The University of Osaka
Institutional Knowledge Archive

Title	The 3-canonical system on 3-folds of general type
Author(s)	Zhou, Yiqun
Citation	Osaka Journal of Mathematics. 2011, 48(1), p. 91-98
Version Type	VoR
URL	https://doi.org/10.18910/9368
rights	
Note	

The University of Osaka Institutional Knowledge Archive : OUKA

https://ir.library.osaka-u.ac.jp/

The University of Osaka

Zhou, Y.
Osaka J. Math.
48 (2011), 91–98

THE 3-CANONICAL SYSTEM ON 3-FOLDS OF GENERAL TYPE

YIQUN ZHOU

(Received January 13, 2009, revised September 16, 2009)

Abstract

Let X be a projective minimal Gorenstein 3-fold of general type with $\mathbb{Q}$-factorial terminal singularities. We classify minimal Gorenstein 3-folds of general type according to the birationality of 3-canonical system on X.

1. Introduction

Let X be a projective variety on $\mathbb{C}$, and K_X is the canonical divisor on X. One may define the m-canonical map ϕ_m corresponding to the complete linear system $|mK_X|$. To study the behavior of ϕ_m, especially the birationality of ϕ_m, has been one of the most important topics of birational geometry. When $\dim X \leq 2$, the behavior of ϕ_m has been thoroughly studied. However, when $\dim X \geq 3$, many problems still remain open.

Let X be a projective minimal Gorenstein 3-fold of general type with $\mathbb{Q}$-factorial terminal singularities. When $\dim X = 3$, we know that when m is big enough, ϕ_m must be birational, hence it is important to find the universal lower bound N for all 3-folds such that ϕ_m is birational for all $m \geq N$. The known best result is as follows: Chen, Chen and Zhang [5] proved that $m \geq 5$ is enough; Chen and Zhang [6] proved that if X cannot be fibred by a unique family of irreducible curves of geometric genus 2, then $m \geq 4$ is enough; Zhu [12] proved the generic finiteness of ϕ_3. Also in [6], Chen and Zhang proposed an open problem(see [6, 6.4 (2)]): is it possible to characterize the birationality of ϕ_3? We have several examples to illustrate the importance of this problem.

EXAMPLE 1.1 ([6, Example 6.2]). Kobayashi (see [9, Proposition 3.2]) has constructed a family of canonically polarized smooth threefolds Y satisfying the equality

$$K_Y^3 = \frac{4}{3}p_g(Y) - \frac{10}{3}$$

where $p_g(Y) = 7, 10, 13, \ldots$, Chen and Zhang proved that all examplxes above have non-birational 4-canonical maps, thus they have non-birational 3-canonical maps.

2000 Mathematics Subject Classification. 14E05, 14J10, 14J30.

EXAMPLE 1.2. Let S be a surface of type $(K_S^2, p_g(S)) = (2, 3)$. Let C be a smooth curve of genus ≥ 2. Take $X = S \times C$. Since ϕ_1 is generically finite, $\dim \phi_1(X) = 3$. When $g(C)$ is big enough, $p_g(X)$ is big enough, hence ϕ_4 is birational by [6, 4.2]. But ϕ_3 is obviously not birational.

This paper makes an effort to answer this open problem, the main theorem is as follows. Notice that the Example 1.1 corresponds to the $d = 2$ situation, and the Example 1.2 corresponds to the $d = 1$ situation.

Theorem 1.1. *Let X be a projective minimal Gorenstein 3-fold of general type with $\mathbb{Q}$-factorial terminal singularities. Denote $d = \dim \phi_1(X)$. If ϕ_3 on X is not birational, then X must be one of the following types*:
(1) $p_g(X) \leq 7$;
(2) $p_g(X) \geq 8$, *while*:
(1) $d = 3$, *X contains a surface which has a family of curves of genus* 2;
(2) $d = 2$, *X contains a surface which has a family of curves of genus* ≤ 3;
(3) $d = 1$, *X contains a surface S such that either S has a family of curves of genus ≤ 3 or S satisfies $p_g(S) = 1$, $K_{S_0}^2 \leq 9$, where S_0 is the minimal model of S.*

2. The key technical results

2.1. Brief review on curves.

FACT 2.1. Let C be a smooth curve of genus ≥ 2. Assume D is a divisor on C with $\deg(D) \geq 3$. Then the rational map corresponding to $|K_C + D|$ gives a birational morphism onto its image.

