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ON SEMI-CONTINUITY OF DILATATIONAL FUNCTIONALS

Dedicated to Professor Yukinari Tôki on his sixtieth birthday

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0. Introduction

So far as mere analogies to the analytic functions or conformal mappings were pursued in its earliest stage of the studies on quasiconformal mappings, it only mattered whether the range of dilatations is bounded or not. With the growth of the proper theory of quasiconformal mappings such as the extremal quasiconformality due to Grötzsch, Lavrentieff's mapping problem etc., the dilatation as continuous point-function has inevitably entered into consideration. The modern definition of quasiconformality which dispenses with the continuous differentiability has undoubtedly brought a rich variety of consequences into the global theory thereof to say nothing of the local one. The generalized quasiconformality, however, stated in terms of global nature, determines the dilatation-quotient as an essentially bounded point-function only almost everywhere. One seems to know very little about its behaviour, when the mappings converge, e.g., in the topology of uniform convergence.

In 1959 Lehto, Virtanen and Väisälä introduced the notion of maximal dilatation at a point (cf. [4]), which turns out, by its very definition, upper semi-continuous function. The present paper aims to develop their investigation on the dilatation-like quantity which is well-determined everywhere and majorizes the dilatation-quotient at almost every point: emphasis is laid on seeing how they behave as functionals.

§1 resumes notations, terminologies and known results designed and arranged so as to fit our present setting. In §2 we define *minimal dilatation* by analogy with the Lehto-Virtanen-Väisälä's maximal dilatation and show that it provides a good estimate for dilatation-quotient from below: the semi-continuity of those extreme dilatations is set up, which will play an important role in solving some extremum problem elsewhere. §3 deals with semi-continuities of the weighted average of those dilatations. In the course of refining the lower semi-continuity of the Dirichlet's functional we encounter in §4 a strong convergence of derivatives. The totality of quasiconformal disk-homeomorphisms forms a group, the operation being superposition of mappings: at the unit element the convergence of the global maximal dilatation is stronger than the mean square convergence of derivatives. The basic theorem on boundary correspondence together with the existence theorem in the quasiconformal mappings applies in quite a natural way to the study on such topology to produce a new approximation theorem.

1. Preliminaries

Let Ω be a closed Jordan region lying in $\boldsymbol{C}=\{z: |z|<\infty\}$ with four different boundary points z_1, z_2, z_3, z_4 specified: these four points shall be located in this order on the positively oriented boundary curve $\partial\Omega$. Such configuration is termed *quadrilateral* (or *topological rectangle*) and is denoted by $\Omega(z_1, z_2, z_3, z_4)$. An orientation-preserving topological mapping of the plane transforms quadrilaterals into quadrilaterals.

Map the interior of the region Ω conformally onto a rectangular domain $R=\{\zeta: 0<\operatorname{Re}\zeta<1, 0<\operatorname{Im}\zeta<M\}$, in such a way that z_1, z_2, z_3, z_4 correspond to the vertices $\zeta=0, 1, 1+iM, iM$ respectively. The positive quantity $1/M$ is named *modulus* of the quadrilateral $\Omega(z_1, z_2, z_3, z_4)$ and is denoted customarily by the symbol mod $\Omega(z_1, z_2, z_3, z_4)$. Though two figures $\Omega(z_1, z_2, z_3, z_4)$ and $\Omega(z_2, z_3, z_4, z_1)$ are identical as point-sets, they should be distinguished from one another as quadrilaterals in general, because mod $\Omega(z_2, z_3, z_4, z_1)=1/$mod $\Omega(z_1, z_2, z_3, z_4)$: the abbreviation Ω will be used only when no misunderstanding can occur.

Throughout the following we shall make effective use of the two simple facts, monotony and continuity of modulus:

(A) Monotony of modulus. Let γ be a cross-cut of a quadrilateral $\Omega(z_1, z_2, z_3, z_4)$ whose end-points z_2', z_3' are located on the side $\widehat{z_1, z_2}, \widehat{z_3, z_4}$ respectively. Then there holds
$$\operatorname{mod}\Omega(z_1, z_2, z_3, z_4) \geq \operatorname{mod}\Omega(z_1, z_2', z_3', z_4) + \operatorname{mod}\Omega(z_2', z_2, z_3, z_3')$$
and in particular
$$\operatorname{mod}\Omega(z_1, z_2, z_3, z_4) > \operatorname{mod}\Omega(z_1, z_2', z_3', z_4)\,.$$

(B) Continuity of modulus. Let the sequence of quadrilaterals $\{\Omega^{(n)}(z_1^{(n)}, z_2^{(n)}, z_3^{(n)}, z_4^{(n)})\}_{n=1,2,\dots}$ tend to a quadrilateral $\Omega(z_1, z_2, z_3, z_4)$ in the sense that the *écart* between the arcs $\widehat{z_j^{(n)}, z_{j+1}^{(n)}}$ and $\widehat{z_j, z_{j+1}}$ $(j=1, 2, 3, 4;\ \text{mod}\ 4)$ converges to zero as $n\to\infty$. Then we have
$$\lim_{n\to\infty}\operatorname{mod}\Omega^{(n)}(z_1^{(n)}, z_2^{(n)}, z_3^{(n)}, z_4^{(n)}) = \operatorname{mod}\Omega(z_1, z_2, z_3, z_4)\,.$$

Let G be a bounded domain in $\boldsymbol{C}$ and let $T(z)$ denote an orientation-preserving topological mapping defined in G. Let Ω be a quadrilateral comprised in G together with its boundary. The maximal dilatation of the homeomorphism T in the domain G is defined as $\bar{D}_T[G]=\sup_{\Omega\subset G}\{\operatorname{mod} T(\Omega)/\operatorname{mod}\Omega\}$. T is said *quasionformal* in G, if $\bar{D}_T[G]$ is finite. Since our special interest centres around the quasiconformal homeomorphisms between domains, we assume henceforth that a finite constant K exists which dominates $\bar{D}_T[G]$. Such quasiconformal mapping will often be referred to as K-quasiconformal (or briefly K-q.c.), when no ambiguity can result. If T is K-quasiconformal in G, its inverse mapping T^{-1} is also K-q.c. in the domain $T(G)$: the notation T^{-1} will sometimes be replaced by S in the sequel only in the interest of typography.

