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ALMOST QF RINGS WITH $J^3=0$

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In this paper we always assume that $R$ is a two-sided artinian ring with identity. In [3] we have defined right almost QF rings and showed that those rings coincided with rings satisfying $(*)^*$ in [2], which K. Oshiro [5] called co-H rings. We shall show in Section 2 that right almost QF rings are nothing but direct sums of serial rings and QF rings, provided $J^3=0$. Further in Section 5 we show that if $R$ is a two-sided almost QF ring and $1=e_1+e_2+e_3$, then $R$ has the above structure, provided $J^3=0$, where $\{e_i\}$ is a complete set of mutually orthogonal primitive idempotents. Moreover if $1=e_1+e_2+e_3+e_4$, we have the same result except one case. We shall study, in Section 3, right almost QF rings with homogeneous socles $W^*_i(Q)$ [7] and give certain conditions on the nilpotency $m$ of the radical of $W^*_i(Q)$, under which $W^*_i(Q)$ is left almost QF or serial. In particular if $m\leq 2n$, $W^*_i(Q)$ is serial. We observe a special type of almost QF rings such that every indecomposable projective is uniserial or injective in Section 4.

1. Almost QF rings

In this paper we always assume that $R$ is a two-sided artinian ring with identity and that every module $M$ is a unitary right $R$-module. By $\bar{M}$ we denote $M/J(M)$, where $J(M)$ is the Jacobson radical of $M$. We use the same notations in [3]. We call $R$ a right almost QF ring if $R$ is right almost injective as a right $R$-module [3] and [4]. We can define similarly a left almost QF ring. If $R$ is a two-sided almost QF ring, we call it simply an almost QF ring. It is clear that $R$ is right almost QF if and only if every finitely generated projective $R$-module is right almost injective. Hence the concept of almost QF rings is preserved under Morita equivalence and we may assume that $R$ is basic.

In this section we shall give some results which we use later. First we give a property of any right almost QF rings.

**Proposition 1.** Assume that $R$ is right almost QF. Let $e_iR$ be injective, $e_iJ^i$ be projective, i.e., $e_iJ^i\approx e_{\sigma(i)}R$ for all $i\leq (some k)$ and $e_iJ^{k+1}\approx e_{a_i}R\oplus \cdots$. 
Then if \( e_aR \) is not injective, \( e_1J^{k+1} \cong e_aR \), and hence \( |e_1J^{k+1}|/|e_1J^{k+2}| = 1 \), where \( e_aR = e_aR/e_aJ \).

**Proof.** Let \( x_aR \) be a submodule in \( e_1J^{k+1} \) such that \( (x_aR + e_1J^{k+2})/e_1J^{k+2} \cong e_aR \) (\( x_a = x_a \)). Suppose that \( e_aR \) is not injective. Then \( e_aR \subset e_pR \) (isomorphically) for some \( p = a \), which is injective by [3], Corollary to Theorem 1. Let \( \rho: e_aR \twoheadrightarrow x_aR \subset e_1R; \rho(e_a) = x_a \), be the natural epimorphism. Since \( e_1R \) is injective, there exists \( \rho': e_pR \twoheadrightarrow e_1R \), which is an extension of \( \rho \). Put \( y = \rho'(e_p); (y = ye_p) \) and \( e_a = e_\rho r; r \in R \). We note that the \( e_iJ^i \) are all waists for \( i \leq k+1 \) by assumption. If \( y \subseteq e_1J^{k+1} \), then \( x_a = yr = ye_\rho r = 0 \) in \( e_1J^{k+1}/e_1J^{k+2} \), a contradiction. Accordingly \( yR = e_iJ^i \) for some \( t \leq k \). However \( e_iJ^i \) is projective, and hence \( \rho' \) is a monomorphism. Consequently \( e_1J^{k+1} \) contains isomorphically the projective module \( e_aR \), and \( eJ^{k+1} \) is local form [3], Corollary to Theorem 1.

**Proposition 2.** Let \( R \) be right almost QF. If \( R \) is either a local ring or \( J^2 = 0 \), then \( R \) is serial or QF.

**Proof.** \( R \) is a QF ring in the first case from [3], Corollary to Theorem 1. Assume \( J^2 = 0 \) and \( R \) is basic. If \( eR \) is injective for a primitive idempotent \( e \), then \( |eR| \leq 2 \) and \( eR \) is uniserial. Hence \( fR \) is injective and uniserial provided \( fJ \neq 0 \) by [3], Corollary to Theorem 1. Hence \( R \) is right serial and so \( R \) is serial by [5], Theorem 6.1.

Let \( kR \) (or \( Rg \)) be a simple module which appears in the factor modules of composition series of \( eR \) (or \( Re \)), where \( g \) is a primitive idempotent. In this case we say that \( g \) belongs to \( eR \) (or \( Re \)).

**Lemma 1.** Let \( R \) be basic and let \( \{e_iR\}_{i \leq s} \) be a set of injective and projective modules. Assume that every primitive idempotent belonging to \( e_iR \) is equal to some \( e_{p(j)} \) (in \( \{e_i\} \) such that \( e_jR \cong \text{Soc}(e_{p(j)}R) \). Put \( E = \sum_{i \leq s} e_i \) and \( F = 1 - E = \sum_{s \leq i} f_s \), where the \( f_s \) are primitive idempotents. Then \( ERF = 0 \) from the assumption. Let \( \theta: e_1R \rightarrow f_1R \) be a homomorphism. If \( \theta \neq 0 \), there exist a simple submodule \( S \) of \( f_1R \) and a submodule \( T \) of \( e_1R \) such that \( S \subset \theta(e_1R) \) and \( T/\theta^{-1}(0) \cong S \). We may assume \( S \cong e_jR \) for some \( e_j \) in \( \{e_i\} \) by assumption. Accordingly \( S \cong \text{Soc}(e_{p(j)}R) \) by the initial remark, and hence we obtain a non-zero homomorphism of \( f_1R \) to \( e_{p(j)}R \), since \( e_{p(j)}R \) is injective. Therefore \( f_s \notin \{e_i\} \) by assumption, a contradiction. As a consequence \( \theta = 0 \), i.e., \( FRE = 0 \) and \( R = ER \oplus FR = ER \oplus FRF \).

The following lemma is essential in this paper.

