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INTERPOLATION SETS FOR LOGMODULAR BANACH ALGEBRAS

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1. Introduction. Let A_0 be the disk algebra, i.e., the uniform closure of polynomials on the unit circumference in the complex plane, H^∞ the algebra of bounded analytic functions on the open unit disk, and $\{z_k\}_{k=1}^\infty$ a sequence of distinct points in the open unit disk. We shall call $\{z_k\}_{k=1}^\infty$ an interpolating sequence if, for any bounded sequence $\{w_k\}_{k=1}^\infty$ of complex numbers, there exists a function $f \in H^\infty$ such that $f(z_k) = w_k$ for every k. Then the following two statements are known to be equivalent (cf. Hoffman [6; p. 208]): (a) if g is any continuous function on the closed unit disk, there exists $\varphi \in A_0$ such that $\varphi(z_k) = g(z_k)$ for every k; (b) $\{z_k\}_{k=1}^\infty$ is an interpolating sequence for H^∞, and the set of accumulation points of $\{z_k\}_{k=1}^\infty$ on the unit circumference has Lebesgue measure zero. Recently, Wada [8; Theorem 3.2] observed a similar fact for some uniform algebras generated by a single function. We know that such an algebra is isometrically isomorphic to the uniform closure of polynomials on the boundary ∂K of a compact set K with connected complement in the complex plane. As the latter is one of the best known examples of so-called Dirichlet algebras (cf. Wermer [9]), the question naturally arises as to whether this phenomenon is common to general Dirichlet algebras. The answer is in a sense affirmative and indeed we shall show in this paper that the theorem mentioned above can be extended to arbitrary logmodular Banach algebras.

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2. Statement of the theorem. Let X be a compact Hausdorff space. A *uniform algebra* on X is a uniformly closed subalgebra A of the space $C(X)$ of continuous complex functions on X which contains the function 1 and separates the points of X. A^{-1} denotes the set of all functions φ in A such that $\varphi^{-1} = 1/\varphi$ is also in A and $\log|A^{-1}|$ denotes the set of logarithms of moduli of elements of A^{-1}. A uniform algebra A is called *logmodular* if $\log|A^{-1}|$ is uniformly dense in the space $C_R(X)$ of continuous real functions on X. A detailed discussion on logmodular algebras is given by Hoffman [7]. $\mathfrak{M}(A)$ denotes the maximal ideal space of A and $\hat{A}$ denotes the Gelfand representation of A. $M(X)$ denotes the space of Radon measures on X.

Let A be a logmodular Banach algebra on X. Then there exists, for each $m \in \mathfrak{M}(A)$, a unique representing measure $\mu_m \in M(X)$. μ_m is a probability measure on X such that $m(\varphi) = \int_X \varphi(x) d\mu_m(x)$ for $\varphi \in A$. It is also known that μ_m is an Arens-Singer measure, meaning that

$$\log|m(\varphi)| = \log\left|\int_X \varphi(x) d\mu_m(x)\right| = \int_X |\log \varphi(x)| \, d\mu_m(x) \tag{1}$$

