Harada, M.
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ALMOST QF RINGS WITH $J^3=0$

MANABU HARADA

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In this paper we always assume that R is a two-sided artinian ring with identity. In [3] we have defined right almost QF rings and showed that those rings coincided with rings satisfying $(*)^*$ in [2], which K. Oshiro [5] called co-H rings. We shall show in Section 2 that right almost QF rings are nothing but direct sums of serial rings and QF rings, provided $J^3=0$. Further in Section 5 we show that if R is a two-sided almost QF ring and $1=e_1+e_2+e_3$, then R has the above structure, provided $J^4=0$, where $\{e_i\}$ is a complete set of mutually orthogonal primitive idempotents. Moreover if $1=e_1+e_2+e_3+e_4$, we have the same result except one case. We shall study, in Section 3, right almost QF rings with homogeneous socles $W_k^n(Q)$ [7] and give certain conditions on the nilpotency m of the radical of $W_k^n(Q)$, under which $W_k^n(Q)$ is left almost QF or serial. In particular if $m\leqq 2n$, $W_k^n(Q)$ is serial. We observe a special type of almost QF rings such that every indecomposable projective is uniserial or injeative in Section 4.

1. Almost QF rings

In this paper we always assume that R is a two-sided artinian ring with identity and that every module M is a unitary right R-module. By $\bar{M}$ we denote $M/\mathrm{J}(M)$, where $\mathrm{J}(M)$ is the Jacobson radical of M. We use the same notations in [3]. We call R *a right almost* QF *ring* if R is right almost injective as a right R-module [3] and [4]. We can define similarly *a left almost* QF *ring*. If R is a two-sided almost QF ring, we call it simply *an almost* QF *ring*. It is clear that R is right almost QF if and only if every finitely generated projective R-modlue is right almost injective. Hence the concept of almost QF rings is preserved under Morita equivalence and we may assume that R is basic.

In this section we shall give some results which we use later. First we give a property of any right almost QF rings.

Proposition 1. *Assume that R is right almost* QF. *Let e_1R be injective, e_1J^i be projective, i.e., $e_1J^i\approx e_{\rho(i)}R$ for all $i\leqq$(some k) and $e_1J^{k+1}/e_1J^{k+2}\approx \bar{e}_a\bar{R}\oplus\cdots$.*

Then if e_aR is not injective, $e_1J^{k+1}\approx e_aR$, and hence $|e_1J^{k+1}/e_1J^{k+2}|=1$, where $\bar{e}_a\bar{R}=e_aR/e_aJ$.

Proof. Let x_aR be a submodule in e_1J^{k+1} such that $(x_aR+e_1J^{k+2})/e_1J^{k+2}\approx \bar{e}_a\bar{R}$ $(x_ae_a=x_a)$. Suppose that e_aR is not injective. Then $e_aR\subset e_pR$ (isomorphically) for some $p\neq a$, which is injective by [3], Corollary to Theorem 1. Let ρ: $e_aR\rightarrow x_aR\subset e_1R$; $\rho(e_a)=x_a$, be the natural epimorphism. Since e_1R is injective, there exists ρ': $e_pR\rightarrow e_1R$, which is an extension of ρ. Put $y=\rho'(e_p)$; $(y=ye_p)$ and $e_a=e_pr$; $r\in R$. We note that the e_1J^i are all waists for $i\leqq k+1$ by assumption. If $y\in e_1J^{k+1}$, then $\bar{x}_a=\bar{y}r=\bar{y}e_pre_a=\bar{o}$ in e_1J^{k+1}/e_1J^{k+2}, a contradiction. Accordingly $yR=e_1J^t$ for some $t\leqq k$. However e_1J^t is projective, and hence ρ' is a monomorphism. Consequently e_1J^{k+1} contains isomorphically the projective module e_aR, and eJ^{k+1} is local form [3], Corollary to Theorem 1.

Proposition 2. *Let R be right almost* QF. *If R is either a local ring or $J^2=0$, then R is serial or* QF.

Prof. R is a QF ring in the first case from [3], Corollary to Theorem 1. Assume $J^2=0$ and R is basic. If eR is injective for a primitive idempotent e, then $|eR|\leqq 2$ and eR is uniserial. Hence fR is injective and uniserial provided $fJ\neq 0$ by [3], Corollary to Theorem 1. Hence R is right serial and so R is serial by [5], Theorem 6.1.

Let $\bar{k}\bar{R}$ (or $\bar{R}\bar{g}$) be a simple module which appears in the factor modules of composition series of eR (or Re), where g is a primitive idempotent. In this case we say that *g belongs to eR (or Re).*

Lemma 1. *Let R be basic and let $\{e_iR\}_{i\leqq s}$ be a set of injective and projective modules. Assume that every primitive idempotent belonging to e_iR is equal to some $e_{\mu(i)}\in\{e_i\}$ for each e_i. Then $\Sigma_{i\leqq s}\oplus e_iR$ is a direct summand of R as rings.*

Proof. We note from the asumption that for each $e_j\in\{e_i\}$ there exists $e_{\rho(j)}$ in $\{e_i\}$ such that $\bar{e}_j\bar{R}\approx \mathrm{Soc}(e_{\rho(j)}R)$. Put $E=\Sigma_{i\leqq s}e_i$ and $F=1-E=\Sigma_{k\leqq p}f_k$, where the f_k are primitive idempotents. Then $ERF=0$ from the assumption. Let θ: $e_1R\rightarrow f_kR$ be a homomorphism. If $\theta\neq o$, there exist a simple submodule S of f_kR and a submodule T of e_1R such that $S\subset\theta(e_1R)$ and $T/\theta^{-1}(0)\approx S$. We may assume $S\approx\bar{e}_j\bar{R}$ for some e_j in $\{e_i\}$ by assumption. Accordingly $S\approx\mathrm{Soc}(e_{\rho(j)}R)$ by the initial remark, and hence we obtain a non-zero homomorphism of f_kR to $e_{\rho(j)}R$, since $e_{\rho(j)}R$ is injective. Therefore $f_k\in\{e_i\}$ by asumption, a contradiction. As a consequence $\theta=o$, i.e., $FRE=0$ and $R=ER\oplus FR=ERE\oplus FRF$.

The following lemma is essential in this paper.

Lemma 2. *Let R be artinian and F a uniform R-modlue. Assume that i): eR is injective, ii): eJ is a local quasi-projective module and iii):* $\mathrm{Soc}_2(F)/\mathrm{Soc}(F)$

$\approx \bar{e}\bar{R}\oplus A_2\oplus A_3\oplus\cdots$, *where e is a primitive idempotent and the* A_i *are simple. Then* $A_i\not\approx\bar{e}\bar{R}$ *for all* i.

