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THE 3-CANONICAL SYSTEM ON 3-FOLDS OF GENERAL TYPE

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Abstract

Let X be a projective minimal Gorenstein 3-fold of general type with \mathbb{Q} -factorial terminal singularities. We classify minimal Gorenstein 3-folds of general type according to the birationality of 3-canonical system on X .

1. Introduction

Let X be a projective variety on \mathbb{C} , and K_X is the canonical divisor on X . One may define the m -canonical map ϕ_m corresponding to the complete linear system $|mK_X|$. To study the behavior of ϕ_m , especially the birationality of ϕ_m , has been one of the most important topics of birational geometry. When $\dim X \leq 2$, the behavior of ϕ_m has been thoroughly studied. However, when $\dim X \geq 3$, many problems still remain open.

Let X be a projective minimal Gorenstein 3-fold of general type with \mathbb{Q} -factorial terminal singularities. When $\dim X = 3$, we know that when m is big enough, ϕ_m must be birational, hence it is important to find the universal lower bound N for all 3-folds such that ϕ_m is birational for all $m \geq N$. The known best result is as follows: Chen, Chen and Zhang [5] proved that $m \geq 5$ is enough; Chen and Zhang [6] proved that if X cannot be fibred by a unique family of irreducible curves of geometric genus 2, then $m \geq 4$ is enough; Zhu [12] proved the generic finiteness of ϕ_3 . Also in [6], Chen and Zhang proposed an open problem(see [6, 6.4 (2)]): is it possible to characterize the birationality of ϕ_3 ? We have several examples to illustrate the importance of this problem.

EXAMPLE 1.1 ([6, Example 6.2]). Kobayashi (see [9, Proposition 3.2]) has constructed a family of canonically polarized smooth threefolds Y satisfying the equality

$$K_Y^3 = \frac{4}{3}p_g(Y) - \frac{10}{3}$$

where $p_g(Y) = 7, 10, 13, \dots$, Chen and Zhang proved that all examples above have non-birational 4-canonical maps, thus they have non-birational 3-canonical maps.

EXAMPLE 1.2. Let S be a surface of type $(K_S^2, p_g(S)) = (2, 3)$. Let C be a smooth curve of genus ≥ 2 . Take $X = S \times C$. Since ϕ_1 is generically finite, $\dim \phi_1(X) = 3$. When $g(C)$ is big enough, $p_g(X)$ is big enough, hence ϕ_4 is birational by [6, 4.2]. But ϕ_3 is obviously not birational.

This paper makes an effort to answer this open problem, the main theorem is as follows. Notice that the Example 1.1 corresponds to the $d = 2$ situation, and the Example 1.2 corresponds to the $d = 1$ situation.

Theorem 1.1. *Let X be a projective minimal Gorenstein 3-fold of general type with \mathbb{Q} -factorial terminal singularities. Denote $d = \dim \phi_1(X)$. If ϕ_3 on X is not birational, then X must be one of the following types:*

- (1) $p_g(X) \leq 7$;
- (2) $p_g(X) \geq 8$, while:
 - (1) $d = 3$, X contains a surface which has a family of curves of genus 2;
 - (2) $d = 2$, X contains a surface which has a family of curves of genus ≤ 3 ;
 - (3) $d = 1$, X contains a surface S such that either S has a family of curves of genus ≤ 3 or S satisfies $p_g(S) = 1$, $K_{S_0}^2 \leq 9$, where S_0 is the minimal model of S .

2. The key technical results

2.1. Brief review on curves.

FACT 2.1. Let C be a smooth curve of genus ≥ 2 . Assume D is a divisor on C with $\deg(D) \geq 3$. Then the rational map corresponding to $|K_C + D|$ gives a birational morphism onto its image.

2.2. Brief review of relevant results on surfaces. Let S be a smooth minimal surface of general type, according to [2], one has

FACT 2.2. For all $m \geq 5$, ϕ_m is a birational morphism onto its image.

FACT 2.3. For $m = 4$, ϕ_m is birational if and only if $(K_S^2, p_g(S)) \neq (1, 2)$.

FACT 2.4. For $m = 3$, ϕ_m is birational if and only if $(K_S^2, p_g(S)) \neq (1, 2)$ and $(2, 3)$.

