

Title	Almost QF rings with J^(3=0)
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Citation	Osaka Journal of Mathematics. 1993, 30(4), p. 893–908
Version Type	VoR
URL	https://doi.org/10.18910/9385
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Harada, M. Osaka J. Math. 30 (1993), 893–908

ALMOST QF RINGS WITH $J^3=0$

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(Received June 2, 1992)

(Revised September 16, 1992)

In this paper we always assume that R is a two-sided artinian ring with identity. In [3] we have defined right almost QF rings and showed that those rings coincided with rings satisfying (*)* in [2], which K. Oshiro [5] called co-H rings. We shall show in Section 2 that right almost QF rings are nothing but direct sums of serial rings and QF rings, provided $J^3=0$. Further in Section 5 we show that if R is a two-sided almost QF ring and $1=e_1+e_2+e_3$, then R has the above structure, provided $J^4=0$, where $\{e_i\}$ is a complete set of mutually orthogonal primitive idempotents. Moreover if $1=e_1+e_2+e_3+e_4$, we have the same result except one case. We shall study, in Section 3, right almost QF rings with homogeneous socles $W_k^*(Q)$ [7] and give certain conditions on the nilpotency m of the radical of $W_k^*(Q)$, under which $W_k^*(Q)$ is left almost QF or serial. In particular if $m \leq 2n$, $W_k^*(Q)$ is serial. We observe a special type of almost QF rings such that every indecomposable projective is uniserial or injeative in Section 4.

1. Almost QF rings

In this paper we always assume that R is a two-sided artinian ring with identity and that every module M is a unitary right R-module. By \overline{M} we denote M/J(M), where J(M) is the Jacobson radical of M. We use the same notations in [3]. We call R a right almost QF ring if R is right almost injective as a right R-module [3] and [4]. We can define similarly a left almost QF ring. If R is a two-sided almost QF ring, we call it simply an almost QF ring. It is clear that R is right almost QF if and only if every finitely generated projective R-module is right almost injective. Hence the concept of almost QF rings is preserved under Morita equivalence and we may assume that R is basic.

In this section we shall give some results which we use later. First we give a property of any right almost QF rings.

Proposition 1. Assume that R is right almost QF. Let e_1R be injective, e_1J^i be projective, i.e., $e_1J^i \approx e_{\rho(i)}R$ for all $i \leq (\text{some } k)$ and $e_1J^{k+1}/e_1J^{k+2} \approx \bar{e}_a \bar{R} \oplus \cdots$.

Then if $e_a R$ is not injective, $e_1 J^{k+1} \approx e_a R$, and hence $|e_1 J^{k+1}/e_1 J^{k+2}| = 1$, where $\bar{e}_a \bar{R} = c_a R/e_a J$.

Proof. Let $x_a R$ be a submodule in $e_1 J^{k+1}$ such that $(x_a R + e_1 J^{k+2})/e_1 J^{k+2} \approx \bar{e}_a \bar{R} (x_a e_a = x_a)$. Suppose that $e_a R$ is not injective. Then $e_a R \subset e_p R$ (isomorphically) for some $p \neq a$, which is injective by [3], Corollary to Theorem 1. Let $\rho: e_a R \to x_a R \subset e_1 R$; $\rho(e_a) = x_a$, be the natural epimorphism. Since $e_1 R$ is injective, there exists $\rho': e_p R \to e_1 R$, which is an extension of ρ . Put $y = \rho'(e_p)$; $(y = ye_p)$ and $e_a = e_p r; r \in R$. We note that the $e_1 J^i$ are all waists for $i \leq k+1$ by assumption. If $y \in e_1 J^{k+1}$, then $\bar{x}_a = \bar{y}r = \bar{y}e_p re_a = \bar{o}$ in $e_1 J^{k+1}/e_1 J^{k+2}$, a contradiction. Accordingly $yR = e_1 J^i$ for some $t \leq k$. However $e_1 J^i$ is projective, and hence ρ' is a monomorphism. Consequently $e_1 J^{k+1}$ contains isomorphically the projective module $e_a R$, and $e J^{k+1}$ is local form [3], Corollary to Theorem 1.

Proposition 2. Let R be right almost QF. If R is either a local ring or $J^2=0$, then R is serial or QF.

Prof. R is a QF ring in the first case from [3], Corollary to Theorem 1. Assume $J^2=0$ and R is basic. If eR is injective for a primitive idempotent e, then $|eR| \leq 2$ and eR is uniserial. Hence fR is injective and uniserial provided $fJ \neq 0$ by [3], Corollary to Theorem 1. Hence R is right serial and so R is serial by [5], Theorem 6.1.

Let \overline{kR} (or \overline{Rg}) be a simple module which appears in the factor modules of composition series of eR (or Re), where g is a primitive idempotent. In this case we say that g belongs to eR (or Re).

Lemma 1. Let R be basic and let $\{e_iR\}_{i\leq s}$ be a set of injective and projective modules. Assume that every primitive idempotent belonging to e_iR is equal to some $e_{\mu(i)} \in \{e_i\}$ for each e_i . Then $\sum_{i\leq s} \oplus e_iR$ is a direct summand of R as rings.

Proof. We note from the asumption that for each $e_j \in \{e_i\}$ there exists $e_{\rho(j)}$ in $\{e_i\}$ such that $\bar{e}_j \bar{R} \approx \operatorname{Soc}(e_{\rho(j)}R)$. Put $E = \sum_{i \leq s} e_i$ and $F = 1 - E = \sum_{k \leq p} f_k$, where the f_k are primitive idempotents. Then ERF=0 from the assumption. Let $\theta: e_1R \rightarrow f_kR$ be a homomorphism. If $\theta \neq o$, there exist a simple submodule S of f_kR and a submodule T of e_1R such that $S \subset \theta(e_1R)$ and $T/\theta^{-1}(0) \approx S$. We may assume $S \approx \bar{e}_j \bar{R}$ for some e_j in $\{e_i\}$ by assumption. Accordingly $S \approx \operatorname{Soc}(e_{\rho(j)}R)$ by the initial remark, and hence we obtain a non-zero homomorphism of f_kR to $e_{\rho(j)}R$, since $e_{\rho(j)}R$ is injective. Therefore $f_k \in \{e_i\}$ by asumption, a contradiction. As a consequence $\theta = o$, i.e., FRE=0 and $R = ER \oplus FR = ERE \oplus FRF$.