2.2. Brief review of relevant results on surfaces. Let S be a smooth minimal surface of general type, according to [2], one has

FACT 2.2. For all $m \geq 5$, ϕ_m is a biratinal morphism onto its image.

FACT 2.3. For $m = 4$, ϕ_m is birational if and only if $(K_S^2, p_g(S)) \neq (1, 2)$.

FACT 2.4. For $m = 3$, ϕ_m is birational if and only if $(K_S^2, p_g(S)) \neq (1, 2)$ and $(2, 3)$.

FACT 2.5. For $m = 2$, if $K_S^2 \geq 10$, then ϕ_m is birational if and only if S cannot be fibred by a family of irreducible curves of geometric genus 2.

2.3. General method. Let X be a projective minimal Gorenstein 3-fold of general type with $\mathbb{Q}$-factorial terminal singularities. Under the assumption $p_g(X) \geq 2$, we study the canonical map ϕ_1 which is a rational map. Take the modification $\pi\colon X' \to X$, according to Hironaka, such that

(i) X' is smooth.
(ii) The movable part of $|K_{X'}|$ is basepoint free.
(iii) $\pi^*(K_X)$ is linearly equivalent to a divisor with normal crossing support.

Denote by g the composition $\phi_1 \circ \pi$. So $g\colon X' \to W' \subseteq \mathbb{P}^{p_g(X)-1}$ is a morphism. Let $X' \xrightarrow{f} B \xrightarrow{s} W'$ be the Stein factorization of g. We get a commutative diagram as below:

$$\begin{tikzcd} X' \arrow[r, "f"] \arrow[d, "\pi"'] \arrow[dr, "g"] & B \arrow[d, "s"] \\ X \arrow[r, "\phi_1"'] & W'. \end{tikzcd}$$

We may write $K'_X \sim \pi^*(K_X) + E_\pi \sim M_1 + Z_1$, where M_1 is the movable part of $|K'_X|$, Z_1 is the fixed part, and E_π is a sum of distinct exceptional divisors. We may write $\pi^*(K_X) \sim M_1 + E'_1$, where $E'_1 = Z_1 - E_\pi$ is an effective divisor.

If $\dim \phi_1(X) = 2$, we see that a general fiber S of f is a smooth projective curve C of genus ≥ 2. We say that X is canonically fibred by curves of genus $g = g(C)$.

If $\dim \phi_1(X) = 1$, we see that a general fiber S of f is a smooth projective surface of general type. We say that X is canonically fibred by surfaces with invariants $(c_1^2(S_0), p_g(S))$, where S_0 is the minimal model of S. We may write $M_1 \equiv a_1 S$, where $a_1 \geq p_g(X) - 1$.

A generic irreducible element S of $|M_1|$, means either a general member of $|M_1|$ when $\dim \phi_1(X) \geq 2$ or, otherwise, a general fiber of f.

REMARK 2.1. Throughout the paper $D_1 \sim D_2$ (resp. $=_{\mathbb{Q}}$, or $D_1 \equiv D_2$) means that divisors D_1 and D_2 are linearly equivalent (resp. rD_1 and rD_2 are linearly equivalent for some positive integer r, or D_1 and D_2 are numerically equivalent).

