N_{\gamma,\beta}N_{-\alpha,\beta+\gamma}+N_{-\alpha,\gamma}N_{\beta,\gamma-\alpha})\omega^{\alpha-\beta-\gamma}=N_{-\alpha,\beta}N_{\gamma,\beta-\alpha}\omega^{\alpha-\beta-\gamma}$$

i.e. $(N_{\gamma,\beta}N_{-\alpha,\beta+\gamma}-N_{-\alpha,\beta}N_{\gamma,\beta-\alpha})\omega^{\alpha-\beta-\gamma}-N_{-\alpha,\gamma}N_{\gamma-\alpha,\beta}\omega^{\alpha-\beta-\gamma}=0.$

From (6.21)–(6.24), it follows that

$$\begin{aligned}\mathfrak{R}_\tau\cdot e(\omega^{-\gamma}) &= e(\omega^{-\gamma})\cdot\mathfrak{R}_\tau+\frac{(2\rho-\gamma,\gamma)}{(\gamma,\tau)}e(\omega^{-\gamma})\\ &+\sum_{\beta\in\Delta_+}\frac{1}{(\beta,\tau)}e(\omega^{-\beta})e((N_{\gamma,\beta}N_{-\beta,\beta+\gamma}-N_{-\beta,\gamma}N_{\gamma-\beta,\beta})\omega^{-\gamma})i(\omega^{-\beta})\\ &-\sum_{\beta\in\Delta_+}\frac{1}{(\gamma,\tau)}e(\omega^{-\gamma})e((N_{\beta,\gamma}N_{-\gamma,\beta+\gamma}-N_{-\gamma,\beta}N_{\beta-\gamma,\gamma})\omega^{-\beta})i(\omega^{-\beta}).\end{aligned}$$

On the other hand, by (6.18) in the case $\alpha=\beta$ we have

$$\begin{aligned}(N_{\gamma,\beta}N_{-\beta,\beta+\gamma}-N_{-\beta,\gamma}N_{\gamma-\beta,\beta})\omega^{-\gamma} &= (\gamma,\beta)\omega^{-\gamma}\\ (N_{\beta,\gamma}N_{-\gamma,\beta+\gamma}-N_{-\gamma,\beta}N_{\beta-\gamma,\gamma})\omega^{-\beta} &= (\beta,\gamma)\omega^{-\beta}.\end{aligned}$$

Hence we obtain

$$\begin{aligned}(6.25)\quad \mathfrak{R}_\tau\cdot e(\omega^{-\gamma}) &= e(\omega^{-\gamma})\cdot\mathfrak{R}_\tau+\frac{(2\rho-\gamma,\gamma)}{(\gamma,\tau)}e(\omega^{-\gamma})\\ &-\sum_{\beta\in\Delta_+}(\beta,\gamma)\left(\frac{1}{(\beta,\tau)}+\frac{1}{(\gamma,\tau)}\right)e(\omega^{-\gamma})e(\omega^{-\beta})i(\omega^{-\beta}).\end{aligned}$$

Now, we compute $\mathfrak{R}_\tau\omega^{-A}$. For a 1-form $\omega^{-\gamma}$, by (5.10), (6.17) and (6.19) we have

$$\begin{aligned}\mathfrak{R}_\tau\omega^{-\gamma} &= -\sum_{\alpha,\beta\in\Delta_+}\frac{1}{(\beta,\tau)}{}^t\Lambda_\tau(e_\beta)e(\omega^{-\alpha})i(\omega^{-\beta}){}^t\Lambda_\tau(e_{-\alpha})\omega^{-\gamma}\\ &= -\sum_{\substack{0<\alpha<\gamma\\ \beta\in\Delta_+}}\frac{(\alpha-\gamma,\tau)}{(\beta,\tau)(\gamma,\tau)}N_{-\alpha,\gamma}{}^t\Lambda_\tau(e_\beta)e(\omega^{-\alpha})i(\omega^{-\beta})\omega^{\alpha-\gamma}\\ &= \sum_{0<\alpha<\gamma}\frac{1}{(\gamma,\tau)}N_{-\alpha,\gamma}{}^t\Lambda_\tau(e_{\gamma-\alpha})\omega^{-\alpha}\\ &= \frac{1}{(\gamma,\tau)}\Big(\sum_{0<\alpha<\gamma}N_{-\alpha,\gamma}N_{\gamma-\alpha,\alpha}\Big)\omega^{-\gamma}\\ &= \frac{(2\rho-\gamma,\gamma)}{(\gamma,\tau)}\omega^{-\gamma}.\end{aligned}$$