**Lemma 2.** Let \( R \) be artinian and \( F \) a uniform \( R \)-module. Assume that

1. \( eR \) is injective,
2. \( eJ \) is a local quasi-projective module,
3. \( \text{Soc}_2(F)/\text{Soc}(F) \)


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$\simeq e\bar{R} \oplus A_2 \oplus A_3 \oplus \cdots$, where $e$ is a primitive idempotent and the $A_i$ are simple. Then $A_i \simeq e\bar{R}$ for all $i$.

Proof. Assume $A_2 \simeq e\bar{R}$. Then since $\text{Soc}_2(F)/\text{Soc}(F) \simeq e\bar{R} \oplus e\bar{R} \oplus \cdots$, $\text{Soc}(F)$ is simple and $eJ^2$ is a waist by i) and ii), there exist $x_i, x'_i$ in $\text{Soc}_2(F)$ such that $x_iR + x'_iR, x_iR \simeq x'_iR \simeq eR/eJ^2$. Now let $\rho: x_iR \to eR/eJ^2$ be the isomorphism. Then $\rho(\text{Soc}(x_iR)) = eJ/eJ^2 \simeq e\bar{R}$, where $eJ \simeq eR/D$ and $D$ is a characteristic submodule of $eR$ by ii), where $e_i$ is a primitive idempotent. Take any element $\alpha$ in $\text{End}_R(\text{Soc}(x_iR))$. Then $\alpha$ gives an element $\bar{d}_i$ in $\text{End}_R(e\bar{R})$ via $\rho$. Then $\bar{d}_i$ is induced by an element $d_i$ in $\text{End}_R(e_iR)$. On the other hand, since $D$ is characteristic, $e_iR/D \simeq eJ \subset eR$ and $eR$ is injective, $d_i$ is extendible to $d$ in $\text{End}_R(e_iR)$. Hence $d$ induces an element in $\text{End}_R(eR/eJ^2)$ (and in $\text{End}_R(x_iR)$ via $\rho^{-1}$, cf. the diagram).

Thus we have obtained a mapping $\theta$ by taking extension, which may depend on a choice of $d$

$$\theta: \text{End}(\text{Soc}(x_iR)) \to \text{End}_R(x_iR).$$

Let $t: x_iR \to x'_iR$ be the given isomorphism. Then $t$ induces $\bar{d}_i$ in $\text{End}(\text{Soc}(F)) = \text{End}_R(\text{Soc}(x_iR))$ by taking restriction. Put $t' = \theta(\bar{d}_i) - t: x_iR \to F$. Then $t'(\text{Soc}(x_iR)) = 0$, and hence $t'(x_iR) \subset \text{Soc}(F)$. Then $t(x_iR) = (\theta(\bar{d}_i) - t')(x_iR) \subset x_iR + \text{Soc}(F) = x_iR$, a contradiction.

2. $J^3 = 0$

In this section we shall observe the ring $R$ with following properties: 1) $R$ is a basic and right almost QF ring, 2) $J^2 \neq 0$ and $J^3 = 0$.

Lemma 3. Assume that $fR$ is injective and $J^3 = 0$. Then we have 1) $fJ^3$ is simple or zero and 2) $fR$ is uniserial if $fJ^3 = 0$.

Lemma 4. Let $fR$ and $J$ be as in Lemma 3 and assume that $R$ is right almost QF. If $fR$ contains properly a projective submodule $P \neq 0$, then $fR$ is uniserial and hence $|fR| \leq 3$.

Proof. Since $fR \supset fJ \supset P \supset \text{Soc}(fR)$, $fJ$ is local by [3], Corollary to Theorem 1, and hence $fR$ is uniserial for $fJ^3 = 0$. 
Corollary. Assume that $R$ is right almost QF and $J^3 = 0$. If $|eR| \geq 3$, i.e. $eJ^2 \neq 0$, then $eR$ is injective. Hence $gR$ is injective or uniserial for any primitive idempotent $g$.

Proof. If $eR$ is not injective, $eR \subseteq fR$ for some injective $fR$ by [3], Corollary to Theorem 1, a contradiction to Lemma 4.

Let $e_R$ be an (injective) $R$-module. If $e_1J/e_1J^2 \approx e_2\bar{R} \oplus e_3\bar{R} \oplus \cdots$ and $e_1J^2 \approx e_2\bar{R}$, then we denote this situation by

$$e_1R = (1 \ b \ c) \text{ or } e_1R = (e_1 \ e_2 \ e_3).$$

Lemma 5. Let $e_1R$ be injective and $e_1J^2 \neq 0$ ($\approx e_2\bar{R}$) in the above. Then $e_2J/e_2J^2 \approx e_3\bar{R} \oplus \cdots$.

Proof. There exists $x_2R$ in $e_1J$ such that $x_2R \supseteq \text{Soc}(e_1R)$, $x_2R/\text{Soc}(e_1R) = \bar{e_2R}$ and $x_2R \approx e_2R/A$ for some $A$. Hence we obtain the lemma.

Lemma 6. Let $e_1R$ be a non-uniserial and injective module expressed as above. We assume that $R$ is right almost QF and $J^3 = 0$. Then $e_1R$ is injective. Further if $e_1R$ is uniserial, then $e_1R$ is not.

Proof. First we assume $a \neq b$. Now $e_1R$ is an injective module with $e_1J^2 \neq 0$ by Proposition 1. We have the same for $e_2R$. From Lemma 5 let

$$e_2R = (a \ c_1 \ d) \text{ and } e_2R = (b \ c_2 \ d').$$

Since $e_2R \approx e_2R$, $d \neq d'$. Then $e_2R$ is not uniserial (even though $e_2R$ is uniserial in this case), and hence $e_2R$ is injective by Corollary to Lemma 4. Next assume $a = b$, i.e.

$$e_1R = (1 \ c) \text{ or } e_1R = (1 \ c).$$

If $e_1R$ is not uniserial, $e_1R$ is injective by Lemma 5 and Proposition 1. Hence assume that $e_1R$ is uniserial. If further $e_1R$ is uniserial, then we can derive a contradiction by Lemma 2. Therefore if $e_1R$ is uniserial, then $e_1R$ is not uniserial and hence $e_1R$ is injective by Corollary to Lemma 4.

