$C_{r,p}$-CAPACITY ASSOCIATED WITH HUNT PROCESSES

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1. Introduction

After the $C_{r,p}$-capacity associated with a general Markov semigroup $(T_t)_{t>0}$ on L^p-space was introduced by Fukushima-Kaneko [3], Kaneko [5] constructed a Hunt process associated with $(T_t)_{t>0}$ uniquely up to $C_{r,p}$-equivalence in the case that the underlying space is locally compact and the semigroup is analytic. Fukushima [2] extended Kaneko's result to the non-locally compact case. In this paper we release the restriction of the analyticity of the semigroup. The difficulty to deal with this case is that the transition kernel of process can not be directly constructed from semigroup since $T_t f$ has no quasi-continuous m-version for general $f \in L^p(E; m)$. To overcome this difficulty, we first compactify the underlying space and then use the weak convergence of probability measures on compact space to construct a transition kernel. Similar to [5] and [2] we construct a Hunt process on the compactified space and then use the tightness property of $C_{r,p}$ capacity to restrict the process back to the original space. Therefore in the case $1 \in \mathcal{F}_{r,p}$, we improve the corresponding result of [2]. Furthermore, we mention that the result can be applied to the case in [6].

Let E be a Hausdorff topological space and m be a σ-finite measure on the Borel σ-field $\mathcal{B}(E)$. Let $(T_t)_{t>0}$ be a strongly continuous contraction semigroup on $L^p(E; m)$, which is Markovian, i.e. $0 \le u \le 1$, m-a.e. implies $0 \le T_t u \le 1$, m-a.e.

Let $(V_r)_{r>0}$ be the Γ transformation of $(T_t)_{t>0}$:

$$V_r := \frac{1}{\Gamma(r/2)} \int_0^{+\infty} t^{r/2-1} e^{-t} T_t \, dt$$

and $\mathcal{F}_{r,p} = V_r(L^p)$, $\|u\|_{r,p} = \|f\|_{L^p}$ for $u = V_r f$, $f \in L^p$. It is well known that $(\mathcal{F}_{r,p}, \|\cdot\|)$ is a Banach space. For $r > 0$, $\alpha > 0$, put

$$V_r^\alpha := \frac{1}{\Gamma(r/2)} \int_0^{+\infty} t^{r/2-1} e^{-\alpha t} T_t \, dt.$$

Then $\alpha^{r/2} V_r^\alpha$ is a Markovian contraction operator on $L^p(E; m)$, $\alpha^{r/2} V_r^\alpha \to I$ strongly as $\alpha \to \infty$ and the range of V_r^α equals $\mathcal{F}_{r,p}$ ([6]).

For open subset $U \subset E$, the $C_{r,p}$-capacity is defined by

$$C_{r,p}(U) := \inf\{\|u\|_{r,p}^p : u \in \mathcal{F}_{r,p}, u \geq 1, m\text{-}a.e. \text{ on } U\}$$

and for an arbitrary set $B \subset X$

$$\text{Cap}_{r,p}(B) := \inf\{C_{r,p}(U) : U \text{ open and } U \subset B\}.$$

"$C_{r,p}$-quasi-everywhere" or briefly "$C_{r,p}$-q.e." means that a statement holds except on a $C_{r,p}$-capacity zero set. An increasing sequence of closed sets $\{E_k\}_{k \geq 1}$ is called a $C_{r,p}$-nest if $C_{r,p}(E_k^c) \downarrow 0$ as $k \to \infty$. A function u is called $C_{r,p}$-quasi-continuous if there exists a $C_{r,p}$-nest $\{E_k\}_{k \geq 1}$ such that the restriction of f to E_k is continuous. For a fixed $C_{r,p}$-nest $\{E_k\}_{k \geq 1}$, let $C(\{E_k\}) := \{f : f|_{E_k} \text{ is continuous on } E_k \text{ for } k \geq 1\}$. We denote by $\tilde{u}$ the $C_{r,p}$-quasi-continuous m-version of u. A set of functions D is said to separate the points of E if for any $x, y \in E$, $x \neq y$, there exist $u \in D$ such that $u(x) \neq u(y)$.

In this paper, we make the following basic assumptions:

(A.1) The capacity is tight: there exist increasing compact subsets E_k ($k = 1, \dots$) of E satisfying $C_{r,p}(E_k^c) \downarrow 0$ as $n \to \infty$.

(A.2) There exists a dense subset $\mathcal{F}_{r,p}^0 \subset \mathcal{F}_{r,p}$ such that each element of $\mathcal{F}_{r,p}^0$ has a $C_{r,p}$-quasi-continuous m-version.

(A.3) There exists a countable Q-algebra $D \subset \mathcal{F}_{r,p}^0$ such that $\tilde{D}$ separates points of $E \setminus N$ for some $C_{r,p}$-exceptional set N and $1 \in D$, where $\tilde{D} := \{\tilde{u} : u \in D\}$.

For an operator T on $L^p(E; m)$, $TD := \{Tu : u \in D\}$ $\widetilde{TD} := \{\widetilde{Tu} : u \in D\}$. Define $\tilde{\mathcal{F}}_{r,p}$ as $\{\tilde{u} : u \in \mathcal{F}_{r,p}\}$. The symbol "$\xrightarrow{w}$" means weak convergence of probability measures. A sequence of functions $\{u_n\}_{n \geq 1}$ is called $C_{r,p}$-quasi-uniformly convergence if there exists $u \subset \tilde{\mathcal{F}}_{r,p}$ and a $C_{r,p}$-nest $\{E_k\}_{k \geq 1}$ such that u_n converges to u uniformly on each E_k.

