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<th>Interpolation sets for logmodular Banach algebras</th>
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<td>Author(s)</td>
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<tr>
<td>Citation</td>
<td>Osaka Journal of Mathematics. 3(2) P.303-P.311</td>
</tr>
<tr>
<td>Issue Date</td>
<td>1966</td>
</tr>
<tr>
<td>Text Version</td>
<td>publisher</td>
</tr>
<tr>
<td>URL</td>
<td><a href="https://doi.org/10.18910/9376">https://doi.org/10.18910/9376</a></td>
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1. Introduction. Let $A_0$ be the disk algebra, i.e., the uniform closure of polynomials on the unit circumference in the complex plane, $H^\infty$ the algebra of bounded analytic functions on the open unit disk, and $\{z_k\}_{k=1}^\infty$ a sequence of distinct points in the open unit disk. We shall call $\{z_k\}_{k=1}^\infty$ an interpolating sequence if, for any bounded sequence $\{w_k\}_{k=1}^\infty$ of complex numbers, there exists a function $f \in H^\infty$ such that $f(z_k) = w_k$ for every $k$. Then the following two statements are known to be equivalent (cf. Hoffman [6; p. 208]): (a) if $g$ is any continuous function on the closed unit disk, there exists $\varphi \in A_0$ such that $\varphi(z_k) = g(z_k)$ for every $k$; (b) $\{z_k\}_{k=1}^\infty$ is an interpolating sequence for $H^\infty$, and the set of accumulation points of $\{z_k\}_{k=1}^\infty$ on the unit circumference has Lebesgue measure zero. Recently, Wada [8; Theorem 3.2] observed a similar fact for some uniform algebras generated by a single function. We know that such an algebra is isometrically isomorphic to the uniform closure of polynomials on the boundary $\partial K$ of a compact set $K$ with connected complement in the complex plane. As the latter is one of the best known examples of so-called Dirichlet algebras (cf. Wermer [9]), the question naturally arises as to whether this phenomenon is common to general Dirichlet algebras. The answer is in a sense affirmative and indeed we shall show in this paper that the theorem mentioned above can be extended to arbitrary logmodular Banach algebras.

I wish to thank Professor O. Takenouchi for his careful reading of the manuscript and valuable suggestions.

2. Statement of the theorem. Let $X$ be a compact Hausdorff space. A uniform algebra on $X$ is a uniformly closed subalgebra $A$ of the space $C(X)$ of continuous complex functions on $X$ which contains the function 1 and separates the points of $X$. $A^{-1}$ denotes the set of all functions $\varphi$ in $A$ such that $\varphi^{-1} = 1/\varphi$ is also in $A$ and $\log |A^{-1}|$ denotes the set of logarithms of moduli of elements of $A^{-1}$. A uniform algebra $A$ is called logmodular if $\log |A^{-1}|$ is uniformly dense in the space $C^R(X)$ of continuous real functions on $X$. A detailed discussion on logmodular algebras is given by Hoffman [7]. $\mathcal{M}(A)$ denotes the maximal
ideal space of $A$ and $\hat{A}$ denotes the Gelfand representation of $A$. $M(X)$ denotes the space of Radon measures on $X$.

Let $A$ be a logmodular Banach algebra on $X$. Then there exists, for each $m \in \mathcal{M}(A)$, a unique representing measure $\mu_m \in M(X)$. $\mu_m$ is a probability measure on $X$ such that $m(\varphi) = \int_X \varphi(x) d\mu_m(x)$ for $\varphi \in A$. It is also known that $\mu_m$ is an Arens-Singer measure, meaning that

$$\log |m(\varphi)| = \log \left| \int_X \varphi(x) d\mu_m(x) \right| = \int_X |\log \varphi(x)| d\mu_m(x)$$

