L. Markus &
H. Yamabe
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Global Stability Criteria for Differential Systems

By Lawrence MARKUS and Hidehiko YAMABE

Consider the real differential system

$$\mathcal{S}) \qquad \frac{dx^i}{dt} = f^i(x^1, \cdots, x^n) \qquad i = 1, 2, \cdots, n$$

with the real vector-valued function $f(x)$ in class C^1 in the real vector space R^n. The local stability theorem of A. Liapounov [8, and 2, P. 341] states that if the origin is a critical point,

$$f(0) = 0,$$

and if the eigenvalues of the Jacobian matrix $J(0)$, where

$$J^i_j(x) = \frac{\partial f^i}{\partial x^j}(x),$$

have negative real parts, then each solution of $\mathcal{S}$) which initiates near the origin must approach the origin $t \to +\infty$. We shall extend this result to a global stability criterion which generalizes a theorem of N. N. Krasovski [3, 4].

For the differential system $\mathcal{S}$) consider the local eigenvalues $\lambda_1(x^1, \cdots, x^n), \cdots, \lambda_n(x^1, \cdots, x^n)$ which are the roots of the characteristic polynomial

$$|J(x) - \lambda I|.$$

If

$$\operatorname{Re} \lambda_i(x^1, \cdots, x^n) < 0 \qquad i = 1, 2, \cdots, n$$

everywhere in R^n, then it has been conjectured [1] that each solution curve of $\mathcal{S}$) must approach a critical point of $\mathcal{S}$) as $t \to +\infty$. This is clearly true for $n=1$ (when $\mathcal{S}$) has a critical point) but it has been established in only particular cases when $n \geq 2$.

Note that an affine coordinate change in R^n,

$$y^i = A^i_j x^j + a^i, \qquad \det A^i_j \neq 0,$$

transforms the system $\mathcal{S}$) to

$$\mathcal{S}') \qquad \frac{dy^i}{dt} = A^i_j f^j(A^{-1}(y-a))\,.$$

Thus the Jacobian matrix $J(x)$ is replaced by

$$A^i_k J^k_l A^{-1l}_i \qquad \text{or} \qquad AJ(A^{-1}(y-a))A^{-1}\,,$$

which has the same local eigenvalues as has $J(x)$. Thus, if $\mathcal{S}$) has a critical point P, we can choose affine coordinates to place P at the origin and yet preserve the hypotheses on the local eigenvalues.

Theorem 1. *Let $\mathfrak{M}^n$ be a connected complete Riemannian manifold with a positive definete metric tensor $g_{ij}(x)$, say of class C^∞. Consider*

$$\mathcal{S}) \quad \dot{x}^i = f^i(x^1, \cdots, x^n)\,,$$

a contravariant C^1 vector field on $\mathfrak{M}^n$.

Assume that the tensor

$$f^i{}_{,j} + f_j{}^{,i} = f^i{}_{,j} + g_{lj} f^l{}_{,s} g^{is}$$

has eigenvalues which all satisfy everywhere

$$\lambda(x) < -\nu(\rho(x))\,,$$

where $\rho(x)$ is the distance from $x \in \mathfrak{M}^n$ to a fixed reference point $P \in \mathfrak{M}^n$. Here $\nu(\rho) > 0$ is monotonic decreasing on $0 \leq \rho < \infty$, and

$$\int_0^\infty e^{-\varepsilon \int_0^\rho \nu(\sigma) d\sigma} d\rho < \infty\,,$$

for each constant $\varepsilon > 0$. Then $\mathcal{S}$) has a unique critical point in $\mathfrak{M}^n$ and $\mathfrak{M}^n$ is homeomorphic with R^n. Then each solution curve $x(t)$ of $\mathcal{S}$) is defined in $\mathfrak{M}^n$ for all large $t \to +\infty$ and the positive limit set of $x(t)$ is the unique critical point of $\mathcal{S}$).