Let us denote by $\mathfrak{T}[G]$ the space of K-quasiconformal mappings of the domain G onto other bounded domain G' normalized by the condition $w_0=T(z_0)$ ($z_0\in G$, $w_0\in G'$ being fixed) and endowed with the topology of normal convergence in G. Suppose $T_n\to T$ in $\mathfrak{T}[G]$ as $n\to\infty$. Every $T_n(n=1,2,\cdots)$ satisfies $\operatorname{mod} T_n(\Omega)\leq K\operatorname{mod}\Omega$ for any quadrilateral $\Omega\subset G$. Then $\operatorname{mod} T(\Omega)\leq K\operatorname{mod}\Omega$ by (B), hence the result known as the lower semi-continuity of the maximal dilatation in a domain:

Proposition 1. *If a sequence* $\{T_n\}_{n=1,2,\dots}$ *of* $\mathfrak{T}[G]$ *converges to a* $T\in\mathfrak{T}[G]$, *we have*

$$\bar{D}_T[G]\leq\liminf_{n\to\infty}\bar{D}_{T_n}[G]\,.$$

In this paper $d\omega$ shall stand for the area-element regardless of the variable employed. By the way we summarize below some convenient notations and terminologies which will be frequently referred to later: in the statements (C) through (G) $w=T(z)$ denotes a K-q.c. homeomorphism belonging to $\mathfrak{T}[G]$.

(C) $T(z)$ possesses the locally square-summable derivatives $p_T(z)=\partial T/\partial z$, $q_T(z)=\partial T/\partial\bar{z}$ at almost every point of G.

(D) $T(z)$ is totally differentiable almost everywhere in G: the estimate

$$|p_T(z)|+|q_T(z)|\leq K(|p_T(z)|-|q_T(z)|)$$

holds at all points where $T(z)$ is totally differentiable.

(E) The Jacobian $|p_T(z)|^2-|q_T(z)|^2$ is positive almost everywhere in G and $T(z)$ transforms every set of 2-dimensional measure zero into another such.

(F) A point $z_1\in G$ shall be named to be *non-singnlar*, if $T(z)$ is totally differentiable at z_1 and further $|p_T(z_1)|^2-|q_T(z_1)|^2\neq 0$: the set of points at which $T(z)$ ceases to be non-singular must necessarily be of 2-dimensional measure zero.

(G) The Beltrami coefficient $h_T(z)=q_T(z)/p_T(z)$ for $T(z)$ can only be determined at non-singular points, which is linked with the classical dilatation-quotient $Q_T(z)$ by $Q_T(z)=[1+|h_T(z)|]/[1-|h_T(z)|]$.

2. Maximal and minimal dilatation at point

Let a point z and $\alpha>0$ be arbitrary. We denote by $N^{\alpha}(z)$ and by $\dot{N}^{\alpha}(z)$ the α-neighbourhood and the deleted α-neighbourhood of z respectively: $N^{\alpha}(z)=\{\zeta: |\zeta-z|<\alpha\}$, $\dot{N}^{\alpha}(z)=\{\zeta: 0<|\zeta-z|<\alpha\}$. For all $z\in G$ we set $\bar{D}_T^{\alpha}(z)=\bar{D}_T[N^{\alpha}(z)\cap G]$.

Theorem 1. *$\bar{D}_T^{\alpha}(z)$ is*
1° *a lower semi-continuous function in $z\in G$,*
2° *a monotone decreasing function in α,*
3° *a lower semi-continuous functional in $T\in\mathfrak{T}[G]$.*

Proof. 2° is trivial and 3° is nothing but a reproduction of Proposition 1. Suppose, contrary to the assertion 1°, that G contain a point z where the lower semi-continuity of $\bar{D}_T^{\alpha}(z)$ is violated: some sequence $\{z_n\}_{n=1,2,\dots}\subset G$ tending to z satisfies $\lim_{n\to\infty}\bar{D}_T^{\alpha}(z_n)<\bar{D}_T^{\alpha}(z)$. There is some constant c and some index n_1 depending on c such that

$$(1)\qquad \bar{D}_T^{\alpha}(z_n)<c<\bar{D}_T^{\alpha}(z)\qquad (n\geq n_1).$$

The inequalities (1) persist in its right half in the existence of some quadrilateral $\Omega\subset N^{\alpha}(z)\cap G$ such that $\mathrm{mod}\, T(\Omega)/\mathrm{mod}\,\Omega<c$. $N^{\alpha}(z_n)\cap G$ will comprise the Ω for all n from some index n_2 on. If $n\geq\max(n_1, n_2)$, we have $\mathrm{mod}\, T(\Omega)/\mathrm{mod}\,\Omega<c$ by the left half of (1). This is a contradiction. q.e.d.

Letting $\alpha\to 0$, we obtain a point-function $\bar{D}_T(z)=\inf_{\alpha>0}\bar{D}_T^{\alpha}(z)$, the maximal dilatation at z, which Lehto-Virtanen-Väisälä [4] first introduced with its basic properties:

Theorem 2. *The estimate*

$$(2)\qquad Q_T(z)\leq\bar{D}_T(z)$$

is valid at every non-singular point z for T. The relation (2) *holds with equality if T is continuously differentiable in a neighbourhood of z.*

The existence proofs for the usual minimization problem of maximal dilatation rest on Proposition 1. We will present here a somewhat different type of statement analogous to Proposition 1:

Theorem 3. $\sup_{z\in G}\bar{D}_T(z)$ *is a lower semi-continuous functional in T of $\mathfrak{T}[G]$.*

Proof. Let α, α' be any positive number such that $\alpha<\alpha'$. Then obviously $\sup_{z\in G}\bar{D}_T^{\alpha}(z)\leq\sup_{z\in G}\bar{D}_T^{\alpha'}(z)$.

Let c be any constant such that $\sup_{z\in G}\bar{D}_T^\alpha(z)<c$: we have $\mathrm{mod}\,T(\Omega)/\mathrm{mod}\,\Omega<c$ for every quadrilateral Ω whose dimaeter is smaller than α. Since the local quasiconformality implies the global one, the estimate $\mathrm{mod}\,T(\Omega)/\mathrm{mod}\,\Omega<c$ holds for every $\Omega\subset N^{\alpha'}(z)$: so $\bar{D}_T^{\alpha'}(z)\leq c$ and $\sup_{z\in G}\bar{D}_T^{\alpha'}(z)\leq c$. Thus we have shown that $\sup_{z\in G}\bar{D}_T^\alpha(z)=\sup_{z\in G}\bar{D}_T^{\alpha'}(z)$.