for each $\varphi \in A^{-1}$. $\mathcal{M}(A)$ is divided into so-called Gleason parts (Gleason [3], Hoffman [7]). We denote by $\mathcal{P}$ the set of all parts $P$ contained in $\mathcal{M}(A) \sim X$. For each part $P \in \mathcal{P}$, let $\mu_P$ be the representing measure for any point in $P$, that is determined by $P$ up to equivalence. So we choose one $\mu_P$ for each $P \in \mathcal{P}$ once for all. $L^\infty$ denotes the subspace of $\Pi_{P \in \mathcal{P}} L^\infty(d\mu_P)$ consisting of all vectors $(f_P)$ such that $\| (f_P) \|_{\infty} = \sup_{P \in \mathcal{P}} \| f_P \|_{\infty} < +\infty$, where $\| \cdot \|_{\infty,P}$ denotes the norm of $L^\infty(d\mu_P)$, and $L^1$ denotes the subspace of $\Pi_{P \in \mathcal{P}} L^1(d\mu_P)$ consisting of all vectors $(f_P)$ such that $\| (f_P) \|_1 = \sum_{P \in \mathcal{P}} \| f_P \|_1 < +\infty$, where $\| \cdot \|_1,P$ denotes the norm of $L^1(d\mu_P)$. If $\mathcal{P}$ is empty, then $L^1$ and $L^\infty$ do not have any meaning so that we may assume $\mathcal{P}$ is non-empty. Then $L^1$ is a Banach space and $L^\infty$ is the dual of $L^1$. For any $f \in C(X)$, the vector $(f_P)$, with $f_P = f$ for all $P \in \mathcal{P}$, belongs to $L^\infty$, so that $C(X)$, or a homomorphic image of $C(X)$, is contained in $L^\infty$ as a subalgebra. (In general, the mapping $f \mapsto (f_P)$ is norm-decreasing.) The algebra $A$, or its homomorphic image, is also viewed as a subalgebra of $L^\infty$. We define $H^\infty$ to be the $\sigma(L^\infty,L^1)$-closure of the latter subalgebra in $L^\infty$. It is easy to see that each function $\psi = (\psi_P)$ in $H^\infty$ has a definite value at each point $m \in P \subseteq \mathcal{M}(A) \sim X$, which we shall denote by $\dot{\psi}(m)$:

$$\dot{\psi}(m) = \int_X \psi_P(x) d\mu_m(x).$$

Consider the following two properties that generalize the properties (a) and (b) in the introduction: for a subset $F_0$ of $\mathcal{M}(A)$,

(I) $\hat{A}|_{F_0} = C(F_0)$,

(II) $A|_{(F_0 \cap X) = C(F_0 \cap X)}$ and $\hat{H}^\infty|_{(F_0 \sim X) = C^b(F_0 \sim X)}$, where

$C^b(F_0 \sim X)$ denotes the space of bounded continuous complex functions on $F_0 \sim X$.

Then the theorem we wish to prove in the present paper is the following:

**Theorem 1.** Let $A$ be logmodular on $X$ and $F_0$ a closed subset of $\mathcal{M}(A)$.

(i) If $F_0 \sim X$ is discrete as a subspace of $\mathcal{M}(A)$, then (I) implies (II).

(ii) If $F_0$ intersects at most countably many parts in $\mathcal{M}(A) \sim X$, then (II) implies (I).
It can be shown that, under the hypothesis on $F_0$ in (ii) of Theorem 1 and the condition $\mathcal{H}^\ast \mid (F_0 \sim X) = C^b(F_0 \sim X)$, $F_0 \sim X$ is countable. We do not know whether our countability hypothesis on $F_0$ can be weakened or omitted at all. We shall make a comment on this matter in the final section of this paper.

3. Lemmas. We collect here some results on uniform algebras that will be needed in the proof of the theorem.

**Lemma 1.** If $A$ is logmodular and $F$ is closed in $X$, then the following are equivalent:

(i) $\mu \in M(X)$ and $\mu \perp A$ imply $\mu \mid F = 0$,

(ii) $A \mid F = C(F)$.

Proof. Let $g$ be any positive continuous function on $X$ and let $\varepsilon > 0$. Then there exists an $h \in C(X)$ such that $g + \varepsilon = e^h$. Since $A$ is logmodular, there exists, for any $\varepsilon' > 0$, $\varphi \in A^{-1}$ such that $|h - \log |\varphi|| < \varepsilon'$. We can take $\varepsilon' > 0$ so small that $|e^h - |\varphi|| < \varepsilon$ and therefore $|g - |\varphi|| < 2\varepsilon$. So the lemma follows from Theorem 4.10 of Glicksberg [4]. Q.E.D.

