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<td><strong>Author(s)</strong></td>
<td>Yamada, Yuichi</td>
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<tr>
<td><strong>Citation</strong></td>
<td>Osaka Journal of Mathematics. 36(3) P.673-P.683</td>
</tr>
<tr>
<td><strong>Issue Date</strong></td>
<td>1999</td>
</tr>
<tr>
<td><strong>Text Version</strong></td>
<td>publisher</td>
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<tr>
<td><strong>URL</strong></td>
<td><a href="https://doi.org/10.18910/9373">https://doi.org/10.18910/9373</a></td>
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<td><strong>DOI</strong></td>
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ON THE GEOMETRIC INTERSECTION NUMBER OF AN IMMERSED MANIFOLD AND A PLANE

YUICHI YAMADA

(Received September 8, 1997)

1. Introduction and Main Theorem

Throughout this article, we work in the $C^\infty$ category. Let $\mathbb{R}^N$ be the $N$-dimensional Euclidean space with the standard orientation and $f : M^{m_1} \to \mathbb{R}^N$ an immersion of an $m_1$-dimensional closed connected (possibly non-orientable) manifold, where $1 \leq m_1 < N$. We let $m_2$ denote the codimension of $f$; $m_2 = N - m_1$. Let $P(N, m_2)$ be the set of oriented $m_2$-planes (lines if $m_2 = 1$) in $\mathbb{R}^N$. We will give a formula on the geometric intersection number $\#f^{-1}(f(M) \cap x)$ of the immersed manifold and an $m_2$-plane $x$ in $\mathbb{R}^N$ in terms of algebraic intersection theory and "chamber-wall" structure on $P(N, m_2)$.

To the author's knowledge, the word "chamber" is used to indicate regions separated from a whole space by a codimension 1 subspace, for example, as "Weyl chamber" in Lie algebra theory (for more recent usage, see [2]). The word "wall" is used for the separating subspace. Following the history, in this paper, we use the two words in the same meanings as above.

The set $P(N, m_2)$ admits a structure of an $(m_2 + 1)m_1$-dimensional $C^\infty$ oriented manifold. In fact, it is homeomorphic to the total space of the orthogonal complement vector bundle to the tautological bundle over the corresponding real oriented Grassmannian manifold $G(N, m_2)$.

In the space $P(N, m_2)$, a subset consisting of $m_2$-planes tangent to the immersed manifold forms "walls" which decompose the space into some "chambers". By giving orientation to the walls, one may associate an number to each chamber by algebraic intersection theory. We will introduce a canonical orientation to the walls and will show that the number associated to a chamber in this way is equal to the geometric intersection number $\#f^{-1}(f(M) \cap x)$ of the immersed manifold and an $m_2$-plane $x$ belonging to the chamber.

In the next section, from the immersion $f$, we will construct an $((m_2 + 1)m_1 - 1)$-
dimensional closed manifold $E(L_f^*(p_1))$ and a map $W_f: E(L_f^*(p_1)) \rightarrow P(N, m_2)$. This map $W_f$ will be constructed such that the image $\text{Im} W_f$ consists of all $m_2$-planes which have at least one non-transverse intersection point with the immersed manifold $f(M)$. Thus the complement $P(N, m_2) \setminus \text{Im} W_f$ consists of $m_2$-planes which have only transverse intersection points with $f(M)$. Since the codimension of $W_f$ is 1, the wall $\text{Im} W_f$ decomposes $P(N, m_2)$ into some chambers.

In the process of the construction of $E(L_f^*(p_1))$, it is canonically oriented. By the orientation, we can regard the map $W_f$ as an $((m_2 + 1)m_1 - 1)$-cycle in the space $P(N, m_2)$.