Next we shall see $\sup_{z\in G}\bar{D}_T(z)=\sup_{z\in G}\bar{D}_T^\alpha(z)$. Suppose that $\sup_{z\in G}\bar{D}_T(z)<\sup_{z\in G}\bar{D}_T^\alpha(z)$ for some α. Then there would be a constant c satisfying $\sup_{z\in G}\bar{D}_T(z)<c<\sup_{z\in G}\bar{D}_T^\alpha(z)$ for all α. The left half of these inequalities implies that for any $z\in G$ there is an α satisfying $\bar{D}_T^\alpha(z)<c$, while the right half implies the presence of some $z'\in G$ satisfying $\bar{D}_T^\alpha(z')>c$ for any α. This is a contradiction. Hence $\sup_{z\in G}\bar{D}_T(z)\geq\sup_{z\in G}\bar{D}_T^\alpha(z)$. We have trivially $\sup_{z\in G}\bar{D}_T(z)\leq\sup_{z\in G}\bar{D}_T^\alpha(z)$, and the assertion is verified. Since $\bar{D}_T^\alpha(z)$ is a lower semi-continuous on $\mathcal{T}[G]$ (Theorem 1), $\sup_{z\in G}\bar{D}_T(z)=\sup_{z\in G}\bar{D}_T^\alpha(z)$ is also a lower semi-continuous functional on $\mathcal{T}[G]$. q.e.d.

Let B denote a square henceforth: but the symbol does not necessarily indicate the same figure at each occurrence.

Theorem 4. $\sup_{B\subset G}\mathrm{mod}\,T(B)=\bar{D}_T[G]$ *for every T of $\mathcal{T}[G]$.*

Proof. Let c be an arbitrary constant greater than $\sup_{B\subset G}\mathrm{mod}\,T(B)$. Taking a non-singular point $z_1\in G$ for T at will, we consider a square $B=B(z_1,z_2,z_3,z_4)\subset G$ with one vertex at z_1 and subject to the requirement $2\arg(z_2-z_1)=\arg h_T(z_1)$. Given any $\varepsilon>0$, there is a $\delta>0$ such that $|\Delta z|<\delta$ implies $|T(z_1+\Delta z)-T(z_1)-p_T(z_1)\Delta z-q_T(z_1)\Delta\bar{z}|<\varepsilon|\Delta z|$. If we specify B so as to be $|z_2-z_1|=\delta$, we see on account of (A)

$$\begin{aligned} c>\mathrm{mod}\,T(B)&>\frac{(|p_T(z_1)|+|q_T(z_1)|)|\Delta z|-2\varepsilon|\Delta z|}{(|p_T(z_1)|-|q_T(z_1)|)|\Delta z|+2\varepsilon|\Delta z|}\\ &>\frac{|p_T(z_1)|+|q_T(z_1)|-2\varepsilon}{|p_T(z_1)|-|q_T(z_1)|+2\varepsilon}.\end{aligned}$$

Letting $\varepsilon\to0$, we have $Q_T(z_1)\leq c$. Hence

$$\operatorname*{ess\,sup}_{z\in G}Q_T(z)\leq\sup_{B\subset G}\mathrm{mod}\,T(B). \tag{3}$$

Next let c' be any constant dominating $\operatorname*{ess\,sup}_{z\in G}Q_T(z)$. Then, since $Q(z)<c'$ a.e. in G, every $\Omega\subset G$ satisfies $\mathrm{mod}\,T(\Omega)/\mathrm{mod}\,\Omega<c'$. Therefore

$$\bar{D}_T[G]\leq\operatorname*{ess\,sup}_{z\in G}Q_T(z). \tag{4}$$

Clearly $\sup_{B\subset G} \operatorname{mod} T(B) \leq \bar{D}_T[G]$, so follows the desired identity from (3) and (4). q.e.d.

Theorem 4 permits us to adopt squares in place of curvilinear quadrilaterals in the definition of the maximal dilatation, which will, in conjunction with Theorem 2, be suggestive of another local concept:

DEFINITION 1. Let $T\in\mathcal{T}[G]$ and $z\in G$. Take any point ζ of $\dot{N}^{\alpha}(z)\cap G$. Let r be a positive number smaller than $\sqrt{2}\min(|\zeta-z|, \alpha-|\zeta-z|)$ and let $B=B(\zeta; r; \theta)\subset\dot{N}^{\alpha}(z)\cap G$ a square centred at ζ with a side of length r and of argument θ. We set

$$\underline{D}^{\alpha}_T(z) = \inf_{\zeta}\,[\inf_{r}\,\{\max_{\theta} \operatorname{mod} T(B(\zeta; r; \theta))\}]\,,$$
$$\underline{D}_T(z) = \lim_{\alpha\to 0} \underline{D}^{\alpha}_T(z)$$

and term the latter of them *minimal dilatation* of T at z.

EXAMPLE 1. The extremal quasiconformal mapping T of closed Riemann surfaces of genus ≥ 2: In terms of local coordinate z we have $\bar{D}_T(z)=\underline{D}_T(z)=$ const. without exception even at the zeros of the analytic quadratic differential associated with T.

EXAMPLE 2. The extremal quasiconformal mappings T of Strebel's chimney-shaped domain with prescribed boundary correspondence: Let $G=\{z\colon \operatorname{Im} z<0\}\cup\{z\colon |\operatorname{Re} z|<1\}$ be the domain in the Gaussian z-plane and let $T(z)$ a quasiconformal homeomorphism of G onto itself with the boundary condition $T(z)=z$ on $\{z\colon \operatorname{Im} z=0,\ |\operatorname{Re} z|\geq 1\}$ and $T(z)=[(K+1)z-(K-1)\bar{z}]/2$ on $\{z\colon \operatorname{Im} z\geq 0,\ |\operatorname{Re} z|=1\}$. In this family of quasiconformal automorphisms of G, the mapping

$$T(z)=\begin{cases}[(K+1)z-(K-1)\bar{z}]/2 & \text{in } \{z\colon \operatorname{Im} z\geq 0,\ |\operatorname{Re} z|<1\}\\ [(K'+1)z-(K'-1)\bar{z}]/2 & \text{in } \operatorname{Im} z<0\end{cases}$$

is extremal quasiconformal for any constant K' such that $1/K\leq K'\leq K$. But $\bar{D}_T(z)=\underline{D}_T(z)=$const. if and only if $K'=K$.