Using the induction on the number of elements in A and applying (6.25), we obtain easily

$$\mathfrak{R}_\tau\omega^{-A}=\left\{\sum_{k=1}^{q}\frac{(2\rho-\alpha_{i_k},\alpha_{i_k})}{(\alpha_{i_k},\tau)}-\sum_{1\leq k<l\leq q}(\alpha_{i_k},\alpha_{i_l})\left(\frac{1}{(\alpha_{i_k},\tau)}+\frac{1}{(\alpha_{i_l},\tau)}\right)\right\}\omega^{-A}$$

$$= \left\{ \sum_{k=1}^{q} \frac{(2\rho, \alpha_{i_k})}{(\alpha_{i_k}, \tau)} - \sum_{1\leq k<l\leq q} \frac{(\alpha_{i_k}, \alpha_{i_l})}{(\alpha_{i_k}, \tau)} \right\} \omega^{-A}$$
$$= \left\{ \sum_{\alpha\in A} \frac{(2\rho, \alpha)}{(\alpha, \tau)} - \sum_{\alpha,\beta\in A} \frac{(\alpha, \beta)}{(\alpha, \tau)} \right\} \omega^{-A}$$

where $A=(\alpha_{i_1}, \cdots, \alpha_{i_q})$. Therefore, by (6.20) the Laplace-Beltrami operator $\square$ is expressed as follows:

$$\square(f\omega^{-A}) = \sum_{\alpha\in\Delta_+} \frac{1}{(\alpha, \tau)} \nabla_{e_\alpha}\nabla_{e_{-\alpha}}(f\omega^{-A}) + C_A\cdot f\omega^{-A}$$

where C_A is a constant depending on A. In fact, we get

$$C_A = \sum_{\alpha\in A} \frac{(\alpha, \lambda+\sum_{\beta\in A}\beta)}{(\alpha, \tau)} + \sum_{\alpha\in A} \frac{(2\rho, \alpha)}{(\alpha, \tau)} - \sum_{\alpha,\beta\in A} \frac{(\alpha, \beta)}{(\alpha, \tau)}$$
$$= \sum_{\alpha\in A} \frac{(\alpha, \lambda+2\rho)}{(\alpha, \tau)}.$$

q.e.d.

7. Vanishing theorems of square-integrable $\bar{\partial}$-cohomology spaces

We retain the notation introduced in the previous sections. Using proposition 2 we will give vanishing theorems of the square-integrable $\bar{\partial}$-cohomology space $H_2^q(\mathcal{L}_\lambda)$. The following lemma is due to [1] Proposition 8.

Lemma. 7. *Let q be an integer such that $0\leqq q\leqq n=\dim D$. If there exists a constant $c>0$ such that for every $\varphi\in A_0^{0,q}(\mathcal{L}_\lambda)$ we have the inequality*

$$(\square\varphi, \varphi)\geqq c(\varphi, \varphi),$$

then we have $$H_2^q(\mathcal{L}_\lambda) = (0).$$

For each character λ, put

$$\Delta_\lambda = \{\alpha\in\Delta \mid \varepsilon_\alpha(\alpha, \lambda)>0\}.$$

Let q_λ be the number of all elements in $\Delta_+\cap\Delta_\lambda$. Then we have the following vanishing theorem about 0-th square-integrable $\bar{\partial}$-cohomology space $H_2^0(\mathcal{L}_\lambda)$.