Theorem 1. Let $R$ be an artinian ring with $J^3 = 0$. Then the following are equivalent:

1) $R$ is right almost QF.
2) $R$ is left almost QF.
3) $R$ is a direct sum of serial rings and QF rings.
Proof. Let \( \{ e_i \}_{i \in I} \) be the complete set of mutually orthogonal primitive idempotents. We shall prove the theorem inductively on \( t \). If every \( e_R \) is uniserial, then \( R \) is right serial. Therefore \( R \) is serial by [5], Theorem 6.1. Hence we assume that there exists an injective but not uniserial module \( e_i R = (1 : c) \). We have shown in Lemma 6

1. If \( e_a \) belongs to \( e_2 R \), then \( e_a R \) is injective, i.e., \( e_a R, e_b R \) and \( e_c R \) are injective.
2. The same result as (1) for those \( e_a R, e_b R, e_c R \).
3. Primitive idempotents \( (\neq e_a) \) belonging to \( e_a R \) belongs to \( e_c R \) if \( e_a R \) is uniserial.

Since \( e_a R \) is not uniserial by Lemma 6, from (3) we obtain again (2) for \( e_a R \). Next consider \( e_2 R \). If \( e_2 R \) is not uniserial, we obtain (2) for \( e_2 R \) from the above (replace \( e_1 R \) by \( e_2 R \)). Suppose \( e_2 R \) is uniserial, and \( e_2 R \) is not uniserial by Lemma 6. Hence we obtain (2) for \( e_2 R \). Thus we have shown (2). Now starting from \( e_1 R \), we get \( e_2 R, e_3 R \) and \( e_4 R \) which belong to \( e_1 R \). Next we take primitive idempotents belonging to \( \{ e_1 R, e_2 R, \ldots, e_r R \} \). Continuing this procedure and gathering all such primitive idempotents (use (1), (2) and (3)), we can find finally a set \( \{ e_i R, e_a R, \ldots \} \) satisfying the condition in Lemma 1. Hence \( R = \sum_{i \in I} e_i R \oplus \sum_{j > m} e_j R \) as rings. Now \( \sum_{i \in I} e_i R \) is a QF ring. Thus we can obtain the theorem by induction.

3. Right almost QF rings with homogeneous socles

In this section we shall study rings stated in the title. Let \( \{ e_i \}_{i \in I} \) be a complete set of mutually orthogonal primitive idempotents with \( 1 = \Sigma e_i \) and \( R \) a basic ring.

Let \( Q \) be a local QF ring with \( J \) radical. Put \( Q = Q / \text{Soc}(Q) \) and \( J = J / \text{Soc}(Q) \). According to [7], Theorem 1 we denote a right almost QF ring \( R \) with homogeneous socle by

\[
W^k_n(Q) = \begin{bmatrix}
Q & Q & Q & \cdots & Q & Q & \cdots & Q \\
J & Q & Q & \cdots & Q & Q & \cdots & Q \\
J J & \cdots & J & Q & Q & \cdots & Q \\
J J & \cdots & J & J & Q & \cdots & Q \\
\vdots & \ddots & \vdots & \ddots & \ddots & \ddots & \ddots \\
J & \cdots & J & JJ & \cdots & J Q
\end{bmatrix}
\]
We note from [1] that there is only one projective and injective module $e_i R$ (resp. $Re_i$) in $R$.

**Lemma 7.** Assume $k < n$ on $R = W_n(Q)$. Then if $R$ is left almost QF, $R$ is serial.

Proof. Let $e_i = e_{ii}$ be the matrix unit in $R$. Then $e_i J(R) \approx e_{i+1} R$ for $i < n$ and $e_n J(R) = (J \cdots J \cdots J)$. Now assume $k < n$ and $R$ is left almost QF. Then since $J(R) e_i \approx Re_{i-1}$ for $s \leq k$, $J(R) e_i = (J \cdots J)^t$ is isomorphic to $Re_i = (Q \cdots Q)^t$ for some $p > k$ from the remark before Lemma 7 and [5], Theorem 3.18 (see [3], Corollary to Theorem 1), where $(\cdot)^t$ is the transposed matrix of $(\cdot)$. Hence since $e_i J(R) e_i \approx \bar{Q}$ as left $Q$-modules, $J$ is local and hence $Q$ is serial (cf. Lemma 9 below). Then $J \approx Q/Soc(Q) = \bar{Q}$ and $J/Soc(J) = J/Soc(Q) = J \approx Q/Soc(Q) \approx \bar{Q}/Soc(\bar{Q})$ as right $Q$-modules. Put $A = (\text{Soc}(Q) \text{Soc}(Q) \cdots \text{Soc}(Q) \text{Soc}(\bar{Q}) \cdots \text{Soc}(\bar{Q}))$ in $e_i R$. Then $e_n J(R) \approx e_i R/A$ from the above observation and hence $e_n J(R)$ is local. Therefore $R$ is right serial, and hence $R$ is serial by [5], Theorem 6.1.

**Lemma 8.** Assume $k = n$ on $R = W_n(Q)$. Then $R$ is left almost QF.

Proof. This is clear from (4).

**Theorem 2.** Let $R$ and $n$ be as in the beginning. Assume that $R$ is a right almost QF ring with homogeneous socle and $J(R)^{n-1} = 0$, $J(R)^n = 0$ (and hence $R = W_n(Q)$ and $m \geq n$). Then

1) if $m \leq 2n$, $R$ is serial,
2) if $m = nr$, $r \geq 3$, $R$ is left almost QF, and
3) if $m = nr + k$, $r \geq 2$ and $0 < k < n$, $R$ is left almost QF if and only if $R$ is serial.

Proof. By assumption and [7], Theorem 1 $R = W_n(Q)$ and we have $e_i J(R) \approx e_{i+1} R$ for $i < n - 1$. By a direct computation of $J(R)^p$ we have

i) $e_n J(R)/e_n J(R) \approx e_i \bar{R} \oplus \cdots \oplus e_i \bar{R}$ (cf. Proposition 1).

ii) $e_i J(R)^n = (J' \cdots)$.

1) Since $m \leq 2n$, $0 = e_i J(R)^{2n} = (J' \cdots)$ by ii). Hence $J' = 0$ and so $Q$ is serial. Accordingly $R$ is serial from the proof of Lemma 7.