Proof.[1)] Let $x(t)$ be a solution of $\mathcal{S}$) for an interval $0 \leq t < \tau \leq +\infty$. We shall consider the tangential component of the acceleration along this trajectory. Let $v(t) > 0$ be the speed and then

$$v(t)^2 = g_{ij} f^i f^j$$

and

$$\frac{d}{dt}[v(t)^2] = g_{ij} f^j{}_{,k} f^k f^j + g_{ij} f^i f^j{}_{,k} f^k\,.$$

Thus

$$\frac{d}{dt}[v(t)^2] = (f^i{}_{,j} + f_j{}^{,i}) g_{ik} f^j f^k = f_i (f^i{}_{,j} + f_j{}^{,i}) f^j\,.$$

1) The unicity of the critical point was pointed out to the authors by Professor P. Hartman, cf. [9].

Using Riemann normal coordinates at a point where $g_{ij}=\delta_{ij}$ we compute

$$\frac{d}{dt}[v(t)^2] \leq -\nu(\rho(x(t)))[v(t)^2].$$

Let $v(0)=v_0$ and then

$$v(t) \leq v_0 \exp\left(-(1/2)\int_0^t \nu(\rho(x(\sigma)))d\sigma\right) \quad \text{on} \quad 0 \leq t < \tau.$$

Since $\nu(\rho)>0$, $0 \leq v(t) < v_0$ and hence $x(t)$ has bounded speed and $\tau=+\infty$.

Also, letting $\rho(x(t))=\rho(t)$ and $\rho(0)=\rho_0$,

$$\rho(t) \leq \rho_0 + \int_0^t v(\sigma)d\sigma$$

so

$$\rho(t) \leq \rho_0 + v_0 t.$$

Then

$$\rho(t) \leq \rho_0 + v_0 \int_0^t \exp\left(-(1/2)\int_0^\sigma \nu(\rho(u))du\right)d\sigma.$$

On $0 \leq u \leq \sigma$ we have $\nu(\rho(u)) \geq \nu(\rho_0+v_0u)$.

Then

$$\rho(t) \leq \rho_0 + v_0 \int_0^t \exp\left(-(1/2)\int_0^\sigma \nu(\rho_0+v_0u)du\right)d\sigma,$$

or

$$\rho(t) \leq \rho_0 + v_0 \int_0^t \exp\left(-(1/2v_0)\int_{\rho_0}^{\rho_0+v_0\sigma} \nu(z)dz\right)d\sigma.$$

Thus

$$\rho(t) \leq \rho_0 + \int_{\rho_0}^{\rho_0+v_0t} \exp\left(-(1/2v_0)\int_{\rho_0}^{\lambda} v(z)dz\right)d\lambda.$$

But

$$\int_{\rho_0}^{\rho_0+v_0t} \exp\left(-(1/2\,v_0)\int_{\rho_0}^{\lambda} v(z)dz\right)d\lambda$$
$$< \left[\exp(1/2\,v_0)\int_0^{\rho_0} v(z)\,dz\right]\int_0^\infty \exp\left(-(1/2\,v_0)\int_0^{\lambda} v(z)dz\right)d\lambda < \infty.$$

Thus $\rho(t)$ is bounded on $0 \leq t < \infty$ and $x(t)$ lies in a compact subset K of $\mathfrak{M}^n$.

But in K,

$$\nu(\rho(x(t))) > \eta > 0$$

so

$$\lim_{t\to+\infty} v(t) = 0.$$

Therefore, if $K_c \subset K$ is the set of critical points of $\mathcal{S})$ in K and if N is any neighborhood of K_c, then $x(t)$ lies in N for all sufficiently large t. The positive limit set of $x(t)$ thus consists of a subset of K_c and this is non-empty, compact and connected by the general theory of dynamical systems.

At a critical point O of $\mathcal{S})$ use geodesic coordinates with $g_{ij}=\delta_{ij}$. Then $J^i_j=\dfrac{\partial f^i}{\partial x^j}$ is non-singular at O since $J+J^T$ is negative definite. Thus the critical points of $\mathcal{S})$ are isolated. But the solution curves of $\mathcal{S})$ define a continuous map of $\mathfrak{M}^n$ onto the set of critical points of $\mathcal{S})$. Thus $\mathcal{S})$ has just one critical point in $\mathfrak{M}^n$.