The main result is

Theorem. *For $r \geq 2$, $p > 1$, there exists a Hunt process $M = (\Omega, \mathcal{F}, \mathcal{F}_t, Q_t, X_t, P_z)_{z \in E}$ such that $Q_t f$ is a $C_{r,p}$-quasi-continuous m-version of $T_t f$ for bounded $f \in \mathcal{F}_{r,p}$.*

2. Preparation and a lemma

With the basic assumptions as in Section 1, similarly to the case of [7, III.3], [3], [6] we can show the following properties:

(a)   If u is a $C_{r,p}$-quasi-continuous function and $u \geq 0$ m-a.e. then $u \geq 0$ $C_{r,p}$-q.e.

(b)   For $u \in \mathcal{F}_{r,p}$, there exists $C_{r,p}$-quasi-continuous modification $\tilde{u}$ of u satisfying

$$C_{r,p}(|\tilde{u}| > \lambda) \leq \lambda^{-p} \|u\|_{r,p}^p, \quad \lambda > 0.$$

(c)   The convergence of $C_{r,p}$-quasi-continuous functions in $\mathcal{F}_{r,p}$ implies the $C_{r,p}$-quasi-uniform convergence of some subsequence.

Similar to [7, IV.3], we can select a $C_{r,p}$-nest $\{E_k\}_{k \geq 1}$ of compact metrizable subsets of E such that $Y := \cup_{k \geq 1} E_k$ (Y is a topological Lusin space) and

(i) $\tilde{D}$ separates points of Y,

(ii) $\tilde{D}, \tilde{T}_t D, \tilde{V}_r^\alpha D \subset C(\{E_k\})$ for $\alpha, r, t \in Q^+$,

(iii) $\forall u \in \tilde{D}, \tilde{T}_t u \xrightarrow[t \in Q, t \downarrow 0]{} u$ on Y.

For simplicity, we assume that each element in D is $C_{r,p}$-quasi-continuous. As in [7, IV] we may identify $L^p(E; m)$ with $L^p(Y; m)$ canonically.

Let $D := \{u_i : i \in N\}$. Define

$$d(x, y) := \sum_{k \geq 1}^{\infty} \frac{1}{2^n} |\arctan u_n(x) - \arctan u_n(y)|, \quad x, y \in Y.$$

Since D separates the points of Y, d is a metric on Y. Define $\bar{E} :=$ completion of Y, with respect to the metric d. Let $\bar{d}$ be the natural extension of d. Then similar to [7, IV, Lemma 3.13], we have

(iv) $(\bar{E}, \bar{d})$ is a compact metric space,

(v) $d\text{-}\mathcal{B}(Y) = \mathcal{B}(Y)$,

(vi) $Y \in d\text{-}\mathcal{B}(\bar{E})$,

(vii) If $F \subset E_k$ for some k, and F is a compact subset of Y, then F is the d-compact set and d-topology induced on F coincides with the topology on F inherited from E.

Let $i : Y \to \bar{E}$ be the inclusion map. Set $\bar{m} := m \cdot i^{-1}$. Using the methods of [7, VI], we can introduce an isometry $\bar{i} : L^p(\bar{E}; \bar{m}) \to L^p(Y; m)$.

Define for $\bar{f} \in L^p(\bar{E}; \bar{m})$

$$\bar{T}_t \bar{f} := \bar{i}^{-1}(T_t(\bar{f} \cdot i)).$$

It is easy to show that $(\bar{T}_t)_{t>0}$ is a strongly continuous contraction semigroup on $L^p(\bar{E}; \bar{m})$. Thus we can introduce $\bar{\mathcal{F}}_{r,p}, \bar{V}_{r,p}$ on $L^p(\bar{E}; \bar{m})$ of $(\bar{T}_t)_{t>0}$. It is easy to show that

(viii) $\bar{D} := \{\bar{u} : u \in D\} \subset \bar{\mathcal{F}}_{r,p}$, where $\bar{u}$ is a continuous extension to $\bar{E}$ of u with respect ot the metric $\bar{d}$,

(ix) $\bar{D}$ separates points of $\bar{E}$,

(x) $\bar{D}$ is a Q-algebra.

In fact, for $\bar{u} \in \bar{D}, u \in D$, there exists $w \in L^p(E; m)$ such that $u = V_{r,p} w$. We trivially extend w to $\bar{E}$ and denote it by $\bar{w}$, then $w = \bar{w} \circ i$ $m-a.e.$ on E. Since $\bar{u} = u$ $m-a.e.$ on E, $V_{r,p} \bar{w} \circ i = \bar{V}_{r,p} \bar{w}$ $\bar{m}-a.e.$ on $\bar{E}$. Thus

$$\bar{u} \xrightarrow{m-a.e. \text{ on } E} u = V_{r,p} w \xrightarrow{m-a.e. \text{ on } E} V_{r,p} \bar{w} \circ i = \bar{i}(\bar{V}_{r,p} \bar{w}) \xrightarrow{\bar{m}-a.e. \text{ on } \bar{E}} \bar{V}_{r,p} \bar{w}.$$

Therefore there exists $\bar{w} \in L^p(\bar{E}; \bar{m})$ such that $\bar{u} = \bar{V}_{r,p}\bar{w}$, i.e. $\bar{u} \in \bar{\mathcal{F}}_{r,p}$. The basic assumption (A.3) implies that $\bar{D}$ is a Q-algebra and D separates the points of Y implies $\bar{D}$ separates the points of $\bar{E}$.