Theorem 1. *Assume that q_λ is not zero, i.e. there exists an element $\alpha\in\Delta_+$ such that $\varepsilon_\alpha(\alpha, \lambda)>0$. Then we have $H_2^0(\mathcal{L}_\lambda)=0$.*

Proof. Let f be a section in $A_0^{0,0}(\mathcal{L}_\lambda)$. From Proposition 2, we have

$$\square f = \sum_{\alpha\in\Delta_+} \frac{1}{(\alpha, \tau)} e_\alpha e_{-\alpha} f.$$

Since $h_\alpha=[e_\alpha, e_{-\alpha}]$, we get

$$\square f=\sum_{\alpha\in\Delta_+\cap\Delta_\lambda}\frac{1}{(\alpha,\tau)}h_\alpha f+\sum_{\alpha\in\Delta_+\cap\Delta_\lambda}\frac{1}{(\alpha,\tau)}e_{-\alpha}e_\alpha f+\sum_{\alpha\in\Delta_+-\Delta_\lambda}\frac{1}{(\alpha,\tau)}e_\alpha e_{-\alpha}f.$$

A section $f\colon G\to C$ satisfies the following formula:

$$h_\alpha f=-\lambda(h_\alpha)f=-(\alpha,\lambda)f.$$

By (6.4), the formal adjoint operator of e_α with respect to the inner product $(\ ,\ )_G$ in $C_0^\infty(G)$ is $-\varepsilon_\alpha e_{-\alpha}$ (cf. the proof of Proposition 1). Hence we have

$$(\square f,f)=\sum_{\alpha\in\Delta_+\cap\Delta_\lambda}\Big(-\frac{(\lambda,\alpha)}{(\alpha,\tau)}\Big)(f,f)_G+\sum_{\alpha\in\Delta_+\cap\Delta_\lambda}\Big(\frac{-\varepsilon_\alpha}{(\alpha,\tau)}\Big)(e_\alpha f,e_\alpha f)_G$$
$$+\sum_{\alpha\in\Delta_+-\Delta_\lambda}\Big(\frac{-\varepsilon_\alpha}{(\alpha,\tau)}\Big)(e_{-\alpha}f,e_{-\alpha}f)_G.$$

Here, $\frac{-\varepsilon_\alpha}{(\alpha,\tau)}$ is positive for every positive root α. Therefore we have

$$(\square f,f)\geqq\Big(\sum_{\alpha\in\Delta_+\cap\Delta_\lambda}\Big(-\frac{(\lambda,\alpha)}{(\alpha,\tau)}\Big)\Big)(f,f).$$

If α belongs to $\Delta_+\cap\Delta_\lambda$, we have

$$-\frac{(\alpha,\lambda)}{(\alpha,\tau)}>0.$$

Thus, if we assume that $\Delta_+\cap\Delta_\lambda$ is not empty, the bundle $\mathcal{L}_\lambda$ satisfies the condition of Lemma 7 for $q=0$, and we have $H_2^0(\mathcal{L}_\lambda)=(0)$. q.e.d.

REMARK. This theorem follows also from the expression of $\square$ on p. 282 of [4] instead of our Proposition 2.

For general q-th $\bar\partial$-cohomology spaces, we get the following main theorem.

Theorem 2. *Let q be an integer such that $0<q\leqq n$. Assume that for any q-tuple A of positive roots the scalar $c_A=\sum_{\alpha\in A}\frac{(\alpha,\lambda+2\rho)}{(\alpha,\tau)}$ is positive. Then we have $H_2^q(\mathcal{L}_\lambda)=(0)$.*

Proof. Let $\varphi=\sum_{A\in\mathfrak{A}}f_A\omega^{-A}$ be a form in $A_0^{0,q}(\mathcal{L}_\lambda)$. By (6.4) and (6.6), we have

$$(\nabla_{e_\alpha}\varphi,\varphi)=-\varepsilon_\alpha(\varphi,\nabla_{e_{-\alpha}}\varphi)$$

where $(\ ,\ )$ is the inner product in $C_0^\infty(G)\otimes\overset{q}{\wedge}\mathfrak{n}_-^*$. From Proposition 2, we get

$$(\square\varphi,\varphi)=\sum_{\alpha\in\Delta_+}\Big(-\frac{\varepsilon_\alpha}{(\alpha,\tau)}\Big)(\nabla_{e_{-\alpha}}\varphi,\nabla_{e_{-\alpha}}\varphi)$$
$$+\sum_{A\in\mathfrak{A}}\Big(\sum_{\alpha\in A}\frac{(\alpha,\lambda+2\rho)}{(\alpha,\tau)}\Big)(f_A\omega^{-A},f_A\omega^{-A}).$$