Proof. Assume $A_2\approx\bar{e}\bar{R}$. Then since $\mathrm{Soc}_2(F)/\mathrm{Soc}(F)\approx\bar{e}\bar{R}\oplus\bar{e}\bar{R}\oplus\cdots$, $\mathrm{Soc}(F)$ is simple and eJ^2 is a waist by i) and ii), there exist x_1, x_1' in $\mathrm{Soc}_2(F)$ such that $x_1R\neq x_1'R$, $x_1R\approx x_1'R\approx eR/eJ^2$. Now let ρ: $x_1R\to eR/eJ^2$ be the isomorphism. Then $\rho(\mathrm{Soc}(x_1R))=eJ/eJ^2\approx\bar{e}_1\bar{R}$, where $eJ\approx e_1R/D$ and D is a charateristic submodule of e_1R by ii), where e_1 is a primitive idempotent. Take any element α in $\mathrm{End}_R(\mathrm{Soc}(x_1R))$. Then α gives an element $\bar{d}_1$ in $\mathrm{End}_R(\bar{e}_1\bar{R})$ via ρ. Then $\bar{d}_1$ is induced by an element d_1 in $\mathrm{End}_R(e_1R)$. On the other hand, since D is characteristic, $e_1R/D\approx eJ\subset eR$ and eR is injective, d_1 is extendible to d in $\mathrm{End}_R(eR)$. Hence d induces an element in $\mathrm{End}_R(eR/eJ^2)$ (and in $\mathrm{End}_R(x_1R)$ via ρ^{-1}, cf. the diagram).

$$
\begin{array}{ccccc}
 & & & & D \\
 & & & & \cap \\
 & & e_1R/e_1J & \longleftarrow & e_1R \\
 & \rho & \downarrow\mu & \nu & \downarrow\mu \\
\mathrm{Soc}(x_1R) & \approx & eJ/eJ^2 & \longleftarrow & eJ \\
\cap & \rho & \cap & & \nu\ \cap \\
x_1R & \approx & eR/eJ^2 & \longleftarrow & eR
\end{array}
$$

Thus we have obtained a mapping θ by taking extension, which may depend on a choice of d

$$\theta\colon \mathrm{End}(\mathrm{Soc}(x_1R))\to\mathrm{End}_R(x_1R)\,.$$

Let t: $x_1R\to x_1'R$ be the given isomorphism. Then t induces $\bar{d}_1$ in $\mathrm{End}(\mathrm{Soc}(F))$ $=\mathrm{End}_R(\mathrm{Soc}(x_1R))$ by taking restriction. Put $t'=\theta(\bar{d}_1)-t$: $x_1R\to F$. Then $t'(\mathrm{Soc}(x_1R))=0$, and hence $t'(x_1R)\subset\mathrm{Soc}(F)$. Then $t(x_1R)=(\theta(\bar{d}_1)-t')(x_1R)\subset$ $x_1R+\mathrm{Soc}(F)=x_1R$, a contradiction.

2. $J^3=0$

In this section we shall observe the ring R with following properties: 1) R is a basic and right almost QF ring, 2): $J^2\neq0$ and $J^3=0$.

Lemma 3. *Assume that* fR *is injective and* $J^3=0$. *Then we have* 1)*:* fJ^2 *is simple or zero and* 2)*:* fR *is uniserial if* $fJ^2=0$.

Lemma 4. *Let* fR *and* J *be as in Lemma* 3 *and assume that* R *is right almost* QF. *If* fR *contains properly a projective submodule* $P\neq0$, *then* fR *is uniseria and hence* $|fR|\leqq3$.

Proof. Since $fR\supset fJ\supset P\supset\mathrm{Soc}(fR)$, fJ is local by [3], Corollary to Theorem 1, and hence fR is uniserial for $fJ^3=0$.

Corollay. *Assume that R is right almost* QF *and $J^3=0$. If $|eR|\geqq 3$, i.e. $eJ^2\neq 0$, then eR is injective. Hence gR is injective or uniserial for any primitive idempotent g.*

Proof. If eR is not injective, $eR\subset fR$ for some injective fR by [3], Corollary to Theorem 1, a contradition to Lemma 4.

Let e_1R be an (injective) R-module. If $e_1J/e_1J^2\approx\bar{e}_a\bar{R}\oplus\bar{e}_b\bar{R}\oplus\cdots$ and $e_1J^2\approx\bar{e}_c\bar{R}$, then we denote this situation by

$$e_1R=(1\ \begin{smallmatrix}a\\ b\\ \vdots\end{smallmatrix}\ c)\ \text{or}\ e_1R=(e_1\ \begin{smallmatrix}e_a\\ e_b\\ \vdots\end{smallmatrix}\ e_c)\,.$$

Lemma 5. *Let e_1R be injective and $e_1J^2\neq 0$ $(\approx\bar{e}_c\bar{R})$ in the above. Then $e_aJ/e_aJ^2\approx\bar{e}_c\bar{R}\oplus\cdots$.*

Proof. There exists x_aR in e_1J such that $x_aR\supset \mathrm{Soc}(e_1R)$, $x_aR/\mathrm{Soc}(e_1R)=\bar{e}_a\bar{R}$ and $x_aR\approx e_aR/A$ for some A. Hence we obtain the lemma.

Lemma 6. *Let e_1R be a non-uniserial and injective module expressed as above. We assume that R is right almost* QF *and $J^3=0$. Then e_cR is injective. Further if e_aR is uniserial, then e_cR is not.*

Proof. First we assume $a\neq b$. Now e_aR is an injective module with $e_aJ^2\neq 0$ by Proposition 1. We have the same for e_bR. From Lemma 5 let

$$e_aR=(a\ \begin{smallmatrix}c\\ c_1\\ \vdots\end{smallmatrix}\ d)\ \text{ and }\ e_bR=(b\ \begin{smallmatrix}c\\ c_2\\ \vdots\end{smallmatrix}\ d')\,.$$

Since $e_aR\not\approx e_bR$, $d\neq d'$. Then e_cR is not uniserial (even though e_aR is uniserial in this case), and hence e_cR is injective by Corollary to Lemma 4. Next assume $a=b$, i.e.

$$e_1R=(1\ \begin{smallmatrix}a\\ \vdots\\ a\end{smallmatrix}\ c)$$

If e_aR is not uniserial, e_cR is injective by Lemma 5 and Proposition 1. Hence assume that e_aR is uniserial. If further e_cR is uniserial, then we can derive a contradiction by Lemma 2. Therefore if e_aR is uniserial, then e_cR is not uniserial and hence e_cR is injective by Corollary to Lemma 4.

