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Osaka J. Math.
**48** (2011), 91–98

# THE 3-CANONICAL SYSTEM ON 3-FOLDS OF GENERAL TYPE

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(Received January 13, 2009, revised September 16, 2009)

**Abstract**

Let $X$ be a projective minimal Gorenstein 3-fold of general type with $\mathbb{Q}$-factorial terminal singularities. We classify minimal Gorenstein 3-folds of general type according to the birationality of 3-canonical system on $X$.

## 1. Introduction

Let $X$ be a projective variety on $\mathbb{C}$, and $K_X$ is the canonical divisor on $X$. One may define the $m$-canonical map $\phi_m$ corresponding to the complete linear system $|mK_X|$. To study the behavior of $\phi_m$, especially the birationality of $\phi_m$, has been one of the most important topics of birational geometry. When $\dim X \leq 2$, the behavior of $\phi_m$ has been thoroughly studied. However, when $\dim X \geq 3$, many problems still remain open.

Let $X$ be a projective minimal Gorenstein 3-fold of general type with $\mathbb{Q}$-factorial terminal singularities. When $\dim X = 3$, we know that when $m$ is big enough, $\phi_m$ must be birational, hence it is important to find the universal lower bound $N$ for all 3-folds such that $\phi_m$ is birational for all $m \geq N$. The known best result is as follows: Chen, Chen and Zhang [5] proved that $m \geq 5$ is enough; Chen and Zhang [6] proved that if $X$ cannot be fibred by a unique family of irreducible curves of geometric genus 2, then $m \geq 4$ is enough; Zhu [12] proved the generic finiteness of $\phi_3$. Also in [6], Chen and Zhang proposed an open problem(see [6, 6.4 (2)]): is it possible to characterize the birationality of $\phi_3$? We have several examples to illustrate the importance of this problem.

EXAMPLE 1.1 ([6, Example 6.2]). Kobayashi (see [9, Proposition 3.2]) has constructed a family of canonically polarized smooth threefolds $Y$ satisfying the equality

$$K_Y^3 = \frac{4}{3}p_g(Y) - \frac{10}{3}$$

where $p_g(Y) = 7, 10, 13, \dots$, Chen and Zhang proved that all examplxes above have non-birational 4-canonical maps, thus they have non-birational 3-canonical maps.

2000 Mathematics Subject Classification. 14E05, 14J10, 14J30.

EXAMPLE 1.2. Let $S$ be a surface of type $(K_S^2, p_g(S)) = (2,3)$. Let $C$ be a smooth curve of genus $\geq 2$. Take $X = S \times C$. Since $\phi_1$ is generically finite, $\dim \phi_1(X) = 3$. When $g(C)$ is big enough, $p_g(X)$ is big enough, hence $\phi_4$ is birational by [6, 4.2]. But $\phi_3$ is obviously not birational.

This paper makes an effort to answer this open problem, the main theorem is as follows. Notice that the Example 1.1 corresponds to the $d = 2$ situation, and the Example 1.2 corresponds to the $d = 1$ situation.

**Theorem 1.1.** *Let $X$ be a projective minimal Gorenstein 3-fold of general type with $\mathbb{Q}$-factorial terminal singularities. Denote $d = \dim \phi_1(X)$. If $\phi_3$ on $X$ is not birational, then $X$ must be one of the following types*:
(1) $p_g(X) \leq 7$;
(2) $p_g(X) \geq 8$, *while*:
  (1) $d = 3$, *$X$ contains a surface which has a family of curves of genus* 2;
  (2) $d = 2$, *$X$ contains a surface which has a family of curves of genus* $\leq 3$;
  (3) $d = 1$, *$X$ contains a surface $S$ such that either $S$ has a family of curves of genus* $\leq 3$ *or $S$ satisfies* $p_g(S) = 1$, $K_{S_0}^2 \leq 9$, *where $S_0$ is the minimal model of $S$.*

## 2. The key technical results

### 2.1. Brief review on curves.

FACT 2.1. Let $C$ be a smooth curve of genus $\geq 2$. Assume $D$ is a divisor on $C$ with $\deg(D) \geq 3$. Then the rational map corresponding to $|K_C + D|$ gives a birational morphism onto its image.

**2.2. Brief review of relevant results on surfaces.** Let $S$ be a smooth minimal surface of general type, according to [2], one has

FACT 2.2. For all $m \geq 5$, $\phi_m$ is a biratinal morphism onto its image.

FACT 2.3. For $m = 4$, $\phi_m$ is birational if and only if $(K_S^2,\ p_g(S)) \neq (1,\ 2)$.

