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ON THE GEOMETRIC INTERSECTION NUMBER OF AN IMMERSED MANIFOLD AND A PLANE

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1. Introduction and Main Theorem

Throughout this article, we work in the C^∞ category. Let $\mathbf{R}^N$ be the N-dimensional Euclidean space with the standard orientation and $f\colon M^{m_1} \to \mathbf{R}^N$ an immersion of an m_1-dimensional closed connected (possibly non-orientable) manifold, where $1 \le m_1 < N$. We let m_2 denote the codimension of f ; $m_2 = N - m_1$. Let $P(N, m_2)$ be the set of oriented m_2-planes (lines if $m_2 = 1$) in $\mathbf{R}^N$. We will give a formula on the geometric intersection number $\#f^{-1}(f(M) \cap x)$ of the immersed manifold and an m_2-plane x in $\mathbf{R}^N$ in terms of algebraic intersection theory and "chamber-wall" structure on $P(N, m_2)$.

To the author's knowledge, the word "chamber" is used to indicate regions separated from a whole space by a codimension 1 subspace, for example, as "Weyl chamber" in Lie algebra theory (for more recent usage, see [2]). The word "wall" is used for the separating subspace. Following the history, in this paper, we use the two words in the same meanings as above.

The set $P(N, m_2)$ admits a structure of an $(m_2 + 1)m_1$-dimensional C^∞ oriented manifold. In fact, it is homeomorphic to the total space of the orthogonal complement vector bundle to the tautological bundle over the corresponding real oriented Grassmannian manifold $G(N, m_2)$.

In the space $P(N, m_2)$, a subset consisting of m_2-planes tangent to the immersed manifold forms "walls" which decompose the space into some "chambers". By giving orientation to the walls, one may associate an number to each chamber by algebraic intersection theory. We will introduce a canonical orientation to the walls and will show that the number associated to a chamber in this way is equal to the geometric intersection number $\#f^{-1}(f(M) \cap x)$ of the immersed manifold and an m_2-plane x belonging to the chamber.

In the next section, from the immersion f, we will construct an $((m_2+1)m_1 - 1)$-

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dimensional closed manifold $E({L_f}^*(p_1))$ and a map $W_f\colon E({L_f}^*(p_1)) \to P(N, m_2)$. This map W_f will be constructed such that the image $\mathrm{Im}W_f$ consists of all m_2-planes which have at least one non-transverse intersection point with the immersed manifold $f(M)$. Thus the complement $P(N, m_2)\setminus \mathrm{Im}W_f$ consists of m_2-planes which have only transverse intersection points with $f(M)$. Since the codimension of W_f is 1, the wall $\mathrm{Im}W_f$ decomposes $P(N, m_2)$ into some chambers.

In the process of the construction of $E({L_f}^*(p_1))$, it is canonically oriented. By the orientation, we can regard the map W_f as an $((m_2+1)m_1-1)$-cycle in the space $P(N, m_2)$.

For a point $x \in P(N, m_2)\setminus \mathrm{Im}W_f$, which is an m_2-plane in $\mathbf{R}^N$, we will associate an (non-negative) integer to a chamber of $P(N, m_2)\setminus \mathrm{Im}W_f$ containing the point x by algebraic intersection number theory. (To the author's knowledge, such an association was first developed in [1 p.277].) For this, in Section 3, we will construct a cohomology class Ω_x in $H^{(m_2+1)m_1-1}(P(N, m_2)\setminus\{x\}\,;\mathbf{Z})$, which measures a linking number of a given $((m_2+1)m_1-1)$-cycle around the point x. In fact, the number $\Omega_x(W_f)$ is essentially equal to the algebraic intersection number of W_f and an oriented arc a_{x_0x} from x_0 to x, intersecting with $\mathrm{Im}W_f$ transversely, where $x_0 \in P(N, m_2)$ is a base point taken sufficiently near to an end of the space $P(N, m_2)$:

$$\Omega_x(W_f) = \mathrm{Int}(W_f, a_{x_0x}).$$

The cohomology class Ω_x is the Poincare dual of the semi-open arc $\lim\limits_{x_0\to end} a_{x_0x}$. This association depends on neither the arc a_{x_0x} nor the point x_0 even if $m_2 = N-1$, in which case $P(N, N-1) \cong S^{N-1}\times\mathbf{R}$ has two ends.