**Lemma 2.** If $A$ is logmodular, $F$ is closed in $X$ and $A \mid F = C(F)$, then we have $\mu_m \mid F = 0$ for the representing measure $\mu_m$ for any point $m \in \mathfrak{M}(A) \sim X$.

Proof. Let $x$ be any point in $X$. Since $m$ is outside of $X$, there exists a function $\varphi \in A$ such that $\varphi(x) \neq 0$ but $\varphi(m) = 0$. Since $\varphi \mu_m$ is orthogonal to $A$, we have $\varphi \mu_m \mid F = 0$ by Lemma 1. As $x$ is arbitrary, we see that $\mu_m \mid F = 0$. Q.E.D.

**Lemma 3.** If $A$ is logmodular, then the correspondence $m \mapsto \mu_m$ is a continuous function from $\mathfrak{M}(A)$ into $M(X)$, where $M(X)$ is equipped with the weak* topology $\sigma(M(X), C(X))$.

Proof. Let $g$ be any function in $C_0(X)$ and let $\varepsilon > 0$. Since $A$ is logmodular, there exists a function $\varphi \in A^{-1}$ such that $|g - \log |\varphi|| < \varepsilon$ on $X$. If a net $\{m_\alpha\}$ converges to $m$ in $\mathfrak{M}(A)$ and $\mu_\alpha$ (resp. $\mu$) denote the representing measures of $m_\alpha$ (resp. $m$), then

$$\left| \int_X g(d\mu_\alpha - d\mu) \right| \leq \left| \int_X (g - \log |\varphi|)(d\mu_\alpha - d\mu) \right| + \int_X \log |\varphi| (d\mu_\alpha - d\mu)$$

$$\leq 2\varepsilon + \left| \log |m_\alpha(\varphi)| - \log |m(\varphi)| \right|$$

because of the equality (1). As $\varphi \in A^{-1}$, we have $\log |m_\alpha(\varphi)| \to \log |m(\varphi)|$. Hence $\int_X g(d\mu_\alpha - d\mu) \to 0$. It follows immediately that $\int_X g(d\mu_\alpha - d\mu) \to 0$ for any $g \in C(X)$. Q.E.D.

The following lemma tells us the structure of a measure $\tau \in M(X)$ which
is orthogonal to a logmodular algebra $A$ on $X$. This was first proved by Glicksberg and Wermer [5] for a Dirichlet algebra on $X$, but a close inspection of their proof reveals that the same is still valid for a logmodular algebra.

**Lemma 4.** If $A$ is logmodular and $\tau \in M(X)$ is orthogonal to $A$, then there exists an at most countable set of parts $P_i$ in $\mathcal{B}$, for each $i$ some $k_i \in H_0^s(d\mu)$ with $\mu_i = \mu_{P_i}$, and a measure $\sigma \in M(X)$ which is orthogonal to $A$ and is singular with respect to all $\mu_m$ with $m \in \mathcal{M}(A)$ such that, with the series converging in total variation, we have

$$\tau = \sum_{i=1}^{\infty} k_i \mu_i + \sigma.$$  

We omit the proof because it is similar to Glicksberg and Wermer's. We also need the following result due to Glicksberg [4; Corollary 3.2].

**Lemma 5.** Let $C$ be a closed subalgebra of $C(Y)$ on a compact Hausdorff space $Y$ and let $F$ be a closed subset of $Y$. Then $C|F = C(F)$ if and only if, for some $c \geq 1,$

$$||\mu|F|| \leq c||\mu|(Y \sim F)||$$

for all $\mu \in M(Y), \mu \perp C$.

4. Proof of Theorem 1.

The case (i). Since any continuous function on $F_0 \cap X$ can be extended to a continuous function on $F_0$, $A| (F_0 \cap X) = C(F_0 \cap X)$ is a direct consequence of (I).

