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Osaka J. Math.
20 (1983), 675-679

A NOTE ON CARTAN INTEGERS FOR *p*-SOLVABLE GROUPS

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(Received September 10, 1981)

1. Introduction

Let B be a p-block of a finite group G. As is well-known, if the Cartan integer $c_{\varphi\varphi}=1$ for some $\varphi\in \mathrm{IBr}(B)$, then B must be a block of defect zero. On the other hand there are various blocks in which the second smallest case, namely $c_{\varphi\varphi}=2$ for some φ occurs, though they do not seem generally to have specific natures in common. However in such blocks of p-solvable groups we can show the following, which is the purpose of this paper.

Theorem. *Let G be a p-solvable group and B a p-block (ideal) of $R[G]$ with defect group D. If the Cartan integer $c_{\varphi\varphi}=2$ for some $\varphi\in IBr(B)$, there exists a group T which is involved in G and satisfies*:

T has a normal Sylow p-subgroup Q isomorphic to D and if H is a p-complement of T, then H acts faithfully and transitively on $Q^{\#}$. B is isomorphic to the full matrix ring $M(n, R[T])$ over $R[T]$ of degree $n=deg\ \varphi$ as R-algebras. In particular D is elementary and $|H|=(|D|-1)(k_B-l_B)$, where $k_B=|Irr(B)|$ and $l_B=|IBr(B)|$.

Here "T is involved in G" means that T is isomorphic to a homomorphic image of a subgroup of G and $Q^{\#}$ is the set of non-identy elements of Q. Note that the above T has a double coset decomposition $T=H\cup HgH$ $(g\in Q)$, so it can be represented as a (p-solvable) doubly transitive permutation group and it holds that $c_{\varphi\varphi}=2$ for every linear character φ of T. Such permutation groups were classified by Huppert [3] and Passman [5] and as a matter of fact the result will take an essential role in the proof of the above Theorem.

NOTATION. G will denote a finite group and p a fixed prime integer. We fix a p-modular system (L, R, F), namely R is a valuation ring of rank one with quotient field L of characteristic zero and residue field F of characteristic p. We assume that L contains a primitive $|G|$-th root of unity. All modular representations will be considered over F and by a p-block of G we mean a block ideal of the group ring $R[G]$. As usual $\mathrm{Irr}(B)$ and $\mathrm{IBr}(B)$ denote the sets of irreducible L-characters and irreducible Brauer characters of B

respectively. Finally for a positive integer n, n_p and n' denote the p-part and the p'-part of n respectively.

The author would like to thank Professor G. Michler for his preprint [4] which motivated this work and also Professors K. Iizuka, T. Okuyama and A. Watanabe for their valuable advice.

2. Preliminary lemmas

First of all we mention the following which is a direct consequence of the classifying theorems of p-solvable doubly transitive permutation groups due to Huppert [3] and Passman [5].

Lemma 1. *Let* (G, Ω) *be a p-solvable doubly transitive permutation group with non-trivial normal p-subgroup. Then the stabilizer* $G_{a,b}$ *of* $a, b\in\Omega(a\neq b)$ *has a normal Sylow p-subgroup and its complement is cyclic.*

Proof. We may assume that G is the semidirect product $G=NQ$, $N\cap Q=1$, in which Q is a minimal normal p-subgroup of G of order $|\Omega|=p^n$ and N acts transitively on $Q^\#$. So $G_{a,b}\underset{G}{=}N_x=C_N(x)$ for some $x\in Q^\#$. In case of "semilinear transformations", Q is identified with the Galois field $GF(p^n)$ and then $N_x\underset{G}{\subset}$ the Galois group of $GF(p^n)$, which is cyclic (Take x from the prime field). In exceptional cases, we have $|Q|=p^2$ or 3^4. If $|Q|=p^2$, then $N\subset GL(2, p)$ and our assertion is obvious (consider the stabilizer in $GL(2, p)$ of the vector $(1, 0)\in(\boldsymbol{Z}/(p))^2$). If $|Q|=3^4$, then "case by case" arguments prove easily our assertion. For example, the cyclic group of order eight generated by $\begin{pmatrix} 1 & 0 & 1 & 0 \\ 0 & 0 & 0 & -1 \\ -1 & 0 & 1 & 0 \\ 0 & -1 & 0 & 0 \end{pmatrix}$ is the stabilizer of the vector $(0\ 1\ 0\ -1)\in(\boldsymbol{Z}/(p))^4$ in the first one of the three groups listed on pp. 127 of Huppert [3].