FACT 2.5. For $m = 2$, if $K_S^2 \geq 10$, then ϕ_m is birational if and only if S cannot be fibred by a family of irreducible curves of geometric genus 2.

2.3. General method. Let X be a projective minimal Gorenstein 3-fold of general type with \mathbb{Q} -factorial terminal singularities. Under the assumption $p_g(X) \geq 2$, we study the canonical map ϕ_1 which is a rational map. Take the modification $\pi: X' \rightarrow X$, according to Hironaka, such that

- (i) X' is smooth.
- (ii) The movable part of $|K_{X'}|$ is basepoint free.
- (iii) $\pi^*(K_X)$ is linearly equivalent to a divisor with normal crossing support.

Denote by g the composition $\phi_1 \circ \pi$. So $g: X' \rightarrow W' \subseteq \mathbb{P}^{p_g(X)-1}$ is a morphism.

Let $X' \xrightarrow{f} B \xrightarrow{s} W'$ be the Stein factorization of g . We get a commutative diagram as below:

$$\begin{array}{ccc} X' & \xrightarrow{f} & B \\ \pi \downarrow & \searrow g & \downarrow s \\ X & \xrightarrow{\phi_1} & W'. \end{array}$$

We may write $K'_X \sim \pi^*(K_X) + E_\pi \sim M_1 + Z_1$, where M_1 is the movable part of $|K'_X|$, Z_1 is the fixed part, and E_π is a sum of distinct exceptional divisors. We may write $\pi^*(K_X) \sim M_1 + E'_1$, where $E'_1 = Z_1 - E_\pi$ is an effective divisor.

If $\dim \phi_1(X) = 2$, we see that a general fiber S of f is a smooth projective curve C of genus ≥ 2 . We say that X is canonically fibred by curves of genus $g = g(C)$.

If $\dim \phi_1(X) = 1$, we see that a general fiber S of f is a smooth projective surface of general type. We say that X is canonically fibred by surfaces with invariants $(c_1^2(S_0), p_g(S))$, where S_0 is the minimal model of S . We may write $M_1 \equiv a_1 S$, where $a_1 \geq p_g(X) - 1$.

A generic irreducible element S of $|M_1|$, means either a general member of $|M_1|$ when $\dim \phi_1(X) \geq 2$ or, otherwise, a general fiber of f .

REMARK 2.1. Throughout the paper $D_1 \sim D_2$ (resp. $=_{\mathbb{Q}}$, or $D_1 \equiv D_2$) means that divisors D_1 and D_2 are linearly equivalent (resp. rD_1 and rD_2 are linearly equivalent for some positive integer r , or D_1 and D_2 are numerically equivalent).

Lemma 2.1 (Chen [6, Theorem 3.6]). *Let X be a minimal projective 3-fold of general type with \mathbb{Q} -factorial terminal singularities and assume $p_g(X) \geq 2$. Keep the same notation as above. Pick a generic irreducible element S of $|M_1|$. Suppose, on the smooth surface S , there is a movable linear system $|G|$ and denote by C a generic irreducible element of $|G|$. Set $\xi := (\pi^*(K_X) \cdot C)_{X'}$, and*

$$p := \begin{cases} 1 & \text{If } \dim \phi_1(X) \geq 2, \\ a_1 & \text{otherwise.} \end{cases}$$

Assume there is a rational number $\beta \geq 0$, such that $\pi^(K_X)|_S - \beta C$ is numerically equivalent to an effective \mathbb{Q} -divisor. Denote $\alpha := (m - 1 - 1/p - 1/\beta)\xi$, set $\alpha_0 = \lceil \alpha \rceil$, then*

(1) Assume there is a positive integer m such that the linear system $|K_S + \lceil (m-2)\pi^*(K_X)|_S \rceil|$ separates different generic irreducible elements of $|G|$, then ϕ_m is birational if one of the following conditions is satisfied:

- i. $\alpha > 2$;
 - ii. $\alpha_0 \geq 2$ and C is non-hyperelliptic;
 - iii. $\alpha > 0$, C is non-hyperelliptic and C is an even divisor of S .
- (2) $\xi \geq (2g(C) - 2 + \alpha_0)/m$ if one of the following conditions is satisfied:
- iv. $\alpha > 1$;
 - v. $\alpha > 0$, and C is an even divisor.