Theorem 1.1. *Let $f\colon M \to \mathbf{R}^N$ be an immersion and W_f the map constructed from the immersion f (For the definition of W_f, see Section 2). We regard W_f as an $((m_2+1)m_1-1)$-cycle in $P(N, m_2)$. Let x be a point in the complement of W_f in $P(N, m_2)$. Then*

(A) $$\#f^{-1}(\{f(M)\cap x\}) \text{ is finite,}$$

and

(B) $$\#f^{-1}(\{f(M)\cap x\}) = \Omega_x(W_f).$$

We call the number $\#f^{-1}(\{f(M)\cap x\})$ the *geometric intersection number* of the immersed manifold $f(M)$ and the m_2-plane x.

In Section 4, we will develop a method of counting geometric intersection numbers in terms of cohomology. The proof of Theorem 1.1 will be given in Section 5 and 6.

Throughout this paper, we assume that the total space of a fiber bundle is oriented so that the local orientation of the base space followed by that of a fiber gives the orientation of the total space.

2. A Decomposition of $P(N, m_2)$

In this secion, from a given immersion $f\colon M \to \mathbf{R}^N$, we construct an $((m_2+1)m_1-1)$-dimensional closed oriented manifold $E(L_f{}^*(p_1))$ and a map $W_f\colon E(L_f{}^*(p_1)) \to P(N, m_2)$, which is also regarded as a cycle.

First we introduce a canonical double fibration. For $0 < n_1 \leq n_2 < N$, we let $P(N; n_1, n_2)$ denote the set defined as follows:

$$P(N; n_1, n_2) = \{\, (x, X) \in P(N, n_1) \times P(N, n_2) \mid x \subset X \,\}.$$

This set admits a structure of an orientable C^∞ manifold whose dimension is $(n_1+1)(N-n_1)+(n_2-n_1)(N-n_2)$. It is homeomorphic to the total space of a certain vector bundle over a corresponding real flag manifold.

Each of the first projection p_1 and the second projection p_2 is a fibration:

$$\begin{array}{rcl} p_1\colon P(N; n_1, n_2) & \to & P(N, n_1) \\ (x, X) & \mapsto & x, \end{array} \quad \text{and} \quad \begin{array}{rcl} p_2\colon P(N; n_1, n_2) & \to & P(N, n_2) \\ (x, X) & \mapsto & X. \end{array}$$

The fiber of p_1 is homeomorphic to the oriented Grassmannian manifold $G(N-n_1, n_2-n_1)$, on the other hand, the fiber of p_2 is homeomorphic to the space $P(n_2, n_1)$.

REMARK 2.1. In the case $n_1 = n_2$, the space $P(N; n_1, n_2)$ is a disconnected double covering of $P(N; n_1)$. The one component is the diagonal Δ and the other is the anti-diagonal $\bar{\Delta} = \{(x, -x)\}$. The latter space $\bar{\Delta}$ has an opposite orientation from the one induced from the former Δ by a homeomorphism $(p_1|_{\bar{\Delta}})^{-1} \circ p_1|_{\Delta}$.

REMARK 2.2. We define an orientation of the space $P(N; n_1, n_2)$ as a manifold by the bundle structure of p_1.

Next, from the immersion $f\colon M \to \mathbf{R}^N$, we define a map $L_f\colon S(TM) \to P(N, 1)$. Here $S(TM)$ is defined as follows:

$$\begin{aligned} S(TM) &= (TM \setminus \{\text{the zero section}\}) \,/ \sim, \\ &\qquad \text{where } v_1 \sim v_2 \Leftrightarrow v_1 = \lambda v_2 \text{ for some } \lambda > 0. \end{aligned}$$

We define an orientation of $S(TM)$ by a local orientation of U (in M) followed by that of the fiber over U. Thus $S(TM)$ is an oriented manifold even if M is non-orientable, because if we change the local orientation of M, the orientation of the fiber over the local base is also changed. The dimension of the manifold $S(TM)$ is $(2m_1 - 1)$. Now we define L_f as:

$$\begin{array}{llll} L_f\colon & S(TM) & \to & P(N,1), \\ & v\ (\in S(T_pM)) & \mapsto & \text{A straight line whose vector is } df(v) \\ & & & \text{and which passes through the point } f(p). \end{array}$$

REMARK 2.3. $P(N,1) \cong TS^{N-1}$.