Put

$$c=\min_{A\in\mathfrak{A}} c_A=\min_{A\in\mathfrak{A}}\left(\sum_{\alpha\in A}\frac{(\alpha,\lambda+2\rho)}{(\alpha,\tau)}\right).$$

Since $-\frac{\varepsilon_\alpha}{(\alpha,\tau)}$ is positive for every positive root α, we have

$$(\Box\varphi,\varphi)\geqq c(\varphi,\varphi).$$

From the assumption, c is positive. Therefore, by Lemma 7 we obtain the theorem. q.e.d.

We note that the criterion for the vanishing in this theorem depends on the choice of τ.

Corollary 1. *Assume that*

$$\begin{cases}(\alpha,\lambda+2\rho)>0 & \text{for}\quad \alpha\in\Delta_+\cap\Delta_{\mathfrak{k}}\\ (\alpha,\lambda+2\rho)<0 & \text{for}\quad \alpha\in\Delta_+\cap\Delta_{\mathfrak{p}}.\end{cases}$$

Then we have $H_2^q(\mathcal{L}_\lambda)=(0)$ *for all* $q\geqq 1$.

Proof. By the assumption, we have

$$c_\alpha=\frac{(\alpha,\lambda+2\rho)}{(\alpha,\tau)}>0 \qquad \text{for all}\quad \alpha\in\Delta_+.$$

Since $c_A=\sum_{\alpha\in A}c_\alpha$, c_A is positive for any q-tuple A provided that $q\geqq 1$. By Theorem 2, we obtain the corollary. q.e.d.

We consider the case $q=0$. From Corollary 1, if we have $(\alpha,\lambda)<-(\alpha,2\rho)$ for all $\alpha\in\Delta_+\cap\Delta_{\mathfrak{p}}$, we obtain $H_2^q(\mathcal{L}_\lambda)=0$ for $q\neq 0$.

Now, let $\mathcal{L}_\lambda^*$ be the dual line bundle of $\mathcal{L}_\lambda$ and $\Theta^*(D)$ be the dual bundle of the holomorphic tangent bundle of D. By Theorem 1.2 in [8], we obtain the Seere's duality

$$H_2^q(\mathcal{L}_\lambda)\cong H_2^{n-q}(\mathcal{L}_\lambda^*\otimes\overset{n}{\wedge}\Theta^*(D)). \tag{7.1}$$

On the other hand, the bundle $\mathcal{L}_\lambda^*$ is the homogeneous line bundle associated with the charactor λ^{-1} of H. The bundle $\overset{n}{\wedge}\Theta^*(D)$ is the homogeneous line bundle $G\times_H\overset{n}{\wedge}\mathfrak{n}_+^*$ associated with the representation $\overset{n}{\wedge}Ad_+^*$ induced from the adjoint representation Ad_+ of H in $\mathfrak{n}_+$. Therefore

$$\mathcal{L}_\lambda^*\otimes\overset{n}{\wedge}\Theta^*(D)=\mathcal{L}_{\lambda^{-1}\otimes\overset{n}{\wedge}Ad_+^*}.$$

The differential of the charactor $\lambda^{-1}\otimes\overset{n}{\wedge}Ad_+^*$ is $-\lambda-2\rho$. By Theorem 1, 2 and (7.1) we obtain the following corollaries.

Corollary 2. *If we assume that* $q_{-\lambda-2\rho}$ *is not zero i.e. there exist a root* $\alpha\in\Delta_+$ *such that* $\varepsilon_\alpha(\alpha, \lambda+2\rho)<0$, *we have* $H_2^n(\mathcal{L}_\lambda)=(0)$.

Corollary 3. *Let q be an integer such that* $0\leqq q<n$. *Assume that for any q-tuple A of positive roots the scalar* $d_A=\sum_{\alpha\in A}\frac{(\alpha, \lambda)}{(\alpha, \tau)}$ *is negative. Then we have* $H_2^{n-q}(\mathcal{L}_\lambda)=(0)$.