Similar to [7, VI], the tightness of $C_{r,p}$-capacity implies that $i : E \to \bar{E}$ is quasi-homeomorphic. Moreover, the following properties hold:

(xi)   If $(F_k)_{k \in N}$ is a $\bar{C}_{r,p}$-nest, then $(F_k \cap E_k)_{k \in N}$ is a $C_{r,p}$-nest,

(xii) $\bar{N} \subset \bar{E}$ is $\bar{C}_{r,p}$-exceptional if and only if $\bar{N} \cap E$ is $C_{r,p}$-exceptional. In particular,
$\bar{C}_{r,p}(\bar{E} \setminus E) = 0$,

(xiii) A function $\bar{u} : \bar{E} \to R$ is $\bar{C}_{r,p}$-quasi-continuous if and only if $\bar{u} \circ i^{-1}$ is $C_{r,p}$-quasi-continuous,

(xiv) $\bar{C}_{r,p}(\bar{A}) = C_{r,p}(A \cap E)$ for $\bar{A} \subset \bar{E}$.

Furthermore, the basic assumptions (A.1), (A.2) and (A.3) hold with respect to $\bar{C}_{r,p}$-capacity.

From now on, we restrict the discussion on $(\bar{E}, \bar{d})$. We first construct a kernel semigroup $\bar{P}_t$ on $(\bar{E}, \bar{d})$ which is a m-version of $\bar{T}_t$. Similar to [5] and [4] we construct a Hunt process $(\bar{X}_t)_{t>0}$ with state space $\bar{E}$, then we show that the process $(\bar{X}_t)_{t>0}$ can be restricted to E.

Denote $Q^+(Q)$ be the (positive) rational number and R^+ be the positive real number.

Lemma. *Let* $r > 2$, $t \in Q^+$ *and* $p > 1$. *There exists a Markovian transition kernel* $\bar{P}_t$ *on* $(\bar{E}_\Delta, \mathcal{B}(\bar{E}_\Delta))$ *(where* $\bar{E}_\Delta := E \cup \{\Delta\}$, Δ *is an isolated point of* $\bar{E}$*) satisfying the following properties*:

(A) $\bar{P}_t \bar{f}$ *is a* $\bar{C}_{r,p}$*-quasi-continuous* m*-version of* $\bar{T}_t \bar{f}$ *for* $\bar{f} \in \bar{D}$,

(B) $0 \le \bar{P}_t(z, \bar{E}) \le 1$ *for all* $z \in \bar{E}$.

If there exists another kernel $(\bar{Q}_t)_{t \in Q^+}$ *satisfying* (A) *and* (B)*, then* $\bar{Q}_t(x, \cdot) = \bar{P}_t(x, \cdot)$, $\bar{C}_{r,p}$*-q.e. on* $\bar{E}$.

Proof.   For $t, \alpha \in Q^+$, $u \in D$, by using the methods as in [4, Lemma 7.2], we can choose a $C_{r,p}$-nest $(F_k)_{k \ge 1}$ consisting of compact metrizable sets of E such that $\bar{T}_t \bar{V}_r^\alpha \bar{u} \in C(\{F_k\})$, $\widetilde{\bar{T}}_t \bar{u} \in C(\{F_k\})$, and for each $x \in Y' := \cup_{k=1}^\infty F_k$, there exists a unique positive linear functional L_x on $C(\bar{E})$ satisfying:

$$L_x^\alpha(\bar{u}) = \alpha^{r/2} \widetilde{\bar{T}}_t \bar{V}_r^\alpha \bar{u}(x), \quad \forall \bar{u} \in \bar{D},$$

$$|L_x^\alpha(\bar{u})| \le \|\bar{u}\|_\infty, \quad \forall x \in Y'.$$

Thus L_x^α admits a positive measure $\widetilde{P}_t^\alpha(x, \cdot)$ on $\bar{E}$ satisfying $L_x^\alpha(\bar{u}) = \widetilde{P}_t^\alpha \bar{u}(x)$, $0 \le \widetilde{P}_t^\alpha(x, \bar{E}) \le 1$, $x \in Y'$.

Denote $\mathcal{B}(\bar{E}_\Delta) = \sigma(\mathcal{B}(\bar{E}), \{\Delta\})$. We extend $(\widetilde{P}_t^\alpha)_{t \in Q_+}$ to $(\bar{E}_\Delta, \mathcal{B}(\bar{E}_\Delta))$ by putting

$$\widetilde{\widetilde{P}}_t^\alpha(x, A) := \begin{cases} \widetilde{P}_t^\alpha(x, A \setminus \Delta) + (1 - \widetilde{P}_t^\alpha(x, \bar{E}))\delta_\Delta(A), & x \in \bar{E}, \\ \delta_\Delta(A) & x = \Delta, \end{cases}$$

where $t \in Q^+$, $A \in \mathcal{B}(\bar{E}_\Delta)$. For simplicity, we still denote $\overset{\approx\alpha}{P_t}$ by $\widetilde{P}_t^\alpha$.