3. Proof of the main theorem

We now study the birationality of ϕ_3 . Set $d = \dim \phi_1(X)$.

3.1. The case $d = 3$.

Proposition 3.1. *Let X be a projective minimal Gorenstein 3-fold of general type with \mathbb{Q} -factorial terminal singularities. Assume $p_g(X) \geq 8$, and $d = 3$. Then ϕ_3 is birational unless X contains a surface which has a family of curves of genus ≤ 2 .*

Proof. According to our general method, a generic irreducible element S of $|M_1|$ is a smooth projective surface of general type. It is sufficient to verify the birationality for $\phi_3|_S$ by virtue of the Matsuki–Tankeev principle [3, 2.1]. We consider the subsystem $|K_{X'} + \pi^*(K_X) + S| \subset |3K'_{X'}|$. By the Kawamata–Viehweg vanishing theorem [8], we have the surjective map

$$H^0(X', K_{X'} + \pi^*(K_X) + S) \rightarrow H^0(S, K_S + L)$$

where $L = \pi^*(K_X)|_S$ is a nef and big divisor on S . If $|L|$ gives a birational map, then so does $|K_S + L|$. Otherwise, $|L|$ gives a generically finite map of degree ≥ 2 . Noting that $h^0(S, L) \geq p_g(X) - 1 \geq 7$, we have $L^2 \geq 2(h^0(S, L) - 2) \geq 10$. If $|K_S + L|$ doesn't give a birational map, then according to Reider's result [10], there is a free pencil of curves on S with a generic irreducible element C , such that $L \cdot C = 1$ or 2 . On the other hand, $L \cdot C \geq 2$ since $|L|$ gives a generically finite map on C and C is a curve of genus ≥ 2 . Therefore we have $L \cdot C = 2$. Moreover, by the Riemann–Roch fomular and Clifford's theorem, we have $h^0(C, L|_C) = 2$. By

$$(3.1) \quad H^0(S, L - C) \rightarrow H^0(S, L) \rightarrow H^0(C, L|_C),$$

we have

$$\begin{aligned} h^0(S, L - C) + h^0(C, L|_C) &\geq h^0(S, L), \\ h^0(S, L) &\leq h^0(S, L - C) + 2. \end{aligned}$$

We can replace L in (3.1) by $L - nC$ and have n sequences relatively. By $C^2 = 0$ we have $(L - C)|_C = L|_C$. Since $h^0(S, L) \geq p_g(X) - 1$, we have $h^0(S, L) \leq h^0(S, L - nC) + 2n$ when $p_g(X) > 2n + 1$. So there exists an effective divisor E such that $L =_{\mathbb{Q}} nC + E$.

Set $F = C + (1/n)E$, so $L =_{\mathbb{Q}} nF$. Since $p_g(X) \geq 8$, we can choose $n = 3$, so $\deg F|_C = (1 - 1/n)L \cdot C > 1$.

By the Kawamata–Viehweg vanishing theorem, we have the following surjective map

$$H^0\left(K_S + \left\lceil L - \frac{1}{n}E \right\rceil\right) \rightarrow H^0(C, K_C + \lceil (n-1)F \rceil|_C).$$

Let M_3 be the movable part of $|K_{X'} + \pi^*(K_X) + S|$, N be the movable part of $|K_S + \lceil L - (1/n)E - C \rceil|$, then $M_3|_S \geq N$ by [3, Lemma 2.7].

So we have

$$3L \cdot C = 3\pi^*(K_X)|_S \cdot C \geq \deg(K_C + 2F|_C) \geq 2g(C) - 2 + \frac{8}{3} = 2g(C) + \frac{2}{3},$$

then we can derive that $g(C) \leq 2$.