FACT 2.4. For $m = 3$, $\phi_m$ is birational if and only if $(K_S^2,\ p_g(S)) \neq (1,2)$ and $(2,3)$.

FACT 2.5. For $m = 2$, if $K_S^2 \geq 10$, then $\phi_m$ is birational if and only if $S$ cannot be fibred by a family of irreducible curves of geometric genus 2.

**2.3. General method.** Let $X$ be a projective minimal Gorenstein 3-fold of general type with $\mathbb{Q}$-factorial terminal singularities. Under the assumption $p_g(X) \geq 2$, we study the canonical map $\phi_1$ which is a rational map. Take the modification $\pi\colon X' \to X$, according to Hironaka, such that

(i) $X'$ is smooth.
(ii) The movable part of $|K_{X'}|$ is basepoint free.
(iii) $\pi^*(K_X)$ is linearly equivalent to a divisor with normal crossing support.

Denote by $g$ the composition $\phi_1 \circ \pi$. So $g\colon X' \to W' \subseteq \mathbb{P}^{p_g(X)-1}$ is a morphism. Let $X' \xrightarrow{f} B \xrightarrow{s} W'$ be the Stein factorization of $g$. We get a commutative diagram as below:

$$\begin{array}{ccc} X' & \xrightarrow{\ f\ } & B \\ {\scriptstyle \pi}\downarrow & \searrow^{g} & \downarrow{\scriptstyle s} \\ X & \xrightarrow[\ \phi_1\ ]{} & W'. \end{array}$$

We may write $K'_X \sim \pi^*(K_X) + E_\pi \sim M_1 + Z_1$, where $M_1$ is the movable part of $|K'_X|$, $Z_1$ is the fixed part, and $E_\pi$ is a sum of distinct exceptional divisors. We may write $\pi^*(K_X) \sim M_1 + E'_1$, where $E'_1 = Z_1 - E_\pi$ is an effective divisor.

If $\dim \phi_1(X) = 2$, we see that a general fiber $S$ of $f$ is a smooth projective curve $C$ of genus $\geq 2$. We say that $X$ is canonically fibred by curves of genus $g = g(C)$.

If $\dim \phi_1(X) = 1$, we see that a general fiber $S$ of $f$ is a smooth projective surface of general type. We say that $X$ is canonically fibred by surfaces with invariants $(c_1^2(S_0), p_g(S))$, where $S_0$ is the minimal model of $S$. We may write $M_1 \equiv a_1 S$, where $a_1 \geq p_g(X) - 1$.

A generic irreducible element $S$ of $|M_1|$, means either a general member of $|M_1|$ when $\dim \phi_1(X) \geq 2$ or, otherwise, a general fiber of $f$.

REMARK 2.1. Throughout the paper $D_1 \sim D_2$ (resp. $=_{\mathbb{Q}}$, or $D_1 \equiv D_2$) means that divisors $D_1$ and $D_2$ are linearly equivalent (resp. $rD_1$ and $rD_2$ are linearly equivalent for some positive integer $r$, or $D_1$ and $D_2$ are numerically equivalent).

**Lemma 2.1** (Chen [6, Theorem 3.6]). *Let $X$ be a minimal projective 3-fold of general type with $\mathbb{Q}$-factorial terminal singularities and assume $p_g(X) \geq 2$. Keep the same notation as above. Pick a generic irreducible element $S$ of $|M_1|$. Suppose, on the smooth surface $S$, there is a movable linear system $|G|$ and denote by $C$ a generic irreducible element of $|G|$. Set $\xi := (\pi^*(K_X) \cdot C)_{X'}$, and*

$$p := \begin{cases} 1 & \textit{If} \quad \dim \phi_1(X) \geq 2, \\ a_1 & \textit{otherwise.} \end{cases}$$

*Assume there is a rational number $\beta \geq 0$, such that $\pi^*(K_X)|_S - \beta C$ is numerically equivalent to an effective $\mathbb{Q}$-divisor. Denote $\alpha := (m - 1 - 1/p - 1/\beta)\xi$, set $\alpha_0 = \lceil \alpha \rceil$, then*

(1) *Assume there is a positive integer* $m$ *such that the linear system* $|K_S + \lceil (m-2)\pi^*(K_X)|_S \rceil|$ *seperates different generic irreducible elements of* $|G|$, *then* $\phi_m$ *is birational if one of the following conditions is satisfied*:
   i. $\alpha > 2$;
   ii. $\alpha_0 \geq 2$ *and* $C$ *is non-hyperelliptic*;
   iii. $\alpha > 0$, $C$ *is non-hyperelliptic and* $C$ *is an even divisor of* $S$.
(2) $\xi \geq (2g(C) - 2 + \alpha_0)/m$ *if one of the following conditions is satisfied*:
   iv. $\alpha > 1$;
   v. $\alpha > 0$, *and* $C$ *is an even divisor.*

## 3. Proof of the main theorem

We now study the birationality of $\phi_3$. Set $d = \dim \phi_1(X)$.