DEFINITION 2.4. Using the map L_f and the double fibration p_1, p_2 of $P(N;1,m_2)$, we construct a manifold $E({L_f}^*(p_1))$ and a map W_f by Diagram 1. W_f is the composite $p_2 \circ \tilde{L}_f$.

$$\begin{tikzcd} E({L_f}^*(p_1)) \arrow[r, "\tilde{L}_f"'] \arrow[d, "\tilde{p}_1"] & P(N;1,m_2) \arrow[dl, "p_1"'] \arrow[dr, "p_2"] \arrow[r, phantom, "W_f"] & \\ S(TM) \arrow[r, "L_f"'] & P(N,1) & P(N,m_2) \end{tikzcd}$$

Diagram 1

We define $E({L_f}^*(p_1))$ as the total space of the pull-back of the fibration p_1 over $S(TM)$ by L_f. It is a manifold, whose dimension is $2m_1 - 1 + (m_2 - 1)(N - m_2)$ $= (m_2 + 1)m_1 - 1$:

$$E({L_f}^*(p_1)) = \{\, (v,(x,X)) \in S(TM) \times P(N;1,m_2) \mid L_f(v) = x \,\}.$$

The image $\mathrm{Im}W_f$ consists of all m_2-planes which have at least one non-transverse intersection point with the immersed manifold $f(M)$.

Lemma 2.6. *The manifold $E({L_f}^*(p_1))$ is closed and oriented.*

Proof. Note that $E({L_f}^*(p_1))$ is a total space of a fiber bundle over $S(TM)$ whose fiber is $G(N-1, m_2-1)$. Both the basespace and the fiber are closed and oriented manifolds. We have the lemma. □

By the orientation of $E({L_f}^*(p_1))$ induced from the bundle structure, we can regard the map W_f as an $((m_2+1)m_1 - 1)$-cycle in the space $P(N,m_2)$.

Since the codimension of the map W_f is 1, the image of W_f decompose the space $P(N,m_2)$ into some chambers.

3. Algebraic Intersection Theory in $P(N,m_2)$

In this section, for a point $x \in P(N, m_2)$, we construct a cohomology class Ω_x in $H^{(m_2+1)m_1-1}(P(N, m_2) \setminus \{x\}; \mathbf{Z})$, which essentially measures a linking number of an $((m_2+1)m_1 - 1)$-cycle around the point x. We also construct a map from the set of chambers $P(N, m_2) \setminus \mathrm{Im}W_f$ to the set of non-negative integers by using Ω_x.

First, we consider the exact sequence in Diagram 2.

$$\begin{array}{lcl}
H^{Top-1}(P(N, m_2); \mathbf{Z}) & & \\
\qquad\quad\downarrow & & \\
H^{Top-1}(P(N, m_2) \setminus \{x\}; \mathbf{Z}) & \supset & \mathbf{Z}[\Omega_x] \\
\qquad\quad\downarrow \delta & & \\
H^{Top}(P(N, m_2), P(N, m_2) \setminus \{x\}; \mathbf{Z}) & = & \mathbf{Z}[o_x] \\
\qquad\quad\downarrow & & \\
\qquad\quad 0 & &
\end{array}$$

Diagram 2

where Top is $(m_2+1)m_1$, the dimension of the space $P(N, m_2)$ as an oriented manifold, δ is the connecting homomorphism and o_x is a generator corresponding to the orientation of $P(N, m_2)$.

DEFINITION 3.1. We take a cohomology class Ω_x in the preimage $\delta^{-1}(o_x)$. In the case $N > 2$ or $m_2 > 1$, the space $P(N, m_2)$ is 1-connected ([3,p.35 and p.134]) and $H^{Top-1}(P(N, m_2)) \cong 0$, thus δ is an isomorphism and Ω_x is uniquely determined by $\Omega_x = \delta^{-1}(o_x)$. In the exceptional case $N = 2$ and $m_2 = 1$, the space $P(2, 1)$ is identified with an oriented open annulus $\mathbf{C} \setminus \{0\}$ in $\mathbf{C}$. Using this identification, we define the cohomology class Ω_x directly as $\Omega_x(c) =$ "the rotation number of c around x".