Note that $(\bar{E}_\Delta, \bar{d})$ is a compact metric space, $\{\widetilde{P}_t^\alpha(x, \cdot)\}_{\alpha \in Q^+}$ is tight. Since each E_k is a compact metrizable space, there exists a countable dense subset $\{x_n^k\}_{n \geq 1} \subset E_k$. Let $S := \cup_{n > 1} \cup_{k \geq 1} \{x_n^k\}$, then S is the dense subset of Y'. By the usual diagonal argument, we can select a sequence of positive numbers $\{\alpha_n\}_{n \geq 1}$ which tend to infinity as $n \to \infty$ such that for any $t \in Q$, $x \in S \subset Y'$

$$\widetilde{P}_t^{\alpha_n}(x, \cdot) \overset{w}{\to} \widetilde{P}_t(x, \cdot),$$

where S is a countable dense subset of (Y', d) and $\widetilde{P}_t(x, \cdot)$ is a probability measure on $(\bar{E}_\Delta, \mathcal{B}(\bar{E}_\Delta))$.

Since $\bar{D} \subset C(\bar{E})$, for $x \in S$, $\bar{u} \in \bar{D}$

$$\lim_{n \to \infty} \widetilde{P}_t^{\alpha_n} \bar{u}(x) = \widetilde{P}_t \bar{u}(x),$$

therefore for any $x \in S \subset Y'$, $\widetilde{P}_t \bar{u}(x) = \overset{\approx}{T}_t \bar{u}(x)$.

For any $x \in Y'$, there exists $\{x_n\}_{n \geq 1} \subset S$ such that x_n, $x \in E_k$ for some k and $x_n \to x$ as $n \to \infty$, so we have

$$\lim_{n \to \infty} \overset{\approx}{T}_t \bar{u}(x_n) = \overset{\approx}{T}_t \bar{u}(x).$$

Since $\{\widetilde{P}_t(x_n, \cdot)\}_{n \geq 1}$ is tight on $(\bar{E}_\Delta, \bar{d})$, there exists a subsequence $\{x_{n_k}\}_{k \geq 1}$ and a probability measure $\widetilde{P}_t(x, \cdot)$ on $(\bar{E}_\Delta, \mathcal{B}(\bar{E}_\Delta))$ such that

$$\widetilde{P}_t(x_{n_k}, \cdot) \overset{w}{\to} \widetilde{P}_t(x, \cdot) \quad (k \to \infty).$$

Thus for any $x \in Y'$, $\bar{u} \in \bar{D}$

(1) $$\overset{\approx}{T}_t \bar{u}(x) = \widetilde{P}_t \bar{u}(x).$$

Since $\bar{D}$ separates points of $\bar{E}$, $\mathcal{B}(\bar{E}) = \sigma(\bar{D})$ by [8, Proposition 1.4] and $\widetilde{P}_t(\cdot, A)$ is $\mathcal{B}(\bar{E}) \cap Y$-measurable for $A \in \mathcal{B}(\bar{E})$ by monotone class theorem. Thus $\widetilde{P}_t(\cdot, \cdot)$ is a probability kernel from $(Y', \mathcal{B}(\bar{E}) \cap Y')$ to $(\bar{E}, \mathcal{B}(\bar{E}))$.

In order to prove the semigroup property, we need to extend (1) to all quasi-continuous function $\bar{u} \in \bar{\mathcal{F}}_{r,p}$. To this end we first show the following

(2) $$\widetilde{P}_t(x, N) = 0 \quad q.e. \text{ on } \bar{E}.$$

Fix an $\bar{N} \in \mathcal{B}(\bar{E})$ with $\bar{C}_{r,p}(\bar{N}) = 0$, we can choose a decreasing sequence of open subsets $\bar{U}_n$ of $\bar{E}$ such that $\bar{U}_n \supset \bar{N}$ and $\bar{C}_{r,p}(\bar{U}_n) \downarrow 0$. For such $\bar{U}_n$, by the property of $\bar{D}$ and $\bar{E}$, using Urysohn lemma and Stone-Weierstrass theorem we can find $\bar{g}_n \in \bar{D}$ satisfying $0 < \bar{g}_n < 1/n$ on $\bar{E} \setminus \bar{U}_{n-1}$ and $\bar{g}_n = 1$ on $\bar{U}_n$. Then for any $x \in Y'$

(3) $$\widetilde{P}_t(x, N) = \int_{\bar{E}} I_{\bar{N}} \widetilde{P}_t(x, dy)$$

$$\leq \int_{\bar{E}} \bar{g}_n(x) \widetilde{P}_t(x, dy)$$
$$= \widetilde{T}_t \bar{g}_n(x).$$

Because for $n \geq 2$, $\bar{g}_n(x) \leq (1/n)I_{\bar{E} \setminus U_{n-1}} + e_{U_{n-1}}$ m-a.e. and $e_{U_{n-1}}$ is 1-excessive, we have $\bar{T}_t \bar{g}_n(x) \leq 1/n + \tilde{e}_{U_{n-1}}$ m-a.e. This implies that

$$(4) \qquad\qquad \widetilde{T}_t \bar{g}_n(x) \leq \frac{1}{n} + e_{\bar{U}_{n-1}} \quad q.e. \text{ on } \bar{E}.$$

Letting n tends to infinity, by (3) we have (2).