From Corollary 3, we have also the following.

Corollary 4. *We assume that*

$$
\begin{aligned}
(\alpha, \lambda)&<0 \quad \text{for all} \quad \alpha\in\Delta_+\cap\Delta_{\mathfrak{k}}\\
(\alpha, \lambda)&>0 \quad \text{for all} \quad \alpha\in\Delta_+\cap\Delta_{\mathfrak{p}}
\end{aligned}
$$

i.e. $q_\lambda=n$. *Then, we have* $H_2^q(\mathcal{L}_\lambda)=(0)$ *for all* $q\leqq n-1$.

EXAMPLE. Let $G=SU(2, 1)$ and T be the subgroup of G consisting of all matrices

$$
U=\begin{pmatrix} u_1 & 0 & 0\\ 0 & u_2 & 0\\ 0 & 0 & u_3\end{pmatrix}
$$

where $u_i\in U(1)$ $(i=1, 2, 3)$ and $\det U=1$. We denote by $K=S(U(2)\times U(1))$ the subgroup of G consisting of all matrices

$$
\begin{pmatrix} \multicolumn{2}{c}{\multirow{2}{*}{U}} & 0\\ & & 0\\ 0 & 0 & v\end{pmatrix}
$$

where $U\in U(2)$, $v\in U(1)$ and $\det U\cdot v=1$. Then, H is a compact Cartan subgroup of G and K is a maximal compact subgroup containing H. The complexification of the Lie algebra of G is $\mathfrak{g}=\mathfrak{sl}(3, C)$ and the subalgebra $\mathfrak{h}$ is given by

$$
\mathfrak{h}=\left\{\begin{pmatrix} \lambda_1 & 0 & 0\\ 0 & \lambda_2 & 0\\ 0 & 0 & \lambda_3\end{pmatrix}\middle|\, \lambda_i\in C,\ \lambda_1+\lambda_2+\lambda_3=0\right\}.
$$

The root system of $\mathfrak{g}$ with respect to the Cartan subalgebra $\mathfrak{h}$ is given by

$$
\Lambda=\{\lambda_i-\lambda_j\,|\,i\neq j,\ 1\leqq i, j\leqq 3\}\ .
$$

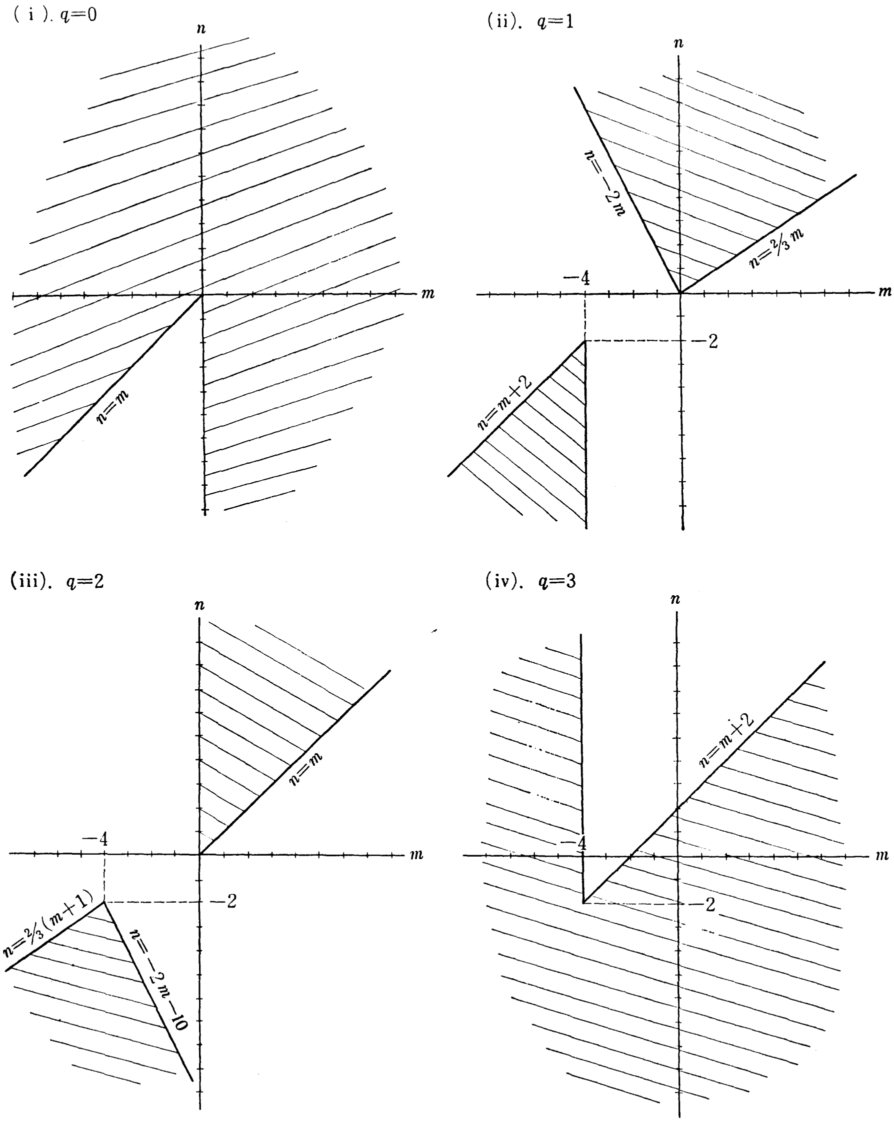
We choose a fundamental root system $\{\lambda_1-\lambda_2, \lambda_2-\lambda_3\}$ and take an ordering for the roots corresponding to this system. Then, the positive root set is $\Delta_+=\{\lambda_1-\lambda_2, \lambda_2-\lambda_3, \lambda_1-\lambda_3\}$ and we have