By the definition, if the point x is in $P(N, m_2) \setminus \mathrm{Im}W_f$, the complement of the image of W_f in $P(N, m_2)$, then the number $\Omega_x(W_f)$ is essentially equal to the algebraic intersection number of W_f as a cycle and an oriented arc $a_{x_0 x}$ from x_0 to x in $P(N, m_2)$ intersecting with $\mathrm{Im}W_f$ transversely,

$$\Omega_x(W_f) = \mathrm{Int}(W_f, a_{x_0 x}),$$

where $x_0 \in P(N, m_2)$ is a point taken sufficiently near to an end of the space.

We have two lemmas.

Lemma 3.2. *If the point x is sufficiently near to an end of the space $P(N, m_2)$, then $\Omega_x(W_f) = 0$. This holds even if $m_2 = N - 1$, in which case $P(N, N-1) \cong S^{N-1} \times \mathbf{R}$ has two ends, i.e., it holds to whichever end the point x is taken near.*

Proof. The first half of the lemma follows from the compactness of the cycle W_f. To verify the second half, we will show that the cycle W_f is a boundary, i.e., the class $[W_f]$ vanishes in the homology $H_{N-1}(P(N,N-1);\mathbf{Z}) = H_{N-1}(S^{N-1}\times\mathbf{R};\mathbf{Z}) \cong \mathbf{Z}$. Since $m_2 = N-1$, the dimension m_1 of the manifold M is 1. Thus M is S^1 and $S(TM) = {S_+}^1 \cup {S_-}^1$ (disjoint union). Note that the two components ${S_+}^1$ and ${S_-}^1$ have opposite orientations if we compare them by the projection over S^1. According to the disjoint union and by the construction of the manifold $E({L_f}^*(p_1))$ (Definition 2.4), $E({L_f}^*(p_1))$ also decomposes as a disjoint union $E_+ \cup E_-$ of two mutually homeomorphic manifolds. The image of the restrictions $W_f|_{E_+}$ and $W_f|_{E_-}$ in the space $P(N,N-1)$ coincide, but their signs as cycles are opposite, because of the opposite orientations of ${S_+}^1$ and ${S_-}^1$ and our assumption on the orientation of the total space of a fiber bundle in Section 1 (see also Remark 2.1 in the case $N=2$). Thus $[W_f] = [W_f|_{E_+}] + [W_f|_{E_-}] = [W_f|_{E_+}] + (-[W_f|_{E_+}]) = 0$ in the homology. We have the lemma. $\square$

Lemma 3.3. *If two points x, y are in a same chamber of $P(N,m_2)\setminus \mathrm{Im}W_f$, the numbers $\Omega_x(W_f)$ and $\Omega_y(W_f)$ agree, i.e., the following map is well-defined.*

$$\pi_0(P(N,m_2)\setminus \mathit{Im}W_f) \to \mathbf{Z} \quad (x \mapsto \Omega_x(W_f)\,).$$

This lemma is easy to see by the property of algebraic intersection numbers. We omit the proof.

4. Counting geometric intersection numbers

In this secion, we develop a method to count geometric intersection numbers in terms of cohomology.

By $\mathbf{R}^{2m_1}$, we denote the $2m_1$-dimensional Euclidean space with a fixed oriented coordinates $(y_1, y_2, \cdots, y_{m_1}, q_1, \cdots, q_{m_1})$. By "0" we denote the origin of $\mathbf{R}^{2m_1}$.

Let o_0 be a generator of $H^{2m_1}(\mathbf{R}^{2m_1}, \mathbf{R}^{2m_1}\setminus\{0\};\mathbf{Z}) \cong \mathbf{Z}$ which is corresponding to the orientation. We take a cohomology class ω_0 in $H^{2m_1-1}(\mathbf{R}^{2m_1}\setminus\{0\};\mathbf{Z})$ which satisfies $\delta(\omega_0) = o_0$, where $\delta\colon H^{2m_1-1}(\mathbf{R}^{2m_1}\setminus\{0\};\mathbf{Z}) \to H^{2m_1}(\mathbf{R}^{2m_1}, \mathbf{R}^{2m_1}\setminus\{0\};\mathbf{Z})$ is the connecting homomorphism. The class ω_0 is uniquely determined by $\omega_0 = \delta^{-1}(o_0)$, because the homomorphism δ is an isomorphism.