The next Lemma was essentially noted by Brauer and Nesbitt [1].

Lemma 2. *The Cartan integers* $c_{\varphi\varphi}$ *are same for all the linear characters* φ *of* G. *If* G *has a p-complement, say* H, *then this common integer is equal to the number of* (H, H)*-cosets of* G.

Proof. Let η be the principal indecomposable Brauer character corresponding to the trivial character 1_G of G. If φ is linear, then $\varphi\eta$ is the principal indecomposable Brauer character corresponding to φ and hence we have $c_{\varphi\varphi}=(\varphi\eta, \varphi\eta)=(\eta, \eta)=c_{11}$. In case G has a p-complement H, we have $\eta=G\otimes_H 1_H$ and so c_{11} is equal to the number of (H, H)-cosets of G by Mackey decomposition.

Lemma 3. *Suppose that there exists $\varphi \in IBr(G)$ such that $c_{\varphi\varphi}=2$. If $O_p(G)$ is not trivial, then it is a unique normal p-subgroup and φ belongs to a block of $G/O_p(G)$ of defect zero.*

Proof. Let Q be any non-trivial normal p-subgroup of G and let I denote the (nilpotent) ideal of $A=F[G]$ generated by $\{x-1;\ x\in Q\}$. Let e be a primitive idempotent of A such that φ is afforded by the socle of Ae. Suppose that Ae/Ie is reducible. Since it is a principal indecomposable $F[G/Q]$-module, we have $c_{\varphi\varphi}\geq 2$ (as $F[G/Q]$ is a symmetric algebra). On the other hand we see that Ie is not zero and contains the socle of Ae. In fact it is easy to see that the right annihilator ideal of I in A is the principal ideal σA with $\sigma=\sum x$ $(x\in Q)$, which is square zero and contains no idempotent. Combining with the above, we get $c_{\varphi\varphi}\geq 3$, contradicting the assumption. Therefore Ae/Ie is irreducible and φ belongs to a block of G/Q of defect zero. In particular it follows that $\varphi(1)_p=[G:Q]_p$. Since Q is arbitary, we have $|O_p(G)|=|Q|$, or $O_p(G)=Q$, completing the proof of Lemma 3.

Lemma 4. *Let G be a p-solvable group and assume that $O_{p'}(G)$ is central. If there exists $\varphi\in IBr(G)$ such that $c_{\varphi\varphi}=2$, then φ must be linear. In particular, G has a normal Sylow p-subgroup S and if H is a p-complement, it acts on $S^\#$ transitively.*

Proof. By the assumption and Lemma 3, $Q=O_p(G)$ is a (non-trivial) minimal normal subgroup. Hence it has a complement in G, that is, if we let N be the normalizer in G of a p-complement of $O_{pp'}(G)$, we get $G=NQ$ by Frattini argument and $N\cap Q=1$ by the minimality of Q. By Lemma 3, φ belongs to a block of N of defect zero, so it can be regarded as a principal indecomposable Brauer character of N and then has the form $\varphi=N\otimes_H\lambda$ for some $\lambda\in \mathrm{Irr}(H)$ by Fong[2], where H is a p-complement of N (and necessarily of G). Put $\eta=G\otimes_N\varphi=G\otimes_H\lambda$. We claim that η is the principal indecomposable Brauer character of G corresponding to φ. In fact if f is a primitive idempotent of $F[H]$ such that $F[H]f$ affords the Brauer character λ, then using the same notation as in the proof of Lemma 3 we see that $F[G]f/If\cong F[G]/I\otimes_H F[H]f\cong F[N]\otimes_H F[H]f$ affords the Brauer character φ. Therefore $c_{\varphi\varphi}=(\eta,\eta)=\sum_g(\varphi, g\otimes_N\varphi)$ by Mackey decomposition, where g runs through a set of representatives of (N,N)-cosets of G and the inner product $(\varphi, g\otimes_N\varphi)$ is taken over $gNg^{-1}\cap N$. Since $G=NQ=NQN$, any (N,N)-coset is represented by an element of Q. Moreover if $g\in Q$, then $gNg^{-1}\cap N=C_N(g)$ and $\varphi=g\otimes_N\varphi$ on it. So $c_{\varphi\varphi}=2$ forces that $G=N\cup NgN$ and $1+(\varphi,\varphi)_{C_N(g)}=2$ for (all) $g\in Q^\#$. This means of course that N acts transitively on $Q^\#$, having $O_{p'}(G)$ as its kernel and $\varphi_{C_N(g)}$ is irreducible. On the other hand noting that $O_{p'}(G)$ is central, we see from Lemma 1 that $C_N(g)$ has a normal Sylow p-subgroup and its complement is abelian. Combining the aboves, we conclude that $(p, |C_N(g)|)=1$ and φ is linear. In particular we have $G=H\cup HgH, g\in Q$ by Lamme 2 and it follows from this that Q is a Sylow p-subgroup of G. This completes the proof of Lemma 4.