Let a point z_1 be of G. Suppose that $\limsup_{z\to z_1} \underline{D}^{\alpha}_T(z)>\underline{D}^{\alpha}_T(z_1)$. Then there would be a constant c such that $\limsup_{z\to z_1} \underline{D}^{\alpha}_T(z)>c>\underline{D}_T(z_1)$: $\dot{N}^{\alpha}(z_1)$ must contain some point ζ, such that $\operatorname{mod} T(B(\zeta; r; \theta))<c$ for some r whatever θ may be. On the other hand if z_2 is sufficiently close to z_1, $\dot{N}^{\alpha}(z_2)$ comprises the squares $B(\zeta; r; \theta)$ $(0\leq\theta\leq 2\pi)$ and yet we must have $\underline{D}^{\alpha}_T(z_2)>c$. It is impossible in view of Definition 1. Hence we have proved $\limsup_{z\to z_1} \underline{D}^{\alpha}_T(z)\leq\underline{D}^{\alpha}_T(z_1)$.

Let $\{T_n\}_{n=1,2,\ldots}$ be a sequence of $\mathcal{T}[G]$ convergent to $T\in\mathcal{T}[G]$. Let $z'\in G$

be arbitrary and let c any constant such that $\underline{D}^{\alpha}_{T_n}(z')>c$ $(n=1, 2, \cdots)$. Take a point $z\in \dot{N}^{\alpha}(z')$ at will which shall be fixed for a moment. If r denote any constant smaller than $\sqrt{2}\min(|z-z'|, \alpha-|z-z'|)$, we have $\operatorname{mod} T_n(B(z; r; \theta_n))>c$ for some θ_n $(n=1, 2, \cdots)$: $\{\theta_n\}_{n=1,2,\dots}$ clusteres at some θ. It follows from (B) that $\operatorname{mod} T(B(z; r; \theta))\geq c$, whence $\underline{D}^{\alpha}_{T}(z')\geq c$. Thus $\limsup_{n\to\infty} \underline{D}^{\alpha}_{T_n}(z)\leq \underline{D}^{\alpha}_{T}(z)$.

We summarize the above results in

Theorem 5. *$\underline{D}^{\alpha}_{T}(z)$ is*

1° *an upper semi-continuous function in $z\in G$,*

2° *a monotone increasing function in α,*

3° *an upper semi-continuous functional in $T\in \mathfrak{T}[G]$.*

Just corresponding to Theorem 2 we shall have

Theorem 6[1]. *The estimate*

$$Q_T(z) \geq \underline{D}_T(z) \tag{5}$$

is valid at every non-singular point z for T. The relation (5) *holds with equality if T is continuously differentiable in a neighbourhood of z.*

Proof. Consider a non-singular point z_1 for T together with $\dot{N}^{\alpha}(z_1)$, Describe a square $B=B(z_1, z_2, z_3, z_4)$ comprised in $N^{\alpha}(z_1)$ with one vertex at z_1, whose side-vector $\overrightarrow{z_1, z_2}$ points to the direction $[\arg h_T(z_1)]/2$. By Definition 1 and (B) we see

$$\underline{D}^{\alpha}_{T}(z_1) \leq \max \{\operatorname{mod} T(B), 1/\operatorname{mod} T(B)\}\ . \tag{6}$$

For any $\varepsilon>0$ there is some δ_1 such that $|\Delta z|<\delta_1$ implies $|T(z_1+\Delta z)-T(z_1)-p_T(z_1)\Delta z-q_T(z_1)\overline{\Delta z}|<\varepsilon|\Delta z|$. Therefore, if the side of B is smaller than δ_1 in length, we have

$$\frac{|p_T(z_1)|-|q_T(z_1)|-2\varepsilon}{|p_T(z_1)|+|q_T(z_1)|+2\varepsilon} < \operatorname{mod} T(B) < \frac{|p_T(z_1)|+|q_T(z_1)|+2\varepsilon}{|p_T(z_1)|-|q_T(z_1)|-2\varepsilon}$$

in view of (A), (B). Substituting these into (6) and letting $\alpha\to 0$ after $\varepsilon\to 0$ we complete the proof of the first assertion.

Suppose that T is continuously differentiable in G and that G contain a non-singular point z_1 for T satisfying $Q_T(z_1)>\underline{D}_T(z_1)$. Then there would be a constant c such that

$$Q_T(z_1) > c > \underline{D}_T(z_1)\ . \tag{7}$$

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1) M. Mohri collaborated in studying the minimal dilatation.

$T(z)$ is totally differntiable uniformly on a closed subregion G_0 of G containing z_1: hence for any $\varepsilon>0$ there is some $\delta>0$ such that $|T(z')-T(z)-p_T(z)(z'-z)-q_T(z)(\overline{z'-z})|\leq\varepsilon|z'-z|$ whenever the points $z, z'\in\bar{G}_0$ fulfill $|z'-z|<\delta$. The right half of (7) requires that the $\dot{N}^{\alpha}(z_1)$ comprises at least one square $B=B(\zeta; r; \theta)=B(z_1', z_2', z_3', z_4')$ such that mod $T(B)<c$: we may assume that the $\arg(z_2'-z_1')=\theta$ attains the $\max\limits_{\theta}$ mod $T(B(\zeta; r; \theta))$. We have then

$$\operatorname{mod} T(B) > \frac{|p_T(z_1')|+|q_T(z_1')|-2\varepsilon|z-z_1'|}{|p_T(z_1')|-|q_T(z_1')|+2\varepsilon|z-z_1'|}, \quad (z\in B).$$

Therefore

$$\frac{|p_T(z_1')|+|q_T(z_1')|}{|p_T(z_1')|-|q_T(z_1')|}<c+\frac{2\varepsilon\delta(K+1)}{|p_T(z_1')|-|q_T(z_1')|}.$$

Letting $\varepsilon\to 0$, we arrive at $Q_T(z_1)\leq c$ by (B), which contradicts (7). q.e.d.

The maximal and minimal dilatation at a point are named generically *extreme dilatation* at the point.

3. Integral mean of dilatations

This section begins by recalling a few locutions as well as introductory propositions in the mass distribution theory.

A mass distribution on the domain G should be interpreted as the completely additive real-valued set-function defined on all the Borel subsets of G.