REMARK 4.1. They are the classes which contain the following cochains ${c_o}^{2m_1}$, ${c_w}^{2m_1-1}$ respectively: $o_0 = [c_o], \omega_0 = [c_\omega]$.

(c_o) If a $2m_1$-chain f_C in $\mathbf{R}^{2m_1}$ is represented by a $2m_1$-dimensional compact oriented manifold C and a smooth map $f\colon C \to \mathbf{R}^{2m_1}$ which is regular at $f^{-1}(0)$, then $c_o(f_C) = \underset{alg}{\#} f^{-1}(0)$, the algebraic sum of the signed isolated points $f^{-1}(0)$.

(c_ω) If a $(2m_1-1)$-cycle f_C is represented by an $(2m_1-1)$-dimensional closed oriented manifold C and a continuous map $f\colon C \to \mathbf{R}^{2m_1}\setminus\{0\}$, then $c_\omega(f_C) = \deg(p\circ$

f), the degree of the composite map $p \circ f\colon C \to S^{2m_1-1}$, where p is the natural projection from $\mathbf{R}^{2m_1}\backslash\{0\}$ to S^{2m_1-1}.

Under the identification $T\mathbf{R}^{m_1} = \mathbf{R}^{m_1} \times \mathbf{R}^{m_1} = \mathbf{R}^{2m_1}$ by $\sum_i q_i \frac{\partial}{\partial y_i}|_{(y_1,\cdots,y_{m_1})} \leftrightarrow (y_1, \cdots, y_{m_1}, q_1, \cdots, q_{m_1})$, the zero vector "$\vec{0}$"at the origin 0 in $T\mathbf{R}^{m_1}$ is corresponding to the origin of $\mathbf{R}^{2m_1}$. Using the identification, we take cohomology classes $o_{\vec{0}}$ and $\omega_{\vec{0}}$ on $T\mathbf{R}^{m_1}$:

$$o_{\vec{0}} \in H^{2m_1}(T\mathbf{R}^{m_1}, T\mathbf{R}^{m_1}\backslash\{\vec{0}\}; \mathbf{Z}), \qquad \omega_{\vec{0}} \in H^{2m_1-1}(T\mathbf{R}^{m_1}\backslash\{\vec{0}\}; \mathbf{Z}).$$

They are generators corresponding to the orientation of the cohomologies and satisfy the equality $\delta\omega_{\vec{0}} = o_{\vec{0}}$.

Let M^{m_1} be a closed connected manifold and $f\colon M \to \mathbf{R}^{m_1}$ a smooth map. We fix a Riemannian metric on M and regard $S(TM)$ as a subset $\{\, v \in TM | \, |v| = 1 \,\}$ of TM. By $D(TM)$, we denote the corresponding disk bundle $\{\, v \in TM | \, |v| \le 1 \,\}$ of TM. Note that $D(TM)$ is a compact oriented $2m_1$-dimensional manifold, where the orientation is induced from the bundle structure. We denote restrictions of $df\colon TM \to T\mathbf{R}^{m_1}$ to $D(TM)$, $S(TM)$ or to some other subsets of TM by the same notation df.

Lemma 4.2. *Let $f\colon M^{m_1} \to \mathbf{R}^{m_1}$ be a smooth map which is regular at the preimage $f^{-1}(0)$ of the origin. Then,*

$$\#f^{-1}(0) = df^*((-1)^{m_1}\omega_{\vec{0}}) \cdot [S(TM)], \tag{4.1}$$

where the dot $\cdot$ means the evaluation.