3. Proof of the Theorem

Let $K=O_{p'}(G)$. If B is a principal block, we may assume that $K=1$. Then $T=G$ satisfies the conclusion of the Theorem by Lemma 4. In general there exists $\xi\in\mathrm{Irr}(K)$ such that $\mathrm{Irr}(B)\subset\mathrm{Irr}(G|\xi)$. If I is the inertia group of ξ in G, there exists a p-block B_I of I which has the same Cartan matrix as that of B and $B\cong M(m, B_I)$ for some $m\geq 1$ (Fong [2], Tsushima [7], more notably B is isomorphic to the corestriction algebra $Cores_I^G B_I$ or the induced algebra $Ind_I^G B_I$ in the languages introduced by M. Broué and L. Puig). So if $G>I$, we get our assertion by the induction on the order of G. Therefore we may assume that $G=I$. Following Fong, we consider a central extension

$$1\to Z\to G^*\to G/K\to 1$$

, where Z is a cyclic group of order prime to p. Moreover from the construction we see that $Z\subset[G^* G^*]$ (Reynolds [6]). There exists a p-block B^* of G^* which has the same Cartan matrix as that of B and $B\cong M(m, B^*)$ for some $m\geq 1$ (Fong [2], Tsushima [7]). If $c_{\varphi\varphi}=2$ for some $\varphi\in\mathrm{IBr}(B^*)$, then φ must be linear by Lemma 4 and hence $\ker\varphi\subset Z=O_{p'}(G^*)$. Therefore B^* must be the principal block of G^*, which coincides with that of $G^*/Z\cong G/K$. So it remains only to show that $|H|=(|D|-1)(k_B-l_B)$. By Brauer's Permutation Lemma, H acts transitively on $\mathrm{Irr}(Q)^\#$. In particular for any $\psi\in\mathrm{Irr}(Q)^\#$, its intertia group H_ψ in H is cyclic by Lemma 1 and hence $|\mathrm{Irr}(T|\psi)|$ $|\mathrm{Irr}(T_\psi|\psi)|=|H_\psi|$ by Clifford's Theorem. Using this, we have $k_B=|\mathrm{Irr}(T)|=|\mathrm{Irr}(H)|+|\mathrm{Irr}(T|\psi)|=l_B+|H_\psi|$. Therefore $|H|=(|Q|-1)|H_\psi|=(|D|-1)(k_B-l_B)$. This completes the proof of the Theorem.

References

[1] R. Brauer and C. Nesbitt: *On the modular characters of groups,* Ann. of Math. **42** (1941), 556–590.

[2] P. Fong: *Solvable groups and modular representation theory,* Trans. Amer. Math. Soc. **102** (1962), 484–494.

[3] B. Huppert: *Zweifach transitive, auflösbare permutationsgruppen,* Math. Z. **68** (1957), 126–150.

[4] G. Michler: *On blocks with multiplicity one,* preprint.

[5] D.S. Passman: *p-Solvable doubly transitive permutation groups,* Pacific J. Math. **26** (1968), 555–577.

[6] W.F. Reynolds: *Projective representations of finite groups in cyclotomic fields,* Illinois J. Math. **9** (1965), 191–198.
[7] Y. Tsushima: *On the second reduction theorem of P. Fong,* Kumamoto J. Sci. Math. **13** (1978), 1–5.

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