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A NOTE ON CARTAN INTEGERS FOR p -SOLVABLE GROUPS

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1. Introduction

Let B be a p -block of a finite group G . As is well-known, if the Cartan integer $c_{\varphi\varphi}=1$ for some $\varphi \in \text{IBr}(B)$, then B must be a block of defect zero. On the other hand there are various blocks in which the second smallest case, namely $c_{\varphi\varphi}=2$ for some φ occurs, though they do not seem generally to have specific natures in common. However in such blocks of p -solvable groups we can show the following, which is the purpose of this paper.

Theorem. *Let G be a p -solvable group and B a p -block (ideal) of $R[G]$ with defect group D . If the Cartan integer $c_{\varphi\varphi}=2$ for some $\varphi \in \text{IBr}(B)$, there exists a group T which is involved in G and satisfies:*

T has a normal Sylow p -subgroup Q isomorphic to D and if H is a p -complement of T , then H acts faithfully and transitively on Q^ . B is isomorphic to the full matrix ring $M(n, R[T])$ over $R[T]$ of degree $n=\deg \varphi$ as R -algebras. In particular D is elementary and $|H|=(|D|-1)(k_B-l_B)$, where $k_B=|\text{Irr}(B)|$ and $l_B=|\text{IBr}(B)|$.*

Here " T is involved in G " means that T is isomorphic to a homomorphic image of a subgroup of G and Q^* is the set of non-identity elements of Q . Note that the above T has a double coset decomposition $T=H \cup HgH$ ($g \in Q$), so it can be represented as a (p -solvable) doubly transitive permutation group and it holds that $c_{\varphi\varphi}=2$ for every linear character φ of T . Such permutation groups were classified by Huppert [3] and Passman [5] and as a matter of fact the result will take an essential role in the proof of the above Theorem.

NOTATION. G will denote a finite group and p a fixed prime integer. We fix a p -modular system (L, R, F) , namely R is a valuation ring of rank one with quotient field L of characteristic zero and residue field F of characteristic p . We assume that L contains a primitive $|G|$ -th root of unity. All modular representations will be considered over F and by a p -block of G we mean a block ideal of the group ring $R[G]$. As usual $\text{Irr}(B)$ and $\text{IBr}(B)$ denote the sets of irreducible L -characters and irreducible Brauer characters of B

respectively. Finally for a positive integer n , n_p and n' denote the p -part and the p' -part of n respectively.

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2. Preliminary lemmas

First of all we mention the following which is a direct consequence of the classifying theorems of p -solvable doubly transitive permutation groups due to Huppert [3] and Passman [5].

Lemma 1. *Let (G, Ω) be a p -solvable doubly transitive permutation group with non-trivial normal p -subgroup. Then the stabilizer $G_{a,b}$ of $a, b \in \Omega (a \neq b)$ has a normal Sylow p -subgroup and its complement is cyclic.*

Proof. We may assume that G is the semidirect product $G = NQ$, $N \cap Q = 1$, in which Q is a minimal normal p -subgroup of G of order $|\Omega| = p^n$ and N acts transitively on $Q^\#$. So $G_{a,b} = N_x = C_N(x)$ for some $x \in Q^\#$. In case of "semilinear transformations", Q is identified with the Galois field $GF(p^n)$ and then $N_x \subseteq$ the Galois group of $GF(p^n)$, which is cyclic (Take x from the prime field). In exceptional cases, we have $|Q| = p^2$ or 3^4 . If $|Q| = p^2$, then $N \subseteq GL(2, p)$ and our assertion is obvious (consider the stabilizer in $GL(2, p)$ of the vector $(1, 0) \in (Z/(p))^2$). If $|Q| = 3^4$, then "case by case" arguments prove easily our assertion. For example, the cyclic group of order eight generated by
$$\begin{pmatrix} 1 & 0 & 1 & 0 \\ 0 & 0 & 0 & -1 \\ -1 & 0 & 1 & 0 \\ 0 & -1 & 0 & 0 \end{pmatrix}$$
 is the stabilizer of the vector $(0 \ 1 \ 0 \ -1) \in (Z/(p))^4$ in the

first one of the three groups listed on pp. 127 of Huppert [3].

The next Lemma was essentially noted by Brauer and Nesbitt [1].

Lemma 2. *The Cartan integers $c_{\varphi\varphi}$ are same for all the linear characters φ of G . If G has a p -complement, say H , then this common integer is equal to the number of (H, H) -cosets of G .*

Proof. Let η be the principal indecomposable Brauer character corresponding to the trivial character 1_G of G . If φ is linear, then $\varphi\eta$ is the principal indecomposable Brauer character corresponding to φ and hence we have $c_{\varphi\varphi} = (\varphi\eta, \varphi\eta) = (\eta, \eta) = c_{11}$. In case G has a p -complement H , we have $\eta = G \otimes_H 1_H$ and so c_{11} is equal to the number of (H, H) -cosets of G by Mackey decomposition.

Lemma 3. *Suppose that there exists $\varphi \in \text{IBr}(G)$ such that $c_{\varphi\varphi} = 2$. If $O_p(G)$ is not trivial, then it is a unique normal p -subgroup and φ belongs to a block of $G/O_p(G)$ of defect zero.*

Proof. Let Q be any non-trivial normal p -subgroup of G and let I denote the (nilpotent) ideal of $A = F[G]$ generated by $\{x-1; x \in Q\}$. Let e be a primitive idempotent of A such that φ is afforded by the socle of Ae . Suppose that Ae/Ie is reducible. Since it is a principal indecomposable $F[G/Q]$ -module, we have $c_{\varphi\varphi} \geq 2$ (as $F[G/Q]$ is a symmetric algebra). On the other hand we see that Ie is not zero and contains the socle of Ae . In fact it is easy to see that the right annihilator ideal of I in A is the principal ideal σA with $\sigma = \sum x$ ($x \in Q$), which is square zero and contains no idempotent. Combining with the above, we get $c_{\varphi\varphi} \geq 3$, contradicting the assumption. Therefore Ae/Ie is irreducible and φ belongs to a block of G/Q of defect zero. In particular it follows that $\varphi(1)_p = [G:Q]_p$. Since Q is arbitrary, we have $|O_p(G)| = |Q|$, or $O_p(G) = Q$, completing the proof of Lemma 3.