Fix a bounded function $\bar{u} \in \bar{\mathcal{F}}_{r,p}$, we choose a $C_{r,p}$-nest $\{E_k\}_{k \geq 1}$ consisting of compact sets of $\bar{E}$ such that $\bar{u} \in C(\{E_k\})$. Since $\bar{D}$ is a Q-algebra, $\bar{D}$ separates points of $\cup_{k \geq 1} E_k$ for any fixed E_k, by Stone-Weierstrass theorem and the usual diagonal argument, we can select a sequence of functions $\{\bar{u}_n\}_{n \geq 1} \subset \bar{D}$ such that $\forall \varepsilon > 0$, $\exists N(\varepsilon, E_k)$, $\forall n > N(\varepsilon, E_k)$

$$\sup_{x \in E_k} |\bar{u}_n(x) - \bar{u}(x)| < \frac{\varepsilon}{2}.$$

Set $M_n := \sup_{x \in \bar{E}} |\bar{u}_n(x) - \bar{u}(x)|$. Then

$$\begin{aligned} T_t |\bar{u}_n - \bar{u}| &= T_t |\bar{u}_n - \bar{u}| I_{E_k} + T_t |\bar{u}_n - \bar{u}| I_{E_k^c} \\ &\leq \varepsilon/2 + M_n T_t e_{E_k^c} \\ &\leq \varepsilon/2 + M_n e_{E_k^c}(x) \quad \bar{m}\text{-a.e.} \end{aligned}$$

Therefore $1 \wedge (\widetilde{T}_t |\bar{u}_n - \bar{u}|) \leq \varepsilon/2 + (M_n \tilde{e}_{E_k^c}) \wedge 1$ q.e. on $\bar{E}$. Since $\tilde{e}_{E_k^c} \downarrow 0$ uniformly q.e. on $\bar{E}$, by (3) of section 2,

$$(5) \qquad\qquad \lim_{n \to \infty} \widetilde{T}_t \bar{u}_n = \widetilde{T}_t \bar{u} \quad q.e. \text{ on } \bar{E}.$$

From (2) we have $\widetilde{P}_t(x, E_k^c) \to \widetilde{P}_t(x, (\cup E_k^c)) = 0$ q.e. on $\bar{E}$. Similarly we can get

$$(6) \qquad\qquad \lim_{n \to \infty} \widetilde{P}_t \bar{u}_n = \widetilde{P}_t \bar{u} \quad q.e. \text{ on } \bar{E}.$$

From (1), (5) and (6), we conclude that for any bounded $\bar{C}_{r,p}$-quasi-continuous function $\bar{u} \in \bar{\mathcal{F}}_{r,p}$, $t \in Q^+$,

$$(7) \qquad\qquad \widetilde{T}_t \bar{u} = \widetilde{P}_t \bar{u} \quad q.e. \text{ on } \bar{E}.$$

Thus for any $\bar{u} \in \bar{D}$, it holds that

$$(8) \qquad\qquad \widetilde{P}_{t+s} \bar{u}(x) = \widetilde{T}_{t+s} \bar{u}(x)$$

$$= \widetilde{\overline{T}}_t \widetilde{\overline{T}}_s \bar{u}(x)$$
$$= \widetilde{\overline{T}}_t \widetilde{\overline{P}}_s \bar{u}(x)$$
$$= \widetilde{\overline{P}}_t \widetilde{\overline{P}}_s \bar{u}(x), \quad q.e. \text{ on } \bar{E}$$

since $\widetilde{\overline{P}}_t \bar{u}(x) = \widetilde{\overline{T}}_t \bar{u}(x) \in \bar{\mathcal{F}}_{r,p}$ and $\widetilde{\overline{T}}_t \bar{u}(x)$ is quasi-continuous.

From (2), similar to [7, IV lemma 3.11], we can find $Y_1 \in \mathcal{B}(\bar{E})$ such that $\bar{E} \setminus Y_1$ is $\bar{C}_{r,p}$-exceptional and

$$\widetilde{\overline{P}}_t(x, \bar{E} \setminus Y_1) = 0 \quad \forall x \in Y_1, \ t \in Q^+.$$

For $t \in Q^+$, $A \in \mathcal{B}(\bar{E})$, set

$$\widetilde{\widetilde{\overline{P}}}_t(x, A) := \begin{cases} \widetilde{\overline{P}}_t(x, A), & x \in Y_1, \\ \delta_x(A), & x \in \bar{E} \setminus Y_1. \end{cases}$$

Thus $(\widetilde{\widetilde{\overline{P}}}_t)_{t \in Q^+}$ is a Markovian transition kernel on $(\bar{E}_\Delta, \mathcal{B}(\bar{E}_\Delta))$. Then $(\bar{P}_t)_{t \in Q^+}$ satisfies the properties(A), (B) of the lemma.

Assume $(\bar{Q}_t)_{t \in Q^+}$ is another kernel satisfying (A) and (B). Then for any $\bar{u} \in \bar{D}$, $t \in Q^+$

$$\bar{P}_t \bar{u}(x) = \bar{Q}_t \bar{u}(x), \quad q.e. \text{ on } \bar{E}.$$

Since $1 \in \bar{D}$, and $\bar{D}$ is a Q-algebra, for any $\bar{u} \in \mathcal{B}(\bar{E})$,

$$\bar{P}_t \bar{u}(x) = \bar{Q}_t \bar{u}(x), \quad q.e. \text{ on } \bar{E}$$

by monotone class theorem. This implies that $\bar{P}_t(x, \cdot) = \bar{Q}_t(x, \cdot)$ q.e. on $\bar{E}$. $\square$

3. Proof of Theorem

From the above section we know that $(\bar{P}_t)_{t \in Q^+}$ is a Markovian semigroup on $(\bar{E}_\Delta, \mathcal{B}(\bar{E}_\Delta))$ satisfying

(a) $\bar{D} \subset C(\bar{E})$, $\bar{D}$ separates the points of $\bar{E}$ and $\bar{D}$ is a Q-algebra.