We put $B = C(F_0)(F_0 \sim X)$. Then it is easy to see that $B$ is closed in $C^b(F_0 \sim X)$ with respect to the supremum norm. If $\pi$ denotes the restriction map of $\hat{A}$ to $F_0 \sim X$, then (I) implies that $\pi$ maps $\hat{A}$ onto $B$. $\pi$ is clearly continuous so that it is a homomorphism by a theorem of Banach [1; Chap. III], i.e., we can find a constant $c > 0$ such that, for any $f \in B$, there exists a $\varphi \in \hat{A}$ which satisfies $\pi(\varphi) = f$ and $||\varphi|| \leq c|f|$. Take any $g \in C^b(F_0 \sim X)$. Then there exists a bounded net $\{f_{\alpha}\}$ in $B$ which converges to $g$ pointwise on $F_0 \sim X$. For each $\alpha$, we can find $\varphi_{\alpha} \in \hat{A}$ such that $\pi(\varphi_{\alpha}) = f_{\alpha}$ and $||\varphi_{\alpha}|| \leq c|f_{\alpha}|$. The $\{\varphi_{\alpha}\}$ can be regarded as a bounded net in $L^\infty$. Since $L^\infty$ is the dual of $L^1$, there exists a subnet $\{\varphi_{\alpha'}\}$ of $\{\varphi_{\alpha}\}$ which converges to some $\psi \in H^\infty$ with respect to the weak* topology $\sigma(L^\infty, L^1)$. Let $m \in F_0 \sim X$. If $m$ belongs to a part $P$, then the representing measure $\mu_m$ for $m$ is absolutely continuous with respect to $\mu_P$ and indeed there exists a positive function $k_m \in L^\infty(d\mu_P)$ such that $\mu_m = k_m \mu_P$. Then

$$f_{\alpha'}(m) = \varphi_{\alpha'}(m) = \int_X \varphi_{\alpha'}(x) d\mu_m(x) = \int_X \varphi_{\alpha'}(x) k_m(x) d\mu_P(x)$$

$$\rightarrow \int_X \psi_P(x) k_m(x) d\mu_P(x) = \int_X \psi_P(x) d\mu_m(x) = \hat{\psi}(m)$$
where \( \psi = (\psi_p) \). Hence \( \hat{\psi} |(F_0 \sim X) = g \). This shows that \( C^b(F_0 \sim X) \subseteq \hat{H}^\infty |(F_0 \sim X) \). Since \( F_0 \sim X \) is discrete, the converse inclusion is obvious. This proves the case (i).

**The case (ii).** We now suppose the property (II). We first consider the property

(5) \[ \hat{H}^\infty |(F_0 \sim X) = C^b(F_0 \sim X). \]

The topology of \( F_0 \sim X \) induced from that of \( \mathcal{M}(H^\infty) \) is the weakest topology of \( F_0 \sim X \) that makes each function in \( H^\infty \) continuous. Similarly, the original topology of \( F_0 \sim X \) as a subset of \( \mathcal{M}(A) \) is the weakest topology of \( F_0 \sim X \) that makes each function in \( C^b(F_0 \sim X) \) continuous. Thus the equality (5) implies that these two topologies on \( F_0 \sim X \) coincide. Therefore the relation (2) defines a topological imbedding of \( F_0 \sim X \) into \( \mathcal{M}(H^\infty) \).

We know that \( L^\infty \) is isometrically and algebraically isomorphic to the Banach algebra \( C(\Omega) \) for some compact Hausdorff space \( \Omega \). We have the natural mappings \( A \rightarrow C(X) \rightarrow L^\infty = C(\Omega) \) and \( A \rightarrow H^\infty \rightarrow L^\infty = C(\Omega) \), where the mappings are bounded. Therefore we have the natural mappings, among their maximal ideal spaces, that are continuous: \( \Omega \rightarrow X \rightarrow \mathcal{M}(A) \) and \( \pi_1 \rightarrow \mathcal{M}(L^\infty) \rightarrow \mathcal{M}(A) \). It follows immediately that \( \pi_2 \circ \pi_1 (\Omega) \subseteq X \).

Let \( \Omega_1 = \pi_1 (\Omega) \). Then we have \( \pi_1 (\Omega) \subseteq X \). On the other hand, the set \( \mathcal{M}(A) \sim X \) can be identified as a subset of \( \mathcal{M}(H^\infty) \) by means of the formula (2). It follows from the fact \( \pi_2 (\mathcal{M}(A) \sim X) = \mathcal{M}(A) \sim X \) that we have \( (\mathcal{M}(A) \sim X) \cap \Omega_1 = \emptyset \) in \( \mathcal{M}(H^\infty) \) and in particular \( (F_0 \sim X) \cap \Omega_1 = \emptyset \).