$\mathscr{M}[G]$: the class of uniformly bounded non-negative mass distributions μ on G

$\mathscr{M}'[G]$: the class of uniformly bounded non-negative mass distributions μ' with continuous density on G

A sequence $\{\mu_n\}_{n=1,2,\dots}$ of $\mathscr{M}[G]$ is termed convergent towards a $\mu\in\mathscr{M}[G]$ if and only if $\lim\limits_{n\to\infty}\mu_n(e)=\mu(e)$ for every Borel subset e of G which is regular with respect to μ.

Lemma 1. (Vallée-Poussin [8], p. 42). *If $\phi(z)$ is a continuous function on G and a sequence $\{\mu_n\}_{n=1,2,\dots}$ of $\mathscr{M}[G]$ tends to μ as $n\to\infty$, we have*

$$\lim_{n\to\infty}\int_G\phi(z)d\mu_n(z)=\int_G\phi(z)d\mu(z).$$

Lemma 2. (Schwartz [6], t. II, pp. 21–22). *Let $\bar{G}_0$ be an arbitrary closed subregion of the domain G. Given any $\mu\in\mathscr{M}[G]$, there exists $\mu_n'\in\mathscr{M}'[G]$ such that*

$$\lim_{n\to\infty}\int_G\psi(z)d\mu_n'(z)=\int_G\psi(z)d\mu(z)$$

for every function $\psi(z)$ *of class* C^∞ *with a support comprised in* G.

The extreme dilatation introduced in the preceding section has an advantage of enjoying the semi-continuity as point-functions unlike the more familiar dilatation-quotient, our knowledge about which seems no more than the bounded measurability.

Theorem 7. *The extreme dilatation* $\bar{D}_T(z)$ (*resp.* $\underline{D}_T(z)$) *is an upper semi-continuous* (*res p. alower semi-continuous*) *function in* z.

Proof. Upper semi-continuity of the maximal dilatation was shown in Lehto-Virtanen-Väisälä [4]: we have only to prove the lower semi-continuity of $\underline{D}_T(z)$.

Suppose G contain a point z_1 such that $\liminf_{z\to z_1} \underline{D}_T(z) < \underline{D}_T(z_1)$: a constant c exists satisfying $\liminf_{z\to z_1} \underline{D}_T(z) < c < \underline{D}_T(z_1)$. The right half of these inequalities asserts the presence of some such α that $\underline{D}_T^\alpha(z_1) > c$, while from the left half it follows that the $\dot{N}^\alpha(z_1)$ contain a z satisfying $\underline{D}_T(z) < c$. Let $\alpha' > 0$ be smaller than $\min(|z-z_1|, \alpha - |z-z_1|)$: then $\underline{D}_T^{\alpha'}(z) < c$, so $\dot{N}^{\alpha'}(z)$ contain a point ζ such that $\max_\theta \operatorname{mod} T(B(\zeta; r; \theta)) < c$ for some $r < \sqrt{2} \min(|\zeta - z|, \alpha' - |\zeta - z|)$. But the squares $B(\zeta; r; \theta)$ $(0 \leq \theta \leq 2\pi)$ lie in $N^\alpha(z_1)$, which is a contradiction. q.e.d.

DEFINITION 2. We set for $T \in \mathfrak{T}[G]$, $\mu \in \mathcal{M}[G]$

$$\bar{a}[T; \mu; G] = \int_G \frac{\bar{D}_T(z)^2+1}{2\bar{D}_T(z)} d\mu(z), \quad \underline{a}[T; \mu; G] = \int_G \frac{\underline{D}_T(z)^2+1}{2\underline{D}_T(z)} d\mu(z)$$

and analogously

$$a[T; \mu; G] = \int_G \frac{Q_T(z)^2+1}{2Q_T(z)} d\mu(z) .$$

Theorem 7 yields immediately the two corollaries:

Corollary 1. $\bar{a}[T; \mu; G]$ (*resp.* $\underline{a}[T; \mu; G]$) *is an upper semi-continuous* (*resp. a lower semi-continuous*) *functional in* $\mu \in \mathcal{M}[G]$.

Proof. Let $\mu_n \to \mu$ $(n \to \infty)$ on $\mathcal{M}[G]$. On account of the upper semi-continuity of $\bar{D}_T(z)$ in G there exists a bounded continuous function $\phi_m(z)$ majorizing $[\bar{D}_T(z)^2+1]/2\bar{D}_T(z)$ in G. From Lemma 1 it follows that

$$\limsup_{n\to\infty} \bar{a}[T; \mu_n; G] \leq \lim_{n\to\infty} \int_G \phi_m(z) d\mu_n(z) = \int_G \phi_m(z) d\mu(z) .$$

Since $[\bar{D}_T(z)^2+1]/2\bar{D}_T(z)$ can be expressed as the monotone decreasing limit of such $\phi_m(z)$ $(m=1, 2, \cdots)$, we have

$$\limsup_{n\to\infty} \bar{a}[T;\ \mu_n;\ G] \leq \bar{a}[T;\ \mu;\ G]\,.$$

q.e.d.

Corollary 2. *Let $\bar{G}_0$ be an arbitrary compact subregion of G. For any $\mu\in\mathcal{M}[\bar{G}_0]$ there exists a sequence $\{\mu_n'\}_{n=1,2,\cdots}$ of $\mathcal{M}'[G]$, such that*

$$\bar{a}[T;\ \mu;\ G] \geq \limsup_{n\to\infty} \bar{a}[T;\ \mu_n';\ G]\,,$$
$$a[T;\ \mu;\ G] \leq \liminf_{n\to\infty} a[T;\ \mu_n';\ G]\,.$$

Proof. Take a continuous function $\phi(z)$ majorizing the upper semi-continuous function $[\bar{D}_T(z)^2+1]/2\bar{D}_T(z)$ on G. Given any $\varepsilon_n>0$ tending to zero as $n\to\infty$, there are functions $\psi_n(z)$ of class C^∞ with a support $\bar{G}_0'\subset G$ (G_0' being a domain comprising $\bar{G}_0$) such that $|\phi(z)-\psi_n(z)|<\varepsilon_n/3$ $(n=1, 2, \cdots)$ uniformly on $\bar{G}_0$ Lemma 2 assures the existence of $\mu_n'\in\mathcal{M}'[\bar{G}_0']$ such that

$$\left|\int_{\bar{G}_0'}\psi_n(z)\,d\mu(z)-\int_{\bar{G}_0'}\psi_n(z)d\mu_n'(z)\right| < \varepsilon_n/3, \quad (n=1, 2, \cdots)\,,$$

whence

$$\left|\int_G\phi(z)d\mu(z)-\int_G\phi(z)d\mu_n'(z)\right| < \varepsilon_n\,.$$

Therefore for any index n we have

$$\bar{a}[T;\ \mu_n'\ ;G] \leq \int_G\phi(z)\,d\mu_n'(z) < \int_G\phi(z)d\mu(z) + \varepsilon_n\,.$$