2) and 3). From i) we know

\[ e_i R = (1 \ 2 \ 3 \ \cdots) . \]

Further $J^n = 0$ if and only if $e_i J(R)^n = 0$. Hence $\text{Soc}(e_i R) \approx e_i \bar{R}$ if $m = nr$ and $\text{Soc}(e_i R) \approx e_i \bar{R}$ if $m = nr + k$, $0 < k < n$. Therefore $R = W_n(Q)$ if $m = nr$ and $R \approx W_n(Q)$ if $k = 0$. As a consequence we obtain the theorem from Lemmas 7 and 8.
Corollary. Assume \( n=2 \) and \( R \) is right almost QF. Then if \( J(R)^{2m-1} \neq 0 \), \( J(R)^{2m} = 0 \), \( R \) is left almost QF. If \( J(R)^{2m} \neq 0 \), \( J(R)^{2m+1} = 0 \), \( R \) is QF or serial if and only if \( R \) is left almost QF. Further if \( J(R)^{t} = 0 \), \( R \) is QF or serial.

Proof. If \( R \) is QF or serial, the corollary is clear by [5], Theorem 4.5. Assume that \( R \) is not QF. Since \( n=2 \), we can suppose that \( e_i R \) is injective and \( e_i J(R) \approx e_i R \). Hence we obtain the corollary from Theorem 2.

4. Rings with (♯-i)

In the previous sections we have observed a ring which is a direct sum of QF rings and serial rings. In this case

(♯-1) \( eR \) is injective or uniserial for each primitive idempotent \( e \).

We consider two more conditions. Let \( eR \) be injective but not uniserial. Then we may assume that there exists an integer \( s \) such that \( eJ^i/eJ^{i+1} \) is simple for all \( i \) (\( 0 \leq i \leq s-1 \)) and \( eJ^i/eJ^{i+1} \approx \sum_{j=1}^{k} f_j R \); \( k \geq 2 \), where the \( f_j \) are primitive idempotents. Here we consider the second condition

(♯-2) the \( f_j R \) is injective for all \( j \).

Assume that \( R \) is a right almost QF ring with (♯-1). In the above we put \( eJ^i/eJ^{i+1} \approx g_i R \); \( g_i \) is a primitive idempotent. Since \( eR \) is not uniserial, \( g_i R \) is injective by (♯-1). In particular \( eJ^i/eJ^{i+1} \approx g_{s-1} R/A \) for some \( A \) in an injective \( g_{s-1} R \) and hence \( eJ^i/eJ^{i+1} \approx g_{s-1} J/(g_{s-1} J^2 + A) \leftarrow g_{s-1} J/g_{s-1} J^2 \). Since \( |eJ^i/eJ^{i-1}| \geq 2 \), (♯-2) is satisfied from Propostion 1. From the above observation we know that

Assume that \( R \) is right almost QF, the (♯-1) is satisfied if and only if every non-injective projective \( gR \) is contained in a uniserial injective \( eR \) and in this case (♯-2) and (♯-3) below are satisfied.

Taking some non-serial right serial rings, we can get rings with (♯-1, 2) which are not right almost QF. Hence we consider the third condition. Here we assume temporarily that \( R \) is an algebra over a field \( K \) with finite dimension. We further assume that \( R \) satisfies (♯-1) as right as well as left \( R \)-modules. Let \( gR \) be not injective, and hence uniserial. Then \( E(gR) \) is indecomposable. Take \( E(gR)^* = \text{Hom}_K(E(gR), K) \). Then \( E(gR)^* \) is indecomposable and projective. Therefore \( E(gR) \approx E(gR)^* \) is local. We consider this property for any ring.

(♯-3) \( E(gR) \) is local for each primitive idempotent \( g \).

Now we study rings with (♯-1, 2, 3). We always assume that \( R \) is basic.

Lemma 9. Assume \( eJ^i/eJ^{i+1} \approx \bar{e}_1 R \oplus \bar{e}_2 R \oplus \cdots \oplus \bar{e}_s R \). Then \( eJ^{i+1}/eJ^{i+2} \) is a homomorphic image of \( eJ \oplus \bar{e}_1 J \oplus \cdots \oplus \bar{e}_s J \).

Proof. We can express \( eJ^i \) as \( x_1 R + x_2 R + \cdots + x_s R + eJ^{i+1} \), where \( x_i e_j = x_j \). Hence \( eJ^{i+1} = x_1 e_1 J + \cdots + x_s e_j J + eJ^{i+2} \). Thus we obtain the lemma.
Lemma 10. We assume that (♯-3) is satisfied. Suppose that \(eR\) is injective and \(eJ/eJ^{j} \cong g_{i}R, g_{i}J^{j} \cong g_{i}R, \ldots, g_{i-1}J^{j} \cong g_{i}R, \) where the \(g_{i}\) is a primitive idempotent and \(g_{i}R\) is not injective for all \(i\). Then \(eR \supset g_{i}R \supset \cdots \supset g_{i}R\) isomorphically.

Proof. We shall show \(eJ^{i} \cong g_{i}R\) for all \(i\) by induction on \(i\). Assume \(eJ^{i} \cong g_{i}R\) if \(i \leq \text{some } k - 1\). Then \(eJ^{i}/eJ^{i+1} \cong g_{i-1}J^{i}/g_{i}J^{i} \cong g_{i}R\) by assumption. Let \(eJ^{i} = x_{i}R(x_{i}g_{i} = x_{i})\) and \(\rho: g_{i}R \rightarrow eJ^{i}(\rho(g_{i}) = x_{i})\) the natural epimorphism. Take a diagram

\[
\begin{array}{c}
0 \rightarrow g_{i}R \rightarrow E(g_{i}R) \\
\downarrow \rho \\
x_{i}R \\
\cap \rho' \\
eR
\end{array}
\]

Since \(eR\) is injective, we have \(\rho': E(g_{i}R) \rightarrow eR\) which commutes the diagram. \(E(g_{i}R)\) being local from (♯-3), \(\rho'(E(g_{i}R)) \cong x_{i}R = \rho'(g_{i}R)\) for \(g_{i}R \cong E(g_{i}R)\). Further \(eJ^{i}\) is a waist for all \(i \leq k\) by induction hypothesis. Consequently \(\rho'(E(g_{i}R))\) is projective. Therefore \(\rho'\) is a monomorphism, and hence so is \(\rho\).

Lemma 11. We assume that (♯-1), (♯-2) and (♯-3) are satisfied and that \(eR\) is injective and \(g_{i}\) belongs to \(eR\). If \(g_{i}R\) is not injective, then \(g_{i}R\) is contained isomorphically in an injective and uniserial module \(eR\).

