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## A NOTE ON CARTAN INTEGERS FOR $p$ -SOLVABLE GROUPS

YUKIO TSUSHIMA

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### 1. Introduction

Let  $B$  be a  $p$ -block of a finite group  $G$ . As is well-known, if the Cartan integer  $c_{\varphi\varphi}=1$  for some  $\varphi \in \text{IBr}(B)$ , then  $B$  must be a block of defect zero. On the other hand there are various blocks in which the second smallest case, namely  $c_{\varphi\varphi}=2$  for some  $\varphi$  occurs, though they do not seem generally to have specific natures in common. However in such blocks of  $p$ -solvable groups we can show the following, which is the purpose of this paper.

**Theorem.** *Let  $G$  be a  $p$ -solvable group and  $B$  a  $p$ -block (ideal) of  $R[G]$  with defect group  $D$ . If the Cartan integer  $c_{\varphi\varphi}=2$  for some  $\varphi \in \text{IBr}(B)$ , there exists a group  $T$  which is involved in  $G$  and satisfies:*

*$T$  has a normal Sylow  $p$ -subgroup  $Q$  isomorphic to  $D$  and if  $H$  is a  $p$ -complement of  $T$ , then  $H$  acts faithfully and transitively on  $Q^*$ .  $B$  is isomorphic to the full matrix ring  $M(n, R[T])$  over  $R[T]$  of degree  $n = \deg \varphi$  as  $R$ -algebras. In particular  $D$  is elementary and  $|H| = (|D| - 1)(k_B - l_B)$ , where  $k_B = |\text{Irr}(B)|$  and  $l_B = |\text{IBr}(B)|$ .*

Here " $T$  is involved in  $G$ " means that  $T$  is isomorphic to a homomorphic image of a subgroup of  $G$  and  $Q^*$  is the set of non-identity elements of  $Q$ . Note that the above  $T$  has a double coset decomposition  $T = H \cup HgH$  ( $g \in Q$ ), so it can be represented as a ( $p$ -solvable) doubly transitive permutation group and it holds that  $c_{\varphi\varphi}=2$  for every linear character  $\varphi$  of  $T$ . Such permutation groups were classified by Huppert [3] and Passman [5] and as a matter of fact the result will take an essential role in the proof of the above Theorem.

**NOTATION.**  $G$  will denote a finite group and  $p$  a fixed prime integer. We fix a  $p$ -modular system  $(L, R, F)$ , namely  $R$  is a valuation ring of rank one with quotient field  $L$  of characteristic zero and residue field  $F$  of characteristic  $p$ . We assume that  $L$  contains a primitive  $|G|$ -th root of unity. All modular representations will be considered over  $F$  and by a  $p$ -block of  $G$  we mean a block ideal of the group ring  $R[G]$ . As usual  $\text{Irr}(B)$  and  $\text{IBr}(B)$  denote the sets of irreducible  $L$ -characters and irreducible Brauer characters of  $B$ .

respectively. Finally for a positive integer  $n$ ,  $n_p$  and  $n'$  denote the  $p$ -part and the  $p'$ -part of  $n$  respectively.

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## 2. Preliminary lemmas

First of all we mention the following which is a direct consequence of the classifying theorems of  $p$ -solvable doubly transitive permutation groups due to Huppert [3] and Passman [5].

**Lemma 1.** *Let  $(G, \Omega)$  be a  $p$ -solvable doubly transitive permutation group with non-trivial normal  $p$ -subgroup. Then the stabilizer  $G_{a,b}$  of  $a, b \in \Omega (a \neq b)$  has a normal Sylow  $p$ -subgroup and its complement is cyclic.*

Proof. We may assume that  $G$  is the semidirect product  $G = NQ$ ,  $N \cap Q = 1$ , in which  $Q$  is a minimal normal  $p$ -subgroup of  $G$  of order  $|\Omega| = p^n$  and  $N$  acts transitively on  $Q^*$ . So  $G_{a,b} = N_x = C_N(x)$  for some  $x \in Q^*$ . In case of “semilinear transformations”,  $Q$  is identified with the Galois field  $GF(p^n)$  and then  $N_x \subset_G$  the Galois group of  $GF(p^n)$ , which is cyclic (Take  $x$  from the prime field). In exceptional cases, we have  $|Q| = p^2$  or  $3^4$ . If  $|Q| = p^2$ , then  $N \subset GL(2, p)$  and our assertion is obvious (consider the stabilizer in  $GL(2, p)$  of the vector  $(1, 0) \in (\mathbb{Z}/(p))^2$ ). If  $|Q| = 3^4$ , then “case by case” arguments prove easily our assertion. For example, the cyclic group of order eight generated

by  $\begin{pmatrix} 1 & 0 & 1 & 0 \\ 0 & 0 & 0 & -1 \\ -1 & 0 & 1 & 0 \\ 0 & -1 & 0 & 0 \end{pmatrix}$  is the stabilizer of the vector  $(0 \ 1 \ 0 \ -1) \in (\mathbb{Z}/(p))^4$  in the

first one of the three groups listed on pp. 127 of Huppert [3].