Lemma 2.1 (Chen [6, Theorem 3.6]). *Let X be a minimal projective 3-fold of general type with $\mathbb{Q}$-factorial terminal singularities and assume $p_g(X) \geq 2$. Keep the same notation as above. Pick a generic irreducible element S of $|M_1|$. Suppose, on the smooth surface S, there is a movable linear system $|G|$ and denote by C a generic irreducible element of $|G|$. Set $\xi := (\pi^*(K_X) \cdot C)_{X'}$, and*

$$p := \begin{cases} 1 & \text{If} \quad \dim \phi_1(X) \geq 2, \\ a_1 & \text{otherwise.} \end{cases}$$

Assume there is a rational number $\beta \geq 0$, such that $\pi^*(K_X)|_S - \beta C$ is numerically equivalent to an effective $\mathbb{Q}$-divisor. Denote $\alpha := (m - 1 - 1/p - 1/\beta)\xi$, set $\alpha_0 = \lceil \alpha \rceil$, then

(1) *Assume there is a positive integer* m *such that the linear system* $|K_S + \lceil (m-2)\pi^*(K_X)|_S \rceil|$ *seperates different generic irreducible elements of* $|G|$, *then* ϕ_m *is birational if one of the following conditions is satisfied*:

i. $\alpha > 2$;

ii. $\alpha_0 \geq 2$ *and* C *is non-hyperelliptic*;

iii. $\alpha > 0$, C *is non-hyperelliptic and* C *is an even divisor of* S.

(2) $\xi \geq (2g(C) - 2 + \alpha_0)/m$ *if one of the following conditions is satisfied*:

iv. $\alpha > 1$;

v. $\alpha > 0$, *and* C *is an even divisor.*

3. Proof of the main theorem

We now study the birationality of ϕ_3. Set $d = \dim \phi_1(X)$.

3.1. The case $d = 3$.

Proposition 3.1. *Let* X *be a projective minimal Gorenstein* 3*-fold of general type with* $\mathbb{Q}$*-factorial terminal singularities. Assume* $p_g(X) \geq 8$, *and* $d = 3$. *Then* ϕ_3 *is birational unless* X *contains a surface which has a family of curves of genus* ≤ 2.

Proof. According to our general method, a generic irreducible element S of $|M_1|$ is a smooth projective surface of general type. It is sufficient to verify the birationality for $\phi_3|_S$ by virtue of the Matsuki–Tankeev principle [3, 2.1]. We consider the subsystem $|K_{X'} + \pi^*(K_X) + S| \subset |3K'_X|$. By the Kawamata–Viehweg vanishing theorem [8], we have the surjective map

$$H^0(X', K_{X'} + \pi^*(K_X) + S) \to H^0(S, K_S + L)$$

where $L = \pi^*(K_X)|_S$ is a nef and big divisor on S. If $|L|$ gives a birational map, then so does $|K_S + L|$. Otherwise, $|L|$ gives a generically finite map of degree ≥ 2. Noting that $h^0(S, L) \geq p_g(X) - 1 \geq 7$, we have $L^2 \geq 2(h^0(S, L) - 2) \geq 10$. If $|K_S + L|$ doesn't give a birational map, then according to Reider's result [10], there is a free pencil of curves on S with a generic irreducible element C, such that $L \cdot C = 1$ or 2. On the other hand, $L \cdot C \geq 2$ since $|L|$ gives a generically finite map on C and C is a curve of genus ≥ 2. Therefore we have $L \cdot C = 2$. Moreover, by the Riemann–Roch fomular and Clifford's theorem, we have $h^0(C, L|_C) = 2$. By

$$H^0(S, L - C) \to H^0(S, L) \to H^0(C, L|_C), \tag{3.1}$$

we have

$$h^0(S, L - C) + h^0(C, L|_C) \geq h^0(S, L),$$
$$h^0(S, L) \leq h^0(S, L - C) + 2.$$

We can replace L in (3.1) by $L-nC$ and have n sequences relatively. By $C^2=0$ we have $(L-C)|_C = L|_C$. Since $h^0(S,L) \geq p_g(X)-1$, we have $h^0(S,L) \leq h^0(S, L-nC)+2n$ when $p_g(X) > 2n+1$. So there exists an effective divisor E such that $L =_{\mathbb{Q}} nC+E$.

Set $F = C + (1/n)E$, so $L =_{\mathbb{Q}} nF$. Since $p_g(X) \geq 8$, we can choose $n=3$, so $\deg F|_C = (1-1/n)L\cdot C > 1$.