### 3.1. The case $d = 3$.

**Proposition 3.1.** *Let* $X$ *be a projective minimal Gorenstein* 3*-fold of general type with* $\mathbb{Q}$*-factorial terminal singularities. Assume* $p_g(X) \geq 8$, *and* $d = 3$. *Then* $\phi_3$ *is birational unless* $X$ *contains a surface which has a family of curves of genus* $\leq 2$.

Proof. According to our general method, a generic irreducible element $S$ of $|M_1|$ is a smooth projective surface of general type. It is sufficient to verify the birationality for $\phi_3|_S$ by virtue of the Matsuki–Tankeev principle [3, 2.1]. We consider the sub-system $|K_{X'} + \pi^*(K_X) + S| \subset |3K'_X|$. By the Kawamata–Viehweg vanishing theorem [8], we have the surjective map

$$H^0(X', K_{X'} + \pi^*(K_X) + S) \to H^0(S, K_S + L)$$

where $L = \pi^*(K_X)|_S$ is a nef and big divisor on $S$. If $|L|$ gives a birational map, then so does $|K_S + L|$. Otherwise, $|L|$ gives a generically finite map of degree $\geq 2$. Noting that $h^0(S, L) \geq p_g(X) - 1 \geq 7$, we have $L^2 \geq 2(h^0(S, L) - 2) \geq 10$. If $|K_S + L|$ doesn't give a birational map, then according to Reider's result [10], there is a free pencil of curves on $S$ with a generic irreducible element $C$, such that $L \cdot C = 1$ or 2. On the other hand, $L \cdot C \geq 2$ since $|L|$ gives a generically finite map on $C$ and $C$ is a curve of genus $\geq 2$. Therefore we have $L \cdot C = 2$. Moreover, by the Riemann–Roch fomular and Clifford's theorem, we have $h^0(C, L|_C) = 2$. By

$$H^0(S, L - C) \to H^0(S, L) \to H^0(C, L|_C), \tag{3.1}$$

we have

$$h^0(S, L - C) + h^0(C, L|_C) \geq h^0(S, L),$$
$$h^0(S, L) \leq h^0(S, L - C) + 2.$$

We can replace $L$ in (3.1) by $L-nC$ and have $n$ sequences relatively. By $C^2=0$ we have $(L-C)|_C = L|_C$. Since $h^0(S,L)\geq p_g(X)-1$, we have $h^0(S,L)\leq h^0(S,L-nC)+2n$ when $p_g(X)>2n+1$. So there exists an effective divisor $E$ such that $L=_{\mathbb{Q}} nC+E$.

Set $F=C+(1/n)E$, so $L=_{\mathbb{Q}} nF$. Since $p_g(X)\geq 8$, we can choose $n=3$, so $\deg F|_C = (1-1/n)L\cdot C > 1$.

By the Kawamata–Viehweg vanishing theorem, we have the following surjective map

$$H^0\left(K_S+\left\lceil L-\frac{1}{n}E\right\rceil\right)\to H^0(C, K_C+\lceil (n-1)F\rceil|_C).$$

Let $M_3$ be the movable part of $|K_{X'}+\pi^*(K_X)+S|$, $N$ be the movable part of $|K_S+\lceil L-(1/n)E-C\rceil|$, then $M_3|_S\geq N$ by [3, Lemma 2.7].

So we have

$$3L\cdot C = 3\pi^*(K_X)|_S\cdot C\geq \deg(K_C+2F|_C)\geq 2g(C)-2+\frac{8}{3}=2g(C)+\frac{2}{3},$$

then we can derive that $g(C)\leq 2$.