Letting n tend to infinity we get

$$(8)\qquad \limsup_{n\to\infty} \bar{a}[T;\ \mu_n';\ G] \leq \int_G\phi(z)d\mu(z)\,.$$

Since $\phi(z)$ is taken as close to $[\bar{D}_T(z)^2+1]/2\bar{D}_T(z)$ as one pleases, the left hand side in (8) cannot be larger than $\bar{a}[T;\ \mu;\ G]$. q.e.d.

Let us write $\Delta_z=\{z\colon |z|<1\}$: the set-function $\omega(e)=\int_e d\omega$ belongs to $\mathcal{M}[\Delta_z]$ if e is a Borel subset of Δ_z. The space $\mathfrak{T}[\Delta_z]$ with which we deal henceforth shall consist of the normalized K-quasiconformal homeomorphisms $T(z)$ between Δ_z and Δ_w such that $T(1)=1$, $T(i)=i$, $T(-1)=-1$: $\mathcal{L}^2[\Delta_z]$ denotes the space of all linear differential $\tau=p(z)dz+q(z)d\bar{z}$ of summable square in Δ_z with the inner product $(\tau_1, \tau_2)=\int_{\Delta_z}[p_1(z)\overline{p_2(z)}+q_1(z)\overline{q_2(z)}]d\omega(z)$ $(\tau_j=p_j(z)dz+q_j(z)d\bar{z}\in\mathcal{L}^2[\Delta_z];\ j=1, 2)$ and the norm $\|\tau\|=\sqrt{(\tau, \tau)}$. $T\in\mathfrak{T}[\Delta_z]$ implies $dT\in\mathcal{L}^2[\Delta_z]$, since $\|dT\|^2 = a[S;\ \omega;\ \Delta_w] \leq \pi(K^2+1)/2K$ by (G).

Lemma 3. *If $\{T_n\}_{n=1,2\cdots}$ converges to T in $\mathcal{Q}[\Delta_z]$, then $\{T_n^{-1}\}_{n=1,2,\cdots}$ also converges to T^{-1} in $\mathcal{Q}[\Delta_w]$.*

Proof. Set $z=T^{-1}(w)$ for an arbitrary $w\in\Delta_w$. If we write $w_n=T_n(z)$, we see $|w_n-w|\to 0$. The equicontinuity of $\{T_n^{-1}\}_{n=1,2,\cdots}$ yields $|T^{-1}(w)-T_n^{-1}(w)| =|T_n^{-1}(w_n)-T_n^{-1}(w)|\to 0$ $(n\to\infty)$. Therefore $\lim_{n\to\infty} T_n^{-1}=T^{-1}$. If this convergence were not uniform, there would be some sequence $\{w_n\}_{n=1,2,\cdots}$ on Δ_w such that $|T_n^{-1}(w_n)-T^{-1}(w_n)|\geq c$ for some constant $c>0$. But it is a contradiction, since $\{T_n^{-1}\}_{n=1,2,\cdots}$ contains a subsequence $\{T_{n_k}^{-1}\}_{k=1,2,\cdots}$ convergent uniformly on Δ_w, which implies $|T_{n_k}^{-1}(w_{n_k})-T^{-1}(w_{n_k})|\to 0$ $(k\to\infty)$. q.e.d.

Next suppose a sequence of $\mathcal{Q}[G]$ converges: then the behaviour of their derivatives naturally comes into question. The first step in such direction was perhaps be made by a theorem included in Ahlfors [1]:

Proposition 2. *When a sequence $\{T_n\}_{n=1,2,\cdots}$ converges to T in the sapce $\mathcal{Q}[\Delta_z]$ in its intrinsic topology, $\{p_{T_n}\}_{n=1,2,\cdots}$ (resp. $\{q_{T_n}\}_{n=1,2,\cdots}$) converges to p_T (resp. q_T) weakly in the space $\mathcal{L}^2[\Delta_z]$.*

This proposition provides us with the clearest and firmest background for semi-continuity of the Dirichlet integral in its mapping-theoretic version:

Theorem 8. *For any $\mu'\in\mathcal{M}'[\Delta_z]$, $a[T;\mu';\Delta_z]$ is a lower semi-continuous functional on $\mathcal{Q}[\Delta_z]$.*

Proof. Let $\lim_{n\to\infty} T_n=T$ in $\mathcal{Q}[G]$ and let $\sigma(z)=d\mu'(z)/d\omega(z)$ be the density of the smooth mass distribution μ'. On setting $\tau_n=\sqrt{\sigma(S_n(w))}[p_{S_n}(w)dw+q_{S_n}(w)d\overline{w}]$, $\tau=\sqrt{\sigma(S(w))}[p_S(w)dw+q_S(w)d\overline{w}]$, we see by Lemma 3 and Proposition 2 that $\{\tau_n\}_{n=1,2,\cdots}$ converges weakly to τ in $\mathcal{L}^2[\Delta_w]$. Hence we have $\liminf_{n\to\infty} ||\tau_n||^2\geq||\tau||^2$ (cf. Riesz-Nagy [5], p. 200), which was to be proved by virtue of (G).

4. Strong convergence in the space of quasiconformal disk-homeomorphisms

In the final section we are concerned with different kinds of topologies in a fixed family of quasiconformal homeomorphisms of a disk. Henceforth we denote the variables in $\boldsymbol{C}$ by z, w, Z, W and write $H_Z=\{Z: \operatorname{Im} Z>0\}$; though the space $\mathcal{Q}[\Delta_z]$ is mainly treated, we need auxiliarily also the space $\mathcal{Q}[H_Z]$ of the K-quasiconformal automorphisms of H_Z which leaves 0, 1, ∞ fixed. The passage to limit are all referred to the index n which grows indefinitely, unless otherwise mentioned.