Proof. Since \(g_{1}\) belongs to \(eR\), we may suppose \(eJ^{s}/eJ^{s+1} \cong g_{i}R \oplus \cdots\) for some \(s\), \(g_{s}R\) being not injective, \(s \neq 0\). If \(s = 1\), then \(|eJ/eJ^{j}| = 1\) by (♯-2) and \(g_{s}R \cong eJ\) from Lemma 10 and \(eR\) is uniserial by (♯-1). Hence assume \(s \geq 1\). From Lemma 9 there exists \(g_{2}\) such that \(eJ^{s-1}/eJ^{s} \cong g_{2}R \oplus \cdots\) and \(g_{s}J/g_{s}J^{s} \cong g_{i}R \oplus \cdots\). If \(g_{2}R\) is not uniserial, \(g_{2}R\) is injective by (♯-1), and then \(g_{s}R\) is injective by (♯-2), a contradiction (cf. the remark after (♯-2)). Accordingly \(g_{2}R\) is uniserial and hence \(g_{2}J/g_{2}J^{s} \cong g_{i}R\). Next assume that \(g_{2}R\) is not injective. Then \(g_{2}R\) satisfies the same condition as on \(g_{s}R\), and hence similarly to the above we can find \(g_{3}R\) such that \(eJ^{s-2}/eJ^{s-1} \cong g_{3}R \oplus \cdots\) and \(g_{s}J/g_{s}J^{s} \cong g_{3}R \oplus \cdots\). Repeating this process, we obtain finally an injective and uniserial module \(eR\) such that \(eJ/eJ^{j} \cong g_{i}R\) for some \(i\) (and \(g_{s}J/g_{s}J^{s} \cong g_{i}R, \cdots g_{2}J/g_{2}J^{s} \cong g_{i}R\)). Hence \(eR\) contains isomorphically \(g_{i}R\) from Lemma 10.

Proposition 3. (♯-1), (♯-2) and (♯-3) are satisfied if and only if \(R\) is right almost QF and every non-injective projective \(gR\) is contained in a uniserial and injective \(eR\).

Proof. We assume (♯-1, 2, 3). First we shall show that \(R\) is right QF-3. Let \(eR\) be not injective. Then \(E(gR)\) is local by (♯-3), i.e., \(E(gR) \cong fR/A\) and
Almost QF Rings with \( J^3 = 0 \)

\( fR \) is uniform from (\#-1). Further \( fR/A \supseteq gR \) and \( gR \approx B/A \) for some \( B (\supset A) \) in \( fR \). Therefore since \( gR \) is projective and \( fR \) is uniform, \( A = 0 \) and \( fR = E(gR) \supseteq gR \). Accordingly \( R \) is right QF-3. Let \( hR \) be injective and suppose \( hR \supseteq k_1R \), where \( h \) and \( k_1 \) are primitive idempotents. Then form the last part of the proof of Lemma 11 there exists a uniserial and injective module \( h_1R \) such that \( h_1R = (h_1, k_1, \cdots, k_1, \cdots) \) and \( h_1R \supseteq k_1R \). Hence \( hR \approx h_1R \). Thus \( R \) is right almost QF by [3], Corollary to Theorem 1. The converse is clear from the remark before Lemma 9.

5. \( J^4 = 0 \)

In this section we assume that \( R \) is an (basic) artinian ring with \( J^4 = 0 \). Let \( 1 = \sum_{i \in \Lambda} e_i \) be as in §3. We studied almost QF rings with \( n = 2 \) in Corollary to Theorem 2. We study almost QF rings with \( n = 3 \) or 4 in this section.

**Lemma 12.** Let \( R \) be two-sided almost QF. If \( R \) is not QF, then there exists an injective and projective \( eR \) such that \( eR/Soc(eR) \) is again injective.

**Proof.** \( R \) is right almost QF by [6], Theorem 3.7. Hence we obtain the lemma from [2], Theorem 2.3.

**Theorem 3.** Let \( R \) be an (basic) artinian ring. Assume that \( J^4 = 0 \) and \( n \leq 3 \), where \( \{e_i\}_{i \in \Lambda} \) is a complete set of mutually orthogonal primitive idempotents. Then the following are equivalent:

1) (\#-1), (\#-2) and (\#-3) are satisfied as right as well as left \( R \)-modules.
2) \( R \) is a two-sided almost QF ring.
3) \( R \) is a direct sum of serial rings and QF rings.

**Proof.** 1) \( \rightarrow \) 2). This is given by Proposition 3.

2) \( \rightarrow \) 3). From Corollary to Theorem 2 and Theorem 1 we can suppose \( n = 3 \) and \( J^3 \neq 0 \). First we note that if \( R \) is a direct sum of two rings, then \( R \) is a direct sum of serial rings and QF rings from Proposition 2 and Corollary to Theorem 2. We call this situation \( R \) splits. Let \( R \) be two-sided indecomposable and neither serial nor QF. Then we shall derive a contradiction for all possible situations. If \( e_iR \supseteq e_2R \supseteq e_3R \), \( R \) is serial by Theorem 2. Thus we may suppose from [3], Theorem 1

(5) \( e_iR, e_3R \) are injective and \( e_iJ \approx e_3R \).

First we assume that \( e_iR \) is uniserial.

i) \( e_iR \) is uniserial and \( e_iJ \) is local.

Then \( e_iJ/e_iJ^2 \) is uniserial for all \( i \). Hence \( R \) is right serial, and \( R \) is serial by [5], Theorem 6.1.