The following lemma is essential in this paper.

Lemma 2. Let R be artinian and F a uniform R-module. Assume that i): eR is injective, ii): eJ is a local quasi-projective module and iii): $Soc_2(F)/Soc(F)$

 $\approx \bar{e}\bar{R} \oplus A_2 \oplus A_3 \oplus \cdots$, where e is a primitive idempotent and the A_i are simple. Then $A_i \approx \bar{e}\bar{R}$ for all i.

Proof. Assume $A_2 \approx \bar{e}\bar{R}$. Then since $\operatorname{Soc}_2(F)/\operatorname{Soc}(F) \approx \bar{e}\bar{R} \oplus \bar{e}\bar{R} \oplus \cdots$, Soc(F) is simple and eJ^2 is a waist by i) and ii), there exist x_1, x_1' in Soc₂(F) such that $x_1R \neq x_1'R, x_1R \approx x_1'R \approx eR/eJ^2$. Now let $\rho: x_1R \to eR/eJ^2$ be the isomorphism. Then $\rho(\operatorname{Soc}(x_1R)) = eJ/eJ^2 \approx \bar{e}_1\bar{R}$, where $eJ \approx e_1R/D$ and D is a charateristic submodule of e_1R by ii), where e_1 is a primitive idempotent. Take any element α in $\operatorname{End}_R(\operatorname{Soc}(x_1R))$. Then α gives an element \bar{d}_1 in $\operatorname{End}_R(\bar{e}_1\bar{R})$ via ρ . Then \bar{d}_1 is induced by an element d_1 in $\operatorname{End}_R(e_1R)$. On the other hand, since D is characteristic, $e_1R/D \approx eJ \subset eR$ and eR is injective, d_1 is extendible to d in $\operatorname{End}_R(eR)$. Hence d induces an element in $\operatorname{End}_R(eR/eJ^2)$ (and in $\operatorname{End}_R(x_1R)$ via ρ^{-1} , cf. the diagram).

$$D$$

$$e_1R/e_1J \longleftarrow e_1R$$

$$\rho \qquad \downarrow \mu \qquad \nu \qquad \downarrow \mu$$

$$\operatorname{Soc}(x_1R) \approx eJ/eJ^2 \longleftarrow eJ$$

$$\bigcap \qquad \rho \qquad \cap \qquad \nu \qquad \cap$$

$$x_1R \qquad \approx eR/eJ^2 \longleftarrow eR$$

Thus we have obtained a mapping θ by taking extension, which may depend on a choice of d

 θ : End(Soc($x_1 R$)) \rightarrow End_R($x_1 R$).

Let $t: x_1R \to x'_1R$ be the given isomorphism. Then t induces \bar{d}_1 in End(Soc(F)) = End_R(Soc(x_1R)) by taking restriction. Put $t' = \theta(\bar{d}_1) - t: x_1R \to F$. Then t'(Soc(x_1R))=0, and hence $t'(x_1R) \subset$ Soc(F). Then $t(x_1R) = (\theta(\bar{d}_1) - t')(x_1R) \subset x_1R +$ Soc(F)= x_1R , a contradiction.

2. $J^3 = 0$

In this section we shall observe the ring R with following properties: 1) R is a basic and right almost QF ring, 2): $J^2 \pm 0$ and $J^3 = 0$.

Lemma 3. Assume that fR is injective and $J^3=0$. Then we have 1): fJ^2 is simple or zero and 2): fR is uniserial if $fJ^2=0$.

Lemma 4. Let fR and J be as in Lemma 3 and assume that R is right almost QF. If fR contains properly a projective submodule $P \neq 0$, then fR is uniseria and hence $|fR| \leq 3$.

Proof. Since $fR \supset fJ \supset P \supset Soc(fR)$, fJ is local by [3], Corollary to Theorem 1, and hence fR is uniserial for $fJ^3=0$.

Corollay. Assume that R is right almost QF and $J^3=0$. If $|eR| \ge 3$, i.e. $eJ^2 \ne 0$, then eR is injective. Hence gR is injective or uniserial for any primitive idempotent g.

Proof. If eR is not injective, $eR \subset fR$ for some injective fR by [3], Corollary to Theorem 1, a contradition to Lemma 4.

Let e_1R be an (injective) *R*-module. If $e_1J/e_1J^2 \approx \bar{e}_a\bar{R} \oplus \bar{e}_b\bar{R} \oplus \cdots$ and $e_1J^2 \approx \bar{e}_c\bar{R}$, then we denote this situation by

$$e_1R = (1 \begin{array}{c} a \\ b \end{array}) \text{ or } e_1R = (e_1 \begin{array}{c} e_a \\ e_b \end{array})$$

Lemma 5. Let e_1R be injective and $e_1J^2 \neq 0$ ($\approx \bar{e}_c\bar{R}$) in the above. Then $e_a J/e_a J^2 \approx \bar{e}_c\bar{R} \oplus \cdots$.

Proof. There exists $x_a R$ in $e_1 J$ such that $x_a R \supset \operatorname{Soc}(e_1 R)$, $x_a R / \operatorname{Soc}(e_1 R) = \bar{e}_a \bar{R}$ and $x_a R \approx e_a R / A$ for some A. Hence we obtain the lemma.

Lemma 6. Let e_1R be a non-uniserial and injective module expressed as above. We assume that R is right almost QF and $J^3=0$. Then e_cR is injective. Further if e_aR is uniserial, then e_cR is not.