For a point $x \in P(N, m_2) \setminus \text{Im} W_f$, which is an $m_2$-plane in $\mathbb{R}^N$, we will associate an (non-negative) integer to a chamber of $P(N, m_2) \setminus \text{Im} W_f$ containing the point $x$ by algebraic intersection number theory. (To the author's knowledge, such an association was first developed in [1 p.277].) For this, in Section 3, we will construct a cohomology class $\Omega_x$ in $H^{(m_2+1)m_1 - 1}(P(N, m_2) \setminus \{x\}; \mathbb{Z})$, which measures a linking number of a given $((m_2 + 1)m_1 - 1)$-cycle around the point $x$. In fact, the number $\Omega_x(W_f)$ is essentially equal to the algebraic intersection number of $W_f$ and an oriented arc $a_{x_0x}$ from $x_0$ to $x$, intersecting with $\text{Im} W_f$ transversely, where $x_0 \in P(N, m_2)$ is a base point taken sufficiently near to an end of the space $P(N, m_2)$:

$$\Omega_x(W_f) = \text{Int}(W_f, a_{x_0x}).$$

The cohomology class $\Omega_x$ is the Poincare dual of the semi-open arc $\lim_{x_0 \to \text{end}} a_{x_0x}$. This association depends on neither the arc $a_{x_0x}$ nor the point $x_0$ even if $m_2 = N - 1$, in which case $P(N, N - 1) \cong S^{N-1} \times \mathbb{R}$ has two ends.

**Theorem 1.1.** Let $f: M \rightarrow \mathbb{R}^N$ be an immersion and $W_f$ the map constructed from the immersion $f$ (For the definition of $W_f$, see Section 2). We regard $W_f$ as an $((m_2 + 1)m_1 - 1)$-cycle in $P(N, m_2)$. Let $x$ be a point in the complement of $W_f$ in $P(N, m_2)$. Then

(A) \hspace{1cm} \#f^{-1}\left(\{f(M) \cap x\}\right) \text{ is finite,}

and

(B) \hspace{1cm} \#f^{-1}\left(\{f(M) \cap x\}\right) = \Omega_x(W_f).

We call the number $\#f^{-1}\left(\{f(M) \cap x\}\right)$ the geometric intersection number of the immersed manifold $f(M)$ and the $m_2$-plane $x$.

In Section 4, we will develop a method of counting geometric intersection numbers in terms of cohomology. The proof of Theorem 1.1 will be given in Section 5 and 6.
Throughout this paper, we assume that the total space of a fiber bundle is oriented so that the local orientation of the base space followed by that of a fiber gives the orientation of the total space.

2. A Decomposition of $P(N, m_2)$

In this section, from a given immersion $f: M \to \mathbb{R}^N$, we construct an $((m_2 + 1)m_1 - 1)$-dimensional closed oriented manifold $E(L_f^*(p_1))$ and a map $W_f: E(L_f^*(p_1)) \to P(N, m_2)$, which is also regarded as a cycle.

First we introduce a canonical double fibration. For $0 < n_1 \leq n_2 < N$, we let $P(N; n_1, n_2)$ denote the set defined as follows:

$$P(N; n_1, n_2) = \{(x, X) \in P(N, n_1) \times P(N, n_2) \mid x \subset X\}.$$ 

This set admits a structure of an orientable $C^\infty$ manifold whose dimension is $(n_1 + 1)(N - n_1) + (n_2 - n_1)(N - n_2)$. It is homeomorphic to the total space of a certain vector bundle over a corresponding real flag manifold.

Each of the first projection $p_1$ and the second projection $p_2$ is a fibration:

$$p_1: P(N; n_1, n_2) \to P(N, n_1) \quad \text{and} \quad p_2: P(N; n_1, n_2) \to P(N, n_2)$$

$$(x, X) \mapsto x, \quad (x, X) \mapsto X.$$ 

The fiber of $p_1$ is homeomorphic to the oriented Grassmannian manifold $G(N - n_1, n_2 - n_1)$, on the other hand, the fiber of $p_2$ is homeomorphic to the space $P(n_2, n_1)$.

**Remark 2.1.** In the case $n_1 = n_2$, the space $P(N; n_1, n_2)$ is a disconnected double covering of $P(N; n_1)$. The one component is the diagonal $\Delta$ and the other is the anti-diagonal $\tilde{\Delta} = \{(x, -x)\}$. The latter space $\tilde{\Delta}$ has an opposite orientation from the one induced from the former $\Delta$ by a homeomorphism $(p_1|\tilde{\Delta})^{-1} \circ p_1|\Delta$.