Now choose local coordinates around the critical point O so that $\mathcal{S})$ has a negative radial velocity on the coordinate unit sphere centered at O. Each solution curve of $\mathcal{S})$, other than O, intersects this unit sphere in exactly one point. We use this intersection point on the coordinate unit sphere and the t-parameter along the solution curves of $\mathcal{S})$ to map $\mathfrak{M}^n$ onto an open subset of R^n. We map O to the origin of R^n and every other solution of $\mathcal{S})$ is mapped onto a ray in R^n. Thus $\mathfrak{M}^n$ is homeomorphic with an open subset T of R^n which is star-convex from the origin.

Now the radial distance function from the origin to the boundary of T is lower-semi-continuous as defined on the unit sphere in R^n. But a lower-semi-continuous function is the limit of a monotonic increasing sequence of continuous functions. The concentric shells defined by this monotonic sequence of continuous functions can be mapped homeomorphically (with fixed angular coordinates) onto shells between concentric spheres. Thus $\mathfrak{M}^n$ is homeomorphic with all R^n. Q. E. D.

REMARK. The function $\nu(\rho)=\dfrac{\mu}{(1+\rho)^\alpha}$,
for each constant $\mu>0$ and $0\leq\alpha<1$, satisfies the requirements of the theorem.

Corollary. *Let $\mathfrak{M}^n=R^n$ be the number space with the flat metric $g_{ij}=B_{ij}$ where $(B_{ij})=B$ is a constant, positive definite, symmetric matrix. Write*

$$f^i{}_{,j}=\frac{\partial f^i}{\partial x^j}=J^i_j(x)\,,$$

and assume that each eigenvalue of

$$M=J^TB+BJ$$

everywhere satisfies

$\lambda(x) < -\nu(\rho(x))$, $\rho(x)^2 = \sum_{i=1}^n (x^i)^2$, $\nu(\rho)$ *as in the theorem. Then each solution curve of* $\mathcal{S}$) *is bounded, as* $t \to +\infty$, *and approaches the critical point of* $\mathcal{S}$).

Proof. Now

$$f^i{}_{,j} + f_j{}^{,i} = f^i{}_{,j} + B_{lj} f^l{}_{,s} B^{is}.$$

By this is just the matrix

$$J + B^{-1} J^T B = B^{-1}(BJ + J^T B).$$

Since B^{-1} and $(BJ + J^T B)$ are symmetric and B^{-1} is positive definite, these two matrices can be simultaneously diagonalized at each point $x \in R^n$. Thus the eigenvalues of

$$f^i{}_{,j} + f_j{}^{,i}$$

everywhere satisfy

$$\lambda(x) < -\varepsilon\nu(\rho(x)), \quad \text{for some} \quad \varepsilon > 0.$$

Using the Theorem 1, we obtain the corollary. Q. E. D.

In the case where $\nu(\rho)$ is constant we obtain the theorem of Krasovski [3, 4].

We now utilize only the Euclidean metric in R^n and state a result which does not require the computation of the eigenvalues $\lambda_i(x^1, \cdots, x^n)$ of $J(x) + J^T(x)$.

Theorem 2. *Consider*

$$\mathcal{S}) \qquad \dot{x}^i = f^i(x^1, \cdots, x^n) \quad in\ C^1\ in\ R^n.$$

Assume that each eigenvalue of

$$M(x) = J(x) + J^T(x), \quad where \quad J^i_j(x) = \frac{\partial f^i}{\partial x^j},$$

is negative, and assume there exist constant bounds $\beta_1 > 0$, $\beta_2 > 0$ *for*

$$|\operatorname{Trace} M(x)| < \beta_1$$
$$|\det \ M(x)| > \beta_2.$$

Then each solution $x(t)$ *of* $\mathcal{S}$) *is bounded in* R^n, *as* $t \to +\infty$, *and* $x(t)$ *approaches the critical point of* $\mathcal{S}$).