(b) For $\bar{f} \in \bar{D}$, $\bar{P}_t \bar{f}(x) \xrightarrow[t \in Q^+, t \downarrow 0]{} \bar{f}(x)$ $\bar{C}_{r,p}$-$q.e.$ $x \in \bar{E}$.

Similar to [5] and [4] we can construct a Hunt process $\bar{M} = (\bar{\Omega}, \bar{\mathcal{F}}, \bar{\mathcal{F}}_t, Q_t, \bar{X}_t, \bar{P}_x)_{x \in \bar{E}_\Delta}$ with state space $(\bar{E}_\Delta, \mathcal{B}(\bar{E}_\Delta))$ and time parameter R^+ such that the transition function $\bar{Q}_t \bar{u} = \bar{P}_t \bar{u}$, $t \in Q^+$.

Let $(E_k)_{k \geq 1}$ be a $C_{r,p}$-nest consisting of compact sets of E, then $\bar{E} \setminus E_k$ is an open set in $(\bar{E}, \bar{d})$. For any $\bar{E} \setminus E_k$, there exists $e_{\bar{E} \setminus E_k} \in \bar{\mathcal{F}}_{r,p}$ such that its $\bar{C}_{r,p}$-quasi-continuous m-version $\tilde{e}_{\bar{E} \setminus E_k} \geq 1$ $\bar{C}_{r,p}$-q.e. and by (7) $e^{-t} \bar{E}_x \tilde{e}_{\bar{E} \setminus E_k}(\bar{X}_t) \leq \tilde{e}_{\bar{E} \setminus E_k}(x)$ $\bar{C}_{r,p}$-q.e. on $\bar{E}$, where $\bar{E}$ is the expectation with respect to $\bar{P}$. Thus there exists a $\bar{C}_{r,p}$-exceptional set $\bar{N}_1 \subset \mathcal{B}(E)$ such that for any $x \in \bar{N}_1^c$

$$\tilde{e}_{\bar{E} \setminus E_k}(x) \geq 1, \quad x \in \bar{E} \setminus E_k, \ k \geq 1,$$

and

$$e^{-t}\bar{E}_x\tilde{e}_{\bar{E}\backslash E_k}(\bar{X}_t) \leq \tilde{e}_{\bar{E}\backslash E_k}(x), \quad t \in Q^+, \ k \geq 1.$$

By (2) and the definition of $\bar{P}_x$, $\bar{P}_x(\bar{X}_t \in \bar{N}_1) = \bar{P}_t(x, \bar{N}_1) = 0$ $\bar{C}_{r,p}$-q.e. on $\bar{E}$, we conclude that

$$\bar{P}_x(\bar{X}_t \in \bar{N}_1 \text{ for some } t \in Q^+) = 0 \quad \bar{C}_{r,p}\text{-}q.e. \text{ on } \bar{E},$$

therefore there exist a $\bar{C}_{r,p}$-exceptional set $\bar{N}_2 \subset \mathcal{B}(\bar{E})$ such that for any $x \in \bar{N}_2^c$

$$\bar{P}_x(\bar{X}_t \in \bar{N}_1 \text{ for some } t \in Q^+) = 0.$$

For $\bar{N}_1 \cup \bar{N}_2$, then there exists a $\bar{C}_{r,p}$-exceptional set $\bar{N}_3 \in \mathcal{B}(\bar{E})$ such that for any $x \in \bar{N}_3^c$

$$\bar{P}_x(\bar{X}_t \in \bar{N}_1 \cup \bar{N}_2 \quad \text{for some } t \in Q^+) = 0.$$

Repeating this argument we obtain an increasing sequence $(\bar{N}_n)_{n\geq 1}$ in $\mathcal{B}(\bar{E})$ such that $\bar{E} \backslash \bar{N}_n$ is $\bar{C}_{r,p}$-exceptional and for any $x \in \bar{N}_{n+1}^c$

$$\bar{P}_x(\bar{X}_t \in \bar{N}_1 \cup \cdots \cup \bar{N}_n \text{ for some } t \in Q^+) = 0.$$

Set $\bar{N} = \cup_{n\geq 1}\bar{N}_n$, then for $x \in \bar{N}^c$

(9) $$\tilde{e}_{\bar{E}\backslash E_k}(x) \geq 1 \quad x \in \bar{E} \backslash E_k, \quad k \geq 1,$$

(10) $$e^{-t}\bar{E}_x\tilde{e}_{\bar{E}\backslash E_k}(\bar{X}_t) \leq \tilde{e}_{\bar{E}\backslash E_k}(x), \quad t \in Q^+, \ k \geq 1,$$

(11) $$\bar{P}_x(\bar{X}_t \in N, \text{ for some } t \in Q^+) = 0.$$

Let $Y_t = e^{-t}\tilde{e}_{\bar{E}\backslash E_k}(\bar{X}_t)$. It is easy to check that $(Y_t)_{t\in Q^+}$ is a $\bar{P}_x$ supermartingale for $x \in \bar{N}^c$.