for each $\varphi \in A^{-1}$. $\mathfrak{M}(A)$ is divided into so-called Gleason parts (Gleason [3], Hoffman [7]). We denote by $\mathcal{P}$ the set of all parts P contained in $\mathfrak{M}(A) \sim X$. For each part $P \in \mathcal{P}$, let μ_P be the representing measure for any point in P, that is determined by P up to equivalence. So we choose one μ_P for each $P \in \mathcal{P}$ once for all. L^∞ denotes the subspace of $\Pi_{P\in\mathcal{P}} L^\infty(d\mu_P)$ consisting of all vectors (f_P) such that $\|(f_P)\|_\infty = \sup_{P\in\mathcal{P}} \|f_P\|_{\infty,P} < +\infty$, where $\|\cdot\|_{\infty,P}$ denotes the norm of $L^\infty(d\mu_P)$, and L^1 denotes the subspace of $\Pi_{P\in\mathcal{P}} L^1(d\mu_P)$ consisting of all vectors (f_P) such that $\|(f_P)\|_1 = \sum_{P\in\mathcal{P}} \|f_P\|_{1,P} < +\infty$, where $\|\cdot\|_{1,P}$ denotes the norm of $L^1(d\mu_P)$. If $\mathcal{P}$ is empty, then L^1 and L^∞ do not have any meaning so that we may assume $\mathcal{P}$ is non-empty. Then L^1 is a Banach space and L^∞ is the dual of L^1. For any $f \in C(X)$, the vector (f_P), with $f_P = f$ for all $P \in \mathcal{P}$, belongs to L^∞, so that $C(X)$, or a homomorphic image of $C(X)$, is contained in L^∞ as a subalgebra. (In general, the mapping $f \to (f_P)$ is norm-decreasing.) The algebra A, or its homomorphic image, is also viewed as a subalgebra of L^∞. We define H^∞ to be the $\sigma(L^\infty, L^1)$-closure of the latter subalgebra in L^∞. It is easy to see that each function $\psi = (\psi_P)$ in H^∞ has a definite value at each point $m \in P \subseteq \mathfrak{M}(A) \sim X$, which we shall denote by $\hat{\psi}(m)$:

$$\hat{\psi}(m) = \int_X \psi_P(x) d\mu_m(x) \,. \tag{2}$$

Consider the following two properties that generalize the properties (a) and (b) in the introduction: for a subset F_0 of $\mathfrak{M}(A)$,

(I) $\hat{A} | F_0 = C(F_0)$,

(II) $A | (F_0 \cap X) = C(F_0 \cap X)$ and $\hat{H}^\infty | (F_0 \sim X) = C^b(F_0 \sim X)$, where $C^b(F_0 \sim X)$ denotes the space of bounded continuous complex functions on $F_0 \sim X$.

Then the theorem we wish to prove in the present paper is the following:

Theorem 1. *Let A be logmodular on X and F_0 a closed subset of $\mathfrak{M}(A)$.*

(i) *If $F_0 \sim X$ is discrete as a subspace of $\mathfrak{M}(A)$, then* (I) *implies* (II).

(ii) *If F_0 intersects at most countably many parts in $\mathfrak{M}(A) \sim X$, then* (II) *implies* (I).

It can be shown that, under the hypothesis on F_0 in (ii) of Theorem 1 and the condition $\hat{H}^\infty | (F_0 \sim X) = C^b(F_0 \sim X)$, $F_0 \sim X$ is countable. We do not know whether our countability hypothesis on F_0 can be weakened or omitted at all. We shall make a comment on this matter in the final section of this paper.

3. Lemmas. We collect here some results on uniform algebras that will be needed in the proof of the theorem.

Lemma 1. *If A is logmodular and F is closed in X, then the following are equivalent*:

(i) $\mu \in M(X)$ *and* $\mu \perp A$ *imply* $\mu | F = 0$,

(ii) $A | F = C(F)$.

Proof. Let g be any positive continuous function on X and let $\varepsilon > 0$. Then there exists an $h \in C(X)$ such that $g + \varepsilon = e^h$. Since A is logmodular, there exists, for any $\varepsilon' > 0$, $\varphi \in A^{-1}$ such that $|h - \log|\varphi|| < \varepsilon'$. We can take $\varepsilon' > 0$ so small that $|e^h - |\varphi|| < \varepsilon$ and therefore $|g - |\varphi|| < 2\varepsilon$. So the lemma follows from Theorem 4.10 of Glicksberg [4]. Q.E.D.

Lemma 2. *If A is logmodular, F is closed in X and $A | F = C(F)$, then we have $\mu_m | F = 0$ for the representing measure μ_m for any point $m \in \mathfrak{M}(A) \sim X$.*

Proof. Let x be any point in X. Since m is outside of X, there exists a function $\varphi \in A$ such that $\varphi(x) \neq 0$ but $\hat{\varphi}(m) = 0$. Since $\varphi \mu_m$ is orthogonal to A, we have $\varphi \mu_m | F = 0$ by Lemma 1. As x is arbitrary, we see that $\mu_m | F = 0$. Q.E.D.