Now let \( Y = F_0 \cup X \). Then \( \hat{A} | Y \) is a uniformly closed subalgebra of \( C(Y) \), because \( Y \) contains the Šilov boundary \( X \) of the algebra \( A \). We suppose that a measure \( \nu \in M(Y) \) is orthogonal to \( \hat{A} | Y \). It follows from Lemma 3 that

\[ \tau = \int_Y \mu_m d\nu (m) \]

is well-defined and belongs to \( M(X) \). Then, for any \( \varphi \in A \),

\[ \int_X \varphi (x) d\tau (x) = \int_Y (\mu_m, \varphi) d\nu (m) = \int_Y \phi (m) d\nu (m) = 0. \]

So \( \tau \perp A \). By Lemma 4, \( \tau \) is expressed as a series converging in total variation:

(3) \[ \tau = \sum_{i=1}^{\infty} k_i \mu_i + \sigma = \xi + \sigma, \] where \( \mu_i \) come from distinct parts in \( \mathcal{M}(A) \sim X \), \( k_i \in H_b^0 (d \mu_i) \), and \( \sigma \) is completely singular. For later convenience, we admit here those \( \mu_i \) with \( k_i = 0 \).

We set \( \nu_0 = \nu - \tau \), where we regard \( \tau \) as a measure on \( Y \). Then we have

(6) \[ \int_Y \mu_m d\nu_0 (m) = 0. \]
In fact, as we know that $\mu_x$ for $x \in X$ is the evaluation measure $\delta_x$ at $x$,

$$\int_X \mu_x d\nu(m) = \int_X \mu_x d\tau(x) = \tau - \int_X \delta_x d\tau(x) = \tau - \tau = 0.$$ 

We transfer $\nu_0$ and $\xi$ to $\mathfrak{M}(H^\omega)$ as follows. As $F_0 \sim X$ is topologically imbedded in $\mathfrak{M}(H^\omega) \sim \Omega_1$, the measure $\nu_0 | (F_0 \sim X)$ is directly transferred to $\mathfrak{M}(H^\omega) \sim \Omega_1$. We denote it by $\nu'$. In order to transfer $\nu_0 | X$, we use the following immediate consequence of (6):

$$\nu_0 | X = - \int_{F_0 \sim X} \mu_m d\nu_0(m).$$

Now, $F_0$ intersects at most countably many parts in $\mathfrak{M}(A) \sim X$. By the convention we adopted before, we may suppose that these parts are already in the set of parts used in the expression (3). Thus,

$$v_0 | X = - \sum_{i=1}^\infty \int_{F_0 \cap P_i} \mu_m d\nu_0(m) = \sum_{i=1}^\infty u_i \mu_i$$

where $u_i \in L^1(d\mu_i)$ for $i = 1, 2, \ldots$. The first equality in (7) is obvious. The second equality can be seen as follows. It is enough to show that $F_0 \cap P_i$ is at most countable for each $i$. This is trivial if $P_i$ consists of a single point. So we may assume that $P_i$ contains more than one point. It follows from the construction of our $H^\omega$ that $\hat{H}^\omega | P_i = \hat{H}^\omega(d\mu_i) | P_i$ where $H^\omega(d\mu_i)$ denotes the weak* closure of $A$ in the space $L^\omega(d\mu_i)$. It is also known (cf. Hoffman [7; Section 7]) that there exists a continuous univalent mapping $\kappa$ of the open unit disk $D$ onto $P_i$ such that $\psi \circ \kappa$ is an analytic function on $D$ for every $\psi \in H^\omega(d\mu_i)$. We now suppose, on the contrary, that $F_0 \cap P_i$ is uncountable. Then we can find disjoint compact subsets $K_1$ and $K_2$ of $D$ in such a way that $F_0 \cap P_i$ and $F_0 \cap \kappa(K_2)$ is uncountable. As both $F_0 \cap \kappa(K_1)$ and $F_0 \cap \kappa(K_2)$ are also disjoint compact sets in $P_i$ (and therefore in $\mathfrak{M}(A)$), we can find a function $g \in C^0(F_0 \sim X)$ such that

$$g | (F_0 \cap \kappa(K_1)) = 1 \quad \text{and} \quad g | (F_0 \cap \kappa(K_2)) = 0.$$ 