Proof. First we assume $a \neq b$. Now $e_a R$ is an injective module with $e_a J^2 \neq 0$ by Proposition 1. We have the same for $e_b R$. From Lemma 5 let

$$e_{a}R = (a \begin{array}{c} c \\ c_{1} \end{array} d) \text{ and } e_{b}R = (b \begin{array}{c} c \\ c_{2} \end{array} d').$$

 \vdots

Since $e_a R \approx e_b R$, $d \neq d'$. Then $e_c R$ is not uniserial (even though $e_a R$ is uniserial in this case), and hence $e_c R$ is injective by Corollary to Lemma 4. Next assume a=b, i.e.

$$e_1 R = (1 \stackrel{a}{\vdots} c)$$

If $e_a R$ is not uniserial, $e_c R$ is injective by Lemma 5 and Proposition 1. Hence assume that $e_a R$ is uniserial. If further $e_c R$ is uniserial, then we can derive a contradiction by Lemma 2. Therefore if $e_a R$ is uniserial, then $e_c R$ is not uniserial and hence $e_c R$ is injective by Corollary to Lemma 4.

Theorem 1. Let R be an arithmian ring with $J^3=0$. Then the following are equivalent:

- 1) R is right almost QF.
- 2) R is left almost QF.
- 3) R is a direct sum of serial rings and QF rings.

Proof. Let $\{e_i\}_{i\leq t}$ be the complete set of mutually orthogonal primitive idempotents. We shall prove the theorem inductively on t. If every eR is uniserial, then R is right serial. Therefore R is serial by [5], Theorem 6.1. Hence we assume that there exists an injective but not uniserial module

$$e_1 R = (1 \stackrel{a}{\vdots} c)$$
. We have shown in Lemma 6

(1) if e_g belongs to e_1R , then e_gR is injective, i.e., e_aR , e_bR and e_cR are injective. We shall show that if we replace e_1R with e_aR , e_bR and e_cR , then we obtain

(2) the same result as (1) for those $e_a R$, $e_b R$, $e_c R$.

If $e_a R$ is not uniserial, we obtain (2) for $e_a R$. Suppose $e_a R$ is uniserial. Then $e_a J \approx e_c R/B$. Hence

(3) primitive idempotents $(\pm e_a)$ belonging to $e_a R$ belongs to $e_c R$ if $e_a R$ is uniserial.

Since e_cR is not uniserial by Lemma 6, from (3) we obtain again (2) for e_aR . Next consider e_cR . If e_aR is not uniserial, we obtain (2) for e_cR from the above (replace e_1R by e_aR). Suppose e_aR is uniserial, and e_cR is not uniserial by Lemma 6. Hence we obtain (2) for e_cR . Thus we have shown (2). Now starting from e_1R , we get e_aR , e_bR and e_cR which belong to e_1R . Next we take primitive idempotents belonging to $\{e_aR, e_bR, \cdots, e_cR\}$. Continuing this procedure and gathering all such primitive idempotents (use (1), (2) and (3)), we can find finally a set $\{e_1R, e_aR, \cdots\}$ satisfying the condition in Lemma 1. Hence $R = \sum_{i \leq m} \bigoplus e_iR \bigoplus \sum_{j > m} \bigoplus e_jR$ as rings. Now $\sum_{i \leq m} \bigoplus e_iR$ is a QF ring. Thus we can obtain the theorem by induction.

3. Right almost QF rings with homogeneous socles

In this section we shall study rings stated in the title. Let $\{e_i\}_{i\leq n}$ be a complete set of mutually orghogonal promitive idempotents with $1=\sum e_i$ and R a basic ring.

Let Q be a local QF ring with J radical. Put $\overline{Q}=Q/\operatorname{Soc}(Q)$ and $\overline{J}=J/\operatorname{Soc}(Q)$. According to [7], Theorem 1 we denote a right almost QF ring R with homogeneous socle by

We note from [1] that there is only one projective and injective module e_1R (resp. Re_k) in R.

Lemma 7. Assume k < n on $R = W_k^n(Q)$. Then if R is left almost QF, R is serial.

Proof. Let $e_i = e_{ii}$ be the matrix unit in R. Then $e_i J(R) \approx e_{i+1}R$ for i < nand $e_n J(R) = (J \cdots J \overline{J} \overline{J} \cdots \overline{J})$. Now assume k < n and R is left almost QF. Then since $J(R) e_s \approx Re_{s-1}$ for $s \leq k$, $J(R) e_1 = (J J \cdots J)^t$ is isomorphic to $Re_q = (\overline{Q} \ \overline{Q} \cdots \overline{J})^t$ for some p > k from the remark before Lemma 7 and [5], Theorem 3.18 (see [3], Corollary to Theorem 1), where ()^t is the transposed matrix of (). Hence since $e_1 J(R) e_1 \approx \overline{Q}$ as left Q-modules, J is local and hence Q is serial (cf. Lemma 9 below). Then $J \approx Q/\operatorname{Soc}(Q) = \overline{Q}$ and $J/\operatorname{Soc}(J) = J/\operatorname{Soc}(Q) = \overline{J} \approx Q/\operatorname{Soc}_2(Q) \approx \overline{Q}/\operatorname{Soc}(\overline{Q})$ as right Q-modules. Put $A = (\operatorname{Soc}(Q) \operatorname{Soc}(Q) \cdots \operatorname{Soc}(Q) \operatorname{Soc}(\overline{Q}) \cdots$ $\operatorname{Soc}(\overline{Q})$) in $e_1 R$. Then $e_n J(R) \approx e_1 R/A$ from the above observation and hence $e_n J(R)$ is local. Therefore R is right serial, and hence R is serial by [5], Theorem 6.1.

Lemma 8. Assume k=n on $R=W_k^n(Q)$. Then R is left almost QF.

Proof. This is clear from (4)

Theorem 2. Let R and n be as in the begining. Assume that R is a right almost QF ring with homogeneous socle and $J(R)^{m-1} \neq 0$, $J(R)^m = 0$ (and hence $R = W_k^n(Q)$ and $m \ge n$). Then

1) if $m \leq 2n$, R is serial,

2) if m=nr, $r \ge 3$, R is left almost QF, and

3) if m=nr+k, $r\geq 2$ and o < k < n, R is left almost QF if and only if R is serial.