$$\Delta_+ \cap \Delta_{\mathfrak{k}} = \{\lambda_1 - \lambda_2\}$$
$$\Delta_+ \cap \Delta_{\mathfrak{p}} = \{\lambda_2 - \lambda_3,\ \lambda_1 - \lambda_3\} .$$

Let a be a positive real constant, and put

$$h_\tau = \begin{pmatrix} 0 & 0 & 0 \\ 0 & -a & 0 \\ 0 & 0 & a \end{pmatrix} \in \mathfrak{h}_{\boldsymbol{R}} .$$

Figure 1.



Let τ be an element of $\mathfrak{h}_R{}^*$ corresponding to $h_\tau \in \mathfrak{h}_R$ with respect to the Killing form of $\mathfrak{g}$. Then, the element τ satisfies the condition (3.4). Hence, we have the homogeneous complex manifold $D=G/H$ with the invariant Kähler metric g_τ. The space $\mathfrak{h}_R{}^*$ is generated by λ_1 and λ_2 over R. The set of all elements of $\mathfrak{h}_R{}^*$ which are the differentials of charactors of H is given by

$$\left\{\lambda \in \mathfrak{h}_R{}^* \,\middle|\, \frac{2(\lambda, \alpha)}{(\alpha, \alpha)} \in Z \quad \text{for all} \quad \alpha \in \Delta\right\}$$

$$\text{i.e.} \quad \{\lambda = m\lambda_1 + n\lambda_2 \mid m, n \in Z\}.$$

We can consider a character λ of H as a lattice point in R^2. As for the vanishing of the cohomology space $H^q_2(\mathcal{L}_\lambda)$, our theorems give the following figures (cf. Figure 1). Here, the space $H^q_2(\mathcal{L}_\lambda)$ vanishes for all characters belonging to the shadowed domains.

On the other hand, the vanishing theorems in [4] are written as follows: There exists a positive constant η such that, if the character λ statisfies $|(\lambda, \alpha)| > \eta$ for every $\alpha \in \Delta$, the space $H^q_2(\mathcal{L}_\lambda)$ vanishes for all $q \neq q_\lambda$. In the case of this example, we can see that a positive constant η must be larger than 12 and the above condition on λ is equivalent to the following inequalities:

$$|m| > 6\eta, \quad |n| > 6\eta, \quad |m-n| > 6\eta\,.$$

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