Proof. By the compactness of M and regularity of the map f, the preimage $f^{-1}(0)$ consists of finite points $\{p_i \ (1 \le i \le K)\}$. Taking a preimage $f^{-1}(B)$ of a sufficiently small closed oriented m_1-ball B around the origin 0 in $\mathbf{R}^{m_1}$, we can take closed mutually disjoint oriented m_1-balls B_i $(1 \le i \le K)$ in M each of which satisfies the following two conditions:

(1) The differential $df_x\colon T_xM \to T_{f(x)}\mathbf{R}^{m_1}$ is an orientation-preserving isomorphism at any points $x \in \mathrm{Int}B_i$, and
(2) $B_i \cap f^{-1}(0) = \{p_i\}$.

According to the fact that $S(TM)$ is the boundary of $D(TM)$, the cycle $df\colon S(TM) \to T\mathbf{R}^{m_1}$ is $(-1)^{m_1}$ times the boundary of the $2m_1$-chain $df\colon D(TM) \to T\mathbf{R}^{m_1}$ (the change of the sign derives from the difference between the orientation of $S(TM)$ induced as the boundary and that induced from the bundle structures). Next, the cycle $[S(TM)]$ is decomposed as a sum of $(K+1)$ cycles according to the decomposition of M as a union of the K balls B_i and their exterior $M\backslash\mathrm{Int}(\cup B_i)$:

$$\begin{aligned}[S(TM)] &= (-1)^{m_1}\partial[D(TM)] \\ &= (-1)^{m_1}\partial[D(TM)|_{M\backslash \mathrm{Int}(\cup B_i)}] + (-1)^{m_1}\sum_{i=1}^{K}\partial[D(TM)|_{B_i}],\end{aligned}$$

where we wrote only the domain of each cycle (the corresponding map is the restriction of df to each domain) and each $D(TM)|_X$ is the restriction of the disk bundle $D(TM)$ to the corresponding subset X of M. Thus the right-hand side of (4.1) becomes

$$df^*(\omega_{\vec{0}})\cdot\partial[(D(TM)|_{M\backslash \mathrm{Int}(\cup B_i)})] + \sum_{i=1}^{K} df^*(\omega_{\vec{0}})\cdot\partial[(D(TM)|_{B_i})]. \tag{4.2}$$

By the definition of B_i, the image $df(\partial(D(TM)|_{M\backslash \mathrm{Int}(\cup B_i)}))$ is disjoint from $T_0\mathbf{R}^{m_1} = \{0\}\times R^{m_1}$. Thus the vanishing of the first term follows the characterization of ω_0 $(= [c_\omega])$ in Remark 4.1.

Finally, we see that each term $df^*(\omega_{\vec{0}})\cdot\partial[(D(TM)|_{B_i})]$ of the summation in (4.2) is equal to 1. Because of the regularity of the map f at p_i, the definition of B_i and its orientation, the smooth map $df\colon D(TM)|_{B_i}\to T\mathbf{R}^{m_1}$ is regular at the zero vector $\vec{0}_{p_i}$ at p_i and preserves orientation. In fact, df is a perturbation of the isomorphism $(f, df_{p_i})\colon B_i\times T_{p_i}M\to\mathbf{R}^{m_1}\times\mathbf{R}^{m_1}$ near the point $\vec{0}_{p_i}$. Thus

$$\begin{aligned} df^*(\omega_{\vec{0}})\cdot\partial[(D(TM)|_{B_i})] &= \delta df^*(\omega_{\vec{0}})\cdot[D(TM)|_{B_i})] \\ &= (df^*\delta\omega_{\vec{0}})\cdot[D(TM)|_{B_i})] \\ &= (df^*o_{\vec{0}})\cdot[D(TM)|_{B_i})] \\ &= 1.\end{aligned}$$

The final line follows the characterization of o_0 $(= [c_o])$ in Remark 4.1. We have the lemma. □

5. Proof of Main Theorem

In this section, we give a proof of Theorem 1.1 except for a step of a calculation.

For the given point x in $P(N, m_2)$, by changing an affine coordinate $(y_1, y_2, \cdots, y_N)$ of $\mathbf{R}^N$ if needed, we can assume that the m_2-plane x is an oriented m_2-plane defined by $\{(y_1,\cdots,y_N)\,|\,y_i = 0\,(i > m_2)\}$. Let $\pi_2\colon\mathbf{R}^N\to\mathbf{R}^{m_1}$ be the projection defined by $\pi_2(y_1,\cdots,y_N) = (y_{m_2+1},\cdots,y_N)$, and by f_2, we denote the composition $\pi_2\circ f\colon M\to\mathbf{R}^{m_1}$. Then the plane x is represented as ${\pi_2}^{-1}(0)$ and we have

$$\#f^{-1}(\{f(M)\cap x\}) = \#{f_2}^{-1}(0), \tag{5.1}$$

where 0 is the origin of $\mathbf{R}^{m_1}$.