Besides the original topology with which $\mathcal{Q}[\Delta_z]$ is endowed intrincically there are some other topologies, which may be stronger and expressed in terms of the convergence of derivatives or of Beltrami coefficient in addition to the uniform convergence of mapping itself. Since it is known that the weak convergence in the conclusion of Proposition 2 cannot be replaced by the strong one, what is then the condition for those derivatives to converge strongly? One of the answers reads

Theorem 9. *Let $\{T_n\}_{n=1,2,\dots}$ converge to T in $\mathfrak{T}[\Delta_z]$ in the uniform topology. Then $\{p_{T_n}\}_{n=1,2,\dots}$, $\{q_{T_n}\}_{n=1,2,\dots}$ converges to the respective derivative p_T, q_T of the limit mapping T in the topology of $\mathcal{L}^2[\Delta_z]$ if and only if*

$$\lim_{n\to\infty}(\|p_{T_n}\|^2+\|q_{T_n}\|^2)=\|p_T\|^2+\|q_T\|^2\,,$$

or equivalently

$$\lim_{n\to\infty} a[S_n;\,\omega;\,\Delta_w]=a[S;\,\omega;\,\Delta_w]\,.$$

Proof. Since

$$\begin{aligned}0&\le\|p_T-p_{T_n}\|^2+\|q_T-q_{T_n}\|^2\\&=\|p_T\|^2+\|q_T\|^2+\|p_{T_n}\|^2+\|q_{T_n}\|^2-2\mathrm{Re}\,\{(p_{T_n},p_T)+(q_{T_n},q_T)\}\,,\end{aligned}$$

it follows from Proposition 2 that

$$\liminf_{n\to\infty}(\|p_{T_n}\|^2+\|q_{T_n}\|^2)\ge\|p_T\|^2+\|q_T\|^2\,. \tag{9}$$

If $\lim_{n\to\infty}(\|p_T-p_{T_n}\|^2+\|q_T-q_{T_n}\|^2)=0$, not only the limit subsists in (9) but also the equality holds there, and vice versa. q.e.d.

DEFINITION 3. If a sequence $\{T_n\}_{n=1,2,\dots}$ of $\mathfrak{T}[\Delta_z]$ converges to $T\in\mathfrak{T}[\Delta_z]$ with the additional condition that the derivative p_{T_n}, q_{T_n} $(n=1,2,\cdots)$ tends to the respective derivative p_T, q_T of the limit mapping T in $\mathcal{L}^2[\Delta_z]$, $\{T_n\}_{n=1,2,\dots}$ shall be said to converge to T in the S_1-topology.

Theorem 10. *In order that $\lim_{n\to\infty}T_n=T$ on $T[\Delta_z]$ in the S_1-topology it is necessary and sufficient that the sequence $\{T_n\circ T^{-1}\}_{n=1,2,\dots}$ converges to the identity on $\mathfrak{T}[\Delta_w]$ in the same topology.*

Proof. Assume that $T_n\to T$ on $\mathfrak{T}[\Delta_z]$ in the S_1-topology. Then $P_n=T_n\circ T^{-1}\to id.$ (Lemma 3). Substitution $p_{P_n}=(\bar{p}_Tp_{T_n}-q_T\bar{q}_{T_n})/(|p_T|^2-|q_T|^2)$, $q_{P_n}=(\bar{p}_Tq_{T_n}-\bar{p}_{T_n}q_T)/(|p_T|^2-|q_T|^2)$ gives $\|p_{P_n}-1\|^2+\|q_{P_n}\|^2\le 3(K^2+1)(\|p_T-p_{T_n}\|^2+\|q_T-q_{T_n}\|^2)/2K\to 0$. Verification of the converse will follow the similar line of argument. q.e.d.

Restricted to our purpose at hand, i.e., the convergence problem of the normalized K-quasiconformal automorphisms, it suffices to consdier only a neighbourhood of the identity owing to Lemma 3, Theorem 10 and Proposition 2.

DEFINITION 4. Let a sequence $\{T_n\}_{n=1,2,\dots}$ of the normalized K-quasiconformal automorphisms of the domain G converge normally to the identity. If further the essential upper bound of $|h_{T_n}|$ in G tends to zero, we shall say that T_n $(n=1, 2, \cdots)$ is convergent on G in the *S_2-topology*.

Theorem 11. *In the space $\mathfrak{T}[\Delta_z]$ of K-quasiconformal automorphisms the S_2-topology is not weaker than the S_1-topology.*

Proof. If $T_n \to id.$ on $\mathfrak{T}[\Delta_z]$ in the S_2-topology, we have

$$\pi \leq \|p_{T_n}\|^2 + \|q_{T_n}\|^2 \leq \pi \cdot \frac{1+\operatorname*{ess\,sup}_{z\in G} |h_{T_n}(z)|}{1-\operatorname*{ess\,sup}_{z\in G} |h_{T_n}(z)|} \to \pi .$$

Hence $\|p_{T_n}\|^2 + \|q_{T_n}\|^2 \to \pi = \|p_{id}\|^2 + \|q_{id}\|^2$, so $\|p_{T_n} - p_{id}\|^2 + \|q_{T_n} - q_{id}\|^2 \to 0$ by Theorem 9. q.e.d.

A deep result due to Beurling and Ahlfors enables us to reduce the investigation on S_2-topology of $\mathfrak{T}[\Delta_z]$ to that on boundary correspondence in $\mathfrak{T}[H_Z]$. Let $\mathfrak{N}$ be a collection of real-valued monotone increasing continuous function $\nu = \nu(X)$ defined for $-\infty < X < \infty$ such that $\lim_{X\to\pm\infty} \nu(X) = \pm\infty$.

DEFINITION 5. For any $\nu \in \mathfrak{N}$ we set

$$\rho[\nu] = \sup \frac{\nu(X+t)-\nu(X)}{\nu(X)-\nu(X-t)},$$

where the supremum is referred to all X, t varying over the whole real axis $(-\infty, \infty)$.

Suppose the upper half plane $\operatorname{Im} Z > 0$ be mapped by means of a $T \in \mathfrak{T}[H_Z]$ K-quasiconformally onto the upper half plane $\operatorname{Im} W > 0$; such T induces a boundary correspondence $T(X) = \nu \in \mathfrak{N}$. $\rho[T(X)]$ is known to be bounded.