**Remark 2.2.** We define an orientation of the space $P(N; n_1, n_2)$ as a manifold by the bundle structure of $p_1$.

Next, from the immersion $f: M \to \mathbb{R}^N$, we define a map $L_f: S(TM) \to P(N, 1)$. Here $S(TM)$ is defined as follows:

$$S(TM) = (TM \setminus \{\text{the zero section}\}) / \sim,$$

where $v_1 \sim v_2 \iff v_1 = \lambda v_2$ for some $\lambda > 0$.

We define an orientation of $S(TM)$ by a local orientation of $U$ (in $M$) followed by that of the fiber over $U$. Thus $S(TM)$ is an oriented manifold even if $M$ is non-orientable, because if we change the local orientation of $M$, the orientation of the fiber over the
local base is also changed. The dimension of the manifold $S(TM)$ is $(2m_1 - 1)$. Now we define $L_f$ as:

$$L_f: \quad S(TM) \to P(N,1),$$

$$v (\in S(T_pM)) \mapsto \text{A straight line whose vector is } df(v)$$

and which passes through the point $f(p)$.

**Remark 2.3.** $P(N,1) \cong TS^{N-1}$.

**Definition 2.4.** Using the map $L_f$ and the double fibration $p_1, p_2$ of $P(N;1,m_2)$, we construct a manifold $E(L_f^*(p_1))$ and a map $W_f$ by Diagram 1. $W_f$ is the composite $p_2 \circ L_f$.

We define $E(L_f^*(p_1))$ as the total space of the pull-back of the fibration $p_1$ over $S(TM)$ by $L_f$. It is a manifold, whose dimension is $2m_1 - 1 + (m_2 - 1)(N - m_2) = (m_2 + 1)m_1 - 1$:

$$E(L_f^*(p_1)) = \{(v, (x, X)) \in S(TM) \times P(N;1,m_2) | L_f(v) = x \}.$$

The image $\text{Im}W_f$ consists of all $m_2$-planes which have at least one non-transverse intersection point with the immersed manifold $f(M)$.

**Lemma 2.6.** The manifold $E(L_f^*(p_1))$ is closed and oriented.

Proof. Note that $E(L_f^*(p_1))$ is a total space of a fiber bundle over $S(TM)$ whose fiber is $G(N - 1, m_2 - 1)$. Both the basespace and the fiber are closed and oriented manifolds. We have the lemma.

By the orientation of $E(L_f^*(p_1))$ induced from the bundle structure, we can regard the map $W_f$ as an $((m_2 + 1)m_1 - 1)$-cycle in the space $P(N, m_2)$.

Since the codimension of the map $W_f$ is 1, the image of $W_f$ decompose the space $P(N, m_2)$ into some chambers.

3. **Algebraic Intersection Theory in $P(N,m_2)$**
In this section, for a point \( x \in P(N, m_2) \), we construct a cohomology class \( \Omega_x \) in \( H^{(m_2 + 1)m_1 - 1}(P(N, m_2) \setminus \{x\}; \mathbb{Z}) \), which essentially measures a linking number of an \(((m_2 + 1)m_1 - 1)\)-cycle around the point \( x \). We also construct a map from the set of chambers \( P(N, m_2) \setminus \text{Im} W_f \) to the set of non-negative integers by using \( \Omega_x \).

First, we consider the exact sequence in Diagram 2.

\[
\begin{array}{c}
H^{\text{Top}}(P(N, m_2); \mathbb{Z}) \\
\downarrow \\
H^{\text{Top}}(P(N, m_2) \setminus \{x\}; \mathbb{Z}) \supset \mathbb{Z}[\Omega_x] \\
\downarrow \delta \\
H^{\text{Top}}(P(N, m_2), P(N, m_2) \setminus \{x\}; \mathbb{Z}) = \mathbb{Z}[o_x] \\
\downarrow \\
0
\end{array}
\]

Diagram 2

where \( \text{Top} \) is \((m_2 + 1)m_1\), the dimension of the space \( P(N, m_2) \) as an oriented manifold, \( \delta \) is the connecting homomorphism and \( o_x \) is a generator corresponding to the orientation of \( P(N, m_2) \).