By the Kawamata–Viehweg vanishing theorem, we have the following surjective map

$$H^0\left(K_S + \left\lceil L - \frac{1}{n}E \right\rceil\right) \to H^0(C, K_C + \lceil (n-1)F\rceil|_C).$$

Let M_3 be the movable part of $|K_{X'} + \pi^*(K_X) + S|$, N be the movable part of $|K_S + \lceil L-(1/n)E - C\rceil|$, then $M_3|_S \geq N$ by [3, Lemma 2.7].

So we have

$$3L\cdot C = 3\pi^*(K_X)|_S \cdot C \geq \deg(K_C + 2F|_C) \geq 2g(C) - 2 + \frac{8}{3} = 2g(C) + \frac{2}{3},$$

then we can derive that $g(C) \leq 2$.

Therefore, ϕ_3 is birational if X does not contain a surface which has a family of curves of genus 2. □

3.2. The case $d = 2$.

Proposition 3.2. *Let X be a projective minimal Gorenstein 3-fold of general type with $\mathbb{Q}$-factorial terminal singularities. Assume $p_g(X) \geq 6$, and $d = 2$. Then ϕ_3 is birational unless X contains a surface which has a family of curves of genus ≤ 3.*

Proof. Let S be a generic element of $|M_1|$. So $S|_S \equiv aC$, where $a \geq p_g(X)-2$, and C is a general fiber of f and is a smooth curve. Denote $L = \pi^*(K_X)|_S > S|_S$. By the Kawamata–Viehweg vanishing theorem, we have the following surjective map

$$H^0(X', K_{X'} + \pi^*(K_X) + S) \to H^0(S, K_S + \lceil L\rceil).$$

We may write $L \equiv aC + E''$, so $L - C - (1/a)E''$ is nef and big. By the Kawamata–Viehweg vanishing theorem, we have the following surjective map

$$H^0\left(S, K_S + \left\lceil L - \frac{1}{a}E''\right\rceil\right) \to H^0(C, K_C + D)$$

where $D := \lceil L - C - (1/a)E''\rceil|_C$. If $\deg\lfloor D\rfloor = (1-(1/a))L\cdot C \geq 3$, then $|K_C + \lceil L\rceil|_C - C|$ gives a birational map by Fact 2.1. By the Matsuki–Tankeev principle, we can derive that $|K_{X'} + \pi^*(K_X) + S|$ gives a birational map, and the birationality of ϕ_3 follows.

We now prove that $(1-1/a)L\cdot C>2$.

Let $G=M_1|_S$, G is composed of a pencil and $G\equiv aC$. In Lemma 2, we may take $p=1$, $\beta=2$, and $\xi=L\cdot C$, $\alpha=(m-2.5)\xi$. By [3], ϕ_4 is generically finite, this means hat $|\lfloor 4\pi^*(K_X)\rfloor|$ maps a general C onto a curve. Thus $4\pi^*(K_X)|_S\cdot C\geq 2$, and $\xi\geq 1/2$.

Let $m=7$, we have $(m-4)\xi\geq 1$. By Lemma 2, $\xi\geq(2g(C)-2+\alpha_0)/m$, follows $\xi\geq(2g(C)-2)/2.5$. When $g(C)\geq 4$, we have $\xi\geq 3$. Hence $(1-1/a)\xi>2$, since $a\geq p_g(X)-2\geq 4$.

Therefore, if X does not contain a surface which has a family of curves of genus ≤ 3, then ϕ_3 is birational. □

3.3. The case $d=1$.

Lemma 3.1 ([6, Lemma 3.7]). *Keep the same notation as in* Subsection 2.3 *and with p as in* Lemma 2.1. *Assume $d=1$, and $g(B)=0$. Let S be a general fiber of $g\colon X'\to W'$. Let $\sigma\colon S\to S_0$ be the contraction onto the minimal model. Then*

$$\pi^*(K_X)|_S-\frac{p}{p+1}\sigma^*(K_{S_0})$$

is pseudo-effective.

