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Author(s)	Asami, Takeo
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Asami, Takeo
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On the Conditions of a Stein Variety

By Takeo ASAMI

§1. **Introduction.** The purpose of this paper is to give a criterion for a Stein variety. An analytic space $\mathfrak{V}$ [1] with a countable base is called a *Stein variety*, when:

1. $\mathfrak{V}$ is *holomorph-convex*; that is, a holomorphic convex hull of any compact subset of $\mathfrak{V}$ is compact. The holomorphic convex hull of a subset K is the set of the points P satisfying $|f(P)| \leq \mathrm{Max}|f(K)|$ for all functions holomorphic in $\mathfrak{V}$.

2. For any two points $P, Q \in \mathfrak{V}$ $(P \neq Q)$, there exists a function f holomorphic in $\mathfrak{V}$, such that $f(P) \neq g(Q)$.

3. For any point $P \in \mathfrak{V}$, there exists a finite number of functions holomorphic in $\mathfrak{V}$ which imbed a neighborhood U of P in the following way, i.e., by means of which U is represented as an analytic set[1] S in an open set of the space of complex variables of sufficiently high dimensions such that S has the property that, for arbitrary point P' of S, any function holomorphic in a neighborhood of P' is expressed as a trace of a function of the space[2].

The definition in this form is due to H. Grauert [2].

The problem of simplifying these conditions is treated by H. Grauert [2] and R. Remmert [7]. Grauert proved that a holomorphic convex analytic space (without the assumption of having a countable base) is a Stein variety, if it is K-complete. An analytic space $\mathfrak{R}$ is called *K-complete*, if, for any point $P \in \mathfrak{R}$, there exist a finite number of functions holomorphic in $\mathfrak{R}$ which map a neighborhood of P non degeneratedly at P, i.e., the image of P in the space of complex variables has as an inverse image in U a discrete set. Since, as Remmert remarked, K-completeness follows immediately from the separability condition, so, according to Grauert's result, one of the conditions (2., 3.) implies that a holomorph-convex analytic space is a Stein variety. But a holomorph-

1) Namely the set which is locally the common zeros of a finite number of equations.

2) In this paper, we shall call for convenience the conditions 2. and 3. the separability condition and the coordinate condition respectively.

convex analytic space is not always a Stein variety, as a simple example shows[3].

In the present paper we shall introduce the notion of a positive definite Levi function and derive the coordinate condition from the existence of such a function and the holomorph-convexity. Thus we shall prove the following Theorem:

Theorem. *It is necessary and sufficient for an analytic space to be a Stein variety that it is holomorph-convex and admits a positive definite Levi function.*

§2. **Definition.** In the following we assume an analytic space not to be 0 dimentional unless we mention the contrary.

DEFINITION. Let φ be an upper semicontinuous[4] function in an analytic space $\mathfrak{R}$, which takes real values or $-\infty$. We shall call φ a *positive definite Levi function*, if, for any point $P \in \mathfrak{R}$, there exist a neighborhood U of P and a family $\{\sigma_t\}$ of characteristic surfaces in U such that each σ_t is expressed by the equation $f(Q, t)=0$ ($Q \in U$, $0 \leqq t \leqq 1$), $f(Q, t)$ being univalent and holomorphic for $Q \in U$ and continuous for t in the interval [0, 1] and such that

1. σ_0 passes through P and lies in the part $\varphi > \varphi(P)$, except P.
2. σ_t $(t \neq 0)$ lies in the part $\varphi > \varphi(P)$.

REMARK. When $\mathfrak{R}$ is of 1 dimension, these conditions mean that there exist continuous curves in U which start at P and lie in the part $\varphi > \varphi(P)$, except P.

A pseudoconvex (plurisubharmonic) function with the property (P_1) which is defined in a complex analytic manifold is a positive definite Levi function[5].

§3. **Lemma.** *Let φ be a positive definite Levi function in an analytic space $\mathfrak{R}$ and let $\mathfrak{R}'$ be an analytic subspace contained in $\mathfrak{R}$ ($\mathfrak{R}'$ may have boundaries in $\mathfrak{R}$). If we take the trace φ' of φ on $\mathfrak{R}'$, φ' is a positive definite Levi function in $\mathfrak{R}'$.*

From this Lemma, it is evident that, for any analytic set A contained in $\mathfrak{R}$, φ cannot attain its relative maximums at inner points of

3) For example, the product space of the complex projective space and the space of complex variables.

4) Precisely e^{φ} upper semicontinuous.

5) "No. 13. Propriété pricipale" of Oka [4]. For the meaning of the property (P_1), see p. 125 of Oka [5].

A and that the set of the form $\varphi=$constant is non-dense in A. From both facts it follows that, when A is compact and everywhere regular in $\mathfrak{R}$, A is 0 dimensional and so it is a finite point set.