Lemma 3. *If A is logmodular, then the correspondence $m \to \mu_m$ is a continuous function from $\mathfrak{M}(A)$ into $M(X)$, where $M(X)$ is equipped with the weak* topology $\sigma(M(X), C(X))$.*

Proof. Let g be any function in $C_R(X)$ and let $\varepsilon > 0$. Since A is logmodular, there exists a function $\varphi \in A^{-1}$ such that $|g - \log|\varphi|| < \varepsilon$ on X. If a net $\{m_\alpha\}$ converges to m in $\mathfrak{M}(A)$ and μ_α (resp. μ) denote the representing measures of m_α (resp. m), then

$$\begin{aligned} \left|\int_X g(d\mu_\alpha - d\mu)\right| &\leqslant \left|\int_X (g - \log|\varphi|)(d\mu_\alpha - d\mu)\right| + \left|\int_X \log|\varphi|(d\mu_\alpha - d\mu)\right| \\ &\leqslant 2\varepsilon + \big|\log|m_\alpha(\varphi)| - \log|m(\varphi)|\big| \end{aligned}$$

because of the equality (1). As $\varphi \in A^{-1}$, we have $\log|m_\alpha(\varphi)| \to \log|m(\varphi)|$. Hence $\int_X g(d\mu_\alpha - d\mu) \to 0$. It follows immediately that $\int_X g(d\mu_\alpha - d\mu) \to 0$ for any $g \in C(X)$. Q.E.D.

The following lemma tells us the structure of a measure $\tau \in M(X)$ which is orthogonal to a logmodular algebra A on X. This was first proved by Glicksberg and Wermer [5] for a Dirichlet algebra on X, but a close inspection of their proof reveals that the same is still valid for a logmodular algebra.

Lemma 4. *If* A *is logmodular and* $\tau \in M(X)$ *is orthogonal to* A*, then there exists an at most countable set of parts* P_i *in* $\mathcal{P}$*, for each* i *some* k_i *in* $H^1_0(d\mu_i)$ *with* $\mu_i = \mu_{P_i}$*, and a measure* $\sigma \in M(X)$ *which is orthogonal to* A *and is singular with respect to all* μ_m *with* $m \in \mathfrak{M}(A)$ *such that, with the series converging in total variation, we have*

$$(3) \qquad \tau = \textstyle\sum_{i=1}^{\infty} k_i \mu_i + \sigma \, .$$

We omit the proof because it is similar to Glicksberg and Wermer's. We also need the following result due to Glicksberg [4; Corollary 3.2].

Lemma 5. *Let* C *be a closed subalgebra of* $C(Y)$ *on a compact Hausdorff space* Y *and let* F *be a closed subset of* Y*. Then* $C \mid F = C(F)$ *if and only if, for some* $c \geqslant 1$,

$$(4) \qquad \|\mu \mid F\| \leqslant c \|\mu \mid (Y \sim F)\|$$

for all $\mu \in M(Y)$, $\mu \perp C$.

4. Proof of Theorem 1.

The case (*i*). Since any continuous function on $F_0 \cap X$ can be extended to a continuous function on F_0, $A \mid (F_0 \cap X) = C(F_0 \cap X)$ is a direct consequence of (I).