Proof. By assumption and [7], Theorem 1 $R = W_i^n(Q)$ and we have $e_i J(R) \approx e_{i+1}R$ for i < n-1. By a direct computation of $J(R)^p$ we have

i) $e_n J(R)/e_n J(R)^2 \approx \bar{e}_1 \bar{R} \oplus \cdots \oplus \bar{e}_1 \bar{R}$ (cf. Proposition 1).

ii) $e_1 J(R)^{tn} = (J^t \cdots).$

1). Since $m \leq 2n$, $0 = e_1 J(R)^{2n} = (J^2 \cdots)$ by ii). Hence $J^2 = 0$ and so Q is serial. Accordingly R is seiral from the proof of Lemma 7.

2) and 3). From i) we know

$$e_1R = (1 \ 2 \ 3 \ n \ 1 \ 2 \ 3 \cdots)$$
.

Further $\int^{m} = 0$ if and only if $e_{1}J(R)^{m} = 0$. Hence $\operatorname{Soc}(e_{1}R) \approx \bar{e}_{n}\bar{R}$ if m = nr and $\operatorname{Soc}(e_{1}R) \approx \bar{e}_{k}\bar{R}$ if m = nr + k, o < k < n. Therefore $R \approx W_{n}^{n}(Q)$ if m = nr and $R \approx W_{k}^{n}(Q)$ if $k \neq o$. As a consequence we obtain the theorem from Lemmas 7 and 8.

Corollary. Assume n=2 and R is right almost QF. Then if $J(R)^{2m-1} \neq 0$, $J(R)^{2m} = 0$, R is left almost QF. If $J(R)^{2m} \neq 0$, $J(R)^{2m+1} = 0$, R is QF or serial if and only if R is left almost QF. Further if $J(R)^4 = 0$, R is QF or serial.

Proof. If R is QF or serial, the corollary is clear by [5], Theorem 4.5. Assume that R is not QF. Since n=2, we can suppose that e_1R is injective and $e_1 J(R) \approx e_2 R$. Hence we obtain the corollary from Theorem 2.

4. Rings with (#-i)

In the previous sections we have observed a ring which is a direct sum of QF rings and serial rings. In this case

(#-1) eR is injective or uniserial for each primitive idempotent e.

We consider two more conditions. Let eR be injective but not uniserial. Then we may assume that there exists an integer s such that eJ^i/eJ^{i+1} is simple for all i ($o \le i \le s-1$) and $eJ^s/eJ^{s+1} \approx \sum_{j \le k} \bigoplus \tilde{f}_j \bar{R}$; $k \ge 2$, where the f_j are primitive idempotents. Here we consider the second condition (#-2) the $f_j R$ is injective for all j.

Assume that R is a right almost QF ring with (#-1). In the above we put $eJ^i/eJ^{i+1} \approx \overline{g}_i \overline{R}$; g_i is a primitive ideomptent. Since eR is not uniserial, g_iR is injective by (#-1). In particular $eJ^{s-1} \approx g_{s-1}R/A$ for some A in an injective $g_{s-1}R$ and hence $eJ^s/eJ^{s+1} \approx g_{s-1}J/(g_{s-1}J^2+A) \leftarrow g_{s-1}J/g_{s-1}J^2$. Since $|eJ^s/eJ^{s-1}| \ge 2$, (#-2) is satisfied from Propostion 1. From the above observation we know that

Assume that R is right almost QF, the (#-1) is satisfied if and only if every non-injective projective gR is contained in a uniserial injective eR and in this case (#-2) and (#-3) below are satisfied.

Taking some non-serial right serial rings, we can get rings with (#-1, 2) which are not right almost QF. Hence we consider the third condition. Here we assume temporarily that R is an algebra over a field K with finite dimension. We further assume that R satisfies (#-1) as right as well as left R-modules. Let gR be not injective, and hence uniserial. Then E(gR) is indecomposable. Take $E(gR)^* = \operatorname{Hom}_{K}(E(gR), K)$. Then $E(gR)^*$ is indecomposable and projective. Therefore $E(gR) \approx E(gR)^{**}$ is local. We consider this property for any ring.

(#-3) E(gR) is local for each primitive idempotent g.

Now we study rings with (#-1, 2, 3). We always assume that R is basic.

Lemma 9. Assume $eJ^{i}/eJ^{i+1} \approx \bar{e}_1 \bar{R} \oplus \bar{e}_2 \bar{R} \oplus \cdots \oplus \bar{e}_s \bar{R}$. Then eJ^{i+1}/eJ^{i+2} is a homomorphic image of $\bar{e}_1 \bar{J} \oplus \bar{e}_2 \bar{J} \oplus \cdots \oplus \bar{e}_s \bar{J}$.

Proof. We can express eJ^i as $x_1R+x_2R+\dots+x_sR+eJ^{i+1}$, where $x_je_j=x_j$. Hence $eJ^{i+1}=x_1e_1J+\dots+x_se_sJ+eJ^{i+2}$. Thus we obtain the lemma.

Lemma 10. We assume that $(\sharp -3)$ is satisfied. Suppose that eR is injective and $eJ/eJ^2 \approx \overline{g}_1 \overline{R}, g_1 J/g_1 J^2 \approx \overline{g}_2 \overline{R}, \dots, g_{s-1} J/g_{s-1} J^2 \approx \overline{g}_s \overline{R}$, where the g_i is a primitive idempotent and $g_i R$ is not injective for all *i*. Then $eR \supset g_1 R \supset \dots \supset g_s R$ isomorphically.

Proof. We shall show $eJ^i \approx g_i R$ for all *i* by induction on *i*. Assume $eJ^i \approx g_t R$ if $t \leq (\text{soem } k-1)$. Then $eJ^k/eJ^{k+1} \approx g_{k-1}J/g_{k-1}J^2 \approx \overline{g}_k \overline{R}$ by assumption. Let $eJ^k = x_k R(x_k g_k = x_k)$ and $\rho: g_k R \rightarrow eJ^k(\rho(g_k) = x_k)$ the natural epimorphism. Take a diagram



Since eR is injective, we have $\rho': E(g_k R) \rightarrow eR$ which commutes the diagram. $E(g_k R)$ being local from $(\sharp -3)$, $\rho'(E(g_k R)) \supseteq x_k R = \rho'(g_k R)$ for $g_k R \neq E(g_k R)$. Further eJ^t is a waist for all $t \leq k$ by induction hypothesis. Consequenctly $\rho'(E(g_k R))$ is projective. Therefore ρ' is a monomorphism, and hence so is ρ .