Lemma 4. *Let G be a p -solvable group and assume that $O_{p'}(G)$ is central. If there exists $\varphi \in \text{IBr}(G)$ such that $c_{\varphi\varphi} = 2$, then φ must be linear. In particular, G has a normal Sylow p -subgroup S and if H is a p -complement, it acts on S^* transitively.*

Proof. By the assumption and Lemma 3, $Q = O_p(G)$ is a (non-trivial) minimal normal subgroup. Hence it has a complement in G , that is, if we let N be the normalizer in G of a p -complement of $O_{p'}(G)$, we get $G = NQ$ by Frattini argument and $N \cap Q = 1$ by the minimality of Q . By Lemma 3, φ belongs to a block of N of defect zero, so it can be regarded as a principal indecomposable Brauer character of N and then has the form $\varphi = N \otimes_H \lambda$ for some $\lambda \in \text{Irr}(H)$ by Fong[2], where H is a p -complement of N (and necessarily of G). Put $\eta = G \otimes_N \varphi = G \otimes_H \lambda$. We claim that η is the principal indecomposable Brauer character of G corresponding to φ . In fact if f is a primitive idempotent of $F[H]$ such that $F[H]f$ affords the Brauer character λ , then using the same notation as in the proof of Lemma 3 we see that $F[G]f/If \cong F[G]/I \otimes_H F[H]f \cong F[N] \otimes_H F[H]f$ affords the Brauer character φ . Therefore $c_{\varphi\varphi} = (\eta, \eta) = \sum_g (\varphi, g \otimes_N \varphi)$ by Mackey decomposition, where g runs through a set of representatives of (N, N) -cosets of G and the inner product $(\varphi, g \otimes_N \varphi)$ is taken over $gNg^{-1} \cap N$. Since $G = NQ = NQN$, any (N, N) -coset is represented by an element of Q . Moreover if $g \in Q$, then $gNg^{-1} \cap N = C_N(g)$ and $\varphi = g \otimes_N \varphi$ on it. So $c_{\varphi\varphi} = 2$ forces that $G = N \cup NgN$ and $1 + (\varphi, \varphi)_{C_N(g)} = 2$ for (all) $g \in Q^*$. This means of course that N acts transitively on Q^* , having $O_{p'}(G)$ as its kernel and $\varphi_{C_N(g)}$ is irreducible. On the other hand noting that $O_{p'}(G)$ is central, we see from Lemma 1 that $C_N(g)$ has a normal Sylow p -subgroup and its com-

plement is abelian. Combining the aboves, we conclude that $(p, |C_N(g)|) = 1$ and φ is linear. In particular we have $G = H \cup HgH, g \in Q$ by Lamme 2 and it follows from this that Q is a Sylow p -subgroup of G . This completes the proof of Lemma 4.

3. Proof of the Theorem

Let $K = O_p(G)$. If B is a principal block, we may assume that $K = 1$. Then $T = G$ satisfies the conclusion of the Theorem by Lemma 4. In general there exists $\xi \in \text{Irr}(K)$ such that $\text{Irr}(B) \subset \text{Irr}(G|\xi)$. If I is the inertia group of ξ in G , there exists a p -block B_I of I which has the same Cartan matrix as that of B and $B \cong M(m, B_I)$ for some $m \geq 1$ (Fong [2], Tsushima [7], more notably B is isomorphic to the corestriction algebra $\text{Cores}_I^G B_I$ or the induced algebra $\text{Ind}_I^G B_I$ in the languages introduced by M. Broué and L. Puig). So if $G > I$, we get our assertion by the induction on the order of G . Therefore we may assume that $G = I$. Following Fong, we consider a central extension

$$1 \rightarrow Z \rightarrow G^* \rightarrow G/K \rightarrow 1$$

, where Z is a cyclic group of order prime to p . Moreover from the construction we see that $Z \subset [G^* G^*]$ (Reynolds [6]). There exists a p -block B^* of G^* which has the same Cartan matrix as that of B and $B \cong M(m, B^*)$ for some $m \geq 1$ (Fong [2], Tsushima [7]). If $c_{\varphi\varphi} = 2$ for some $\varphi \in \text{IBr}(B^*)$, then φ must be linear by Lemma 4 and hence $\ker \varphi \subset Z = O_p(G^*)$. Therefore B^* must be the principal block of G^* , which coincides with that of $G^*/Z \cong G/K$. So it remains only to show that $|H| = (|D| - 1)(k_B - l_B)$. By Brauer's Permutation Lemma, H acts transitively on $\text{Irr}(Q)^*$. In particular for any $\psi \in \text{Irr}(Q)^*$, its inertia group H_ψ in H is cyclic by Lemma 1 and hence $|\text{Irr}(T|\psi)| = |\text{Irr}(T_\psi|\psi)| = |H_\psi|$ by Clifford's Theorem. Using this, we have $k_B = |\text{Irr}(T)| = |\text{Irr}(H)| + |\text{Irr}(T|\psi)| = l_B + |H_\psi|$. Therefore $|H| = (|Q| - 1)|H_\psi| = (|D| - 1)(k_B - l_B)$. This completes the proof of the Theorem.

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