Proposition 3.3. *Let X be a projective minimal Gorenstein 3-fold of general type with $\mathbb{Q}$-factorial terminal singularities. Assume $p_g(X)\geq 6$, and $d=1$. Then ϕ_3 is birational unless X contains a surface S which has a family of curves of genus ≤ 3 or which satisfies $p_g(S)=1$, $K_{S_0}^2\leq 9$, where S_0 is the minimal model of S.*

Proof. Let S be a general fiber of f. Then S is a smooth projective surface of general type. We have $M_1\equiv aS$, with $a\geq p_g(X)-1$. Denote $L=\pi^*(K_X)|_S$. Let $\sigma\colon S\to S_0$ be the contraction onto the minimal model. Denote $F'=\pi_*(S)$. Notice that $K_X\cdot F'^2$ is an even integer [5, 2.1] and $K_X\cdot F'^2\geq 0$.

CASE 1. The case $K_X\cdot F'^2>0$.

Notice that $K_X\equiv aF'+E'$, where E' is an effective divisor. We have $L^2=K_X^2F'\geq aK_XF'^2\geq 2(p_g(X)-1)\geq 10$. By Reider's result, if $|K_X+L|$ doesn't give a birational map, there exists a free pencil of curves on S with a generic irreducible element C, such that $L\cdot C\leq 2$. By Lemma 3.1 and [5, Claim 3.3], we always have

$$\left(\pi^*(K_X)|_S-\frac{p}{p+1}\sigma^*(K_{S_0})\right)\cdot C\geq 0,$$

then we can derive

$$\frac{p}{p+1}\sigma^*(K_{S_0})\cdot C\leq 2,$$

so

$$\sigma^*(K_{S_0})\cdot C\leq 2,\ \text{since}\, p\geq p_g(X)-1\geq 5.$$

Let $\bar{C}$ be the image of C under σ, then we have $K_{S_0} \cdot \bar{C} \leq 2$.

If $\bar{C}^2 = 0$, then $K_{S_0} \cdot \bar{C} = 2g(\bar{C}) - 2 \leq 2$, so $g(\bar{C}) \leq 2$, which means S has a family of curves of genus 2.

If $\bar{C}^2 > 0$, by the Hodge index theorem, we can derive that $K_{S_0}^2 \cdot \bar{C}^2 \leq (K_{S_0} \cdot \bar{C})^2 \leq 4$. But $K_{S_0}^2 \geq L^2 > 10$, which is a contradiction.

CASE 2. The case $K_X \cdot F'^2 = 0$.

One always has $\pi^*(K_X)|_S \sim \sigma^*(K_{S_0})$ by [5, Claim 3.3]. Notice that

$$\pi^*(K_X) - S - \frac{1}{a}E_1' =_{\mathbb{Q}} \left(1 - \frac{1}{a}\right)\pi^*(K_X).$$

Applying the vanishing theorem, one has the surjective map:

$$H^0\left(X', K_{X'} + \left\lceil 2\pi^*(K_X) - \frac{1}{a}E_1' \right\rceil\right) \to H^0\left(S, K_S + \left\lceil \left(2 - \frac{1}{a}\right)\pi^*(K_X)\right\rceil\Big|_S\right).$$

Since

$$K_S + \left\lceil \left(2 - \frac{1}{a}\right)\pi^*(K_X)\right\rceil\Big|_S \geq K_S + \sigma^*(K_{S_0}) + \left\lceil \left(1 - \frac{1}{a}\right)\pi^*(K_X)\right\rceil\Big|_S,$$

by Fact 2.5, $|K_S + \sigma^*(K_{S_0}) + \lceil (1 - 1/a)\pi^*(K_X)\rceil|_S|$ doesn't give a birational map only if $K_{S_0}^2 \leq 9$ or S has a family of curves of genus 2.