**Definition 3.1.** We take a cohomology class \( \Omega_x \) in the preimage \( \delta^{-1}(o_x) \). In the case \( N > 2 \) or \( m_2 > 1 \), the space \( P(N, m_2) \) is 1-connected ([3, p.35 and p.134]) and \( H^{\text{Top}}(P(N, m_2)) \cong 0 \), thus \( \delta \) is an isomorphism and \( \Omega_x \) is uniquely determined by \( \Omega_x = \delta^{-1}(o_x) \). In the exceptional case \( N = 2 \) and \( m_2 = 1 \), the space \( P(2, 1) \) is identified with an oriented open annulus \( C \setminus \{0\} \) in \( C \). Using this identification, we define the cohomology class \( \Omega_x \) directly as \( \Omega_x(c) = \text{"the rotation number of } c \text{ around } x\" \).

By the definition, if the point \( x \) is in \( P(N, m_2) \setminus \text{Im} W_f \), the complement of the image of \( W_f \) in \( P(N, m_2) \), then the number \( \Omega_x(W_f) \) is essentially equal to the algebraic intersection number of \( W_f \) as a cycle and an oriented arc \( a_{x_0x} \) from \( x_0 \) to \( x \) in \( P(N, m_2) \) intersecting with \( \text{Im} W_f \) transversely,

\[ \Omega_x(W_f) = \text{Int}(W_f, a_{x_0x}), \]

where \( x_0 \in P(N, m_2) \) is a point taken sufficiently near to an end of the space.

We have two lemmas.

**Lemma 3.2.** If the point \( x \) is sufficiently near to an end of the space \( P(N, m_2) \), then \( \Omega_x(W_f) = 0 \). This holds even if \( m_2 = N - 1 \), in which case \( P(N, N - 1) \cong S^{N-1} \times \mathbb{R} \) has two ends, i.e., it holds to whichever end the point \( x \) is taken near.
Proof. The first half of the lemma follows from the compactness of the cycle $W_f$. To verify the second half, we will show that the cycle $W_f$ is a boundary, i.e., the class $[W_f]$ vanishes in the homology $H_{N-1}(P(N, N-1); \mathbb{Z}) = H_{N-1}(S^{N-1} \times \mathbb{R}; \mathbb{Z}) \cong \mathbb{Z}$. Since $m_2 = N - 1$, the dimension $m_1$ of the manifold $M$ is 1. Thus $M$ is $S^1$ and $S(TM) = S^1_+ \cup S^{-1}_-$ (disjoint union). Note that the two components $S^1_+$ and $S^{-1}_-$ have opposite orientations if we compare them by the projection over $S^1$.

According to the disjoint union and by the construction of the manifold $E(L_f^*(p_1))$ (Definition 2.4), $E(L_f^*(p_1))$ also decomposes as a disjoint union $E_+ \cup E_-$ of two mutually homeomorphic manifolds. The image of the restrictions $W_f|_{E_+}$ and $W_f|_{E_-}$ in the space $P(N, N-1)$ coincide, but their signs as cycles are opposite, because of the opposite orientations of $S^1_+$ and $S^{-1}_-$ and our assumption on the orientation of the total space of a fiber bundle in Section 1 (see also Remark 2.1 in the case $N = 2$). Thus $[W_f] = [W_f|_{E_+}] + [W_f|_{E_-}] = [W_f|_{E_+}] + (-[W_f|_{E_+}]) = 0$ in the homology. We have the lemma. \(\square\)

**Lemma 3.3.** If two points $x$, $y$ are in the same chamber of $P(N, m_2) \setminus \text{Im} W_f$, the numbers $\Omega_x(W_f)$ and $\Omega_y(W_f)$ agree, i.e., the following map is well-defined.

$$\pi_0(P(N, m_2) \setminus \text{Im} W_f) \to \mathbb{Z} \quad (x \mapsto \Omega_x(W_f)).$$

This lemma is easy to see by the property of algebraic intersection numbers. We omit the proof.