Lemma 11. We assume that (#-1), (#-2) and (#-3) are satisfied and that eR is injective and g_1 belongs to eR. If g_1R is not injective, then g_1R is contained isomorphically in an injective and uniserial module e_1R .

Proof. Since g_1 belongs to eR, we may suppose $eJ^{s'}/eJ^{s+1} \approx \overline{g}_1 \overline{R} \oplus \cdots$ for some s. g_1R being not injective, $s \neq o$. If s=1, then $|eJ/eJ^2|=1$ by (#-2) and $g_1R \approx eJ$ from Lemma 10 and eR is uniserial by (#-1). Hence assume s>1. From Lemma 9 there exists g_2 such that $eJ^{s-1}/eJ^s \approx \overline{g}_2 \overline{R} \oplus \cdots$ and $g_2 J/g_2 J^2 \approx \overline{g}_1 \overline{R}$ $\oplus \cdots$. If g_2R is not uniserial, g_2R is injetive by (#-1), and then g_1R is injective by (#-2), a contradiction (cf. the remark after (#-2)). Accordingly g_2R is uniserial and hence $g_2 J/g_2 J^2 \approx \overline{g}_1 \overline{R}$. Next assume that g_2R is not injective. Then g_2R satisfies the same condition as on g_1R , and hence similarly to the above we can find g_3R such that $eJ^{s-2}/eJ^{s-1} \approx \overline{g}_3 \overline{R} \oplus \cdots$ and $g_3 J/g_3 J^2 \approx \overline{g}_2 \overline{R} \oplus \cdots$. Repeating this process, we obtain finally an injetive and uniserial module e_1R such that $e_1 J/e_1 J^2 \approx \overline{g}_s \overline{R}$ for some t (and $g_t J/g_t J^2 \approx \overline{g}_{t-1} \overline{R}, \cdots g_2 J/g_2 J^2 \approx \overline{g}_1 \overline{R}$). Hence e_1R contains isomorphically g_1R from Lemma 10.

Proposition 3. (#-1), (#-2) and (#-3) are satisfied if and only if R is right almost QF and every non-injective projective gR is contained in a uniserial and injective eR.

Proof. We assume (#-1, 2, 3). First we shall show that R is right QF-3. Let eR be not injective. Then E(gR) is local by (#-3), i.e., $E(gR) \approx fR/A$ and

fR is uniform from (\$4-1). Further $fR/A \supset gR$ and $gR \approx B/A$ for some $B (\supset A)$ in fR. Therefore since gR is projective and fR is uniform, A=0 and $fR=E(gR)\supset gR$. Accordingly R is right QF-3. Let hR be injective and suppose $hR\supset k_1R$, where h and k_1 are primitive ideompotents. Then form the last part of the proof of Lemma 11 there exists a uniserial and injective module h_1R such that $h_1R=(h_1 k_s \cdots k_1 \cdots)$ and $h_1R\supset k_1R$. Hence $hR\approx h_1R$. Thus R is right almost QF by [3], Corollary to Theorem 1. The converse is clear from the remark before Lemma 9.

5. $J^4 = 0$

In this section we assume that R is an (basic) artinian ring with $J^4=0$. Let $1=\sum_{i\leq n} e_i$ be as in §3. We studied almost QF rings with n=2 in Corollary to Theorem 2. We study almost QF rings with n=3 or 4 in this section.

Lemma 12. Let R be two-sided almost QF. If R is not QF, then there exists an injective and projective eR such that eR/Soc(eR) is again injective.

Proof. R is right almost QF^{*} by [6], Theorem 3.7. Hence we obtain the lemma from [2], Theorem 2.3.

Theorem 3. Let R be an (basic) artinian ring. Assume that $J^4=0$ and $n \leq 3$, where $\{e_i\}_{i\leq n}$ is a complete set of mutually orthogonal primitive idempotents. Then the following are equivalent:

- 1) (#-1), (#-2) and (#-3) are satisfied as right as well as left R-modules.
- 2) R is a two-sided almost QF ring.
- 3) R is a direct sum of serial rings and QF rings.

Proof. 1) \rightarrow 2). This is given by Proposition 3.

2) \rightarrow 3). From Corollary to Theorem 2 and Theorem 1 we can suppose n=3 and $J^3 \pm 0$. First we note that if R is a direct sum of two rings, then R is a direct sum of serial rings and QF rings from Propostion 2 and Corollary to Theorem 2. We call this situation R splits. Let R be two-sided indecomposable and neither serial nor QF. Then we shall derive a contradiction for all possible situations. If $e_1R \supset e_2R \supset e_3R$, R is serial by Theorem 2. Thus we may suppose from [3], Theorem 1

(5) e_1R , e_3R are injective and $e_1J \approx e_2R$.

First we assume that e_1R is uniserial.

i) e_1R is uniserial and e_3J is local. Then e_iJ/e_iJ^2 is uniserial for all *i*. Hence *R* is right serial, and *R* is serial by [5], Theorem 6.1.

Thus we may assume

ii) e_1R is uniserial, but e_3J is not local, i.e.,

$$e_{3}R = (3 \begin{array}{ccc} a & a' \\ \vdots & \vdots & d \end{pmatrix} \\ b & b' \\ \vdots & \vdots \\ c' \\ \vdots \end{array}$$

Then $\{a, b\} \subset \{1, 3\}$ from Propositon 1. First we note that if a=b=3, then R splits from Lemmas 1 and 9. Hence we can skip the case a=b=3.

 $|e_1R|=2.$ i) $e_1R=(1 2).$

a=1. Let $e_3 J/e_3 J^2 \approx \bar{e}_1 \bar{R} \oplus \cdots$. Then there exists x_1 in $e_3 J$ such that $x_1 e_1 = x_1$ and $(x_1 R + e_3 J^2)/e_3 J^2 \approx \bar{e}_1 \bar{R}$. (We use this notation in the following arguments.) Suppose that $x_1 R$ is simple. Then $x_1 R = \operatorname{Soc}(e_3 R) \subset e_3 J^2$ for $e_3 J^2 \neq 0$, a contradiction. Hence $x_1 R \approx e_1 R$ is injective, again a contradiction.