Therefore, ϕ_3 is birational if X does not contain a surface which has a family of curves of genus 2. \square

3.2. The case $d = 2$.

Proposition 3.2. *Let X be a projective minimal Gorenstein 3-fold of general type with \mathbb{Q} -factorial terminal singularities. Assume $p_g(X) \geq 6$, and $d = 2$. Then ϕ_3 is birational unless X contains a surface which has a family of curves of genus ≤ 3 .*

Proof. Let S be a generic element of $|M_1|$. So $S|_S \equiv aC$, where $a \geq p_g(X) - 2$, and C is a general fiber of f and is a smooth curve. Denote $L = \pi^*(K_X)|_S > S|_S$. By the Kawamata–Viehweg vanishing theorem, we have the following surjective map

$$H^0(X', K_{X'} + \pi^*(K_X) + S) \rightarrow H^0(S, K_S + \lceil L \rceil).$$

We may write $L \equiv aC + E''$, so $L - C - (1/a)E''$ is nef and big. By the Kawamata–Viehweg vanishing theorem, we have the following surjective map

$$H^0\left(S, K_S + \left\lceil L - \frac{1}{a}E'' \right\rceil\right) \rightarrow H^0(C, K_C + D)$$

where $D := \lceil L - C - (1/a)E'' \rceil|_C$. If $\deg[D] = (1 - (1/a))L \cdot C \geq 3$, then $|K_C + \lceil L \rceil|_C - C|$ gives a birational map by Fact 2.1. By the Matsuki–Tankeev principle, we can derive that $|K_{X'} + \pi^*(K_X) + S|$ gives a birational map, and the birationality of ϕ_3 follows.

We now prove that $(1 - 1/a)L \cdot C > 2$.

Let $G = M_1|_S$, G is composed of a pencil and $G \equiv aC$. In Lemma 2, we may take $p = 1$, $\beta = 2$, and $\xi = L \cdot C$, $\alpha = (m - 2.5)\xi$. By [3], ϕ_4 is generically finite, this means that $|\pi^*(K_X)|_S$ maps a general C onto a curve. Thus $4\pi^*(K_X)|_S \cdot C \geq 2$, and $\xi \geq 1/2$.

Let $m = 7$, we have $(m - 4)\xi \geq 1$. By Lemma 2, $\xi \geq (2g(C) - 2 + \alpha_0)/m$, follows $\xi \geq (2g(C) - 2)/2.5$. When $g(C) \geq 4$, we have $\xi \geq 3$. Hence $(1 - 1/a)\xi > 2$, since $a \geq p_g(X) - 2 \geq 4$.

Therefore, if X does not contain a surface which has a family of curves of genus ≤ 3 , then ϕ_3 is birational. \square

3.3. The case $d = 1$.

Lemma 3.1 ([6, Lemma 3.7]). *Keep the same notation as in Subsection 2.3 and with p as in Lemma 2.1. Assume $d = 1$, and $g(B) = 0$. Let S be a general fiber of $g: X' \rightarrow W'$. Let $\sigma: S \rightarrow S_0$ be the contraction onto the minimal model. Then*

$$\pi^*(K_X)|_S - \frac{p}{p+1}\sigma^*(K_{S_0})$$

is pseudo-effective.

Proposition 3.3. *Let X be a projective minimal Gorenstein 3-fold of general type with \mathbb{Q} -factorial terminal singularities. Assume $p_g(X) \geq 6$, and $d = 1$. Then ϕ_3 is birational unless X contains a surface S which has a family of curves of genus ≤ 3 or which satisfies $p_g(S) = 1$, $K_{S_0}^2 \leq 9$, where S_0 is the minimal model of S .*

Proof. Let S be a general fiber of f . Then S is a smooth projective surface of general type. We have $M_1 \equiv aS$, with $a \geq p_g(X) - 1$. Denote $L = \pi^*(K_X)|_S$. Let $\sigma: S \rightarrow S_0$ be the contraction onto the minimal model. Denote $F' = \pi_*(S)$. Notice that $K_X \cdot F'^2$ is an even integer [5, 2.1] and $K_X \cdot F'^2 \geq 0$.

CASE 1. The case $K_X \cdot F'^2 > 0$.

Notice that $K_X \equiv aF' + E'$, where E' is an effective divisor. We have $L^2 = K_X^2 F' \geq aK_X F'^2 \geq 2(p_g(X) - 1) \geq 10$. By Reider's result, if $|K_X + L|$ doesn't give a birational map, there exists a free pencil of curves on S with a generic irreducible element C , such that $L \cdot C \leq 2$. By Lemma 3.1 and [5, Claim 3.3], we always have

$$\left(\pi^*(K_X)|_S - \frac{p}{p+1}\sigma^*(K_{S_0}) \right) \cdot C \geq 0,$$

then we can derive

$$\frac{p}{p+1}\sigma^*(K_{S_0}) \cdot C \leq 2,$$

so

$$\sigma^*(K_{S_0}) \cdot C \leq 2, \text{ since } p \geq p_g(X) - 1 \geq 5.$$

Let \bar{C} be the image of C under σ , then we have $K_{S_0} \cdot \bar{C} \leq 2$.