Proof. Now

$$\lambda^n - TrM\lambda^{n-1} + \cdots + (-1)^n \det M = 0.$$

Since each root $\lambda < 0$, we have

$$|\lambda| < \beta_1 .$$

Also

$$|\lambda_1 \lambda_2 \cdots \lambda_n| = |\det M| > \beta_2 > 0 ,$$

and hence

$$|\lambda_1| > \frac{\beta_2}{\beta_1^{n-1}} .$$

Thus each eigenvalue everywhere satisfies

$$\lambda(x) < -\varepsilon , \quad \text{for some constant } \varepsilon > 0 . \quad \text{Q. E. D.}$$

EXAMPLE. Consider the system in R^2,

$$\dot{x} = -2x + \cos y$$
$$\dot{y} = \sin^2 x - y .$$

Compute

$$M = J + J^T = \begin{pmatrix} -4 & 2\sin x \cos x - \sin y \\ 2 \sin x \cos x - \sin y & -2 \end{pmatrix}$$

Since

$$|\operatorname{Trace} M| = -6$$

and

$$|\det M| \geq 4 ,$$

the Hurwitz criterion assures us that the eigenvalues of M are negative. The corollary show that every solution tends towards the unique critical point as $t \to +\infty$.

Theorem 1 and 2 deal with the symmetric matrix $M = J + J^T$. We now turn to the consideration of stability criteria based directly on J. The next example indicates that considerable caution is required here.

EXAMPLE. Consider

$$x = (-1 + (3/2)\cos^2 t)x + (1 - (3/2)\cos t \sin t)y$$
$$y = (-1 - (3/2)\sin t \cos t)x + (-1 + (3/2)\sin^2 t)y .$$

The instantaneous eigenvalues are

$$\lambda_1(t) = -(1/4) + (7/4)\sqrt{-1} \quad \text{and} \quad \lambda_2 = -(1/4) - (7/4)\sqrt{-1}$$

so

$$\operatorname{Re} \lambda_i(t) = -(1/4) \quad \text{for all } t .$$

Yet there is a solution which grows exponentially

$$x = -(\exp(t/2)) \cos t , \quad y = (\exp(t/2)) \sin t .$$

Now consider

$$\dot{z} = -z+w, \quad \dot{w} = -w$$

with a solution

$$z = te^{-t}, \quad w = e^{-t}.$$

Now form the infinite autonomous system

$$\begin{aligned}
\dot{x}_n &= [-1+(3/2)\cos^2 z_n(x_{n+1}^2+y_{n+1}^2)]x_n \\
&\quad +[-1-(3/2)\cos z_n(x_{n+1}^2+y_{n+1}^2)\sin z_n(x_{n+1}^2+y_{n+1}^2)]y_n \\
\dot{y}_n &= [-1-(3/2)\sin z_n(x_{n+1}^2+y_{n+1}^2)\cos z_n(x_{n+1}^2+y_{n+1}^2)]x_n \\
&\quad +[-1+(3/2)\sin^2 z_n(x_{n+1}^2+y_{n+1}^2)]y_n, \\
\dot{z}_n &= -z_n+w_n \\
\dot{w}_n &= -w_n
\end{aligned}$$

for $n=1, 2, 3, \cdots$.

The eigenvalues of this 'infinite Jacobian matrix' all have negative real parts, yet the solutions do not tend towards the unique critical point as $t \to +\infty$. We cannot, as yet, construct such an example using a finite set of equations.

Theorem 3. *Consider*

$$\mathscr{S}) \qquad \begin{aligned} \dot{x} &= f(x, y) \\ \dot{y} &= g(x, y) \end{aligned}$$

with f, g in C^1 in the (x, y)-plane R^2 and assume that the origin is the unique critical point. Assume that

$$J(x, y) = \begin{pmatrix} f_x & f_y \\ g_x & g_y \end{pmatrix}$$

has eigenvalues which have negative real parts everywhere in R^2. Assume that one of the four functions f_x, f_y, g_x, g_y vanishes identically in R^2. Then every solution of $\mathscr{S}$) approaches the origin as $t \to +\infty$.