Proposition 3. (Beurling-Ahlfors [3]). *If a sequence $\{T_n\}_{n=1,2,\dots}$ of $\mathfrak{T}[H_Z]$ converges to the the identity in the S_2-topology, then $\lim_{n\to\infty} \rho[T_n(X)] = 1$. Conversely, given a sequence $\{\nu_n(X)\}_{n=1,2,\dots}$ of $\mathfrak{N}$ such that $\rho[\nu_n]$ tends to* 1, *there exists some smooth non-singular mapping T_n in $\mathfrak{T}[H_Z]$ $(n=1, 2, \cdots)$ which satisfies $T_n(X) = \nu_n(X)$ and converges to the identity in the S_2-topology.*

Theorem 12. *Suppose a non-smooth K-quasiconformal homeomorphism $T(z)$ is given, which sends the disk $|z|<1$ onto the disk $|w|<1$. Then there* *exists a sequence* $\{T_n(z)\}_{n=1,2,\dots}$ *with the following properties:*

1° $T_n(z)$ *maps* $|z|<1$ K'*-quasiconformally onto* $|w|<1$,

2° $T_n(z)$ *is continuously differentiable and non-singular in* $|z|<1$,[2)]

3° $T_n(z)=T(z)$ on $|z|=1$ $(n=1, 2, \cdots)$.

4° $\{T_n(z)\}_{n=1,2,\dots}$ *converges to* $T(z)$ *uniformly on* $|z|\leq 1$;

5° *though we must be content with the constant* K' *larger than* K, $K'-K$ *can be made as samll as we please.*

Proof. We lose no generality if we assume $T(z)$ belongs to $\mathcal{I}[\Delta_z]$. Fix a monotone-decreasing sequence $\{\rho_n\}_{n=1,2,\dots}$ tending to 1. Then there exists a real sequence $\{\nu_n(U)\}_{n=1,2,\dots}$ of a real variable U satisfying the conditions: (i) for every n, $\nu_n(U)$ is monotone-increasing function for $-\infty<U<\infty$ and $\lim_{U\to\pm\infty}\nu_n(U)=\pm\infty$; (ii) whatever value the real variables U, t may assume, we have

$$\rho_{n+1}\leq \sup\frac{\nu_n(U+t)-\nu_n(U)}{\nu_n(U)-\nu_n(U-t)}\leq \rho_n;$$

(iii) $\nu_n(0)=0$, $\nu_n(1)=1$, $\nu_n(\infty)=\infty$. According to Proposition 3 it is possible to construct a mapping $F_n(W)$ explicitly which belongs to $\mathcal{I}[H_W]$ and fulfills the requirement $F_n(U)=\nu_n(U)$ (Re $W=U$), $|p_{F_n}|-|q_{F_n}|>0$. On setting

$$\Theta_n(w)=\frac{i-F_n\left(i\dfrac{1-w}{1+w}\right)}{i+F_n\left(i\dfrac{1-w}{1+w}\right)},$$

we see that $\Sigma_n(z)=\Theta_n\circ T(z)$ is in $\mathcal{I}[\Delta_z]$. Let $\tilde{h}_{n,m}(z)$ denote the complex-valued function defined and smooth in G with a compact support containing Δ_z such that (i) $|\tilde{h}_{n,m}(z)|\leq|h_{\Sigma_n}(z)|$ a.e. on Δ_z and (ii) $\lim_{m\to\infty}\tilde{h}_{n,m}(z)=h_{\Sigma_n}(z)$ in $\mathcal{L}^2[\Delta_z]$. Existence of such functions is seen, e.g., by averaging $h_{\Sigma_n}(z)$ disk-wise. The Beltrami equation $(\partial w/\partial\bar{z})/(\partial w/\partial z)=\tilde{h}_{n,m}(z)$ has the unique solution $w=\Sigma_{n,m}(z)$ in $\mathcal{I}[\Delta_z]$ which is of class C^1 and non-singular (Ahlfors [2]). Let us examine how an arbitrary quadrilateral $\Omega\subset\Delta_w$ is distorted in modulus by the quasiconformal automorphism $\Theta_{n,m}(w)=\Sigma_{n,m}\circ T^{-1}(w)$ of Δ_w: the condition (i) implies

$$\frac{\mathrm{mod}\,\Theta_{n,m}(\Omega)}{\mathrm{mod}\,\Omega}=\frac{\mathrm{mod}\,\Sigma_{n,m}\circ T^{-1}(\Omega)}{\mathrm{mod}\,T^{-1}(\Omega)}\cdot\frac{\mathrm{mod}\,T^{-1}(\Omega)}{\mathrm{mod}\,\Omega}$$
$$\leq\frac{\mathrm{mod}\,\Sigma_n\circ T^{-1}(\Omega)}{\mathrm{mod}\,T^{-1}(\Omega)}\cdot\frac{\mathrm{mod}\,T^{-1}(\Omega)}{\mathrm{mod}\,\Omega}=\frac{\mathrm{mod}\,\Theta_n(\Omega)}{\mathrm{mod}\,\Omega}\leq D_{\Theta_n}[\Delta_w],$$

whence $D_{\Theta_{n,m}}[\Delta_w]\leq D_{\Theta_n}[\Delta_w]$. Let $\Phi_{n,m}(W)$ be the quasiconformal automorphism which is conformally equivalent to $\Theta_{n,m}^{-1}$ and belongs to $\mathcal{I}[H_W]$. If n, m increase,

2) non-singular at every point of the domain.

the positive quantity $\rho[\Phi_{n,m}(U)]-1$ becomes as close to zero as one pleases. For any $\varepsilon>0$ it is possible to construct a smooth non-singular quasiconformal automorphism $\psi_{n,m}(W)$ of $\mathcal{I}[H_W]$, such that $\psi_{n,m}(U)=\Phi_{n,m}(U)$ and $D_{\psi_{n,m}}[H_W]<\varepsilon$ if $n, m>n_0(\varepsilon)$. The smooth non-singular quasiconformal automorphism

$$\chi_{n,n}(w)=\frac{i-\psi_{n,n}\left(i\dfrac{1-w}{1+w}\right)}{i+\psi_{n,n}\left(i\dfrac{1-w}{1+w}\right)}$$

is of $\mathcal{I}[\Delta_w]$ and satisfies $D_{\chi_{n,n}}[\Delta_w]<\varepsilon$. Therefore $T_n(z)=\chi_{n,n}\circ\Sigma_{n,n}(z)$ $(n=1, 2,\cdots)$ is a desired sequence. q.e.d.

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