Thus we may assume

ii) \( e_iR \) is uniserial, but \( e_iJ \) is not local, i.e.,
Then \( \{a, b\} \subset \{1, 3\} \) from Proposition 1. First we note that if \( a=b=3 \), then \( R \) splits from Lemmas 1 and 9. Hence we can skip the case \( a=b=3 \).

\[ e_3R = \begin{pmatrix} a & a' \\ b & b' \\ c' \\ \vdots \end{pmatrix} \]

\[ e_3R = (3 : 2) \]

Then \( \{a, b\} \subset \{1, 3\} \) from Proposition 1. First we note that if \( a=b=3 \), then \( R \) splits from Lemmas 1 and 9. Hence we can skip the case \( a=b=3 \).

\[ |e_1R| = 2 \]

i) \( e_1R=(1 \ 2) \).

\[ a=1 \]

Let \( e_3J/e_3J^2 \cong e_1R \oplus \cdots \). Then there exists \( x_1 \) in \( e_3J \) such that \( x_1 e_1 = x_1 \) and \( (x_1R + e_3J)/e_3J^2 \cong e_1R \). (We use this notation in the following arguments.) Suppose that \( x_1R \) is simple. Then \( x_1R = \text{Soc}(e_3R) \subset e_3J^2 \) for \( e_3J^2 \neq 0 \), a contradiction. Hence \( x_1R \approx e_3R \) is injective, again a contradiction. \[ |e_1R| = 3 \]

ii) \( e_1R=(1 \ 2 \ 1) \).

\[ a=1 \]

Then we take \( X_x \) in \( e_3R \) such that \( X_x \supset e_3J^2 \) and \( e_3J/X_x \cong e_1R \). Since \( e_3J/X_x \cong \text{Soc}(e_1R) \cong e_1R \) and \( e_1R \) is injective, \( e_2 = e_3 \), a contradiction.

iii) \( e_1R=(1 \ 2 \ 2) \). Then \( e_1J/e_1J^2 \cong \text{Soc}(e_1R) \). Hence \( e_1 = e_2 \), a contradiction.

iv) \( e_1R=(1 \ 2 \ 3) \).

\[ a=1 \]

Then \( x_1R \) in \( e_3J \) is a homomorphic image of \( e_1R \), and hence \( x_1R \cong \text{Soc}(e_1R) \) for \( e_3J \approx e_1J \). If \( x_1R \cong e_1R/e_1J \), \( x_1R \subset \text{Soc}(e_1R) \subset e_3J^2 \), a contradiction. Hence we obtain a homomorphism \( \psi : \text{Soc}_2(x_1R) \rightarrow \text{Soc}_2(x_1R)/\text{Soc}(x_1R) \cong e_3R \rightarrow \text{Soc}(e_1R) \). Since \( e_1R \) is injective, we obtain an extension of \( \psi \), which is a contradiction to the structure of \( e_1R \) and \( e_3R \).

v) \( e_1R=(1 \ 2 \ 1 \ x) \).

Then \( x=2 \). Then \( e_3J/e_1J \cong \text{Soc}(e_1R) \). Hence \( e_1 = e_2 \), a contradiction.

vi) \( e_1R=(1 \ 2 \ 2 \ x) \).

Then \( x=2 \) and \( e_3J/e_1J^2 \cong \text{Soc}(e_1R) \). Hence \( e_1 = e_2 \), a contradiction.

vii) \( e_1R=(1 \ 2 \ 3 \ x) \).

Since \( \{a, b\} \subset \{1, 3\} \), \( x=1 \) or \( 3 \), and \( d \neq 1 \).

vii-i) \( x=1 \) and \( d=2 \). Then \( a=1 \).

\[ b=1 \]

Let \( e_3J/e_3J^2 \cong x_1R \oplus x_1R \oplus \cdots \). Since \( d=2 \), we may assume \( x_1R \approx x_1R \oplus \cdots \) (\( \cong e_1R/e_1J^2 \), which is uniserial). Hence \( x_1R, x_1R \oplus x_1R \oplus \cdots \) are contained in \( \text{Soc}_2(e_3R) \). Therefore \( e_3J = \text{Soc}_2(e_3R) \) for \( \text{Soc}_2(e_3R) \supset e_3J^2 \). As a consequence

\[ e_3R = (3 : 2) \]

Then we obtain a contradiction to Lemma 2.

\( \beta \) \( b=3 \). \( e_3J \) contains a submodule \( x_1R \) isomorphic to \( e_1R/e_1J^2 \) as in \( \alpha \). Hence \( x_1R \supset \text{Soc}_2(e_3R) \) and \( x_1R \subset e_3J^2 \). Since \( b=3 \), \( e_3J^2/e_3J^3 \) has to contain a
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simple submodule isomorphic to \( \tilde{e}_1 \tilde{R} \) by Lemma 9 and its proof. Hence since \( x_1 R \subseteq e_1 J^p \), \( \text{Soc}(e_3 R) / \text{Soc}(e_2 R) \cong \tilde{e}_1 \tilde{R} \oplus \cdots \), a contradiction to Lemma 2.

vii-ii) \( x = 1 \) and \( d = 3 \) (and hence \( a = 1 \)).

\( \alpha ) \) \( b = 1 \). Since \( \text{Soc}(e_1 R / \text{Soc}(e_1 R)) \cong \text{Soc}(e_3 R) \), \( e_3 R / \text{Soc}(e_3 R) (= E) \) is injective by Lemma 12. Further \( \text{Soc}(e_3 R) = e_3 J^p \) and \( \text{Soc}(E) / \text{Soc}(E) \cong e_3 J / e_3 J^p \cong \tilde{e}_1 \tilde{R} \oplus \cdots \), a contradiction to Lemma 2.

\( \beta ) \) \( b = 3 \). From the structure of \( e_3 R \) and Lemma 9 we know \( \text{Soc}(e_3 R) / \text{Soc}(e_3 R) \cong \tilde{e}_2 \tilde{R} \) or \( \cong \tilde{e}_3 \tilde{R} \). Then \( \text{Soc}(e_1 R) / \text{Soc}(e_3 R) \cong \text{Soc}(e_1 R) \) as above, a contradiction.

\( \chi - iii) \) \( x = 3 \), i.e. \( e_1 R = (1 2 3 3) \).

Since \( e_1 R / \text{Soc}(e_1 R) \) is not injective, \( | \text{Soc}(e_2 R) / \text{Soc}(e_3 R) | = 1 \) by Lemma 12. Hence

\[
\begin{array}{c}
e_3 R = (3 : x y) or (3 : x y) \\
1 \\
3
\end{array}
\]

(note \( e_3 J^p \subseteq \text{Soc}(e_3 R) \)). If \( e_3 J^p \neq 0 \), \( y = 3 \), a contradiction. If \( e_3 J^p = 0 \), \( \text{Soc}(e_3 R) / \text{Soc}(e_3 R) \geq 2 \), a contradiction.

Thus we have shown that \( R \) is a direct sum of serial rings and QF rings, provided \( e_i R \) is uniserial.