 $|e_1R| = 3$. ii) $e_1R = (1 \ 2 \ 1)$.

a=1. Then we take X_a in e_3R such that $X_a \supset e_3J^2$ and $e_3J/X_a \approx \bar{e}_a\bar{R}$. Since $e_3J/X_s \approx \operatorname{Soc}(e_1R) \approx \bar{e}_1\bar{R}$ and e_1R is injective, $e_2=e_3$, a contradiction.

iii) $e_1R = (1 \ 2 \ 2)$. Then $e_1J/e_1J^2 \approx \operatorname{Soc}(e_1R)$. Hence $e_1 = e_2$, a contradiction.

iv) $e_1 R = (1 \ 2 \ 3).$

a=3. We obtain the same contradiction as in iii).

a=b=1. $x_a R \approx (e_1 R/\operatorname{Soc}_2(e_1 R) \text{ or } e_1 R/\operatorname{Soc}(e_1 R))$. Hence $(x_a R+e_1 J^2) J^2=0$. Accordingly $0=(\sum_a x_a R+e_1 J^2) J^2=e_3 J^3$, a contradiction to $J^3 \neq 0$.

 $|e_1R| = 4$. v) $e_1R = (1 \ 2 \ 1 \ x)$. Then x = 2.

a=1. Then x_aR in e_3J is a homomorphic image of e_1R , and hence $x_aR \approx (e_1R/e_1J^3 \text{ or } e_1R/e_1J)$. If $x_aR \approx e_1R/e_1J$, $x_aR \subset \operatorname{Soc}(e_3R) \subset e_3J^2$, a contradiction. Hence we obtain a homomorphism $\psi: \operatorname{Soc}_2(x_aR) \to \operatorname{Soc}_2(x_aR)/\operatorname{Soc}(x_aR) \approx \bar{e}_2\bar{R} \to \operatorname{Soc}(e_1R)$. Since e_1R is injective, we obtain an extension of ψ , which is a contradiction to the structure of e_1R and e_3R .

vi) $e_1 R = (1 \ 2 \ 2 \ x)$.

Then x=2 and $e_1 J/e_1 J^2 \approx \text{Soc}(e_1 R)$. Hence $e_1=e_2$, a contradiction.

vii)
$$e_1 R = (1 \ 2 \ 3 \ x)$$
. Since $\{a, b\} \subset \{1, 3\}$, $x = 1$ or 3, and $d \neq x$.

vii-i) x=1 and d=2. Then a=1.

 α) b=1. Let $e_3 J/e_3 J^2 \approx \bar{x}_1 \bar{R} \oplus \bar{x}_1' \bar{R} \oplus \cdots$. Since d=2, we may assume $x_1 R \approx x_1' R \approx \cdots (\approx e_1 R/e_1 J^2)$, which is uniserial). Hence $x_1 R, x_1' R \cdots$ are contained in $\operatorname{Soc}_2(e_3 R)$. Therefore $e_3 J=\operatorname{Soc}_2(e_3 R)$ for $\operatorname{Soc}_2(e_3 R) \supset e_3 J^2$. As a consequence

$$e_3R = (3 \stackrel{1}{\vdots} 2).$$

Then we obtain a contradiction to Lemma 2.

β) b=3. e_3J contains a submodule x_1R isomorphic to e_1R/e_1J^2 as in α). Hence $x_1R \subset \text{Soc}_2(e_3R)$ and $x_1R \subset e_3J^2$. Since b=3, e_3J^2/e_3J^3 has to contain a

simple submodule isomorphic to $\bar{e}_1\bar{R}$ by Lemma 9 and its proof. Hence since $x_1R \oplus e_3 \int^2$, $\operatorname{Soc}_2(e_3R)/\operatorname{Soc}(e_3R) \approx \bar{e}_1\bar{R} \oplus \bar{e}_1\bar{R} \oplus \cdots$, a contradiction to Lemma 2.

vii-ii) x=1 and d=3 (and hence a=1).

 α) b=1. Since $\operatorname{Soc}(e_1R/\operatorname{Soc}(e_1R)) \approx \operatorname{Soc}(e_3R)$, $e_3R/\operatorname{Soc}(e_3R)$ (=E) is injective by Lemma 12. Further $\operatorname{Soc}_2(e_3R) = e_3J^2$ and $\operatorname{Soc}_2(E)/\operatorname{Soc}(E) \approx e_3J/e_3J^2 \approx \bar{e}_1\bar{R} \oplus \bar{e}_1\bar{R} \oplus \cdots$, a contradiction to Lemma 2.

 β) b=3. From the structure of e_3R and Lemma 9 we know $\operatorname{Soc}_2(e_3R)/\operatorname{Soc}(e_3R) \approx \bar{e}_2\bar{R}$ or $\approx \bar{e}_3\bar{R}$. Then $\operatorname{Soc}_2(e_1R)/\operatorname{Soc}(e_1R) \approx \operatorname{Soc}(e_3R)$ as above, a contradiction.

vii-iii) x=3, i.e. $e_1R=(1\ 2\ 3\ 3)$.

Since $e_1R/\operatorname{Soc}(e_1R)$ is not injective, $|\operatorname{Soc}_2(e_3R)/\operatorname{Soc}(e_3R)| = 1$ by Lemma 12. Hence

$$e_3 R = (3 \stackrel{1}{\underset{1}{\vdots}} x y) \text{ or } (3 \stackrel{1}{\underset{3}{\vdots}} x y)$$

(note $e_3 J^2 \subset \operatorname{Soc}_2(e_3 R)$). If $e_3 J^3 \neq 0$, y = 3, a contradiction. If $e_3 J^3 = 0$, $|\operatorname{Soc}_2(e_3 R)/\operatorname{Soc}(e_3 R)| \geq 2$, a contradiction.

Thus we have shown that R is a direct sum of serial rings and QF rings, provided e_1R is uniserial.