We put $B = C(F_0) \mid (F_0 \sim X)$. Then it is easy to see that B is closed in $C^b(F_0 \sim X)$ with respect to the supremum norm. If π denotes the restriction map of $\hat{A}$ to $F_0 \sim X$, then (I) implies that π maps $\hat{A}$ onto B. π is clearly continuous so that it is a homomorphism by a theorem of Banach [1; Chap. III], i.e., we can find a constant $c > 0$ such that, for any $f \in B$, there exists a $\varphi \in \hat{A}$ which satisfies $\pi(\varphi) = f$ and $\|\varphi\| \leqslant c\|f\|$. Take any $g \in C^b(F_0 \sim X)$. Then there exists a bounded net $\{f_\alpha\}$ in B which converges to g pointwise on $F_0 \sim X$. For each α, we can find $\varphi_\alpha \in \hat{A}$ such that $\pi(\varphi_\alpha) = f_\alpha$ and $\|\varphi_\alpha\| \leqslant c\|f_\alpha\|$. $\{\varphi_\alpha\}$ can be regarded as a bounded net in L^∞. Since L^∞ is the dual of L^1, there exists a subnet $\{\varphi_{\alpha'}\}$ of $\{\varphi_\alpha\}$ which converges to some $\psi \in H^\infty$ with respect to the weak* topology $\sigma(L^\infty, L^1)$. Let $m \in F_0 \sim X$. If m belongs to a part P, then the representing measure μ_m for m is absolutely continuous with respect to μ_P and indeed there exists a positive function $k_m \in L^\infty(d\mu_P)$ such that $\mu_m = k_m \mu_P$. Then

$$f_{\alpha'}(m) = \varphi_{\alpha'}(m) = \int_X \varphi_{\alpha'}(x) d\mu_m(x) = \int_X \varphi_{\alpha'}(x) k_m(x) d\mu_P(x)$$

$$\to \int_X \psi_P(x) k_m(x) d\mu_P(x) = \int_X \psi_P(x) d\mu_m(x) = \hat{\psi}(m)$$

where $\psi=(\psi_p)$. Hence $\hat{\psi}\,|(F_0\sim X)=g$. This shows that $C^b(F_0\sim X)\subseteq \hat{H}^\infty\,|(F_0\sim X)$. Since $F_0\sim X$ is discrete, the converse inclusion is obvious. This proves the case (i).

The case (*ii*). We now suppose the property (II). We first consider the property

$$\hat{H}^\infty\,|(F_0\sim X)=C^b(F_0\sim X)\,. \tag{5}$$

The topology of $F_0\sim X$ induced from that of $\mathfrak{M}(H^\infty)$ is the weakest topology of $F_0\sim X$ that makes each function in H^∞ continuous. Similarly, the original topology of $F_0\sim X$ as a subset of $\mathfrak{M}(A)$ is the weakest topology of $F_0\sim X$ that makes each function in $C^b(F_0\sim X)$ continuous. Thus the equality (5) implies that these two topologies on $F_0\sim X$ coincide. Therefore the relation (2) defines a topological imbedding of $F_0\sim X$ into $\mathfrak{M}(H^\infty)$.

We know that L^∞ is isometrically and algebraically isomorphic to the Banach algebra $C(\Omega)$ for some compact Hausdorff space Ω. We have the natural mappings $A\to C(X)\to L^\infty=C(\Omega)$ and $A\to H^\infty\to L^\infty=C(\Omega)$, where the mappings are bounded. Therefore we have the natural mappings, among their maximal ideal spaces, that are continuous: $\Omega\to X\to\mathfrak{M}(A)$ and $\Omega\xrightarrow{\pi_1}\mathfrak{M}(H^\infty)\xrightarrow{\pi_2}\mathfrak{M}(A)$. It follows immediately that $\pi_2\circ\pi_1(\Omega)\subseteq X$.

Let $\Omega_1=\pi_1(\Omega)$. Then we have $\pi_2(\Omega_1)\subseteq X$. On the other hand, the set $\mathfrak{M}(A)\sim X$ can be identified as a subset of $\mathfrak{M}(H^\infty)$ by means of the formula (2). It follows from the fact $\pi_2(\mathfrak{M}(A)\sim X)=\mathfrak{M}(A)\sim X$ that we have $(\mathfrak{M}(A)\sim X)\cap\Omega_1=\emptyset$ in $\mathfrak{M}(H^\infty)$ and in particular $(F_0\sim X)\cap\Omega_1=\emptyset$.