By (5) and $\hat{H}^\omega | P_i = \hat{H}^\omega(d\mu_i) | P_i$, there exists a $\psi \in H^\omega(d\mu_i)$ such that $\psi | (F_0 \cap P_i) = g | (F_0 \cap P_i)$. This $\psi$ then satisfies

$$\psi \circ \kappa | K_1 = 1 \quad \text{and} \quad \psi \circ \kappa | K_2 = 0.$$ 

But, $\psi \circ \kappa$ is a non-constant analytic function on $D$ and has uncountably many zeros. This contradiction shows that $F_0 \cap P_i$ is at most countable.

We now define a functional

$$\Phi(f) = \sum_{i=1}^\infty \int_X (k_i + u_i)(x) f(x) d\mu_i(x)$$
on $L^\infty$ where $f_i$ denotes the $P_i$-th component of the vector $f \in L^\infty$. $\Phi$ is a bounded linear functional on $L^\infty$ and thus defines a measure $\eta \in M(\Omega)$. Since $\pi$ maps $\Omega$ onto $\Omega_1$ continuously, there exists a measure $\nu''$ on $\Omega_1$ so that $\pi_*(\eta) = \nu''$. We finally define a measure $\nu$ on $\mathcal{M}(H^\infty)$ by putting

$$
\nu = \begin{cases} 
\nu' & \text{on } \mathcal{M}(H^\infty) \sim \Omega_1, \\
\nu'' & \text{on } \Omega_1.
\end{cases}
$$

From our construction of $\nu$ follows easily that $\nu \perp H^\infty$. If we denote by $G_0$ the closure of $F_0 \sim X$ in the space $\mathcal{M}(H^\infty)$, then our assumption on $H^\infty$ implies that $\hat{H}^\infty|G_0 = C(G_0)$. Thus, by Lemma 5 with $C=\hat{H}^\infty$, $Y=\mathcal{M}(H^\infty)$, and $F=G_0$, there exists a constant $c \geq 1$ such that

$$
||\nu\big|G_0|| \leq c||\nu\big|\mathcal{M}(H^\infty)|G_0||,
$$

where $c$ does not depend on $\nu$. Consequently, we have

$$
||\nu\big|(F_0 \sim X)|| = ||\nu_0\big|(F_0 \sim X)|| = ||\nu\big|(F_0 \sim X)|| \leq ||\nu\big|G_0|| \\
\leq c||\nu\big|\mathcal{M}(H^\infty)|G_0|| \leq c||\nu\big|\Omega_1|| = c||\nu''\big|\Omega_1|| \\
\leq c\sum_{k=1}^n \int_X |k_i(x) + u_i(x)| \, d\mu_i(x) \\
= c\int_X |(\nu_0 \big| X) + \xi|| \\
\leq c\int_X |(\nu_0 \big| X) + \xi| + ||\sigma|| \\
= c\int_X |(\nu_0 \big| X) + \xi + \sigma|| \\
= c\int_X |\nu| X||,
$$

because $(\nu_0 \big| X) + \xi$ and $\sigma$ are mutually singular on $X$.

Now we use the assumption $A \big|(F_0 \cap X) = C(F_0 \cap X)$. By Lemma 2 and the expression (7), we have $\nu_0 \big|(F_0 \cap X) = 0$. As $\xi$ and $\sigma$ are orthogonal to $A$, Lemma 1 implies that $\xi \big|(F_0 \cap X) = 0$ as well as $\sigma \big|(F_0 \cap X) = 0$. Hence $\nu \big|(F_0 \cap X) = 0$. It follows therefore from (8) that

$$
||\nu\big|F_0|| \leq c||\nu\big|(X \sim F_0)||,
$$

where $c$ is a constant independent of $\nu$. Thus by Lemma 5, $\hat{A} \big|F_0 = C(F_0)$. This proves the case (ii) and the theorem is established.