### 4. Counting geometric intersection numbers

In this section, we develop a method to count geometric intersection numbers in terms of cohomology.

By $\mathbb{R}^{2m_1}$, we denote the $2m_1$-dimensional Euclidean space with fixed oriented coordinates $(y_1, y_2, \ldots, y_{m_1}, q_1, \ldots, q_{m_1})$. By "0" we denote the origin of $\mathbb{R}^{2m_1}$.

Let $o_0$ be a generator of $H^{2m_1}(\mathbb{R}^{2m_1}, \mathbb{R}^{2m_1}\{0\}; \mathbb{Z}) \cong \mathbb{Z}$ which is corresponding to the orientation. We take a cohomology class $\omega_0$ in $H^{2m_1-1}(\mathbb{R}^{2m_1}\{0\}; \mathbb{Z})$ which satisfies $\delta(\omega_0) = o_0$, where $\delta: H^{2m_1-1}(\mathbb{R}^{2m_1}\{0\}; \mathbb{Z}) \to H^{2m_1}(\mathbb{R}^{2m_1}, \mathbb{R}^{2m_1}\{0\}; \mathbb{Z})$ is the connecting homomorphism. The class $\omega_0$ is uniquely determined by $\omega_0 = \delta^{-1}(o_0)$, because the homomorphism $\delta$ is an isomorphism.

**Remark 4.1.** They are the classes which contain the following cochains $c_0^{2m_1}$, $c_w^{2m_1-1}$ respectively: $o_0 = [c_0], \omega_0 = [c_w]$.

$c_0$ If a $2m_1$-chain $f_C$ in $\mathbb{R}^{2m_1}$ is represented by a $2m_1$-dimensional compact oriented manifold $C$ and a smooth map $f: C \to \mathbb{R}^{2m_1}$ which is regular at $f^{-1}(0)$, then $c_0(f_C) = \# f^{-1}(0)$, the algebraic sum of the signed isolated points $f^{-1}(0)$.

$c_w$ If a $(2m_1 - 1)$-cycle $f_C$ is represented by an $(2m_1 - 1)$-dimensional closed oriented manifold $C$ and a continuous map $f: C \to \mathbb{R}^{2m_1}\{0\}$, then $c_w(f_C) = \deg(p \circ$
$f$), the degree of the composite map $p \circ f : C \to S^{2m_1-1}$, where $p$ is the natural projection from $\mathbb{R}^{2m_1} \setminus \{0\}$ to $S^{2m_1-1}$.

Under the identification $TR^{m_1} = R^{m_1} \times R^{m_1} = R^{2m_1}$ by $\sum_{i=1}^{m_1} \frac{\partial \cdot \cdot \cdot \partial}{\partial y_{m_1}} \cdot (y_1, \cdot \cdot \cdot , y_{m_1}) \leftrightarrow (y_1, \cdot \cdot \cdot , y_{m_1}, q_1, \cdot \cdot \cdot , q_{m_1})$, the zero vector "0" at the origin 0 in $TR^{m_1}$ is corresponding to the origin of $R^{2m_1}$. Using the identification, we take cohomology classes $\omega_0$ and $\omega_{\delta}$ on $TR^{m_1}$:

$$\omega_0 \in H^{2m_1}(TR^{m_1}, TR^{m_1}\setminus \{0\}; \mathbb{Z}), \quad \omega_{\delta} \in H^{2m_1-1}(TR^{m_1}\setminus \{0\}; \mathbb{Z}).$$

They are generators corresponding to the orientation of the cohomologies and satisfy the equality $\delta \omega_0 = \omega_{\delta}$.

Let $M^{m_1}$ be a closed connected manifold and $f : M \to R^{m_1}$ a smooth map. We fix a Riemannian metric on $M$ and regard $S(TM)$ as a subset $\{v \in TM | |v| = 1\}$ of $TM$. By $D(TM)$, we denote the corresponding disk bundle $\{v \in TM | |v| \leq 1\}$ of $TM$. Note that $D(TM)$ is a compact oriented 2$m_1$-dimensional manifold, where the orientation is induced from the bundle structure. We denote restrictions of $df : TM \to TR^{m_1}$ to $D(TM)$, $S(TM)$ or to some other subsets of $TM$ by the same notation $df$.