Proof. We can assume that either $f_x \equiv 0$ or $f_y \equiv 0$, for the other cases reduce to these upon interchanging x and y.

Case 1. $\qquad f_x \equiv 0$

Here

$$\begin{aligned} \dot{x} &= f(y) \\ \dot{y} &= g(x, y) \end{aligned}$$

so

$$J=\begin{pmatrix}0 & f_y \\ g_x & g_y\end{pmatrix}$$

and $g_y<0$, $f_y g_x<0$ everywhere.
By replacing x by $-x$, if necessary, we can assume that

$$f_y>0 \quad \text{and} \quad g_x<0 \quad \text{everywhere.}$$

Thus

$$\begin{aligned} g(0, y)>0 &\quad \text{for} \quad y<0 \\ g(0, y)<0 &\quad \text{for} \quad y>0 \\ g(x, 0)>0 &\quad \text{for} \quad x<0 \\ g(x, 0)<0 &\quad \text{for} \quad x>0 \end{aligned}$$

and

$$\begin{aligned} f(y)>0 &\quad \text{for} \quad y>0 \\ f(y)<0 &\quad \text{for} \quad y<0 . \end{aligned}$$

By the classical Liapounov stability criterion, there is a disc D, centered at the origin, such that each solution of $\mathcal{S})$ which intersects D must tend to the origin as $t\to+\infty$.

Now let $S(t)$ be a solution of $\mathcal{S})$ with coordinates $x(t)$, $y(t)$ for $0\leq t<\tau\leq+\infty$.

Suppose that $S(t)$ does not intersect D.

If $S(t)$ passes through a point P_1 in the first quadrant of the (x, y)-plane, then $\dot{x}(t)>0$ and $\dot{y}(t)<0$ as long as $S(t)$ remains in the first quadrant. Also $|\dot{x}(t)|$ is bounded, on $S(t)$ past P_1 in the first quadrant, and $\dot{y}(t)<-\eta<0$ for some $\eta>0$. Thus $S(t)$ must enter the fourth quadrant.

Note that the inequalities obtained for $g(0, y)$, $g(x, 0)$, and $f(y)$ force $S(t)$ to enter the quadrants 1, 4, 3, 2, 1 cyclically unless $S(t)$ remains eventually in just one quadrant or $S(t)$ is unbounded for $t>0$.

If $S(t)$ passes through a point P_4 in the fourth quadrant, then $\dot{x}(t)<-\eta<0$ as long as $S(t)$ remains in the fourth quadrant. But, past P_4 on $S(t)$ in the fourth quadrant, $\dot{y}(t)$ is bounded from below and so $S(t)$ must enter the third quadrant.

If $S(t)$ passes through a point P_3 in the third quadrant, then $\dot{x}(t)<0$ $\dot{y}(t)>0$ as long as $S(t)$ remains in the third quadrant. But $|\dot{x}(t)|$ is bounded, on $S(t)$ past P_3 in the third quadrant, and $\dot{y}(t)>\eta>0$. So $S(t)$ must enter the second quadrant.

If $S(t)$ passes through a point P_2 in the second quadrant, then $\dot{x}(t)>0$

and $\dot{y}(t)$ is bounded above as long as $S(t)$ remains in the second quadrant. Moreover $\dot{x}(t)>\eta>0$ on $S(t)$ past P_2 in the second quadrant. Thus $S(t)$ must enter the first quadrant.

Therefore each solution $S(t)$ of $\mathcal{S})$, which does not intersect D, encircles D clockwise infinitely many times. Let A_1 and A_2 be the y-intercepts of successive intersections of the spiral $S(t)$ with the positive y-axis.

If $A_1=A_2$, then $S(t)$ is a periodic solution of $\mathcal{S})$. This is impossible, by Bendixson's criterion, since the flow is area-decreasing. If $A_2>A_1$, the area bounded by the spiral loop $\widehat{A_1A_2}$ and the line segment $\overline{A_2A_1}$ is mapped by the flow onto an area which properly includes it. This is impossible since the flow is area-decreasing.