Theorem 1. *Let R be an aritnian ring with $J^3=0$. Then the following are equivalent:*

1) *R is right almost* QF.
2) *R is left almost* QF.
3) *R is a direct sum of serial rings and* QF *rings.*

Proof. Let $\{e_i\}_{i\leq t}$ be the complete set of mutually orthogonal primitive idempotents. We shall prove the theorem inductively on t. If every eR is uniserial, then R is right serial. Therefore R is serial by [5], Theorem 6.1. Hence we asaume that there exists an injective but not uniserial module $e_1R=\left(1\ \begin{matrix} a\\ \vdots\\ b\end{matrix}\ c\right)$. We have shown in Lemma 6

(1) if e_g belongs to e_1R, then e_gR is injective, i.e., e_aR, e_bR and e_cR are injective. We shall show that if we replace e_1R with e_aR, e_bR and e_cR, then we obtain

(2) the same result as (1) for those e_aR, e_bR, e_cR.

If e_aR is not uniserial, we obtain (2) for e_aR. Suppose e_aR is uniserial. Then $e_aJ\approx e_cR/B$. Hence

(3) primitive idempotents ($\neq e_a$) belonging to e_aR belongs to e_cR if e_aR is uniserial.

Since e_cR is not uniserial by Lemma 6, from (3) we obtain again (2) for e_aR. Next consider e_cR. If e_aR is not uniserial, we obtain (2) for e_cR from the above (replace e_1R by e_aR). Suppose e_aR is uniserial, and e_cR is not uniserial by Lemma 6. Hence we obtain (2) for e_cR. Thus we have shown (2). Now starting from e_1R, we get e_aR, e_bR and e_cR which belong to e_1R. Next we take primitive idempotents belonging to $\{e_aR, e_bR, \cdots, e_cR\}$. Continuing this procedure and gathering all such primitive idempotents (use (1), (2) and (3)), we can find finally a set $\{e_1R, e_aR, \cdots\}$ satisfying the condition in Lemma 1. Hence $R=\sum_{i\leq m}\oplus e_iR\oplus\sum_{j>m}\oplus e_jR$ as rings. Now $\sum_{i\leq m}\oplus e_iR$ is a QF ring. Thus we can obtain the theorem by induction.

3. Right almost QF rings with homogeneous socles

In this section we shall study rings stated in the title. Let $\{e_i\}_{i\leq n}$ be a complete set of mutually orghogonal promitive idempotents with $1=\sum e_i$ and R a basic ring.

Let Q be a local QF ring with J radical. Put $\bar{Q}=Q/\mathrm{Soc}(Q)$ and $\bar{J}=J/\mathrm{Soc}(Q)$. According to [7], Theorem 1 we denote a right almost QF ring R with homogeneous socle by

$$(4)\quad W_k^n(Q)=\left.\left(\left(\begin{array}{c|c} \overbrace{Q\,Q\,Q\cdots Q}^{k} & \bar{Q}\,\bar{Q}\cdots\bar{Q} \\ J\,Q\,Q\cdots Q & \bar{Q}\,\bar{Q}\cdots\bar{Q} \\ \cdots & \cdots \\ J\,J\cdots J\ Q & \bar{Q}\,\bar{Q}\cdots\bar{Q} \\ \hline J\,J\ \cdots\cdots\ J & \bar{Q}\,\bar{Q}\cdots\bar{Q} \\ J\,J\ \cdots\cdots\ J & \bar{J}\,\bar{Q}\cdots\bar{Q} \\ \cdots & \cdots \\ J\ \cdots\cdots\cdots\ J & \bar{J}\,\bar{J}\cdots\bar{J}\,\bar{Q} \end{array}\right)\right)\right\}n$$

We note from [1] that there is only one projective and injective module e_1R (resp. Re_k) in R.

Lemma 7. *Assume $k<n$ on $R=\mathrm{W}_k^n(Q)$. Then if R is left almost* QF, *R is serial.*

Proof. Let $e_i=e_{ii}$ be the matrix unit in R. Then $e_i\,\mathrm{J}(R)\approx e_{i+1}R$ for $i<n$ and $e_n\,\mathrm{J}(R)=(J\cdots J\,\bar{J}\,\bar{J}\cdots\bar{J})$. Now assume $k<n$ and R is left almost QF. Then since $\mathrm{J}(R)\,e_s\approx Re_{s-1}$ for $s\leqq k$, $\mathrm{J}(R)\,e_1=(J\,J\cdots J)^t$ is isomorphic to $Re_q=(\bar{Q}\,\bar{Q}\cdots\bar{J})^t$ for some $p>k$ from the remark before Lemma 7 and [5], Theorem 3.18 (see [3], Corollary to Theorem 1), where $(\quad)^t$ is the transposed matrix of $(\quad)$. Hence since $e_1\,\mathrm{J}(R)\,e_1\approx\bar{Q}$ as left Q-modules, J is local and hence Q is serial (cf. Lemma 9 below). Then $J\approx Q/\mathrm{Soc}(Q)=\bar{Q}$ and $J/\mathrm{Soc}(J)=J/\mathrm{Soc}(Q)=\bar{J}\approx Q/\mathrm{Soc}_2(Q)\approx\bar{Q}/\mathrm{Soc}(\bar{Q})$ as right Q-modules. Put $A=(\mathrm{Soc}(Q)\,\mathrm{Soc}(Q)\cdots\mathrm{Soc}(Q)\,\mathrm{Soc}(\bar{Q})\cdots\mathrm{Soc}(\bar{Q}))$ in e_1R. Then $e_n\,\mathrm{J}(R)\approx e_1R/A$ from the above observation and hence $e_n\,\mathrm{J}(R)$ is local. Therefore R is right serial, and hence R is serial by [5], Theorem 6.1.

Lemma 8. *Assume $k=n$ on $R=\mathrm{W}_k^n(Q)$. Then R is left almost* QF.

Proof. This is clear from (4)

Theorem 2. *Let R and n be as in the begining. Assume that R is a right almost* QF *ring with homogeneous socle and $\mathrm{J}(R)^{m-1}\neq 0$, $\mathrm{J}(R)^m=0$ (and hence $R=\mathrm{W}_k^n(Q)$ and $m\geqq n$). Then*

1) *if $m\leqq 2n$, R is serial,*

2) *if $m=nr$, $r\geqq 3$, R is left almost* QF, *and*

3) *if $m=nr+k$, $r\geqq 2$ and $o<k<n$, R is left almost* QF *if and only if R is serial.*

Proof. By assumption and [7], Theorem 1 $R=\mathrm{W}_t^n(Q)$ and we have $e_i\mathrm{J}(R)\approx e_{i+1}R$ for $i<n-1$. By a direct computation of $\mathrm{J}(R)^p$ we have

i) $e_n\,\mathrm{J}(R)/e_n\,\mathrm{J}(R)^2\approx\bar{e}_1\bar{R}\oplus\cdots\oplus\bar{e}_1\bar{R}$ (cf. Proposition 1).

ii) $e_1\,\mathrm{J}(R)^{tn}=(J^t\cdots)$.