The next Lemma was essentially noted by Brauer and Nesbitt [1].

**Lemma 2.** *The Cartan integers  $c_{\varphi\varphi}$  are same for all the linear characters  $\varphi$  of  $G$ . If  $G$  has a  $p$ -complement, say  $H$ , then this common integer is equal to the number of  $(H, H)$ -cosets of  $G$ .*

Proof. Let  $\eta$  be the principal indecomposable Brauer character corresponding to the trivial character  $1_G$  of  $G$ . If  $\varphi$  is linear, then  $\varphi\eta$  is the principal indecomposable Brauer character corresponding to  $\varphi$  and hence we have  $c_{\varphi\varphi} = (\varphi\eta, \varphi\eta) = (\eta, \eta) = c_{11}$ . In case  $G$  has a  $p$ -complement  $H$ , we have  $\eta = G \otimes_H 1_H$  and so  $c_{11}$  is equal to the number of  $(H, H)$ -cosets of  $G$  by Mackey decomposition.

**Lemma 3.** *Suppose that there exists  $\varphi \in \text{IBr}(G)$  such that  $c_{\varphi\varphi}=2$ . If  $O_p(G)$  is not trivial, then it is a unique normal  $p$ -subgroup and  $\varphi$  belongs to a block of  $G/O_p(G)$  of defect zero.*

Proof. Let  $Q$  be any non-trivial normal  $p$ -subgroup of  $G$  and let  $I$  denote the (nilpotent) ideal of  $A=F[G]$  generated by  $\{x-1; x \in Q\}$ . Let  $e$  be a primitive idempotent of  $A$  such that  $\varphi$  is afforded by the socle of  $Ae$ . Suppose that  $Ae/Ie$  is reducible. Since it is a principal indecomposable  $F[G/Q]$ -module, we have  $c_{\varphi\varphi} \geq 2$  (as  $F[G/Q]$  is a symmetric algebra). On the other hand we see that  $Ie$  is not zero and contains the socle of  $Ae$ . In fact it is easy to see that the right annihilator ideal of  $I$  in  $A$  is the principal ideal  $\sigma A$  with  $\sigma = \sum x$  ( $x \in Q$ ), which is square zero and contains no idempotent. Combining with the above, we get  $c_{\varphi\varphi} \geq 3$ , contradicting the assumption. Therefore  $Ae/Ie$  is irreducible and  $\varphi$  belongs to a block of  $G/Q$  of defect zero. In particular it follows that  $\varphi(1)_p = [G:Q]_p$ . Since  $Q$  is arbitrary, we have  $|O_p(G)| = |Q|$ , or  $O_p(G) = Q$ , completing the proof of Lemma 3.

**Lemma 4.** *Let  $G$  be a  $p$ -solvable group and assume that  $O_p(G)$  is central. If there exists  $\varphi \in \text{IBr}(G)$  such that  $c_{\varphi\varphi}=2$ , then  $\varphi$  must be linear. In particular,  $G$  has a normal Sylow  $p$ -subgroup  $S$  and if  $H$  is a  $p$ -complement, it acts on  $S^*$  transitively.*