**Lemma 4.2.** Let $f : M^{m_1} \to R^{m_1}$ be a smooth map which is regular at the preimage $f^{-1}(0)$ of the origin. Then,

$$\# f^{-1}(0) = df^*((-1)^{m_1} \omega_{\delta}) \cdot [S(TM)],$$

where the dot \cdot means the evaluation.

**Proof.** By the compactness of $M$ and regularity of the map $f$, the preimage $f^{-1}(0)$ consists of finite points $\{p_i (1 \leq i \leq K)\}$. Taking a preimage $f^{-1}(B)$ of a sufficiently small closed oriented $m_1$-ball $B$ around the origin 0 in $R^{m_1}$, we can take closed mutually disjoint oriented $m_1$-balls $B_i (1 \leq i \leq K)$ in $M$ each of which satisfies the following two conditions:

1. The differential $df_x : T_x M \to T_{f(x)} R^{m_1}$ is an orientation-preserving isomorphism at any points $x \in \text{Int}B_i$, and
2. $B_i \cap f^{-1}(0) = \{p_i\}$.

According to the fact that $S(TM)$ is the boundary of $D(TM)$, the cycle $df : S(TM) \to TR^{m_1}$ is $(-1)^{m_1}$ times the boundary of the $2m_1$-chain $df : D(TM) \to TR^{m_1}$ (the change of the sign derives from the difference between the orientation of $S(TM)$ induced as the boundary and that induced from the bundle structures). Next, the cycle $[S(TM)]$ is decomposed as a sum of $(K + 1)$ cycles according to the decom-
position of $M$ as a union of the $K$ balls $B_i$ and their exterior $M \setminus \text{Int}(\cup B_i)$:

$$[S(TM)] = (-1)^m \partial [D(TM)]$$

$$= (-1)^m \partial [D(TM)|_{M \setminus \text{Int}(\cup B_i)}] + (-1)^m \sum_{i=1}^{K} \partial [D(TM)|_{B_i}],$$

where we wrote only the domain of each cycle (the corresponding map is the restriction of $df$ to each domain) and each $D(TM)|_X$ is the restriction of the disk bundle $D(TM)$ to the corresponding subset $X$ of $M$. Thus the right-hand side of (4.1) becomes

$$df^*(\omega) \cdot \partial [D(TM)|_{M \setminus \text{Int}(\cup B_i)}] + \sum_{i=1}^{K} df^*(\omega) \cdot \partial [D(TM)|_{B_i}].$$

By the definition of $B_i$, the image $df(\partial [D(TM)|_{M \setminus \text{Int}(\cup B_i)}])$ is disjoint from $T_0 R^{m_1} = \{0\} \times R^{m_2}$. Thus the vanishing of the first term follows the characterization of $\omega_0 (= [c_0])$ in Remark 4.1.

Finally, we see that each term $df^*(\omega) \cdot \partial [D(TM)|_{B_i})$ of the summation in (4.2) is equal to 1. Because of the regularity of the map $f$ at $p_i$, the definition of $B_i$ and its orientation, the smooth map $df: D(TM)|_{B_i} \to T R^{m_1}$ is regular at the zero vector $\tilde{0}_{p_i}$ at $p_i$ and preserves orientation. In fact, $df$ is a perturbation of the isomorphism $(f, df_{p_i}): B_i \times T_{p_i} M \to R^{m_1} \times R^{m_1}$ near the point $\tilde{0}_{p_i}$. Thus

$$df^*(\omega) \cdot \partial [D(TM)|_{B_i}] = \delta df^*(\omega) \cdot [D(TM)|_{B_i}]$$

$$= (df^* \delta \omega) \cdot [D(TM)|_{B_i}]$$

$$= (df^* o \tilde{0}) \cdot [D(TM)|_{B_i}]$$

$$= 1.$$

The final line follows the characterization of $\omega_0 (= [c_0])$ in Remark 4.1. We have the lemma. \(\square\)

5. Proof of Main Theorem

In this section, we give a proof of Theorem 1.1 except for a step of a calculation.