1). Since $m\leqq 2n$, $0=e_1J(R)^{2n}=(J^2\cdots)$ by ii). Hence $J^2=0$ and so Q is serial. Accordingly R is seiral from the proof of Lemma 7.

2) and 3). From i) we know

$$e_1R=\left(1\;2\;3\;n\;\begin{matrix}1\;2\;3\cdots\\1\;\;2\;\;3\cdots\\\cdots\end{matrix}\right).$$

Further $J^m=0$ if and only if $e_1\mathrm{J}(R)^m=0$. Hence $\mathrm{Soc}(e_1R)\approx\bar{e}_n\bar{R}$ if $m=nr$ and $\mathrm{Soc}(e_1R)\approx\bar{e}_k\bar{R}$ if $m=nr+k$, $o<k<n$. Therefore $R\approx\mathrm{W}_n^n(Q)$ if $m=nr$ and $R\approx\mathrm{W}_k^n(Q)$ if $k\neq o$. As a consequence we obtain the theorem from Lemmas 7 and 8.

Corollary. *Assume $n=2$ and R is right almost* QF. *Then if* $\mathrm{J}(R)^{2m-1}\neq 0$, $\mathrm{J}(R)^{2m}=0$, *R is left almost* QF. *If* $\mathrm{J}(R)^{2m}\neq 0$, $\mathrm{J}(R)^{2m+1}=0$, *R is* QF *or serial if and only if R is left almost* QF. *Further if $J(R)^4=0$, R is* QF *or serial.*

Proof. If R is QF or serial, the corollary is clear by [5], Theorem 4.5. Assume that R is not QF. Since $n=2$, we can suppose that e_1R is injective and $e_1\mathrm{J}(R)\approx e_2R$. Hence we obtain the corollary from Theorem 2.

4. Rings with (#-i)

In the previous sections we have observed a ring which is a direct sum of QF rings and serial rings. In this case

(#-1) *eR is injective or uniserial for each primitive idempotent e.*

We consider two more conditions. Let eR be injective but not uniserial. Then we may assume that there exists an integer s such that eJ^i/eJ^{i+1} is simple for all i ($o\leqq i\leqq s-1$) and $eJ^s/eJ^{s+1}\approx\Sigma_{j\leqq k}\oplus\bar{f}_j\bar{R}$; $k\geqq 2$, where the f_j are primitive idempotents. Here we consider the second condition

(#-2) *the f_jR is injective for all j.*

Assume that R is a right almost QF ring with (#-1). In the above we put $eJ^i/eJ^{i+1}\approx\bar{g}_i\bar{R}$; g_i is a primitive ideomptent. Since eR is not uniserial, g_iR is injective by (#-1). In particular $eJ^{s-1}\approx g_{s-1}R/A$ for some A in an injective $g_{s-1}R$ and hence $eJ^s/eJ^{s+1}\approx g_{s-1}J/(g_{s-1}J^2+A)\leftarrow g_{s-1}J/g_{s-1}J^2$. Since $|eJ^s/eJ^{s-1}|\geqq 2$, (#-2) is satisfied from Propostion 1. From the above observation we know that

Assume that R is right almost QF, the (#-1) is satisfied if and only if every non-injective projective gR is contained in a uniserial injective eR and in this case (#-2) and (#-3) below are satisfied.

Taking some non-serial right serial rings, we can get rings with (#-1, 2) which are not right almost QF. Hence we consider the third condition. Here we assume temporarily that R is an algebra over a field K with finite dimension. We further assume that R satisfies (#-1) as right as well as left R-modules. Let gR be not injective, and hence uniserial. Then $\mathrm{E}(gR)$ is indecomposable. Take $\mathrm{E}(gR)^*=\mathrm{Hom}_K(E(gR), K)$. Then $\mathrm{E}(gR)^*$ is indecomposable and projective. Therefore $\mathrm{E}(gR)\approx\mathrm{E}(gR)^{**}$ is local. We consider this property for any ring.

(#-3) $\mathrm{E}(gR)$ *is local for each primitive idempotent g.*

Now we study rings with (#-1, 2, 3). We always assume that R is basic.

Lemma 9. *Assume $eJ^i/eJ^{i+1}\approx\bar{e}_1\bar{R}\oplus\bar{e}_2\bar{R}\oplus\cdots\oplus\bar{e}_s\bar{R}$. Then eJ^{i+1}/eJ^{i+2} is a homomorphic image of $\bar{e}_1\bar{J}\oplus\bar{e}_2\bar{J}\oplus\cdots\oplus\bar{e}_s\bar{J}$.*

Proof. We can express eJ^i as $x_1R+x_2R+\cdots+x_sR+eJ^{i+1}$, where $x_je_j=x_j$. Hence $eJ^{i+1}=x_1e_1J+\cdots+x_se_sJ+eJ^{i+2}$. Thus we obtain the lemma.

Lemma 10. *We assume that* ($\sharp$-3) *is satisfied. Suppose that eR is injective and $eJ/eJ^2\approx\bar{g}_1\bar{R}$, $g_1J/g_1J^2\approx\bar{g}_2\bar{R}$, $\cdots$, $g_{s-1}J/g_{s-1}J^2\approx\bar{g}_s\bar{R}$, where the g_i is a primitive idempotent and g_iR is not injective for all i. Then $eR\supset g_1R\supset\cdots\supset g_sR$ isomorphically.*

Proof. We shall show $eJ^i\approx g_iR$ for all i by induction on i. Assume $eJ^t\approx g_tR$ if $t\leqq$(soem $k-1$). Then $eJ^k/eJ^{k+1}\approx g_{k-1}J/g_{k-1}J^2\approx\bar{g}_k\bar{R}$ by assumption. Let $eJ^k=x_kR(x_kg_k=x_k)$ and $\rho\colon g_kR\to eJ^k(\rho(g_k)=x_k)$ the natural epimorphism. Take a diagram

$$\begin{tikzcd}
0 \arrow[r] & g_kR \arrow[r] \arrow[d, "\rho"] & \mathrm{E}(g_kR) \arrow[ddl, dashed, "\rho'"] \\
& x_kR \arrow[d, phantom, "\cap"] & \\
& eR &
\end{tikzcd}$$

Since eR is injective, we have $\rho'\colon E(g_kR)\to eR$ which commutes the diagram. $\mathrm{E}(g_kR)$ being local from ($\sharp$-3), $\rho'(\mathrm{E}(g_kR))\supsetneqq x_kR=\rho'(g_kR)$ for $g_kR\neq\mathrm{E}(g_kR)$. Further eJ^t is a waist for all $t\leqq k$ by induction hypothesis. Consequenctly $\rho'(\mathrm{E}(g_kR))$ is projective. Therefore ρ' is a monomorphism, and hence so is ρ.