Proof of (A). By the assumption of Theorem 1.1 that $x \in P(N, m_2) \setminus \mathrm{Im} W_f$, the map f_2 is regular at the preimage $f_2^{-1}(0)$, because the assumption means that the plane x contains no tangent line or vector of the immersed manifold $f(M)$. Thus the preimage of the geometric intersection points $f^{-1}(\{f(M) \cap x\})$ in M, which agrees to $f_2^{-1}(0)$, consists of isolated points. The finiteness follows from the compactness of M.

$\square$

Proof of (B). First, using the diagram in Definition 2.4, we move the cohomology class Ω_x to a cohomology on the space $P(N, 1)$ by pulling-back and pushing-forward the class. Let $p_1!$: $H^{(m_2+1)m_1-1}(P(N; 1, m_2), *; \mathbf{Z}) \to H^{2m_1-1}(P(N, 1), *'; \mathbf{Z})$ be the push forward, in other words $[Fiber]\cap$ or $\int_{Fiber}$, where $*$ and $*'$ are appropriate subspaces. Here, a fiber of p_1 is the oriented Grassmannian manifold $G(N-1, m_2-1)$, thus the map $p_1!$ decreases the degree of the cohomology by its dimension $m_1(m_2-1)$.

Let $P(x, 1)$ denote a subset $p_1({p_2}^{-1}(x)) = \{l \in P(N, 1) | \, l \subset x\}$ in the space $P(N, 1)$. It is a $2(m_2-1)$-dimensional submanifold. The cohomology class $p_1!{p_2}^*(\Omega_x)$, whose support is contained in the complement $P(N, 1) \setminus P(x, 1)$, is the linking dual of $P(x, 1)$ in $P(N, 1)$:

$$p_1!{p_2}^*(\Omega_x) \in H^{2m_1-1}(P(N, 1) \setminus P(x, 1); \mathbf{Z}). \tag{5.2}$$

By the assumption of Theorem 1.1 and the construction of the map W_f, the image of the map $L_f \colon S(TM) \to P(N, 1)$ is contained in $P(N, 1) \setminus P(x, 1)$, and we can define a number $p_1!{p_2}^*(\Omega_x)(L_f)$ by regarding the map L_f as a $(2m_1-1)$-cycle. We have

$$\begin{aligned} \Omega_x(W_f) &= \Omega_x(p_2 \circ \tilde{L}_f) \\ &= {p_2}^*(\Omega_x)(\tilde{L}_f) \\ &= p_1!{p_2}^*(\Omega_x)(L_f) \\ &= {L_f}^*(p_1!{p_2}^*(\Omega_x)) \cdot [S(TM)], \end{aligned} \tag{5.3}$$

where the dot $\cdot$ in the final line means the evaluation.

Now, it is sufficient to verify

$${L_f}^*(p_1!{p_2}^*(\Omega_x)) \cdot [S(TM)] = d{f_2}^*((-1)^{m_1}\omega_{\vec{0}}) \cdot [S(TM)], \tag{5.4}$$

where $\omega_{\vec{0}} \in H^{2m_1-1}(T\mathbf{R}^{m_1} \backslash \{\vec{0}\}; \mathbf{Z})$ is the cohomology class defined in the last section. But, to show this, we have to define some maps and represent the maps L_f, f_2 by them. Thus we postpone this final step to the next section.

By (5.1), (5.3), Lemma 4.2 and (5.4), which is proved in the next section, we have the theorem. $\square$

6. Calculation on L_f and df_2

In this section, we show (5.4) in the last section.