Let $I := \{t_i : i \in N\}$ be a dense subset of Q^+ and define for $n \in N$

$$\sigma_{\bar{E}\backslash E_k}^n := \inf\{t_i : 1 \leq i \leq n \text{ and } \bar{X}_{t_i} \in \bar{E} \backslash E_k\}$$

(where as usual we set $\inf \emptyset = \infty$). Then by (9)-(11) and the optional sampling theorem for any $x \in \bar{N}^c$,

$$\bar{E}_x e^{-\sigma_{\bar{E}\backslash E_k}^n} \leq \bar{E}_x e^{-\sigma_{\bar{E}\backslash E_k}^n} e_{\bar{E}\backslash E_k}(\bar{X}_{\sigma_{\bar{E}\backslash E_k}^n}) \leq \tilde{e}_{\bar{E}\backslash E_k}(x).$$

Since $\bar{E} \backslash E_k$ is open, $\{\sigma_{\bar{E}\backslash E_k}^n\}_{n\geq 1}$ decreases to $\sigma_{\bar{E}\backslash E_k}$ as $n \to \infty$. Let n tends to infinity, we have for any $x \in \bar{N}^c$

$$\bar{E}_x e^{-\sigma_{\bar{E}\backslash E_k}} \leq \tilde{e}_{\bar{E}\backslash E_k}(x).$$

By (xiv) $\bar{C}_{r,p}(\bar{E} \setminus E_k) \downarrow 0$ as $k \to \infty$, so $\|\tilde{e}_{\bar{E} \setminus E_k}\|_{r,p} \to 0$ by the definition of capacity. Thus by (2) there exists a subsequence $\{k_n\}_{n \geq 1}$ such that $\tilde{e}_{\bar{E} \setminus E_{k_n}} \downarrow 0$ $\bar{C}_{r,p}$-q.e. This implies that

$$\bar{P}_x(\lim_{k \to \infty} \sigma_{\bar{E} \setminus E_k} = \infty) = 1, \quad \bar{C}_{r,p}\text{-}q.e. \text{ on } \bar{E},$$

i.e. there exists a $\bar{C}_{r,p}$-exceptional set $\bar{N}$ such that for any $x \in \bar{N}^c$

(12) $$\bar{P}_x(\lim_{k \to \infty} \sigma_{\bar{E} \setminus E_k} = \infty) = 1.$$

Set $N_1 = \bar{N}$, $Z_1 = Y = \cup_{k \geq 1} E_k$ and

$$\bar{\Omega}_1 := \{w \in \bar{\Omega} | \bar{X}_t(w) \in Z_1 \cup \{\Delta\} \text{ for } t \geq 0 \text{ and } \bar{X}_{t-}(w) \in Z_1 \cup \{\Delta\} \text{ for } t > 0\}.$$

By (11), we have for any $x \in N_1^c$

$$\bar{P}_x(\bar{\Omega}_1) = 1.$$

For such an N_1, there exist $\bar{U}_n \downarrow N_1$ such that $\bar{C}_{r,p}(\bar{U}_n) \downarrow 0$. Set $F_n := (\bar{E} \setminus \bar{U}_n) \cap E_n$, by (xiv) $\{F_n\}_{n \geq 1}$ is a $\bar{C}_{r,p}$-nest. Similar to the above argument, we know that there exists a $\bar{C}_{r,p}$-exceptional set N_2, $N_2 \supset N_1$ such that for any $x \in N_2^c$

$$\bar{P}_x(\lim_{k \to \infty} \sigma_{\bar{E} \setminus F_k} = \infty) = 1.$$

Set $Z_2 = \cup_{k \geq 1} F_k (\subseteq Z_1 \cap N_1^c)$, $\Omega_2 := \{w \in \bar{\Omega}_1 | \bar{X}_t(w) \in Z_2 \cup \{\Delta\} \text{ for } t \geq 0 \text{ and } \bar{X}_{t-1}(w) \in Z_2 \cup \{\Delta\} \text{ for } t > 0\}$, then for any $x \in N_2^c$

$$\bar{P}_x(\bar{\Omega}_2) = 1.$$

Repeating the above argument, we can find a decreasing sequence $(Z_n)_{n \geq 1}$ in $\mathcal{B}(\bar{E}) \cap E$ and an increasing sequence $\{N_n\}_{n \geq 1}$ in $\mathcal{B}(\bar{E})$ satisfying:
(a) $Z_n \subset Z_{n-1} \cap N_{n-1}^c$, $\bar{E} \setminus Z_n$, and N_{n-1} are $\bar{C}_{r,p}$-exceptional sets.
(b) $\forall x \in N_{n-1}^c$, $\bar{P}_x(\bar{\Omega}_n) = 1$, $\bar{\Omega}_n := \{w \in \bar{\Omega}_{n-1}, \bar{X}_t(w) \in Z_n \cup \{\Delta\} \text{ for } t \geq 0,$
$X_{t-}(w) \in Z_n \cup \{\Delta\} \text{ for } t > 0\}$.
   Set $Z := \cap_{n \geq 1} Z_n$, $N := \cup_{n \geq 1} N_n$, $\Omega := \cap_{n \geq 1} \bar{\Omega}_n$, then for any $x \in N^c$

$\bar{P}_x(\bar{\Omega})$

$= \bar{P}_x(\cap_{n \geq 1} \bar{\Omega}_n)$

$= \lim_{n \to \infty} \bar{P}_x(\bar{\Omega}_n)$

$= \lim_{n \to \infty} \bar{P}_x(\{w \in \bar{\Omega}_{n-1}, \bar{X}_t(w) \in Z_n \cup \{\Delta\}, \forall t \geq 0, \bar{X}_{t-}(w) \in Z_n \cup \{\Delta\}, \forall t > 0\})$