Lemma 11. *We assume that* ($\sharp$-1), ($\sharp$-2) *and* ($\sharp$-3) *are satisfied and that eR is injective and g_1 belongs to eR. If g_1R is not injective, then g_1R is contained isomorphically in an injective and uniserial module e_1R.*

Proof. Since g_1 belongs to eR, we may suppose $eJ^s/eJ^{s+1}\approx\bar{g}_1\bar{R}\oplus\cdots$ for some s. g_1R being not injective, $s\neq o$. If $s=1$, then $|eJ/eJ^2|=1$ by ($\sharp$-2) and $g_1R\approx eJ$ from Lemma 10 and eR is uniserial by ($\sharp$-1). Hence assume $s>1$. From Lemma 9 there exists g_2 such that $eJ^{s-1}/eJ^s\approx\bar{g}_2\bar{R}\oplus\cdots$ and $g_2J/g_2J^2\approx\bar{g}_1\bar{R}\oplus\cdots$. If g_2R is not uniserial, g_2R is injetive by ($\sharp$-1), and then g_1R is injective by ($\sharp$-2), a contradiction (cf. the remark after ($\sharp$-2)). Accordingly g_2R is uniserial and hence $g_2J/g_2J^2\approx\bar{g}_1\bar{R}$. Next assume that g_2R is not injective. Then g_2R satisfies the same condition as on g_1R, and hence similarly to the above we can find g_3R such that $eJ^{s-2}/eJ^{s-1}\approx\bar{g}_3\bar{R}\oplus\cdots$ and $g_3J/g_3J^2\approx\bar{g}_2\bar{R}\oplus\cdots$. Repeating this process, we obtain finally an injetive and uniserial module e_1R such that $e_1J/e_1J^2\approx\bar{g}_s\bar{R}$ for some t (and $g_tJ/g_tJ^2\approx\bar{g}_{t-1}\bar{R}$, $\cdots g_2J/g_2J^2\approx\bar{g}_1\bar{R}$). Hence e_1R contains isomorphically g_1R from Lemma 10.

Proposition 3. ($\sharp$-1), ($\sharp$-2) *and* ($\sharp$-3) *are satisfied if and only if R is right almost* QF *and every non-injective projective gR is contained in a uniserial and injective eR.*

Proof. We assume ($\sharp$-1, 2, 3). First we shall show that R is right QF-3. Let eR be not injective. Then $\mathrm{E}(gR)$ is local by ($\sharp$-3), i.e., $\mathrm{E}(gR)\approx fR/A$ and

fR is uniform from (♯-1). Further $fR/A\supset gR$ and $gR\approx B/A$ for some B ($\supset A$) in fR. Therefore since gR is projective and fR is uniform, $A=0$ and $fR=E(gR)\supset gR$. Accordingly R is right QF-3. Let hR be injective and suppose $hR\supset k_1R$, where h and k_1 are primitive ideompotents. Then form the last part of the proof of Lemma 11 there exists a uniserial and injective module h_1R such that $h_1R=(h_1\, k_s\cdots k_1\cdots)$ and $h_1R\supset k_1R$. Hence $hR\approx h_1R$. Thus R is right almost QF by [3], Corollary to Theorem 1. The converse is clear from the remark before Lemma 9.

5. $J^4=0$

In this section we assume that R is an (basic) artinian ring with $J^4=0$. Let $1=\Sigma_{i\leqq n}\, e_i$ be as in §3. We studied almost QF rings with $n=2$ in Corollary to Theorem 2. We study almost QF rings with $n=3$ or 4 in this section.

Lemma 12. *Let R be two-sided almost* QF. *If R is not* QF, *then there exists an injective and projective eR such that $eR/\mathrm{Soc}(eR)$ is again injective.*

Proof. R is right almost QF♯ by [6], Theorem 3.7. Hence we obtain the lemma from [2], Theorem 2.3.

Theorem 3. *Let R be an (basic) artinian ring. Assume that $J^4=0$ and $n\leqq 3$, where $\{e_i\}_{i\leqq n}$ is a complete set of mutually orthogonal primitive idempotents. Then the following are equivalent:*

1) (♯-1), (♯-2) *and* (♯-3) *are satisfied as right as well as left R-modules.*
2) *R is a two-sided almost* QF *ring.*
3) *R is a direct sum of serial rings and* QF *rings.*

Proof. 1)→2). This is given by Proposition 3.

2)→3). From Corollary to Theorem 2 and Theorem 1 we can suppose $n=3$ and $J^3\neq 0$. First we note that if R is a direct sum of two rings, then R is a direct sum of serial rings and QF rings from Propostion 2 and Corollary to Theorem 2. We call this situation R splits. Let R be two-sided indecomposable and neither serial nor QF. Then we shall derive a contradiction for all possible situations. If $e_1R\supset e_2R\supset e_3R$, R is serial by Theorem 2. Thus we may suppose from [3], Theorem 1

(5) e_1R, e_3R are injective and $e_1J\approx e_2R$.

First we assume that e_1R is uniserial.

i) e_1R is uniserial and e_3J is local.

Then e_iJ/e_iJ^2 is uniserial for all i. Hence R is right serial, and R is serial by [5], Theorem 6.1.

Thus we may assume

ii) e_1R is uniserial, but e_3J is not local, i.e.,

$$e_3R=\left(3\ \begin{array}{cc} a & a' \\ \vdots & \vdots \\ b & b' \\ \vdots & \vdots \\ & c' \\ & \vdots \end{array}\ d\right)$$

Then $\{a, b\}\subset\{1, 3\}$ from Propostion 1. First we note that if $a=b=3$, then R splits from Lemmas 1 and 9. Hence we can skip the case $a=b=3$.

$|e_1R|=2$. i) $e_1R=(1\ 2)$.

$a=1$. Let $e_3J/e_3J^2\approx\bar{e}_1\bar{R}\oplus\cdots$. Then there exists x_1 in e_3J such that $x_1 e_1=x_1$ and $(x_1R+e_3J^2)/e_3J^2\approx\bar{e}_1\bar{R}$. (We use this notation in the following arguments.) Suppose that x_1R is simple. Then $x_1R=\mathrm{Soc}(e_3R)\subset e_3J^2$ for $e_3J^2\neq 0$, a contradiction. Hence $x_1R\approx e_1R$ is injective, again a contradiction.

$|e_1R|=3$. ii) $e_1R=(1\ 2\ 1)$.

$a=1$. Then we take X_a in e_3R such that $X_a\supset e_3J^2$ and $e_3J/X_a\approx\bar{e}_a\bar{R}$. Since $e_3J/X_s\approx\mathrm{Soc}(e_1R)\approx\bar{e}_1\bar{R}$ and e_1R is injective, $e_2=e_3$, a contradiction.

iii) $e_1R=(1\ 2\ 2)$. Then $e_1J/e_1J^2\approx\mathrm{Soc}(e_1R)$. Hence $e_1=e_2$, a contradiction.

iv) $e_1R=(1\ 2\ 3)$.