First, we define some maps. We use the fixed coordinate $(y_1, y_2, \cdots, y_N)$ of $\mathbf{R}^N$ such that x is an m_2-plane defined by $\{(y_1, \cdots, y_N) \mid y_i = 0\,(i > m_2)\}$. For a given point $y = (y_1, \cdots, y_N)$ in $\mathbf{R}^N$ and a non-zero vector $q = (q_1, \cdots, q_N)$, a straight line whose vector is q and which passes through the point y is uniquely determined. We denote by $l_{y,q}$ this line. We have

$$\text{(6.1)} \qquad l_{y,q} = l_{y',q'} \text{ in } P(N,1) \iff \begin{array}{l} \text{there exist } \lambda, \mu \in \mathbf{R} \text{ with } \lambda > 0 \\ \text{such that } q' = \lambda q \text{ and } y' - y = \mu q. \end{array}$$

Thus we have a quotient map

$$\text{(6.2)} \qquad \psi\colon\ \mathbf{R}^N \times (\mathbf{R}^N \backslash \{0\}) \underset{/\sim}{\longrightarrow} P(N,1) \qquad (\,(y,q) \mapsto l_{y,q}\,),$$

where we define the equivalence relation $(y', q') \sim (y, q)$ by the right-hand side of (6.1).

Recall that $\pi_2\colon \mathbf{R}^N \to \mathbf{R}^{m_1}$ is the projection $\pi_2(y_1, \cdots, y_N) = (y_{m_2+1}, \cdots, y_N)$. Here, we use a projection $\pi_2 \times \pi_2\colon \mathbf{R}^N \times (\mathbf{R}^N \backslash \{0\}) \to \mathbf{R}^{m_1} \times \mathbf{R}^{m_1}$. Note that

$$\text{(6.3)} \qquad \psi^{-1}(P(x,1)) = (\pi_2 \times \pi_2)^{-1}(0,0),$$

where $P(x,1)$ is the subset defined in the last section.

It is easy to represent the maps $L_f\colon S(TM) \to P(N,1) \setminus P(x,1)$ and $df_2 = d(\pi_2 \circ f)\colon\ S(TM) \to T\mathbf{R}^{m_1} \backslash \{0\}$ by the maps ψ and $\pi_2 \times \pi_2$:

$$\text{(6.4)} \qquad \begin{aligned} L_f(v) &= \psi(df(v)), \\ df_2(v) &= (\pi_2 \times \pi_2)(df(v)), \end{aligned}$$

where we regard $S(TM)$ as a subset in TM and $df(v)$ ($v \in S(TM)$) as an element in $\mathbf{R}^N \times (\mathbf{R}^N \backslash \{0\})$ under the identification $T\mathbf{R}^N = \mathbf{R}^N \times T_0\mathbf{R}^N = \mathbf{R}^N \times \mathbf{R}^N$ by the fixed coordinates, see Diagram 3. Note that $df(v)$ is a non-zero vector everywhere, because f is an immersion.

$$\begin{array}{ccccc}
 & & S(TM) & & \\
 & L_f \swarrow & \downarrow df & \searrow df_2 & \\
P(N,1) & \underset{\psi}{\longleftarrow} & \mathbf{R}^N \times (\mathbf{R}^N \backslash \{0\}) & \underset{\pi_2 \times \pi_2}{\longrightarrow} & \mathbf{R}^{m_1} \times \mathbf{R}^{m_1} \\
\cup & & \cup & & \cup \\
P(x,1) & \leftarrow & \psi^{-1}(P(x,1)) & \rightarrow & \{(0,0)\} \\
 & & = (\pi_2 \times \pi_2)^{-1}(0,0) & &
\end{array}$$

Diagram 3

Now, we verify that (5.4). By the construction, the left-hand side $p_1!p_2{}^*(\Omega_x)$ (L_f) was the linking number of the $(2m_1 - 1)$-cycle L_f and the submanifold $P(x, 1)$. By (6.4), it is equal to the linking number of the map df as a cycle and the subset $\psi^{-1}(P(x,1))$ in $\mathbf{R}^N \times (\mathbf{R}^N \backslash \{0\})$. By (6.3), it is equal to the linking number of $(\pi_2 \times \pi_2) \circ df$ around the origin $(0,0)$ in $\mathbf{R}^{m_1} \times \mathbf{R}^{m_1}$, which is equal to the right-hand side of (5.4) up to sign. The correct sign is checked by some concrete examples (unit spheres, for example). $\square$

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