Proof. By the assumption and Lemma 3,  $Q=O_p(G)$  is a (non-trivial) minimal normal subgroup. Hence it has a complement in  $G$ , that is, if we let  $N$  be the normalizer in  $G$  of a  $p$ -complement of  $O_{pp}(G)$ , we get  $G=NQ$  by Frattini argument and  $N \cap Q=1$  by the minimality of  $Q$ . By Lemma 3,  $\varphi$  belongs to a block of  $N$  of defect zero, so it can be regarded as a principal indecomposable Brauer character of  $N$  and then has the form  $\varphi=N \otimes_H \lambda$  for some  $\lambda \in \text{Irr}(H)$  by Fong[2], where  $H$  is a  $p$ -complement of  $N$  (and necessarily of  $G$ ). Put  $\eta=G \otimes_N \varphi=G \otimes_H \lambda$ . We claim that  $\eta$  is the principal indecomposable Brauer character of  $G$  corresponding to  $\varphi$ . In fact if  $f$  is a primitive idempotent of  $F[H]$  such that  $F[H]f$  affords the Brauer character  $\lambda$ , then using the same notation as in the proof of Lemma 3 we see that  $F[G]f/If \cong F[G]/I \otimes_H F[H]f \cong F[N] \otimes_H F[H]f$  affords the Brauer character  $\varphi$ . Therefore  $c_{\varphi\varphi} = (\eta, \eta) = \sum_g (\varphi, g \otimes_N \varphi)$  by Mackey decomposition, where  $g$  runs through a set of representatives of  $(N, N)$ -cosets of  $G$  and the inner product  $(\varphi, g \otimes_N \varphi)$  is taken over  $gNg^{-1} \cap N$ . Since  $G=NQ=NQN$ , any  $(N, N)$ -coset is represented by an element of  $Q$ . Moreover if  $g \in Q$ , then  $gNg^{-1} \cap N=C_N(g)$  and  $\varphi=g \otimes_N \varphi$  on it. So  $c_{\varphi\varphi}=2$  forces that  $G=N \cup NgN$  and  $1+(\varphi, \varphi)_{C_N(g)}=2$  for (all)  $g \in Q^*$ . This means of course that  $N$  acts transitively on  $Q^*$ , having  $O_p(G)$  as its kernel and  $\varphi_{C_N(g)}$  is irreducible. On the other hand noting that  $O_p(G)$  is central, we see from Lemma 1 that  $C_N(g)$  has a normal Sylow  $p$ -subgroup and its com-

lement is abelian. Combining the aboves, we conclude that  $(p, |C_N(g)|)=1$  and  $\varphi$  is linear. In particular we have  $G=H \cup HgH, g \in Q$  by Lamme 2 and it follows from this that  $Q$  is a Sylow  $p$ -subgroup of  $G$ . This completes the proof of Lemma 4.

### 3. Proof of the Theorem

Let  $K=O_p(G)$ . If  $B$  is a principal block, we may assume that  $K=1$ . Then  $T=G$  satisfies the conclusion of the Theorem by Lemma 4. In general there exists  $\xi \in \text{Irr}(K)$  such that  $\text{Irr}(B) \subset \text{Irr}(G|\xi)$ . If  $I$  is the inertia group of  $\xi$  in  $G$ , there exists a  $p$ -block  $B_I$  of  $I$  which has the same Cartan matrix as that of  $B$  and  $B \cong M(m, B_I)$  for some  $m \geq 1$  (Fong [2], Tsushima [7], more notably  $B$  is isomorphic to the corestriction algebra  $\text{Cores}_I^G B_I$  or the induced algebra  $\text{Ind}_I^G B_I$  in the languages introduced by M. Broué and L. Puig). So if  $G > I$ , we get our assertion by the induction on the order of  $G$ . Therefore we may assume that  $G=I$ . Following Fong, we consider a central extension

$$1 \rightarrow Z \rightarrow G^* \rightarrow G/K \rightarrow 1$$

, where  $Z$  is a cyclic group of order prime to  $p$ . Moreover from the construction we see that  $Z \subset [G^* G^*]$  (Reynolds [6]). There exists a  $p$ -block  $B^*$  of  $G^*$  which has the same Cartan matrix as that of  $B$  and  $B \cong M(m, B^*)$  for some  $m \geq 1$  (Fong [2], Tsushima [7]). If  $c_{\varphi\varphi}=2$  for some  $\varphi \in \text{IBr}(B^*)$ , then  $\varphi$  must be linear by Lemma 4 and hence  $\ker \varphi \subset Z=O_p(G^*)$ . Therefore  $B^*$  must be the principal block of  $G^*$ , which coincides with that of  $G^*/Z \cong G/K$ . So it remains only to show that  $|H| = (|D|-1)(k_B - l_B)$ . By Brauer's Permutation Lemma,  $H$  acts transitively on  $\text{Irr}(Q)^*$ . In particular for any  $\psi \in \text{Irr}(Q)^*$ , its inertia group  $H_\psi$  in  $H$  is cyclic by Lemma 1 and hence  $|\text{Irr}(T|\psi)| \cdot |\text{Irr}(T_\psi|\psi)| = |H_\psi|$  by Clifford's Theorem. Using this, we have  $k_B = |\text{Irr}(T)| = |\text{Irr}(H)| + |\text{Irr}(T|\psi)| = l_B + |H_\psi|$ . Therefore  $|H| = (|Q|-1)|H_\psi| = (|D|-1)(k_B - l_B)$ . This completes the proof of the Theorem.

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Department of Mathematics  
Osaka City University  
Sumiyoshi-ku, Osaka 558  
Japan