$a=3$. We obtain the same contradiction as in iii).

$a=b=1$. $x_aR\approx(e_1R/\mathrm{Soc}_2(e_1R)$ or $e_1R/\mathrm{Soc}(e_1R))$. Hence $(x_aR+e_1J^2)J^2=0$. Accordingly $0=(\Sigma_a\, x_aR+e_1J^2)J^2=e_3J^3$, a contradiction to $J^3\neq 0$.

$|e_1R|=4$. v) $e_1R=(1\ 2\ 1\ x)$. Then $x=2$.

$a=1$. Then x_aR in e_3J is a homomorphic image of e_1R, and hence $x_aR\approx(e_1R/e_1J^3$ or $e_1R/e_1J)$. If $x_aR\approx e_1R/e_1J$, $x_aR\subset\mathrm{Soc}(e_3R)\subset e_3J^2$, a contradiction. Hence we obtain a homomorphism ψ: $\mathrm{Soc}_2(x_aR)\to\mathrm{Soc}_2(x_aR)/\mathrm{Soc}(x_aR)\approx\bar{e}_2\bar{R}\to\mathrm{Soc}(e_1R)$. Since e_1R is injective, we obtain an extension of ψ, which is a contradiction to the structure of e_1R and e_3R.

vi) $e_1R=(1\ 2\ 2\ x)$.

Then $x=2$ and $e_1J/e_1J^2\approx\mathrm{Soc}(e_1R)$. Hence $e_1=e_2$, a contradiction.

vii) $e_1R=(1\ 2\ 3\ x)$. Since $\{a, b\}\subset\{1, 3\}$, $x=1$ or 3, and $d\neq x$.

vii-i) $x=1$ and $d=2$. Then $a=1$.

α) $b=1$. Let $e_3J/e_3J^2\approx\bar{x}_1\bar{R}\oplus\bar{x}_1'\bar{R}\oplus\cdots$. Since $d=2$, we may assume $x_1R\approx x_1'R\approx\cdots(\approx e_1R/e_1J^2$, which is uniserial). Hence $x_1R, x_1'R\cdots$ are contained in $\mathrm{Soc}_2(e_3R)$. Therefore $e_3J=\mathrm{Soc}_2(e_3R)$ for $\mathrm{Soc}_2(e_3R)\supset e_3J^2$. As a consequence

$$e_3R=\left(3\ \begin{array}{c} 1 \\ \vdots \\ 1 \end{array}\ 2\right).$$

Then we obtain a contradiction to Lemma 2.

β) $b=3$. e_3J contains a submodule x_1R isomorphic to e_1R/e_1J^2 as in α). Hence $x_1R\subset\mathrm{Soc}_2(e_3R)$ and $x_1R\not\subset e_3J^2$. Since $b=3$, e_3J^2/e_3J^3 has to containa a simple submodule isomorphic to $\bar{e}_1\bar{R}$ by Lemma 9 and its proof. Hence since $x_1R\not\subset e_3J^2$, $\mathrm{Soc}_2(e_3R)/\mathrm{Soc}(e_3R)\approx\bar{e}_1\bar{R}\oplus\bar{e}_1\bar{R}\oplus\cdots$, a contradiction to Lemma 2.

vii-ii) $x=1$ and $d=3$ (and hence $a=1$).

α) $b=1$. Since $\mathrm{Soc}(e_1R/\mathrm{Soc}(e_1R))\approx\mathrm{Soc}(e_3R)$, $e_3R/\mathrm{Soc}(e_3R)$ $(=E)$ is injective by Lemma 12. Further $\mathrm{Soc}_2(e_3R)=e_3J^2$ and $\mathrm{Soc}_2(E)/\mathrm{Soc}(E)\approx e_3J/e_3J^2\approx\bar{e}_1\bar{R}\oplus\bar{e}_1\bar{R}\oplus\cdots$, a contradiction to Lemma 2.

β) $b=3$. From the structure of e_3R and Lemma 9 we know $\mathrm{Soc}_2(e_3R)/\mathrm{Soc}(e_3R)\approx\bar{e}_2\bar{R}$ or $\approx\bar{e}_3\bar{R}$. Then $\mathrm{Soc}_2(e_1R)/\mathrm{Soc}(e_1R)\approx\mathrm{Soc}(e_3R)$ as above, a contradiction.

vii-iii) $x=3$, i.e. $e_1R=(1\ 2\ 3\ 3)$.

Since $e_1R/\mathrm{Soc}(e_1R)$ is not injective, $|\mathrm{Soc}_2(e_3R)/\mathrm{Soc}(e_3R)|=1$ by Lemma 12. Hence

$$e_3R=(3\ \begin{smallmatrix}1\\ \vdots\\ 1\end{smallmatrix}\ x\ y)\ \text{or}\ (3\ \begin{smallmatrix}1\\ \vdots\\ 3\end{smallmatrix}\ x\ y)$$

(note $e_3J^2\subset\mathrm{Soc}_2(e_3R)$). If $e_3J^3\neq0, y=3$, a contradiction. If $e_3J^3=0$, $|\mathrm{Soc}_2(e_3R)/\mathrm{Soc}(e_3R)|\geqq2$, a contradiction.

Thus we have shown that R is a direct sum of serial rings and QF rings, provided e_1R is uniserial.

Finally we observe the structure of R, when e_1R is not uniserial. Assume that an injective module eR contains a projective proper submodule and is not uniserial. Then eJ is local by [3], Corollary to Theorem 1, and hence

$$eR=(a\ b\ \begin{smallmatrix}c\\ \vdots\\ c'\\ \vdots\end{smallmatrix}\ d);\ eJ^3\neq0\,.$$

Now from **i)**, **ii)**, (5), Proposition 1 and Lemma 12, we may assume

iii) $e_1R=(1\ 2\ \begin{smallmatrix}a\\ \vdots\\ b\\ \vdots\end{smallmatrix}\ g)$ and $e_3R=(3\ \begin{smallmatrix}a'\\ \vdots\\ b'\\ \vdots\end{smallmatrix}\ h\ g')$ are injective, e_1R is not uniserial and $e_2R\approx e_1J$; $\{a, b\}\subset\{1, 3\}$.

From Lemma 12 we have

Lemma 13. *Let R, e_1R and e_3R be as above. Then $e_3R/\mathrm{Soc}(e_3R)$ is injective.*

First we assume that e_3R is not uniserial. We note that if $a'=b'=3$, then R splits from Lemmas 1 and 9.

iii-1) e_1R and e_3R are not uniserial, and hence $e_3J^3\neq0$ from Lemma 13.

i) $a=1$. Then $g=2$.

$a'=1$. Then $h=2$ and $\mathrm{Soc}(e_3R/\mathrm{Soc}(e_3R))\simeq\mathrm{Soc}(e_1R)$, a contradiction from

Lemma 13.

ii) $a=b=3$.