Thus $A_2<A_1$ and hence $S(t)$ spirals towards a limit cycle of $\mathcal{S})$. But $\mathcal{S})$ has no periodic solution (except the origin). Therefore the supposition that $S(t)$ does not intersect D leads to a contradiction and the proof of the theorem is completed in Case 1.

CASE 2. $\qquad f_y\equiv 0\,.$

Here

$$\dot{x} = f(x)$$
$$\dot{y} = g(x, y)$$

so

$$J=\begin{pmatrix} f_x & 0 \\ g_x & g_y \end{pmatrix}$$

and

$$f_x+g_y<0\,, \quad f_xg_y>0\,.$$

Therefore

$$f_x<0 \quad \text{and} \quad g_y<0$$

everywhere.

Again there is a disc D, centered at the origin, such that each solution of $\mathcal{S})$ which intersects D must approach the origin as $t\to+\infty$. Also there is a strip

$$Z:\ |x|<\xi\,,$$

such that a solution of $\mathcal{S})$ which intersects Z must eventually intersect D.

Let $S(t)$ be a solution of $\mathcal{S})$ with coordinates $x(t)$, $y(t)$, for

$$0\leq t<\tau\leq+\infty\,.$$

Suppose $S(t)$ does not intersect $D\cup Z$.

If $S(t)$ passes through a point P_{+} in the right half-plane $x>0$, then $\dot{x}(t)<0$ as long as $S(t)$ remains in the right half-plane. In fact $\dot{x}(t)<-\eta<0$ since $S(t)$ is bounded away from the y-axis. Since $S(t)$ does not intersect Z we must have $y(t)\rightarrow+\infty$ or $y(t)\rightarrow-\infty$ along $S(t)$.

In $0\leq x\leq P_{+}(x)$, $y>0$ we have a finite upper bound for $g(x, y)$, and hence $y(t)$ cannot approach $+\infty$ along $S(t)$. Also for $0\leq x\leq P_{+}(x)$, $y<0$ we have a finite lower bound for $g(x, y)$, and hence $y(t)$ cannot approach $-\infty$ along $S(t)$. Therefore $S(t)$ cannot lie in the right half-plane.

Similary if $S(t)$ passes through a point P_{-} in the left half-plane $x<0$, then $\dot{x}(t)>0$ as long as $S(t)$ remains in the left half-plane. In fact $\dot{x}(t)>\eta>0$ since $S(t)$ is bounded away from the y-axis. Using the same type of bounds on $y(t)$ as above, we see that $y(t)$ remains bounded on $S(t)$. Hence we obtain a contradiction which shows that $S(t)$ must intersect $D\cup Z$.

Therefore the theorem is proved in Case 2. Q. E. D.

Corollary. *Consider*

$$\ddot{x}+g(x, \dot{x}) = 0$$

with $g(x, y)$ *in* C^1 *in* R^2. *Assume*

$$g(0, 0) = 0\,, \quad g_x>0\,, \quad g_y>0\,.$$

Then each solution $x(t)$ *is defined for all large* $t>0$ *and*

$$\lim_{t\to+\infty} x(t) = 0\,, \quad \lim_{t\to+\infty} \dot{x}(t) = 0\,.$$

Theorem 4. *Consider*

$$\mathcal{S})\qquad \dot{x}^i = f^i(x^1, \cdots, x^n) \quad i=1, 2, \cdots, n$$

with $f(x)$ *in* C^1 *in* R^n. *Assume*

1) $f(x) = 0$ *if and only if* $x=0$

and

2) $\dfrac{\partial f^i}{\partial x^j}(x) = 0$ *for* $j<i$

$\dfrac{\partial f^i}{\partial x^i}(x)<0$ *for each* $i=1, 2, \cdots, n$, *everywhere in* R^n.

Then each solution of $\mathcal{S}$) *is defined for all large* t *and tends to the origin as* $t\rightarrow+\infty$.