For the given point $x$ in $P(N, m_2)$, by changing an affine coordinate $(y_1, y_2, \cdots, y_N)$ of $R^N$ if needed, we can assume that the $m_2$-plane $x$ is an oriented $m_2$-plane defined by $\{(y_1, \cdots, y_N) \mid y_i = 0 (i > m_2)\}$. Let $\pi_2: R^N \to R^{m_1}$ be the projection defined by $\pi_2(y_1, \cdots, y_N) = (y_{m_2+1}, \cdots, y_N)$, and by $f_2$, we denote the composition $\pi_2 \circ f: M \to R^{m_1}$. Then the plane $x$ is represented as $\pi_2^{-1}(0)$ and we have

$$\# f^{-1}(\{f(M) \cap x\}) = \# f_2^{-1}(0),$$

where 0 is the origin of $R^{m_1}$. 

Proof of (A). By the assumption of Theorem 1.1 that $x \in P(N, m_2) \setminus \text{Im}W_f$, the map $f_2$ is regular at the preimage $f_2^{-1}(0)$, because the assumption means that the plane $x$ contains no tangent line or vector of the immersed manifold $f(M)$. Thus the preimage of the geometric intersection points $f^{-1}(\{f(M) \cap x\})$ in $M$, which agrees to $f_2^{-1}(0)$, consists of isolated points. The finiteness follows from the compactness of $M$.

Proof of (B). First, using the diagram in Definition 2.4, we move the cohomology class $\Omega_x$ to a cohomology on the space $P(N, 1)$ by pulling-back and pushing-forward the class. Let $p_1! : H^{(m_2+1)m_1-1}(P(N; 1, m_2), *; \mathbb{Z}) \to H^{2m_1-1}(P(N, 1), *'; \mathbb{Z})$ be the push forward, in other words $[\text{Fiber}] \cap \int_{\text{Fiber}}$, where * and *' are appropriate subspaces. Here, a fiber of $p_1$ is the oriented Grassmannian manifold $G(N - 1, m_2 - 1)$, thus the map $p_1!$ decreases the degree of the cohomology by its dimension $m_1(m_2 - 1)$.

Let $P(x, 1)$ denote a subset $p_1(p_2^{-1}(x)) = \{l \in P(N, 1) | l \subset x\}$ in the space $P(N, 1)$. It is a $2(m_2-1)$-dimensional submanifold. The cohomology class $p_1!p_2^{*}(\Omega_x)$, whose support is contained in the complement $P(N, 1) \setminus P(x, 1)$, is the linking dual of $(5.2)$ $p_1!p_2^{*}(\Omega_x) \in H^{2m_1-1}(P(N, 1) \setminus P(x, 1); \mathbb{Z})$.

By the assumption of Theorem 1.1 and the construction of the map $W_f$, the image of the map $L_f : S(TM) \to P(N, 1)$ is contained in $P(N, 1) \setminus P(x, 1)$, and we can define a number $p_1!p_2^{*}(\Omega_x)(L_f)$ by regarding the map $L_f$ as a $(2m_1 - 1)$-cycle. We have

$$\Omega_x(W_f) = \Omega_x(p_2 \circ \bar{L}_f) = p_2^{*}(\Omega_x)(\bar{L}_f) = p_1!p_2^{*}(\Omega_x)(L_f) = L_f^{*}(p_1!p_2^{*}(\Omega_x)) \cdot [S(TM)],$$

where the dot · in the final line means the evaluation.

Now, it is sufficient to verify

$$(5.4) L_f^{*}(p_1!p_2^{*}(\Omega_x)) \cdot [S(TM)] = df_2^{*}((-1)^m \omega_0) \cdot [S(TM)],$$

where $\omega_0 \in H^{2m_1-1}(TR^{m_1}\setminus \{0\}; \mathbb{Z})$ is the cohomology class defined in the last section. But, to show this, we have to define some maps and represent the maps $L_f, f_2$ by them. Thus we postpone this final step to the next section.