### 5. Uniform algebras with a single generator.

In this section we shall be concerned with uniform algebras generated by a single function, which were previously discussed by Wada [8].

Let $A$ be a uniform algebra on a compact Hausdorff space $X$ such that $A$ is generated by a single function $f_0$. Then, denoting by $P(Z)$ the uniform closure of polynomials on a compact set $Z$ in the complex plane, we know that $A$ is isometrically isomorphic to the algebra $P(\partial K)$, where $\partial K$ denotes the boundary
of some compact set $K$ with connected complement in the complex plane (and
trivially conversely). Such a $P(\partial K)$ is known to be a Dirichlet algebra,
meaning that real parts of functions in $P(\partial K)$ are uniformly dense in
$C_R(\partial K)$ (cf. Wermer [9; Theorem 5.1]). So Theorem 1 can be applied to
$P(\partial K)$. It is also known (Hoffman [7]) that a part of a logmodular algebra (or, a fortiori, a Dirichlet
algebra) is either a one-point set or equivalent (but not necessarily homeo-
morphic) to the open unit disk, as was mentioned in the proof of Theorem 1. A
part which is equivalent to the open unit disk is usually called a disk part.
Clearly, our $P(\partial K)$ has an at most countable set of disk parts. If a single point
$z \in K$ forms a part of $P(\partial K)$, then $z$ cannot be in the interior of $K$ and so $z \in \partial K$.
This means that $K \sim \partial K$ consists of at most countably many disk parts and so
does $\mathcal{N}(A) \sim \partial A$, where $\partial A$ denotes the Šilov boundary of $A$. The algebra
$P(\partial K)$ is completely characterized by Mergelyan’s well known theorem as
follows (cf. Wermer [9; Theorem 7.6]): $P(\partial K)$ consists of all functions in $C(K)$
that are analytic at all interior points of $K$. It then follows that the space $H^\infty$
for $P(\partial K)$ consists of all bounded continuous functions on $K \sim \partial K$ that are
analytic on every disk part. So Theorem 1 implies the following:

**Theorem 2.** Let $A$ be a uniform algebra with a single generator on $X$, such
that $X$ is the Šilov boundary of $A$, and $F_0$ a closed subset of $\mathcal{N}(A)$. Then, (I)
and (II) are equivalent.

Proof. As we have seen above, we may assume $A = P(\partial K)$ with a compact
set $K$ having connected complement in the complex plane. Then we have
$\mathcal{N}(A) = K$ and the topology of $\mathcal{N}(A) = K$ as the maximal ideal space of $A$
is equivalent to the usual topology of the plane. If (I) holds, then $F_0 \cap P$ is
discrete for any part $P \subseteq K \sim \partial K$ and consequently $F_0 \sim \partial K$ is discrete. So,
by Theorem 1, (i), we have (II). Conversely, if (II) holds, then (I) is also valid,
because $K \sim \partial K$ has at most countably many parts so that the hypothesis in
Theorem 1, (ii) is automatically satisfied. This proves Theorem 2.

Theorem 2 extends a theorem of Wada mentioned earlier [8; Theorem 3.2],
although our formulation is a little different from his. Of course, the equivalence
of the statements (a) and (b) in the introduction is a special case of Theorem 2.

6. A remark. We wish to make a comment on the countability hypo-
thesis in Theorem 1, (ii). Let $A$ be logmodular on $X$ and $F_0$ a closed subset of
$\mathcal{N}(A)$ such that $H^\infty | (F_0 \sim X) = C^4(F_0 \sim X)$. Let $F$ be any compact subset of
$F_0 \sim X$. By means of the mapping $m \rightarrow \mu_m$, $F$ can be viewed as a weakly* com-
 pact subset of $M(X)$, so that we can define the weakly* closed convex envelope $co(F)$ of $F$ in $M(X)$. We ask the question: Does $co(F)$ contain a non-zero
measure which is singular with respect to all $\mu_m$, $m \in \mathcal{N}(A)$? If the answer is
negative for any compact subset $F$ of $F_0 \sim X$, then we can remove the countability assumption from Theorem 1, (ii). If this is not the case, then the situation may probably be more delicate.

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**References**