Finally we observe the structure of \( R \), when \( e_1 R \) is not uniserial. Assume that an injective module \( e_1 R \) contains a projective proper submodule and is not uniserial. Then \( e_1 J \) is local by [3], Corollary to Theorem 1, and hence

\[
e_1 R = (a b c' d); e_1 J^p \neq 0 .
\]

Now from \( i) \), \( ii) \), (5), Proposition 1 and Lemma 12, we may assume

\[
a \quad a' \\
::
\]

\( iii) \) \( e_1 R = (1 2 b g) \) and \( e_3 R = (3 b' h g') \) are injective, \( e_1 R \) is not uniserial and \( e_2 R \cong e_1 J; \{ a, b \} \subseteq \{ 1, 3 \} \).

From Lemma 12 we have

**Lemma 13.** Let \( R, e_1 R \) and \( e_3 R \) be as above. Then \( e_3 R / \text{Soc}(e_3 R) \) is injective.

First we assume that \( e_3 R \) is not uniserial. We note that if \( a' = b' = 3 \), then \( R \) splits from Lemmas 1 and 9.

\( iii-1) \) \( e_1 R \) and \( e_3 R \) are not uniserial, and hence \( e_3 J^p \neq 0 \) from Lemma 13.

\( i) \) \( a = 1 \). Then \( g = 2 \).

\( a' = 1 \). Then \( h = 2 \) and \( \text{Soc}(e_3 R / \text{Soc}(e_3 R)) = \text{Soc}(e_1 R) \), a contradiction from
Lemma 13.

ii) \( a=b=3 \).

\[ a'=b'=1. \] Then \( h=2 \) and \( g'=3 \), i.e.,

\[ e_1R = (1, 2; 1), e_2R = (2, 1; 1) \text{ and } e_3R = (3, 2; 3). \]

Then \( e_3R/Soc(e_3R) (=E) \) is injective by Lemma 13 and \( Soc(E)/Soc(E) \cong e_1\bar{R} \oplus \cdots \oplus e_1\bar{R}. \) Since \( e_3R \) is not uniserial, \( |Soc(E)/Soc(E)| \geq 2 \), a contradiction to Lemma 2.

\[ \beta \) \( a'=1 \) and \( b'=3 \). Then \( h=2 \) from \( e_1R \) and \( h=1 \) or \( 3 \) from \( e_3R \), a contradiction.

\[ \text{iii-2)} \] \( e_1R \) is not uniserial and \( e_3R \) is uniserial.

\[ \alpha) \] \( a=b=1 \). Then

\[ e_1R = (1, 2; 2), \] which contradicts Lemma 2.

\[ \beta) \] \( a=1, b=3 \). Then

\[ e_1R = (1, 2; 2), \] and hence \( e_3R = (3, 2, c, d) \).

If \( e_3J' = 0 \) (resp. \( e_3J' = 0 \)), \( Soc_2(e_3R)/Soc(e_3R) \cong Soc(e_1R) \) (resp. \( Soc(e_3R) \cong Soc(e_1R) \)), a contradiction from Lemma 13. Assume \( e_3J' \neq 0 \), then \( c=1 \) or \( 3 \), and hence \( d=2 \), a contradiction.

\[ \gamma) \] \( a=b=3 \).

\[ i) \] \( g=1 \). Then

\[ e_1R = (1, 2; 1) \text{ and } e_3R = (3, 1, c, d) \]

We know as above \( e_3J' \neq 0 \), and so \( e_3R = (3, 1, 2, 3) \). Here we shall again make use of the argument in the proof of Lemma 2. Since \( e_3R \) is uniserial, there exist two submodules \( yR, y'R \) in \( e_3J' \) such that \( yR \approx y'R \approx e_3R/e_3J' \). Let \( \alpha \) be an element in \( \text{End}_\mathbb{k}(Soc(yR)) \). We shall find an extension of \( \alpha \) in \( \text{End}_\mathbb{k}(yR) \). Since \( yR \approx e_3R/e_3J' \), \( Soc(yR) \approx Soc(e_3R/e_3J') \approx e_1R/e_1J' \). Hence we may assume that \( \alpha \) is given by an element \( p \) in \( e_1R \) via the above isomorphism. Then \( p \) induces an endomorphism \( \bar{p} \) of \( e_3J' \). Further \( \bar{p} \) is extendible to \( q \) in \( \text{End}_\mathbb{k}(E) \). Finally since \( E/Soc(E) \approx e_1R/e_1J' \), \( \bar{q} \) induces an element in \( \text{End}_\mathbb{k}(e_3R/e_3J') \), which is an extension of \( \alpha \) (see the diagram below)

\[
\begin{array}{c}
E \approx e_3R/e_3J' \xrightarrow{\rho} e_3R/e_3J' \rightarrow 0 \\
\cup \quad \cup \\
\cup \\
e_1R/e_1J' \approx Soc_2(E) \approx X \xrightarrow{\rho} Soc(e_3R/e_3J') \rightarrow 0,
\end{array}
\]
where \( \rho \) is the natural epimorphism. Using this extension, we can derive a contradiction.

(\( \beta \)) \( a' = 1 \) and \( b' = 3 \). Then \( h = 2 \) from \( e_jR \) and \( h = 1 \) or \( 3 \) from \( e_jR \), a contradiction.

3) \( \rightarrow 1 \). This is trivial.

**Theorem 4.** Let \( R \) and \( n \) be as in Theorem 3. Assume that \( R \) is a two-sided almost QF and two-sided indecomposable ring with \( J^3 = 0 \) and \( n = 4 \). Then \( R \) is either serial or QF if and only if \( R \) is not of the following: there exist exactly three injective and projective modules \( e_iR \) and some one among \( e_iR \) is not uniserial.

Proof. Suppose that \( R \) is not QF. Then we have the following four cases:

1) \( e_1R \) is injective and \( e_1R \supset e_2R \supset e_3R \supset e_4R \) (isomorphically).
2) \( e_1R \) and \( e_2R \) are injective and \( e_1R \supset e_3R \supset e_4R \).
3) \( e_1R \) and \( e_2R \) are injective and \( e_1R \supset e_4R, e_2R \supset e_4R \).
4) \( e_1R, e_2R \) and \( e_4R \) are injective and \( e_1R \supset e_2R \).

Case 1) Since \( J^3 = 0 \), \( R \) is serial by Theorem 2.