Proof. The theorem is trivial if $n=1$ and holds by Theorem 3 when $n=2$. Now we proceed by induction to prove the theorem in the general case.

Suppose the theorem holds for differential systems satisfying the hypotheses in R^{n-1}.

Consider

$$\mathcal{S})\qquad \begin{aligned} \dot{x}^1 &= f^1(x^1, x^2, \cdots, x^n) \\ \dot{x}^2 &= f^2(x^2, x^3, \cdots, x^n) \\ &\vdots \\ \dot{x}^n &= f^n(x^n) \end{aligned}$$

which satisfies the hypotheses of the theorem in R^n.

If $(x_0^1, x_0^2, \cdots, x_0^n)$ is a point in R^n at which

$$\begin{aligned} f^2(x_0^2, x_0^3, \cdots, x_0^n) &= 0 \\ &\vdots \\ f^n(x_0^n) &= 0\,, \end{aligned}$$

then $x_0^n=0$. Hence

$$f^{n-1}(x_0^{n-1}, 0) = 0$$

and since

$$f^{n-1}(0, 0) = 0 \quad\text{and}\quad \frac{\partial f^{n-1}}{\partial x^{n-1}} < 0$$

we have $x_0^{n-1}=0$. Similarly

$$x_0^{n-2} = 0\,,\quad x_0^{n-3} = 0\,,\quad \cdots,\quad x_0^2 = 0\,,\quad x_0^1 = 0\,.$$

Thus the last $(n-1)$ equations of $\mathcal{S})$ form a system

$$\mathcal{S}')\qquad \begin{aligned} \dot{x}^2 &= f^2(x^2, x^3, \cdots, x^n) \\ &\vdots \\ \dot{x}^n &= f^n(x^n) \end{aligned}$$

which satisfies the hypotheses of the theorem in the R^{n-1} space $x^1=0$.

Let $S(t)$ with coordinates $x^1(t)$, $x^2(t)$, $\cdots$, $x^n(t)$ on $0 \leq t < \tau \leq +\infty$ be a solution of $\mathcal{S})$ in R^n. Then $x^2(t)$, $x^3(t)$, $\cdots$, $x^n(t)$ form a solution of $\mathcal{S}')$ and so can be extended over $0 \leq t < \infty$.

Moreover

$$|x^2(t)|^2 + |x^3(t)|^2 + \cdots + |x^n(t)|^2 = \rho(t)^2$$

is bounded on $0 \leq t < \infty$ and

$$\lim_{t\to+\infty} x^2(t) = 0\,, \quad \lim_{t\to+\infty} x^3(t) = 0,\ \cdots,$$

and

$$\lim_{t\to+\infty} x^n(t) = 0$$

by the induction hypothesis.

Let K be a compact subset of the R^{n-1} space $x^1=0$ which contains the curve

$$x^2(t),\ x^3(t),\ \cdots,\ x^n(t) \quad \text{for} \quad 0\leq t<\infty\,.$$

Since

$$|f^1(0,\ x^2,\ \cdots,\ x^n)|$$

is bounded in K and since $\dfrac{\partial f^1}{\partial x^1}<0$ in R^n, we find that $x^1(t)$ can be extended over $0\leq t<\infty$, so that solution $S(t)$ of $\mathcal{S})$ exists on $0\leq t<\infty$.

Now there is a ball $\mathcal{B}$, centered at the origin of R^n, such that $S(t)$ approaches the origin if $S(t)$ intersects $\mathcal{B}$. Moreover there is a tube in R^n

$$T:\quad \rho(t)<\rho_0$$

such that $S(t)$ intersects $\mathcal{B}$ if $S(t)$ intersects T.

But

$$\lim_{t\to+\infty} x^2(t) = \lim_{t\to+\infty} x^3(t) = \cdots = \lim_{t\to+\infty} x^n(t) = 0\,.$$

Hence $S(t)$ must intersect T. Therefore

$$\lim_{t\to+\infty} x^1(t) = 0$$

and $S(t)$ approaches the origin of R^n as $t\to+\infty$. Q. E. D.

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