α) $a'=b'=1$. Then $h=2$ and $g'=3$, i.e.,

$$e_1R = (1\ 2\ \overset{3}{\underset{3}{\vdots}}\ 1),\ e_2R = (2\ \overset{3}{\underset{3}{\vdots}}\ 1)\ \text{ and }\ e_3R = (3\ \overset{1}{\underset{1}{\vdots}}\ 2\ 3)\,.$$

Then $e_3R/\mathrm{Soc}(e_3R)$ $(=E)$ is injective by Lemma 13 and $\mathrm{Soc}_2(E)/\mathrm{Soc}(E)\approx\bar{e}_1\bar{R}\oplus\cdots\oplus\bar{e}_1\bar{R}$. Since e_3R is not uniseial, $|\mathrm{Soc}_2(E)/\mathrm{Soc}(E)|\geqq 2$, a contradiction to Lemma 2.

β) $a'=1$ and $b'=3$. Then $h=2$ from e_1R and $h=1$ or 3 from e_3R, a contradiction.

iii-2) e_1R is not uniserial and e_3R is uniserial.

α) $a=b=1$. Then

$$e_1R = (1\ 2\ \overset{1}{\underset{1}{\vdots}}\ 2),\ \text{which contradicts Lemma 2.}$$

β) $a=1$, $b=3$. Then

$$e_1R = (1\ 2\ \overset{1}{\underset{3}{\vdots}}\ 2),\ \text{and hece } e_3R = (3\ 2\ c\ d)\,.$$

If $e_3J^3=0$ (resp, $e_3J^2=0$), $\mathrm{Soc}_2(e_3R)/\mathrm{Soc}(e_3R)\approx\mathrm{Soc}(e_1R)$ (resp. $\mathrm{Soc}(e_3R)\approx\mathrm{Soc}(e_1R)$), a contradiction from Lemma 13. Assume $e_3J^3\neq 0$, then $c=1$ or 3, and hence $d=2$, a contradiction.

γ) $a=b=3$.

i) $g=1$. Then

$$e_1R = (1\ 2\ \overset{3}{\underset{3}{\vdots}}\ 1)\ \text{ and }\ e_3R = (3\ 1\ c\ d)$$

We know as above $e_3J^3\neq 0$, and so $e_3R=(3\ 1\ 2\ 3)$. Here we shall again make use of the argument in the proof of Lemma 2. Since e_3R is uniserial, there exist two submodules yR, $y'R$ in e_1J^2 such that $yR\approx y'R\approx e_3R/e_3J^2$. Let α be an element in $\mathrm{End}_R(\mathrm{Soc}(yR))$. We shall find an extension of α in $\mathrm{End}_R(yR)$. Since $yR\approx e_3R/e_3J^2$, $\mathrm{Soc}(yR)\approx\mathrm{Soc}(e_3R/e_3J^2)\approx e_1R/e_1J$. Hence we may assume that α is given by an element p in e_1R via the above isomorphism. Then p inducees an endomorphism $\bar{p}$ of $e_1R/e_1J^2\approx\mathrm{Soc}_2(E)\subset E$ $(\approx e_3R/e_3J^3)$. Further $\bar{p}$ is extendible to q in $\mathrm{End}_R(E)$. Finally since $E/\mathrm{Soc}(E)\approx e_3R/e_3J^2$, $\bar{q}$ induces an element in $\mathrm{End}_R(e_3R/e_3J^2)$, which is an extendion of α (see the diagram below)

$$\begin{array}{ccccc} & & E\approx e_3R/e_3J^3 & \xrightarrow{\ \rho\ } & e_3R/e_3J^2 \longrightarrow 0 \\ & & \cup \qquad\quad \cup & & \cup \\ e_1R/e_1J^2 & \approx & \mathrm{Soc}_2(E)\approx X & \xrightarrow{\ \rho\ } & \mathrm{Soc}(e_3R/e_3J^2)\rightarrow 0\,, \end{array}$$

where ρ is the natural epimorphism.
Using this extension, we can derive a contradiction.

β) $a'=1$ and $b'=3$. Then $h=2$ from e_1R and $h=1$ or 3 from e_3R, a contradiction.

3)$\rightarrow$1). This is trivial.

Theorem 4. *Let R and n be as in Theorem 3. Assume that R is a two-sided almost* QF *and two-sided indecomposable ring with $J^4=0$ and $n=4$. Then R is either serial or* QF *if and only if R is not of the followihg: there exist exactly three injective and projective modules e_iR and some one among e_iR is not uniserial.*

Proof. Suppose that R is not QF. Then we have the following four cases:

1) e_1R is injective and $e_1R\supset e_2R\supset e_3R\supset e_4R$ (isomorphically).

2) e_1R and e_4R are injective and $e_1R\supset e_2R\supset e_3R$.

3) e_1R and e_3R are injective and $e_1R\supset e_2R$, $e_3R\supset e_4R$.

4) e_1R, e_2R and e_4R are injective and $e_1R\supset e_2R$.

Case 1) Since $J^4=0$, R is serial by Theorem 2.

Case 2) Then e_1R is uniserial by [3], Corollary to Theorem 1, i.e., $e_1R=(1\ 2\ 3\ d)$ (or$=(1\ 2\ 3)$) and e_4R are injective. If e_4J is local, R is right serial. Suppose that e_4J is not local. Then from Proposition 1 we have the following:

$$\text{a) } e_4R=(4\begin{matrix}1\cdots\\ \vdots\\ 1\cdots\end{matrix}),\quad \text{b) } e_4R=(4\begin{matrix}1\cdots\\ \vdots\\ 4\cdots\end{matrix})\text{ or c) } e_4R=(4\begin{matrix}4\cdots\\ \vdots\\ 4\cdots\end{matrix})$$

R splits if c) occurs. Hence we assume a) or b).

i) $e_1R/\mathrm{Soc}(e_1R)$ and $e_1R/\mathrm{Soc}_2(e_1R)$ are injective (see the proof of Lemma 12).

Let xR be a submodule in e_4J with $(xR+e_4J^2)/e_4J^2\approx\bar{e}_1\bar{R}$. Since e_1R is uniserial, $\mathrm{Soc}(e_4R)=\mathrm{Soc}(xR)\approx\bar{e}_2\bar{R}$ or $\bar{e}_3\bar{R}$ if $e_1J^3\neq0$. However $\mathrm{Soc}(e_1R/\mathrm{Soc}(e_1R))\approx\bar{e}_3\bar{R}$ and $\mathrm{Soc}(e_1R/\mathrm{Soc}_2(e_1R))\approx\bar{e}_2\bar{R}$, a contradiction. If $e_1J^3=0$, we obtain the same result as above.

ii) $e_1R/\mathrm{Soc}(e_1R)$ and $e_4R/\mathrm{Soc}(e_4R)$ are injective.