If $\bar{C}^2 = 0$, then $K_{S_0} \cdot \bar{C} = 2g(\bar{C}) - 2 \leq 2$, so $g(\bar{C}) \leq 2$, which means S has a family of curves of genus 2.

If $\bar{C}^2 > 0$, by the Hodge index theorem, we can derive that $K_{S_0}^2 \cdot \bar{C}^2 \leq (K_{S_0} \cdot \bar{C})^2 \leq 4$. But $K_{S_0}^2 \geq L^2 > 10$, which is a contradiction.

CASE 2. The case $K_X \cdot F'^2 = 0$.

One always has $\pi^*(K_X)|_S \sim \sigma^*(K_{S_0})$ by [5, Claim 3.3]. Notice that

$$\pi^*(K_X) - S - \frac{1}{a}E'_1 =_{\mathbb{Q}} \left(1 - \frac{1}{a}\right)\pi^*(K_X).$$

Applying the vanishing theorem, one has the surjective map:

$$H^0\left(X', K_{X'} + \left[2\pi^*(K_X) - \frac{1}{a}E'_1\right]\right) \rightarrow H^0\left(S, K_S + \left[\left(2 - \frac{1}{a}\right)\pi^*(K_X)\right]\right)_S.$$

Since

$$K_S + \left[\left(2 - \frac{1}{a}\right)\pi^*(K_X)\right]\bigg|_S \geq K_S + \sigma^*(K_{S_0}) + \left[\left(1 - \frac{1}{a}\right)\pi^*(K_X)\right]\bigg|_S,$$

by Fact 2.5, $|K_S + \sigma^*(K_{S_0}) + [(1 - 1/a)\pi^*(K_X)]|_S|$ doesn't give a birational map only if $K_{S_0}^2 \leq 9$ or S has a family of curves of genus 2.

When $p_g(S) \geq 2$, there exists a family of curves with its general member $C' \subset |L|$. By the Kawamata–Viehweg vanishing theorem, we have

$$\left|K_S + C' + \left[\left(1 - \frac{1}{a}\right)L\right]\right|_{C'} = |K_{C'} + D|,$$

where $\deg D \geq (1 - 1/a)L \cdot C'$. If $|K_{C'} + D|$ doesn't give a birational map, we have $(1 - 1/a)L \cdot C' \leq 2$, then $L \cdot C' = \sigma^*(K_{S_0}) \cdot C' = K_{S_0} \cdot \sigma_*(C') \leq 2$. By $p_g(S) \geq 2$, we have $K_{S_0}^2 \geq 2p_g(S_0) - 4 \geq 0$.

If $K_{S_0}^2 \leq 1$, we have $(\sigma_*(C'))^2 \leq K_{S_0}^2 \leq 1$. Therefore, by the Riemann–Roch formula, we have $2(p_a(\sigma_*(C')) - 2) = (K_{S_0} + \sigma_*(C')) \cdot \sigma_*(C') \leq 2$, by which we conclude $g(C') \leq p_a(\sigma_*(C')) \leq 2$.

If $K_{S_0}^2 \geq 2$, we have two cases. When $\sigma_*(C')^2 \leq 1$, it is easy to derive $g(C') \leq 2$. When $\sigma_*(C')^2 \geq 2$, by the Hodge index theorem, we have $4 \leq K_{S_0}^2 \cdot \sigma_*(C')^2 \leq (K_{S_0} \cdot \sigma_*(C'))^2 \leq 4$. The equality holds, so $K_{S_0} \equiv \sigma(C')$ and $K_{S_0}^2 = 2$. Therefore by the Riemann–Roch formula, we can conclude that $g(C') \leq 3$. \square

By Proposition 3.1, Proposition 3.2 and Proposition 3.3, we can summarize that the main theorem is proved.

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