Proof of the Lemma. Since the statement is obvious, if $\mathfrak{R}'$ is of the same dimension as $\mathfrak{R}$, we assume that $\mathfrak{R}'$ is lower dimensional. Let $\mathfrak{R}'$ be of n dimensions. It is sufficient to show that, for any point $P\in\mathfrak{R}'$, we dan determine a neighborhood U' and a family $\{\sigma_t'\}$ of characteristic surfaces satisfying the conditions in the definition. Let us regard P as a point of $\mathfrak{R}$ and consider the neighborhood U and the family $\{\sigma_t\}$ of the characteristic surfaces which are already given by the definition of φ. Roughly speaking, it will be shown that U' and σ_s' will be obtained as the section on U and σ_t by $\mathfrak{R}'$ respectively.

In fact, take from the intersection U and $\mathfrak{R}'$ an irreducible component passing through P and denote it U'. As easily seen from the definition of an analytic space, we can represent U' as zeros of an irreducible pseudopolynomial $f(x, y)$[6] in a polycylinder of the space $(x_1, \cdots, x_n, y)$. To be more precise

$$F(x_1, \cdots, x_n, y)=y^m+A_1(x)y^{m-1}+\cdots+A_m(x)$$

and the coefficients are holomorphic in the polycylinder $\gamma:|x_i|<\delta$ $(\delta>0)$, $(i=1, \cdots, n)$, and further we assume that P coincides with the origin. Consider the trace of $f(Q, t)$ on U', which we denote by the same letter. For each t, $f(Q, t)$ is univalent[7] and holomorphic in U' and since σ_0 passes through the origin, $f(0, 0)=0$. Then two cases arise.

1. $f\equiv 0$; this means $\varphi'(Q)>\varphi'(P)$ for Q in U' different from P. Then it is easy to construct the family of characteristic surfaces (or a continuous curve) satisfying the conditions in the definition.
2. $f\not\equiv 0$; then we can prove the existence of a positive number t_0 such that, for any $t\,(0\leqq t\leqq t_0)$, $f(Q, t)$ always takes zeros.

After having proved this, if we set

$$\sigma_t':\ f(Q, t)=0 \qquad Q\in U',\ 0\leqq t\leqq t_0,$$

$\{\sigma_t'\}$ is the required one.

Suppose that such a nositive number t_0 does not exist, then we have a decreasing sequence $\{t_k\}$ $(0\leqq t_k\leqq 1)$ $(k=1, 2, \cdots)$ converging to 0 such that $f_k=f(Q, t_k)$ does not vanish in U'. For each f_k there exists a pseudopolynomial

6) We use freely the theorems in Kap. II, particularly in §12 and §14, of Osgood [6].

7) This is evident if we suppose U be expressed as an analytic set which is imbedded in the space of complex variables in the way explained in the condition 3) in Introduction.

$$G_k(x_1, \cdots, x_n, z) = z^m + B_1^{(k)}(x)z^{m-1} + \cdots + B_m^{(k)}(x)$$

such that $G_k(x_1, \cdots, x_n, f_k) = 0$. Similarly for $f_0 = f(Q, 0)$ we have

$$G(x_1, \cdots, x_n, z) = z^m + B_1(x)z^{m-1} + \cdots + B_m(x)$$

such that $G(x_1, \cdots, x_n, f_0) = 0$. (Here $B_m^{(k)}(x)$ is the m products of the values of at the points of U' which are superposed over the point $(x_1, \cdots, x_n) \in \gamma$. The same fact holds for f_0 and $B_m(x)$.) Since the sequence $f_1, f_2, \cdots$ converges uniformly to f_0 on any campact subset of U', the sequence $B_m^{(1)}(x)$, $B_m^{(2)}(x), \cdots$ converges to $B_m(x)$ in a similar manner in γ of the space $(x_1, \cdots, x_n)$. Every $B_m^{(k)}(x)$ does not vanish in γ from the assumption for f_k, while $B_m(0) = 0$. This contradicts the well known fact (See p. 82 of Julia [3]), q.e.d.

§ 4. Proof of the Theorem.

Necessity. We show that a Stein variety $\mathfrak{V}$ always admits a positive definite Levi function. Owing to Remmert [7], we can suppose that $\mathfrak{V}$ is an analytic set in the space of complex variables $(x_1, \cdots, x_N)$ of sufficiently high dimension. Then $\Phi = |x_1|^2 + \cdots + |x_N|^2$ is a positive definite Levi function in the space $(x_1, \cdots, x_N)$, for Φ is a pseudoconvex function with the property (P_1). Hence, the Lemma implies that the trace of Φ or $\mathfrak{V}$ is also a positive definite Levi function.

Sufficiency. Let $\mathfrak{R}$ be a holomorph-convex analytic space with a positive definite Levi function φ. It is sufficient to show that $\mathfrak{R}$ is K-complete. Take an arbitrary point $P \in \mathfrak{R}$. From the set Σ of points $P' \in \mathfrak{R}$ such that $f(P') = f(P)$ for all functions holomorphic in $\mathfrak{R}$. Being the holomorph-convex hull of P, Σ is compact by the assumption. On the other hand, Σ is an analytic set everywhere regular in $\mathfrak{R}$. Hence Σ is a finite point set, as metioned in the proceeding section. Then we can choose a neighborhood of P and n functions holomorphic in $\mathfrak{R}$ such that the condition of K-complete holds at P.

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