$= 1.$

Since $Z_n \subset Z_{n-1} \cap N_{n-1}^c$, $Z \subset \bar{N}^c$. Thus for any $x \in S$, $\bar{P}_x(\Omega) = 1$. Note that $\bar{E} \setminus Z$ is a $\bar{C}_{r,p}$-exceptional set, so Z is an $\bar{M}$-invariant set (cf. [4, Theorem A. 2.8]). Thus we know that $M_s := (\Omega, \bar{\mathcal{F}} \cap \Omega, \bar{\mathcal{F}}_t \cap \Omega, Q_t|_{Z_\Delta}, \bar{X}_t, (\bar{P}_x)_{x \in Z_\Delta})$ is again a Hunt process with state space $(Z_\Delta, \mathcal{B}(Z_\Delta))$. By [4, Theorem A. 2.9], we extend M_s to $M = (\Omega, \mathcal{F}, \mathcal{F}_t, Q_t, X_t, (P_x)_{x \in E_\Delta})$ which is a Hunt process with state space E and $\bar{Q}_t(\cdot, \cdot) = Q_t(\cdot, \cdot)$ on Z_Δ.

The last step is to show that for bounded function $u \in \mathcal{F}_{r,p}$, $Q_t u$ is a $C_{r,p}$-quasi-continuous m-version of $T_t u$.

Since for $\bar{u} \in \bar{D}$, $t \in Q^+$, $\bar{P}_t \bar{u}(x) = \widetilde{\bar{T}}_t \bar{u}(x)$ by (1), so $\bar{P}_t \bar{u}(x)$ is $\bar{C}_{r,p}$-quasi-continuous. By the property of Hunt process, similar to the above lemma we can extend $\bar{P}_t$ onto R^+ such that for $t \in R^+$

$$(13) \qquad \bar{Q}_t \bar{u} = \bar{P}_t \bar{u}, \quad \bar{P}_{\bar{u}} = \widetilde{\bar{T}}_t \bar{u}(x), \quad q.e. \text{ on } \bar{E}.$$

Furthermore, by the argument similar to that of the lemma we conclude that for any bounded function $\bar{u} \in \bar{\mathcal{F}}_{r,p}$ and $t \in R^+$

$$(14) \qquad \bar{Q}_t \bar{u} = \bar{P}_t \bar{u} = \widetilde{\bar{T}}_t \bar{u} \quad q.e. \text{ on } \bar{E}$$

and $\bar{Q}_t \bar{u}$ is $\bar{C}_{r,p}$-quasi-continuous.

Since $\bar{E}$ is a metrizable space and $\bar{m}$ is finite on $\mathcal{B}(\bar{E})$, $\bar{m}$ is regular [1]. Thus for $\bar{N} \in \mathcal{B}(\bar{E})$, $\bar{m}(\bar{N}) = 0$, there exists a decreasing sequence of open sets $\bar{V}_k$ of E such that $\bar{V}_k \downarrow \bar{N}$. For fixed k, similar to the $\bar{C}_{r,p}$ case, we can find a sequence of functions $\bar{f}_n \in D$ satisfying that $\bar{f}_n = 1$ on $\bar{V}_k$ and $0 < \bar{f}_n \leq 1/n$ on $\bar{E} \setminus \bar{V}_{k-1}$. For $x \in Y \subset \bar{E}$

$$
\begin{aligned}
\bar{P}_t(x, N) &= \int_E I_N \bar{P}_t(x, dy) \\
&\leq \int_{\bar{E}} \bar{f}_n(x) \bar{P}_t(x, dy) \\
&= \widetilde{\bar{T}}_t f_n(x) \\
&\leq \frac{1}{n} + \widetilde{\bar{T}}_t 1_{\bar{V}_k} \\
&\to 0 \quad \bar{m}\text{-a.e. as } n \to \infty, \ k \to \infty,
\end{aligned}
$$

since $\bar{T}_t$ is continuous in $L^p(\bar{E}; \bar{m})$. Thus we have $\bar{P}_t(x, N) = 0$ m-a.e. For bounded function $u \in \mathcal{F}_{r,p}$ there exists a bounded function $u_1 \in \bar{\mathcal{F}}_{r,p}$ such that $u = u_1$ $\bar{m}$-a.e. Thus we have $Q_t u = \bar{P}_t u_1$ $\bar{m}$-a.e. by (13) and (14), then $Q_t u$ is $\bar{C}_{r,p}$-quasi-continuous on $\bar{E}$. Since $Q_t u = \bar{Q}_t u$ on E by (xiv), $Q_t u$ is $C_{r,p}$-quasi-continuous on E. Therefore the process M satisfies the desired request.

REMARK. (a) Let $v_r f = 1/\Gamma(r/2) \int_0^{+\infty} t^{r/2-1} e^{-t} Q_t f \, dt$, for $f \in \mathcal{B}(E)$, $f > 0$. Then $v_r f$ is a $C_{r,p}$-quasi-continuous m–version of $v_t f$ for $f \in L^p(E; m)$.

In fact, let $\mathcal{H} := \{f \in L^p(E;m)|\ v_r f$ is $C_{r,p}$-quasi-continuous, f bounded$\}$. Then it is easy to check that (1) $\mathcal{H} \supset D$; (2) $1 \in H$; (3) $\forall u_1, u_2 \in \mathcal{H}$, $c_1, c_2 \in R, c_1 u_1 + c_2 u_2 \in \mathcal{H}$; (4) $\forall u_n \in \mathcal{H}, u_n \uparrow u$, u bounded, $u \in \mathcal{H}$. By monotone class theorem, $\mathcal{H} \supset L^p(E;m)$. Thus we get the conclusion.

(b) It is easy to see that the conditions in [6] and [5] satisfy the conditions of the above theorem.

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