Now let $Y=F_0\cup X$. Then $\hat{A}\,|\,Y$ is a uniformly closed subalgebra of $C(Y)$, because Y contains the Šilov boundary X of the algebra A. We suppose that a measure $\nu\in M(Y)$ is orthogonal to $\hat{A}\,|\,Y$. It follows from Lemma 3 that

$$\tau=\int_Y \mu_m d\nu(m)$$

is well-defined and belongs to $M(X)$. Then, for any $\varphi\in A$,

$$\int_X \varphi(x)d\tau(x)=\int_Y(\mu_m,\,\varphi)d\nu(m)=\int_Y\hat{\varphi}(m)d\nu(m)=0\,.$$

So $\tau\perp A$. By Lemma 4, τ is expressed as a series converging in total variation:

$$\tau=\textstyle\sum_{i=1}^\infty k_i\mu_i+\sigma=\xi+\sigma\,,\quad\text{say,} \tag{3}$$

where μ_i come from distinct parts in $\mathfrak{M}(A)\sim X$, $k_i\in H_0^1(d\mu_i)$, and σ is completely singular. For later convenience, we admit here those μ_i with $k_i=0$.

We set $\nu_0=\nu-\tau$, where we regard τ as a measure on Y. Then we have

$$\int_Y \mu_m d\nu_0(m)=0\,. \tag{6}$$

In fact, as we know that μ_x for $x\in X$ is the evaluation measure δ_x at x,

$$\int_Y \mu_m d\nu_0(m) = \int_Y \mu_m d\nu(m) - \int_X \mu_x d\tau(x) = \tau - \int_X \delta_x d\tau(x) = \tau - \tau = 0 .$$

We transfer ν_0 and ξ to $\mathfrak{M}(H^\infty)$ as follows. As $F_0 \sim X$ is topologically imbedded in $\mathfrak{M}(H^\infty) \sim \Omega_1$, the measure $\nu_0 | (F_0 \sim X)$ is directly transferred to $\mathfrak{M}(H^\infty) \sim \Omega_1$. We denote it by ν'. In order to transfer $\nu_0 | X$, we use the following immediate consequence of (6):

$$\nu_0 | X = - \int_{F_0 \sim X} \mu_m d\nu_0(m) .$$

Now, F_0 intersects at most countably many parts in $\mathfrak{M}(A) \sim X$. By the convention we adopted before, we may suppose that these parts are already in the set of parts used in the expression (3). Thus,

$$(7) \qquad \nu_0 | X = - \textstyle\sum_{i=1}^\infty \displaystyle\int_{F_0 \cap P_i} \mu_m d\nu_0(m) = \textstyle\sum_{i=1}^\infty u_i \mu_i$$

where $u_i \in L^1(d\mu_i)$ for $i=1, 2, \cdots$. The first equality in (7) is obvious. The second equality can be seen as follows. It is enough to show that $F_0 \cap P_i$ is at most countable for each i. This is trivial if P_i consists of a single point. So we may assume that P_i contains more than one point. It follows from the construction of our H^∞ that $\hat{H}^\infty | P_i = \hat{H}^\infty(d\mu_i) | P_i$ where $H^\infty(d\mu_i)$ denotes the weak* closure of A in the space $L^\infty(d\mu_i)$. It is also known (cf. Hoffman [7; Section 7]) that there exists a continuous univalent mapping κ of the open unit disk D onto P_i such that $\psi \circ \kappa$ is an analytic function on D for every $\psi \in H^\infty(d\mu_i)$. We now suppose, on the contrary, that $F_0 \cap P_i$ is uncountable. Then we can find disjoint compact subsets K_1 and K_2 of D in such a way that $F_0 \cap \kappa(K_1) \neq \emptyset$ and $F_0 \cap \kappa(K_2)$ is uncountable. As both $F_0 \cap \kappa(K_1)$ and $F_0 \cap \kappa(K_2)$ are also disjoint compact sets in P_i (and therefore in $\mathfrak{M}(A)$), we can find a function $g \in C^b(F_0 \sim X)$ such that

$$g | (F_0 \cap \kappa(K_1)) = 1 \quad \text{and} \quad g | (F_0 \cap \kappa(K_2)) = 0 .$$