Finally we observe the structure of R, when e_1R is not uniserial. Assume that an injective module eR contains a projective proper submodule and is not uniserial. Then eJ is local by [3], Corollary to Theorem 1, and hence

$$eR = (a \ b \ c' \ d); \ eJ^3 \neq 0.$$

Now from i), ii), (5), Proposition 1 and Lemma 12, we may assume

iii) $e_1R = (1 \ 2 \ b \ g)$ and $e_3R = (3 \ b' \ h \ g')$ are injective, e_1R is not uniserial and $e_2R \approx e_1 I$; $\{a, b\} \subset \{1, 3\}$.

From Lemma 12 we have

Lemma 13. Let R, e_1R and e_3R be as above. Then $e_3R/Soc(e_3R)$ is injective.

First we assume that e_3R is not uniserial. We note that if a'=b'=3, then R splits from Lemmas 1 and 9.

iii-1) e_1R and e_3R are not uniserial, and hence $e_3J^3 \neq 0$ from Lemma 13. i) a=1. Then g=2.

a'=1. Then h=2 and $\operatorname{Soc}(e_3R/\operatorname{Soc}(e_3R))\simeq \operatorname{Soc}(e_1R)$, a contradiction from

Lemma 13.

ii)
$$a=b=3$$
.
 α) $a'=b'=1$. Then $h=2$ and $g'=3$, i.e.,
 $e_1R = (1\ 2\ \frac{3}{5}\ 1), e_2R = (2\ \frac{3}{5}\ 1)$ and $e_3R = (3\ \frac{1}{5}\ 2\ 3)$.

Then $e_3R/\operatorname{Soc}(e_3R)$ (=E) is injective by Lemma 13 and $\operatorname{Soc}_2(E)/\operatorname{Soc}(E) \approx \overline{e}_1 \overline{R} \oplus \cdots \oplus \overline{e}_1 \overline{R}$. Since e_3R is not unisedal, $|\operatorname{Soc}_2(E)/\operatorname{Soc}(E)| \ge 2$, a contradiction to Lemma 2.

 β) a'=1 and b'=3. Then h=2 from e_1R and h=1 or 3 from e_3R , a contradiction.

iii-2) e_1R is not uniserial and e_3R is uniserial.

 α) a=b=1. Then

$$e_1 R = (1 2 \frac{1}{1} 2)$$
, which contradicts Lemma 2.

 β) a=1, b=3. Then

$$e_1 R = (1 \ 2 \ \frac{1}{3} 2)$$
, and hece $e_3 R = (3 \ 2 \ c \ d)$.

If $e_3J^3=0$ (resp. $e_3J^2=0$), $\operatorname{Soc}_2(e_3R)/\operatorname{Soc}(e_3R) \approx \operatorname{Soc}(e_1R)$ (resp. $\operatorname{Soc}(e_3R) \approx \operatorname{Soc}(e_1R)$), a contradiction from Lemma 13. Assume $e_3J^3 \pm 0$, then c=1 or 3, and hence d=2, a contradiction.

$$\begin{array}{l} \gamma) \quad a=b=3.\\ i) \quad g=1. \quad \text{Then} \end{array}$$

$$e_1R = (1 \ 2 \ \frac{3}{3} \ 1)$$
 and $e_3R = (3 \ 1 \ c \ d)$

We know as above $e_3J^3 \pm 0$, and so $e_3R = (3 \ 1 \ 2 \ 3)$. Here we shall again make use of the argument in the proof of Lemma 2. Since e_3R is uniserial, there exist two submodules yR, y'R in e_1J^2 such that $yR \approx y'R \approx e_3R/e_3J^2$. Let α be an element in $\operatorname{End}_R(\operatorname{Soc}(yR))$. We shall find an extension of α in $\operatorname{End}_R(yR)$. Since $yR \approx e_3R/e_3J^2$, $\operatorname{Soc}(yR) \approx \operatorname{Soc}(e_3R/e_3J^2) \approx e_1R/e_1J$. Hence we may assume that α is given by an element p in e_1R via the above isomorphism. Then p induces an endomorphism \overline{p} of $e_1R/e_1J^2 \approx \operatorname{Soc}_2(E) \subset E (\approx e_3R/e_3J^3)$. Further \overline{p} is extendible to q in $\operatorname{End}_R(E)$. Finally since $E/\operatorname{Soc}(E) \approx e_3R/e_3J^2$, \overline{q} induces an element in $\operatorname{End}_R(e_3R/e_3J^2)$, which is an extendion of α (see the diagram below)

$$\begin{split} E &\approx e_3 R/e_3 J^3 \stackrel{\rho}{\longrightarrow} e_3 R/e_3 J^2 \longrightarrow 0 \\ \cup & \cup & \cup \\ e_1 R/e_1 J^2 &\approx \operatorname{Soc}_2(E) \approx X \stackrel{\rho}{\longrightarrow} \operatorname{Soc}(e_3 R/e_3 J^2) \to 0 , \end{split}$$

where ρ is the natural epimorphism.

Using this extension, we can derive a contradiction.

 β) a'=1 and b'=3. Then h=2 from e_1R and h=1 or 3 from e_3R , a contradiction.

3) \rightarrow 1). This is trivial.

Theorem 4. Let R and n be as in Theorem 3. Assume that R is a twosided almost QF and two-sided indecomposable ring with $J^4=0$ and n=4. Then R is either serial or QF if and only if R is not of the following: there exist exactly three injective and projective modules e_iR and some one among e_iR is not uniserial.

Proof. Suppose that R is not QF. Then we have the following four cases:

1) e_1R is injective and $e_1R \supset e_2R \supset e_3R \supset e_4R$ (isomorphically).

2) e_1R and e_4R are injective and $e_1R \supset e_2R \supset e_3R$.

- 3) e_1R and e_3R are injective and $e_1R \supset e_2R$, $e_3R \supset e_4R$.
- 4) e_1R , e_2R and e_4R are injective and $e_1R \supset e_2R$.
- Case 1) Since $J^4=0$, R is serial by Theorem 2.