Therefore, $\phi_3$ is birational if $X$ does not contain a surface which has a family of curves of genus 2. □

### 3.2. The case $d=2$.

**Proposition 3.2.** *Let $X$ be a projective minimal Gorenstein 3-fold of general type with $\mathbb{Q}$-factorial terminal singularities. Assume $p_g(X)\geq 6$, and $d=2$. Then $\phi_3$ is birational unless $X$ contains a surface which has a family of curves of genus $\leq 3$.*

Proof. Let $S$ be a generic element of $|M_1|$. So $S|_S\equiv aC$, where $a\geq p_g(X)-2$, and $C$ is a general fiber of $f$ and is a smooth curve. Denote $L=\pi^*(K_X)|_S > S|_S$. By the Kawamata–Viehweg vanishing theorem, we have the following surjective map

$$H^0(X', K_{X'}+\pi^*(K_X)+S)\to H^0(S, K_S+\lceil L\rceil).$$

We may write $L\equiv aC+E''$, so $L-C-(1/a)E''$ is nef and big. By the Kawamata–Viehweg vanishing theorem, we have the following surjective map

$$H^0\left(S, K_S+\left\lceil L-\frac{1}{a}E''\right\rceil\right)\to H^0(C, K_C+D)$$

where $D:=\lceil L-C-(1/a)E''\rceil|_C$. If $\deg\lfloor D\rfloor = (1-(1/a))L\cdot C\geq 3$, then $|K_C+\lceil L\rceil|_C - C|$ gives a birational map by Fact 2.1. By the Matsuki–Tankeev principle, we can derive that $|K_{X'}+\pi^*(K_X)+S|$ gives a birational map, and the birationality of $\phi_3$ follows.

We now prove that $(1-1/a)L\cdot C>2$.

Let $G=M_1|_S$, $G$ is composed of a pencil and $G\equiv aC$. In Lemma 2, we may take $p=1$, $\beta=2$, and $\xi=L\cdot C$, $\alpha=(m-2.5)\xi$. By [3], $\phi_4$ is generically finite, this means hat $|\lfloor 4\pi^*(K_X)\rfloor|$ maps a general $C$ onto a curve. Thus $4\pi^*(K_X)|_S\cdot C\geq 2$, and $\xi\geq 1/2$.

Let $m=7$, we have $(m-4)\xi\geq 1$. By Lemma 2, $\xi\geq(2g(C)-2+\alpha_0)/m$, follows $\xi\geq(2g(C)-2)/2.5$. When $g(C)\geq 4$, we have $\xi\geq 3$. Hence $(1-1/a)\xi>2$, since $a\geq p_g(X)-2\geq 4$.

Therefore, if $X$ does not contain a surface which has a family of curves of genus $\leq 3$, then $\phi_3$ is birational. □

### 3.3. The case $d=1$.

**Lemma 3.1** ([6, Lemma 3.7]). *Keep the same notation as in* Subsection 2.3 *and with $p$ as in* Lemma 2.1. *Assume $d=1$, and $g(B)=0$. Let $S$ be a general fiber of $g\colon X'\to W'$. Let $\sigma: S\to S_0$ be the contraction onto the minimal model. Then*

$$\pi^*(K_X)|_S-\frac{p}{p+1}\sigma^*(K_{S_0})$$

*is pseudo-effective.*

**Proposition 3.3.** *Let $X$ be a projective minimal Gorenstein* 3*-fold of general type with $\mathbb{Q}$-factorial terminal singularities. Assume $p_g(X)\geq 6$, and $d=1$. Then $\phi_3$ is birational unless $X$ contains a surface $S$ which has a family of curves of genus $\leq 3$ or which satisfies $p_g(S)=1$, $K_{S_0}^2\leq 9$, where $S_0$ is the minimal model of $S$.*

*Proof.* Let $S$ be a general fiber of $f$. Then $S$ is a smooth projective surface of general type. We have $M_1\equiv aS$, with $a\geq p_g(X)-1$. Denote $L=\pi^*(K_X)|_S$. Let $\sigma: S\to S_0$ be the contraction onto the minimal model. Denote $F'=\pi_*(S)$. Notice that $K_X\cdot F'^2$ is an even integer [5, 2.1] and $K_X\cdot F'^2\geq 0$.

CASE 1. The case $K_X\cdot F'^2>0$.

Notice that $K_X\equiv aF'+E'$, where $E'$ is an effective divisor. We have $L^2=K_X^2F'\geq aK_XF'^2\geq 2(p_g(X)-1)\geq 10$. By Reider's result, if $|K_X+L|$ doesn't give a birational map, there exists a free pencil of curves on $S$ with a generic irreducible element $C$, such that $L\cdot C\leq 2$. By Lemma 3.1 and [5, Claim 3.3], we always have

$$\left(\pi^*(K_X)|_S-\frac{p}{p+1}\sigma^*(K_{S_0})\right)\cdot C\geq 0,$$

then we can derive

$$\frac{p}{p+1}\sigma^*(K_{S_0})\cdot C\leq 2,$$

so

$$\sigma^*(K_{S_0})\cdot C\leq 2,\ \text{since}\, p\geq p_g(X)-1\geq 5.$$

Let $\bar{C}$ be the image of $C$ under $\sigma$, then we have $K_{S_0} \cdot \bar{C} \leq 2$.