α) $e_1J^3\neq0$. $e_1R=(1\ 2\ 3\ d)$.

Assume a) or b). $\mathrm{Soc}(e_4R)$ and $\mathrm{Soc}_2(e_4R)$ are waists by assumption. Since $\mathrm{Soc}(eR/\mathrm{Soc}(e_1R))\approx\bar{e}_3\bar{R}$, there exists a submodule xR in e_4J such that $xR\approx e_1R/e_1J^2$, i.e., $e_4J^3=0$, and hence e_4R is uniserial.

β) $e_1J^3=0$. $e_1R=(1\ 2\ 3)$.

Then xR is simple, i.e. $|e_4R|\leqq2$, a contradiction.

iii) $e_4R/\mathrm{Soc}(e_4R)$ and $e_4R/\mathrm{Soc}_2(e_4R)$ are injective. Then e_4R is uniserial and hence R is serial.

Case 3) i) $e_1R/\mathrm{Soc}(e_1R)$ and $e_1R/\mathrm{Soc}_2(e_1R)$ are injective. Then $e_1R=(1\ 2\ c\ d)$ (or$=(1\ 2\ c)$) and

$$e_3R = (3\ 4 \overset{g}{\underset{h}{\vdots}}\ k) \text{ or } (3\ 4\ g\ k)$$

In the latter case R is serial. Hence assume the former. Then $\{g, h\} \subset \{1, 3\}$. Assume $e_1J^3 \neq 0$.

α) $g=1$. There exists xR in e_4J^2 with $xR \approx e_1R/A$ for some A in e_1R. However $\mathrm{Soc}(e_3R) = \mathrm{Soc}(xR) \approx \bar{e}_2\bar{R}$, a contradiction.

β) $g=h=3$. Then

$$e_4R = (3\ 4 \overset{3}{\underset{3}{\vdots}}\ 4),$$

which is a contradiction to Lemma 2.
We obtain the same result in a case $e_1J^3 = 0$.

ii) $e_1R/\mathrm{Soc}(e_1R)$ and $e_3R/\mathrm{Soc}(e_3R)$ are injective. Then e_1R and e_3R are uiserial, and hence R is serial.

Case 4) If e_1R, e_3R and e_4R are uniserial, R is right serial.

6. Examples

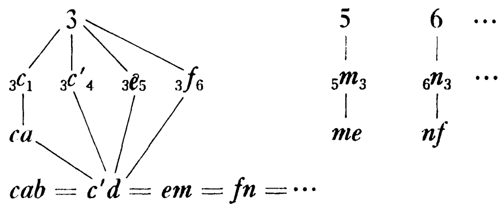
In this section we shall give several examples related to the previous sections.

1. We shall give a two-sided almost QF ring with $J^4=0$ and $n=4$ but neither QF nor serial. This is an example of exceptional algebras in Theorem 4. Let K be a field and $R = \Sigma_{i \leq 4} \oplus e_iR$, where $\{e_i\}$ is a set of mutually orthogonal primitive idempotents with $1 = \Sigma\, e_i$. We define $e_1R = e_1 K \oplus aK \oplus abK \oplus abc'K$, $e_2R = e_2K \oplus bK \oplus bc'K, \cdots$, whose multiplicative structur is given below, where ${}_1a_2$ means $a = e_1\, ae_2$, and so on.
(In the previous sections we expressed horizontally the structure of e_iR, however we shall do vertically here.)

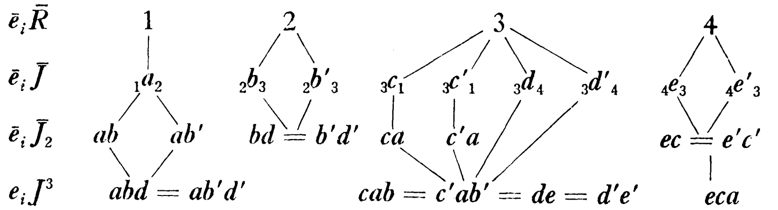
$$\begin{array}{lcccc}
e_iR/e_iJ & 1 & 2 & 3 & 4 \\
 & | & | & / \qquad \backslash & | \\
e_iJ/e_iJ^2 & {}_1a_2 & {}_2b_3 & {}_3c_1 \qquad {}_3c'_4 & {}_4d_3 \\
 & | & | & | \qquad\quad & | \\
e_1J^2/e_iJ^3 & ab & bc' & ca \qquad / & dc \\
 & | & & \backslash \quad & | \\
e_iJ^3 & abc' & & cab = c'd & dca\,,
\end{array}$$

where the other products among $a, b, \cdots$ are zero, e.g. $bc = dc' = o$. Then $(Re_4)^* \approx e_1R$, $(Re_2)^* \approx e_4R$ and $(Re_3)^* \approx e_3R$ are injective and $e_1R \supset e_2R\ (Re_2 \supset Re_1)$. Hence R is the desired algebra, which satisfies (#-1, 2, 3).

In the above example we replace e_3R with

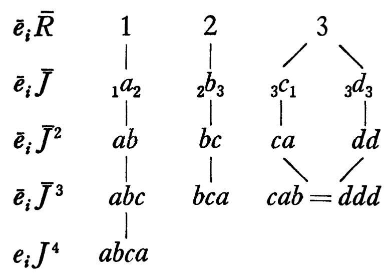


Then we obtain a two-sided almost QF-algebra with $J^4=0$ and any $n\geqq 4$, which is neither QF nor serial. We shall give another type of exceptional algebras, where $e_1R\ (\supset e_2R)$ is not uniserial.



where the other products among $a, b\cdots$ are zero, *e.g.* $\{b, b'\}\ \{c, c'\}=o$, $bde=b'd'e'=o$, $\{e, e'\}\ \{d, d'\}=o$, $dec=d'e'c'=o$ and so on. Then $(Re_4)^*\approx e_1R\supset e_2R$, $(Re_3)^*\approx e_4R$ and $(Re_2)^*\approx e_4R$. This ring is almost QF, but (#-1) is not satisfied.

2. We shall give an algebra which is a two-sided almost QF-algebra with $J^4\neq 0$ and $n=3$, but R is neither QF nor serial (cf. Corollary to Theorem 2). $R=\Sigma_{t\leqq 3}\oplus e_tR$ as above.



Then e_1R, e_3R and Re_2, Re_3 are injective and $e_1R\supset e_2R$, $Re_2\supset Re_1$.

3. There exists a right almost QF algebra with $J^4=0$ and $n=3$, which is not left almost QF (cf. Corollary to Theorem 2). Put $bca=o$ in the above. Then $Re_3\supset Re_2$ and Je_3 is not local.

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Department of Mathematics
Osaka City University
Sugimoto-3, Sumiyoshi-ku
Osaka 558, Japan