Case 2) Then \( e_1R \) is uniserial by [3], Corollary to Theorem 1, i.e., \( e_1R = (1 2 3 d) \) (or = (1 2 3)) and \( e_1R \) are injective. If \( e_4J \) is local, \( R \) is right serial. Suppose that \( e_4J \) is not local. Then from Proposition 1 we have the following:

\[
\begin{align*}
\text{a}) & \quad e_4R = (4 \mid \cdots), \\
\text{b}) & \quad e_4R = (4 \mid \cdots) \text{ or } c}) e_4R = (4 \mid \cdots)
\end{align*}
\]

\( R \) splits if c) occurs. Hence we assume a) or b).

i) \( e_4R/\text{Soc}(e_4R) \) and \( e_4R/\text{Soc}(e_4R) \) are injective (see the proof of Lemma 12).

Let \( xR \) be a submodule in \( e_4J \) with \( (xR + e_4J^3)/e_4J^3 \approx e_2R \). Since \( e_1R \) is uniserial, \( \text{Soc}(e_1R) = \text{Soc}(xR) \approx e_2R \) or \( e_3R \) if \( e_1J^3 = 0 \). However \( \text{Soc}(e_1R/\text{Soc}(e_1R)) \approx e_2R \) and \( \text{Soc}(e_1R/\text{Soc}(e_1R)) \approx e_2R \), a contradiction. If \( e_1J^3 = 0 \), we obtain the same result as above.

ii) \( e_1R/\text{Soc}(e_1R) \) and \( e_1R/\text{Soc}(e_1R) \) are injective.

Assume a) or b). \( \text{Soc}(e_4R) \) and \( \text{Soc}(e_4R) \) are waists by assumption. Since \( \text{Soc}(e_4R/\text{Soc}(e_4R)) \approx e_2R \), there exists a submodule \( xR \) in \( e_4J \) such that \( xR \approx e_1R/e_4J^3 \), i.e., \( e_4J^3 = 0 \), and hence \( e_4R \) is uniserial.

(\( \beta \)) \( e_1J^3 = 0 \). \( e_1R = (1 2 3) \).

Then \( xR \) is simple, i.e., \( |e_4R| \leq 2 \), a contradiction.

iii) \( e_4R/\text{Soc}(e_4R) \) and \( e_4R/\text{Soc}(e_4R) \) are injective. Then \( e_4R \) is uniserial and hence \( R \) is serial.

Case 3) i) \( e_4R/\text{Soc}(e_4R) \) and \( e_4R/\text{Soc}(e_4R) \) are injective. Then \( e_4R = (1 2 c d) \) (or = (1 2 c)) and
In the latter case $R$ is serial. Hence assume the former. Then $\{g, h\} \subset \{1, 3\}$. Assume $e_1J^3 \neq 0$.

\( \alpha \) $g = 1$. There exists $xR$ in $e_1J$ with $xR \approx e_1R/A$ for some $A$ in $e_1R$. However $\operatorname{Soc}(e_1R) = \operatorname{Soc}(xR) \approx e_1\tilde{R}$, a contradiction.

\( \beta \) $g = h = 3$. Then

\[
ed_3R = (3 4 4),
\]

which is a contradiction to Lemma 2.

We obtain the same result in a case $e_1J^3 = 0$.

ii) $e_1R/\operatorname{Soc}(e_1R)$ and $e_3R/\operatorname{Soc}(e_3R)$ are injective. Then $e_1R$ and $e_3R$ are uniserial, and hence $R$ is serial.

Case 4) If $e_1R$, $e_3R$ and $e_4R$ are uniserial, $R$ is right serial.

6. Examples

In this section we shall give several examples related to the previous sections.

1. We shall give a two-sided almost QF ring with $J^* = 0$ and $n = 4$ but neither QF nor serial. This is an example of exceptional algebras in Theorem 4. Let $K$ be a field and $R = \Sigma_{i=4} \oplus e_iR$, where $\{e_i\}$ is a set of mutually orthogonal primitive idempotents with $1 = \Sigma e_i$. We define $e_1R = e_1K \oplus aK \oplus abK \oplus abc'K$, $e_2R = e_2K \oplus bK \oplus bc'K$, $\cdots$, whose multiplicative structure is given below, where $\lambda a$ means $a = e_1ae_2$, and so on.

(In the previous sections we expressed horizontally the structure of $e_iR$, however we shall do vertically here.)

\[
\begin{array}{cccc|c|c|c|c|c|}
e_1R/e_1J & 1 & 2 & 3 & 4 \\
e_1J/e_1J^3 & a_2 & b_3 & c_1 \cap c_4 \cap d_3 \\
e_1J^3/e_1J^3 & ab & bc' & cb = c'd \cap dca ,
\end{array}
\]

where the other products among $a, b, \cdots$ are zero, e.g. $bc = dc' = 0$. Then $(Re_1)^* \approx e_1R$, $(Re_2)^* \approx e_4R$ and $(Re_3)^* \approx e_3R$ are injective and $e_1R \supset e_2R \supset (Re_2 \supset Re_1)$. Hence $R$ is the desired algebra, which satisfies $(\#-1, 2, 3)$.

In the above example we replace $e_3R$ with
Then we obtain a two-sided almost QF-algebra with $J^4=0$ and any $n \geq 4$, which is neither QF nor serial. We shall give another type of exceptional algebras, where $e_3R(=e_2R)$ is not uniserial.

$$e_1R$$

$$e_1J$$

$$e_1J^2$$

$$e_1J^3$$

where the other products among $a, b, c$ are zero, e.g. $\{b, b'\} \{c, c'\} = 0$, $bde = b'd'e' = 0$, $\{e, e'\} \{d, d'\} = 0$, $dec = d'e'c' = 0$ and so on. Then $(Re_3)^* \approx e_1R \supset e_2R$, $(Re_3)^* \approx e_1R$ and $(Re_2)^* \approx e_1R$. This ring is almost QF, but $(#-1)$ is not satisfied.

2. We shall give an algebra which is a two-sided almost QF-algebra with $J^4=0$ and $n=3$, but $R$ is neither QF nor serial (cf. Corollary to Theorem 2). $R=\sum e_3R$ as above.

Then $e_1R, e_2R$ and $Re_3, Re_3$ are injective and $e_1R \supset e_2R, Re_2 \supset Re_3$.

3. There exists a right almost QF algebra with $J^4=0$ and $n=3$, which is not left almost QF (cf. Corollary to Theorem 2). Put $bca=0$ in the above. Then $Re_3 \supset Re_2$ and $J_{e_3}$ is not local.

References


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