By (5.1), (5.3), Lemma 4.2 and (5.4), which is proved in the next section, we have the theorem.
6. Calculation on $L_f$ and $df_2$

In this section, we show (5.4) in the last section.

First, we define some maps. We use the fixed coordinate $(y_1, y_2, \cdots, y_N)$ of $\mathbb{R}^N$ such that $x$ is an $m_2$-plane defined by $\{(y_1, \cdots, y_N) \mid y_i = 0 (i > m_2)\}$. For a given point $y = (y_1, \cdots, y_N)$ in $\mathbb{R}^N$ and a non-zero vector $q = (q_1, \cdots, q_N)$, a straight line whose vector is $q$ and which passes through the point $y$ is uniquely determined. We denote by $l_{y,q}$ this line. We have

$$l_{y,q} = l_{y',q'} \text{ in } P(N,1) \iff \text{there exist } \lambda, \mu \in \mathbb{R} \text{ with } \lambda > 0$$

such that $q' = \lambda q$ and $y' - y = \mu q$.

Thus we have a quotient map

$$(6.2) \quad \psi: \mathbb{R}^N \times (\mathbb{R}^N \setminus \{0\}) \xrightarrow{\approx} P(N,1) \quad ((y,q) \mapsto l_{y,q}),$$

where we define the equivalence relation $(y',q') \sim (y,q)$ by the right-hand side of (6.1).

Recall that $\pi_2: \mathbb{R}^N \to \mathbb{R}^{m_1}$ is the projection $\pi_2(y_1, \cdots, y_N) = (y_{m_2+1}, \cdots, y_N)$. Here, we use a projection $\pi_2 \times \pi_2: \mathbb{R}^N \times (\mathbb{R}^N \setminus \{0\}) \to \mathbb{R}^{m_1} \times \mathbb{R}^{m_1}$. Note that

$$(6.3) \quad \psi^{-1}(P(x,1)) = (\pi_2 \times \pi_2)^{-1}(0,0),$$

where $P(x,1)$ is the subset defined in the last section.

It is easy to represent the maps $L_f: S(TM) \to P(N,1) \setminus P(x,1)$ and $df_2 = d(\pi_2 \circ f): S(TM) \to TR^{m_1} \setminus \{0\}$ by the maps $\psi$ and $\pi_2 \times \pi_2$:

$$(6.4) \quad L_f(v) = \psi(df(v)), \quad df_2(v) = (\pi_2 \times \pi_2)(df(v)),$$

where we regard $S(TM)$ as a subset in $TM$ and $df(v)$ ($v \in S(TM)$) as an element in $\mathbb{R}^N \times (\mathbb{R}^N \setminus \{0\})$ under the identification $TR^N = \mathbb{R}^N \times T_0 \mathbb{R}^N = \mathbb{R}^N \times \mathbb{R}^N$ by the fixed coordinates, see Diagram 3. Note that $df(v)$ is a non-zero vector everywhere, because $f$ is an immersion.

Diagram 3
Now, we verify that (5.4). By the construction, the left-hand side $p_1(p_2^*(\Omega_x)) (L_f)$ was the linking number of the $(2m_1 - 1)$-cycle $L_f$ and the submanifold $P(x, 1)$. By (6.4), it is equal to the linking number of the map $df$ as a cycle and the subset $\psi^{-1}(P(x, 1))$ in $\mathbb{R}^N \times (\mathbb{R}^N \setminus \{0\})$. By (6.3), it is equal to the linking number of $(\pi_2 \times \pi_2) \circ df$ around the origin $(0,0)$ in $\mathbb{R}^{m_1} \times \mathbb{R}^{m_1}$, which is equal to the right-hand side of (5.4) up to sign. The correct sign is checked by some concrete examples (unit spheres, for example).

ACKNOWLEDGEMENT. The author would like to express sincere gratitude to Professors G. Ishikawa, O. Saeki, T. Ozawa, T. Fukui, J. Imai and T. Ohmoto for their valuable advice and encouragement. The author would also like to thank to the referee for his patience in reading manuscripts of this paper carefully and making corrections.

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