When $p_g(S) \geq 2$, there exists a family of curves with its general member $C' \subset |L|$. By the Kawamata–Viehweg vanishing theorem, we have

$$\left|K_S + C' + \left\lceil \left(1 - \frac{1}{a}\right)L\right\rceil\right|\Big|_{C'} = |K_{C'} + D|,$$

where $\deg D \geq (1 - 1/a)L \cdot C'$. If $|K_{C'} + D|$ doesn't give a birational map, we have $(1 - 1/a)L \cdot C' \leq 2$, then $L \cdot C' = \sigma^*(K_{S_0}) \cdot C' = K_{S_0} \cdot \sigma_*(C') \leq 2$. By $p_g(S) \geq 2$, we have $K_{S_0}^2 \geq 2p_g(S_0) - 4 \geq 0$.

If $K_{S_0}^2 \leq 1$, we have $(\sigma_*(C'))^2 \leq K_{S_0}^2 \leq 1$. Therefore, by the Riemann–Roch fomular, we have $2(p_a(\sigma_*(C')) - 2) = (K_{S_0} + \sigma_*(C')) \cdot \sigma_*(C') \leq 2$, by which we conclude $g(C') \leq p_a(\sigma_*(C')) \leq 2$.

If $K_{S_0}^2 \geq 2$, we have two cases. When $\sigma_*(C')^2 \leq 1$, it is easy to derive $g(C') \leq 2$. When $\sigma_*(C')^2 \geq 2$, by the Hodge index theorem, we have $4 \leq K_{S_0}^2 \cdot \sigma_*(C')^2 \leq (K_{S_0} \cdot \sigma_*(C'))^2 \leq 4$. The equality holds, so $K_{S_0} \equiv \sigma(C')$ and $K_{S_0}^2 = 2$. Therefore by the Riemann–Roch formula, we can conclude that $g(C') \leq 3$. $\square$

By Proposition 3.1, Proposition 3.2 and Proposition 3.3, we can summarize that the main theorem is proved.

ACKNOWLEDGMENTS. The author would like to thank Professor Meng Chen for his guidance over this paper. The author would also like to thank Doctor Lei Zhu for her valuable suggestions and comments.

References

[1] W. Barth, K. Hulek, C. Peters and A. Van de Ven: Compact Complex Surfaces, second edition, Ergebnisse der Mathematik und ihrer Grenzgebiete **3**, Springer, Berlin, 2004.

[2] E. Bombieri: *Canonical models of surfaces of general type*, Inst. Hautes Études Sci. Publ. Math. **42** (1973), 171–219.

[3] M. Chen: *Canonical stability in terms of singularity index for algebraic threefolds*, Math. Proc. Cambridge Philos. Soc. **131** (2001), 241–264.

[4] M. Chen: *Canonical stability of* 3*-folds of general type with* $p_g \geq 3$, Internat. J. Math. **14** (2003), 515–528.

[5] J.A. Chen, M. Chen and D.-Q. Zhang: *The* 5*-canonical system on* 3*-folds of general type*, J. Reine Angew. Math. **603** (2007), 165–181.

[6] M. Chen and D.-Q. Zhang: *Characterization of the* 4*-canonical birationality of algebraic threefolds*, Math. Z. **258** (2008), 565–585.

[7] Y. Kawamata, K. Matsuda and K. Matsuki: *Introduction to the minimal model problem*; in Algebraic Geometry, Sendai, 1985, Adv. Stud. Pure Math. **10**, North-Holland, Amsterdam, 1987, 283–360.

[8] Y. Kawamata: *A generalization of Kodaira–Ramanujam's vanishing theorem*, Math. Ann. **261** (1982), 43–46.

[9] M. Kobayashi: *On Noether's inequality for threefolds*, J. Math. Soc. Japan **44** (1992), 145–156.

[10] I. Reider: *Vector bundles of rank* 2 *and linear systems on algebraic surfaces*, Ann. of Math. (2) **127** (1988), 309–316.

[11] E. Viehweg: *Vanishing theorems*, J. Reine Angew. Math. **335** (1982), 1–8.

[12] L. Zhu: *The generic finiteness of the m-canonical map for* 3*-folds of general type*, Osaka J. Math. **42** (2005), 873–884.

School of Mathematical Sciences
Fudan University
Shanghai 200433
P.R. China
e-mail: 062018004@fudan.edu.cn