If $\bar{C}^2 = 0$, then $K_{S_0} \cdot \bar{C} = 2g(\bar{C}) - 2 \leq 2$, so $g(\bar{C}) \leq 2$, which means $S$ has a family of curves of genus 2.

If $\bar{C}^2 > 0$, by the Hodge index theorem, we can derive that $K_{S_0}^2 \cdot \bar{C}^2 \leq (K_{S_0} \cdot \bar{C})^2 \leq 4$. But $K_{S_0}^2 \geq L^2 > 10$, which is a contradiction.

CASE 2. The case $K_X \cdot F'^2 = 0$.

One always has $\pi^*(K_X)|_S \sim \sigma^*(K_{S_0})$ by [5, Claim 3.3]. Notice that

$$\pi^*(K_X) - S - \frac{1}{a}E_1' =_{\mathbb{Q}} \left(1 - \frac{1}{a}\right)\pi^*(K_X).$$

Applying the vanishing theorem, one has the surjective map:

$$H^0\left(X', K_{X'} + \left\lceil 2\pi^*(K_X) - \frac{1}{a}E_1' \right\rceil\right) \to H^0\left(S, K_S + \left\lceil \left(2 - \frac{1}{a}\right)\pi^*(K_X) \right\rceil\Big|_S\right).$$

Since

$$K_S + \left\lceil \left(2 - \frac{1}{a}\right)\pi^*(K_X) \right\rceil\Big|_S \geq K_S + \sigma^*(K_{S_0}) + \left\lceil \left(1 - \frac{1}{a}\right)\pi^*(K_X) \right\rceil\Big|_S,$$

by Fact 2.5, $|K_S + \sigma^*(K_{S_0}) + \lceil (1 - 1/a)\pi^*(K_X) \rceil|_S|$ doesn't give a birational map only if $K_{S_0}^2 \leq 9$ or $S$ has a family of curves of genus 2.

When $p_g(S) \geq 2$, there exists a family of curves with its general member $C' \subset |L|$. By the Kawamata–Viehweg vanishing theorem, we have

$$\left| K_S + C' + \left\lceil \left(1 - \frac{1}{a}\right)L \right\rceil \right|\Big|_{C'} = |K_{C'} + D|,$$

where $\deg D \geq (1 - 1/a)L \cdot C'$. If $|K_{C'} + D|$ doesn't give a birational map, we have $(1 - 1/a)L \cdot C' \leq 2$, then $L \cdot C' = \sigma^*(K_{S_0}) \cdot C' = K_{S_0} \cdot \sigma_*(C') \leq 2$. By $p_g(S) \geq 2$, we have $K_{S_0}^2 \geq 2p_g(S_0) - 4 \geq 0$.

If $K_{S_0}^2 \leq 1$, we have $(\sigma_*(C'))^2 \leq K_{S_0}^2 \leq 1$. Therefore, by the Riemann–Roch fomular, we have $2(p_a(\sigma_*(C')) - 2) = (K_{S_0} + \sigma_*(C')) \cdot \sigma_*(C') \leq 2$, by which we conclude $g(C') \leq p_a(\sigma_*(C')) \leq 2$.

If $K_{S_0}^2 \geq 2$, we have two cases. When $\sigma_*(C')^2 \leq 1$, it is easy to derive $g(C') \leq 2$. When $\sigma_*(C')^2 \geq 2$, by the Hodge index theorem, we have $4 \leq K_{S_0}^2 \cdot \sigma_*(C')^2 \leq (K_{S_0} \cdot \sigma_*(C'))^2 \leq 4$. The equality holds, so $K_{S_0} \equiv \sigma(C')$ and $K_{S_0}^2 = 2$. Therefore by the Riemann–Roch formula, we can conclude that $g(C') \leq 3$. □

By Proposition 3.1, Proposition 3.2 and Proposition 3.3, we can summarize that the main theorem is proved.

ACKNOWLEDGMENTS. The author would like to thank Professor Meng Chen for his guidance over this paper. The author would also like to thank Doctor Lei Zhu for her valuable suggestions and comments.

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