By (5) and $\hat{H}^\infty | P_i = \hat{H}^\infty(d\mu_i) | P_i$, there exists a $\psi \in H^\infty(d\mu_i)$ such that $\psi | (F_0 \cap P_i) = g | (F_0 \cap P_i)$. This ψ then satisfies

$$(\psi \circ \kappa) | K_1 = 1 \quad \text{and} \quad (\psi \circ \kappa) | K_2 = 0 .$$

But, $\psi \circ \kappa$ is a non-constant analytic function on D and has uncountably many zeros. This contradition shows that $F_0 \cap P_i$ is at most countable.

We now define a functional

$$\Phi(f) = \textstyle\sum_{i=1}^\infty \displaystyle\int_X (k_i + u_i)(x) f_i(x) d\mu_i(x)$$

on L^∞ where f_i denotes the P_i-th component of the vector $f \in L^\infty$. Φ is a bounded linear functional on L^∞ and thus defines a measure $\eta \in M(\Omega)$. Since π maps Ω onto Ω_1 continuously, there exists a measure ν'' on Ω_1 so that $\pi_1(\eta)=\nu''$. We finally define a measure $\tilde{\nu}$ on $\mathfrak{M}(H^\infty)$ by putting

$$\tilde{\nu} = \begin{cases} \nu' & \text{on} \quad \mathfrak{M}(H^\infty) \sim \Omega_1\,, \\ \nu'' & \text{on} \quad \Omega_1\,. \end{cases}$$

From our construction of $\tilde{\nu}$ follows easily that $\tilde{\nu} \perp H^\infty$. If we denote by G_0 the closure of $F_0 \sim X$ in the space $\mathfrak{M}(H^\infty)$, then our assumption on H^∞ implies that $\hat{H}^\infty | G_0 = C(G_0)$. Thus, by Lemma 5 with $C=\hat{H}^\infty$, $Y=\mathfrak{M}(H^\infty)$, and $F=G_0$, there exists a constant $c \geqslant 1$ such that

$$\|\tilde{\nu} | G_0\| \leqslant c\|\tilde{\nu} | (\mathfrak{M}(H^\infty) \sim G_0)\|\,,$$

where c does not depend on $\tilde{\nu}$. Consequently, we have

$$\begin{aligned}
(8) \qquad \|\nu | (F_0 \sim X)\| &= \|\nu_0 | (F_0 \sim X)\| = \|\tilde{\nu} | (F_0 \sim X)\| \leqslant \|\tilde{\nu} | G_0\| \\
&\leqslant c\|\tilde{\nu} | (\mathfrak{M}(H^\infty) \sim G_0)\| \leqslant c\|\tilde{\nu} | \Omega_1\| = c\|\nu'' | \Omega_1\| \\
&\leqslant c \textstyle\sum_{i=1}^\infty \displaystyle\int_X |k_i(x)+u_i(x)|\, d\mu_i(x) \\
&= c\|(\nu_0 | X)+\xi\| \\
&\leqslant c(\|(\nu_0 | X)+\xi\| + \|\sigma\|) = c\|(\nu_0 | X)+\xi+\sigma\| \\
&= c\|\nu | X\|\,,
\end{aligned}$$

because $(\nu_0 | X)+\xi$ and σ are mutually singular on X.

Now we use the assumption $A | (F_0 \cap X) = C(F_0 \cap X)$. By Lemma 2 and the expression (7), we have $\nu_0 | (F_0 \cap X) = 0$. As ξ and σ are orthogonal to A, Lemma 1 implies that $\xi | (F_0 \cap X) = 0$ as well as $\sigma | (F_0 \cap X) = 0$. Hence $\nu | (F_0 \cap X) = 0$. It follows therefore from (8) that

$$\|\nu | F_0\| \leqslant c\|\nu | (X \sim F_0)\|\,,$$

where c is a constant independent of ν. Thus by Lemma 5, $\hat{A} | F_0 = C(F_0)$. This proves the case (ii) and the theorem is established.