Case 2) Then e_1R is uniserial by [3], Corollary to Theorem 1, i.e., $e_1R = (1 \ 2 \ 3 \ d)$ (or=(1 2 3)) and e_4R are injective. If e_4J is local, R is right serial. Suppose that e_4J is not local. Then from Proposition 1 we have the following:

a)
$$e_4 R = (4 \stackrel{1}{\underset{1}{\vdots}}),$$
 b) $e_4 R = (4 \stackrel{1}{\underset{4}{\vdots}})$ or c) $e_4 R = (4 \stackrel{1}{\underset{4}{\vdots}})$

R splits if c) occurs. Hence we assume a) or b).

i) $e_1R/Soc(e_1R)$ and $e_1R/Soc_2(e_1R)$ are injective (see the proof of Lemma 12).

Let xR be a submodule in e_4J with $(xR+e_4J^2)/e_4J^2 \approx \bar{e}_1\bar{R}$. Since e_1R is uniserial, $\operatorname{Soc}(e_4R) = \operatorname{Soc}(xR) \approx \bar{e}_2\bar{R}$ or $\bar{e}_3\bar{R}$ if $e_1J^3 \pm 0$. However $\operatorname{Soc}(e_1R/\operatorname{Soc}(e_1R))$ $\approx \bar{e}_3\bar{R}$ and $\operatorname{Soc}(e_1R/\operatorname{Soc}_2(e_1R)) \approx \bar{e}_2\bar{R}$, a contradiction. If $e_1J^3 = 0$, we obtain the same result as above.

- ii) $e_1R/Soc(e_1R)$ and $e_4R/Soc(e_4R)$ are injective.
- α) $e_1 J^3 \neq 0$. $e_1 R = (1 \ 2 \ 3 \ d)$.

Assume a) or b). $\operatorname{Soc}(e_4R)$ and $\operatorname{Soc}_2(e_4R)$ are waists by assumption. Since $\operatorname{Soc}(eR/\operatorname{Soc}(e_1R)) \approx \bar{e}_3 \bar{R}$, there exists a submodule xR in e_4J such that $xR \approx e_1R/e_1J^2$, i.e., $e_4J^3=0$, and hence e_4R is uniserial.

 β) $e_1 J^3 = 0$. $e_1 R = (1 \ 2 \ 3)$.

Then xR is simple, i.e. $|e_4R| \leq 2$, a contradiction.

iii) $e_4R/\text{Soc}(e_4R)$ and $e_4R/\text{Soc}_2(e_4R)$ are injective. Then e_4R is uniserial and hence R is serial.

Case 3) i) $e_1R/Soc(e_1R)$ and $e_1R/Soc_2(e_1R)$ are injective. Then $e_1R = (1 \ 2 \ c \ d)$ (or= $(1 \ 2 \ c)$) and

$$e_{3}R = (3 4 \stackrel{g}{\vdots} k) \text{ or } (3 4 g k)$$

In the latter case R is serial. Hence assume the former. Then $\{g, h\} \subset \{1, 3\}$. Assume $e_1 J^3 \neq 0$.

 α) g=1. There exists xR in e_4J^2 with $xR \approx e_1R/A$ for some A in e_1R . However $\operatorname{Soc}(e_3R) = \operatorname{Soc}(xR) \approx \overline{e}_2\overline{R}$, a contradiction.

 β) g=h=3. Then

$$e_4 R = (3 4 \frac{3}{3} 4),$$

which is a contradiction to Lemma 2.

We obtain the same result in a case $e_1 J^3 = 0$.

ii) $e_1R/\text{Soc}(e_1R)$ and $e_3R/\text{Soc}(e_3R)$ are injective. Then e_1R and e_3R are userial, and hence R is serial.

Case 4) If e_1R , e_3R and e_4R are uniserial, R is right serial.

6. Examples

In this section we shall give several examples related to the previous sections.

1. We shall give a two-sided almost QF ring with $J^4=0$ and n=4 but neither QF nor serial. This is an example of exceptional algebras in Theorem 4. Let K be a field and $R=\sum_{i\leq 4}\oplus e_iR$, where $\{e_i\}$ is a set of mutually orthogonal primitive idempotents with $1=\sum e_i$. We define $e_1R=e_1K\oplus aK\oplus abK\oplus abc'K$, $e_2R=e_2K\oplus bK\oplus bc'K$, ..., whose multiplicative structur is given below, where $_1a_2$ means $a=e_1ae_2$, and so on.

(In the previous sections we expressed horizontally the structure of $e_i R$, however we shall do vertically here.)

where the other products among a, b, \cdots are zero, e.g. bc=dc'=o. Then $(Re_4)^* \approx e_1 R, (Re_2)^* \approx e_4 R$ and $(Re_3)^* \approx e_3 R$ are injective and $e_1 R \supset e_2 R$ $(Re_2 \supset Re_1)$. Hence R is the desired algebra, which satisfies (#-1, 2, 3).

In the above example we replace e_3R with

Almost QF Rings with $J^3=0$



Then we obtain a two-sided almost QF-algebra with $J^4=0$ and any $n \ge 4$, which is neither QF nor serial. We shall give another type of exceptional algebras, where $e_1R (\supset e_2R)$ is not uniserial.



where the other products among $a, b\cdots$ are zero, e.g. $\{b, b'\}$ $\{c, c'\} = o, bde = b'd'e' = o, \{e, e'\}$ $\{d, d'\} = o, dec = d'e'c' = o$ and so on. Then $(Re_4)^* \approx e_1R \supset e_2R$, $(Re_3)^* \approx e_4R$ and $(Re_2)^* \approx e_4R$. This ring is almost QF, but (#-1) is not satisfied.

2. We shall give an algebra which is a two-sided almost QF-algebra with $J^{4} \neq 0$ and n=3, but R is neither QF nor serial (cf. Corollary to Theorem 2). $R=\Sigma_{t\leq 3}\oplus e_t R$ as above.

Then e_1R , e_3R and Re_2 , Re_3 are injective and $e_1R \supset e_2R$, $Re_2 \supset Re_1$.

3. There exists a right almost QF algebra with $J^4=0$ and n=3, which is not left almost QF (cf. Corollary to Theorem 2). Put bca=o in the above. Then $Re_3 \supset Re_2$ and Je_3 is not local.

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