5. Uniform algebras with a single generator.

In this section we shall be concerned with uniform algebras generated by a single function, which were previously discussed by Wada [8].

Let A be a uniform algebra on a compact Hausdorff space X such that A is generated by a single function f_0. Then, denoting by $P(Z)$ the uniform closure of polynomials on a compact set Z in the complex plane, we know that A is isometrically isomorphic to the algebra $P(\partial K)$, where ∂K denotes the boundary of some compact set K with connected complement in the complex plane (and trivially conversely). Such a $P(\partial K)$ is known to be a Dirichlet algebra, meaning that real parts of functions in $P(\partial K)$ are uniformly dense in $C_R(\partial K)$ (cf. Wermer [9; Theorem 5.1]). So Theorem 1 can be applied to $P(\partial K)$. It is also known (Hoffman [7]) that a part of a logmodular algebra (or, a fortiori, a Dirichlet algebra) is either a one-point set or equivalent (but not necessarily homeomorphic) to the open unit disk, as was mentioned in the proof of Theorem 1. A part which is equivalent to the open unit disk is usually called a disk part. Clearly, our $P(\partial K)$ has an at most countable set of disk parts. If a single point $z\in K$ forms a part of $P(\partial K)$, then z cannot be in the interior of K and so $z\in\partial K$. This means that $K\sim\partial K$ consists of at most countably many disk parts and so does $\mathfrak{M}(A)\sim\partial A$, where ∂A denotes the Šilov boundary of A. The algebra $P(\partial K)$ is completely characterized by Mergelyan's well known theorem as follows (cf. Wermer [9; Theorem 7.6]): $P(\partial K)$ consists of all functions in $C(K)$ that are analytic at all interior points of K. It then follows that the space H^∞ for $P(\partial K)$ consists of all bounded continuous functions on $K\sim\partial K$ that are analytic on every disk part. So Theorem 1 implies the following:

Theorem 2. *Let A be a uniform algebra with a single generator on X, such that X is the Šilov boundary of A, and F_0 a closed subset of $\mathfrak{M}(A)$. Then,* (I) *and* (II) *are equivalent.*

Proof. As we have seen above, we may assume $A=P(\partial K)$ with a compact set K having connected complement in the complex plane. Then we have $\mathfrak{M}(A)=K$ and the topology of $\mathfrak{M}(A)=K$ as the maximal ideal space of A is equivalent to the usual topology of the plane. If (I) holds, then $F_0\cap P$ is discrete for any part $P\subseteq K\sim\partial K$ and consequently $F_0\sim\partial K$ is discrete. So, by Theorem 1, (i), we have (II). Conversely, if (II) holds, then (I) is also valid, because $K\sim\partial K$ has at most countably many parts so that the hypothesis in Theorem 1, (ii) is automatically satisfied. This proves Theorem 2.

Theorem 2 extends a theorem of Wada mentioned earlier [8; Theorem 3.2], although our formulation is a little different from his. Of course, the equivalence of the statements (a) and (b) in the introduction is a special case of Theorem 2.

6. A remark. We wish to make a comment on the countability hypothesis in Theorem 1, (ii). Let A be logmodular on X and F_0 a closed subset of $\mathfrak{M}(A)$ such that $\hat{H}^\infty|(F_0\sim X)=C^b(F_0\sim X)$. Let F be any compact subset of $F_0\sim X$. By means of the mapping $m\to\mu_m$, F can be viewed as a weakly* compact subset of $M(X)$, so that we can define the weakly* closed convex envelope $co(F)$ of F in $M(X)$. We ask the question: Does $co(F)$ contain a non-zero measure which is singular with respect to all μ_m, $m\in\mathfrak{M}(A)$? If the answer is negative for any compact subset F of $F_0 \sim X$, then we can remove the countability assumption from Theorem 1, (ii). If this is not the case, then the situation may probably be more delicate.

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