



Title	The Lefschetz number for equivariant maps
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Citation	Osaka Journal of Mathematics. 1987, 24(2), p. 299-305
Version Type	VoR
URL	<a href="https://doi.org/10.18910/9430">https://doi.org/10.18910/9430</a>
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## THE LEFSCHETZ NUMBER FOR EQUIVARIANT MAPS

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(Received March 27, 1986)

### 1. Introduction and results

Let  $G$  be a compact Lie group. A  $G$ -ENR (Euclidean Neighborhood Retract) is a  $G$ -space which is a  $G$ -retract of some  $G$ -invariant open subspace in a Euclidean  $G$ -space. In this paper we will consider the Lefschetz number

$$\lambda(f) = \sum_i (-1)^i \operatorname{trace} f_{*,i} : H_i(X; \mathbb{Z})/\operatorname{Tor} \rightarrow H_i(X; \mathbb{Z})/\operatorname{Tor}$$

of a self  $G$ -map  $f: X \rightarrow X$  of a compact  $G$ -ENR  $X$ .  $f$  restricts to the self map  $f^G: X^G \rightarrow X^G$  of the  $G$ -fixed point set  $X^G$  of  $X$ . Then we will show

**Theorem 1.** *Let  $f: X \rightarrow X$  be a self  $G$ -map of a compact  $G$ -ENR  $X$ .*

(i) *If  $X$  has only one isotropy type  $(H)$ , then  $\lambda(f) \equiv 0 \pmod{\chi(G/H)}$  where  $\chi(\cdot)$  denotes the Euler characteristic.*

(ii) *If the  $G$ -action on  $X$  is semifree, then  $\lambda(f) \equiv \lambda(f^G) \pmod{\chi(G)}$ .*

(iii) *If  $G$  is finite and of prime power order  $p^k$ , then  $\lambda(f) \equiv \lambda(f^G) \pmod{p}$ .*

(iv) *If  $G$  is connected and abelian (i.e., torus), then  $\lambda(f) = \lambda(f^G)$ .*

In section 4 we will prove this theorem by using the fixed point index defined by Dold [2]. (i) of the theorem is a special case of Dold [3; (8.18)]. If  $G$  is finite and the  $G$ -action is free, related results are in Nakaoka [9] and Gottlieb [5]. As a corollary of the theorem we obtain

**Corollary 2.** (i) *If the  $G$ -action on  $X$  is semifree and  $\lambda(f) \equiv 0 \pmod{\chi(G)}$ , then  $f$  has a fixed point in  $X^G$ .*

(ii) *If  $G$  is of prime power order  $p^k$  and  $\lambda(f) \equiv 0 \pmod{p}$ , then  $f$  has a fixed point in  $X^G$ .*

**Proof.** In either case it follows  $\lambda(f^G) \neq 0$  and by the Lefschetz fixed point theorem there exists a fixed point of  $f^G: X^G \rightarrow X^G$ . q.e.d.

If  $G$  is a compact monogenic Lie group (i.e., finite cyclic group, torus and product of these) and  $f \in G$  is its generator, then we may regard  $f$  as a self  $G$ -map of a  $G$ -ENR  $X$ . In this case we can show, as in the proof of Theorem 1, that  $\lambda(f) = \chi(X^G)$ , although this has already appeared in the literature, tom

Dieck [1; (5.3.11)] and Huang [6; Corollary 1] for  $G$  a finite cyclic group, Kobayashi [7; p. 63] for  $X$  a Riemannian manifold. As applications of this we will show the following two results.

**Proposition 3.** *If  $X$  is a compact  $G$ -ENR and  $G$  is monogenic, then*

$$|\chi(X^G)| \leq \sum_i \text{rank } H_i(X; \mathbb{Z}).$$

In connection with this we note that if  $G$  is finite and of prime power order  $p^k$ , Floyd [4] shows

$$|\chi(X^G)| \leq \sum_i \dim H_i(X^G; \mathbb{Z}_p) \leq \sum_i \dim H_i(X; \mathbb{Z}_p).$$

**Proposition 4.** *Let  $G$  be of order 2 and  $f$  be its generator. Let  $M$  be a 2n-dimensional closed smooth  $G$ -manifold and orientable over  $\mathbb{Z}$ . If  $f$  is orientation preserving, then*

$$\chi(M^G) \equiv \text{trace } f_{*,n} \pmod{2}.$$

*If  $f$  is orientation reversing, then*

$$\chi(M^G) = \text{trace } f_{*,n} = 0.$$

*Here  $f_{*,n}$  is the automorphism of  $H_n(M; \mathbb{Z})$  induced from  $f$ .*

These two propositions will be proved in section 5.

## 2. A lemma

If  $M$  is a  $G$ -space and  $x \in M$ , then  $G(x)$  denotes the orbit of  $x$  and  $G_x$  the isotropy subgroup at  $x$ . The conjugacy class  $(G_x)$  of an isotropy subgroup  $G_x$  is called an isotropy type. For a subgroup  $H$  of  $G$  let  $M_{(H)} = \{x \in M \mid (G_x) = (H)\}$ . If  $N$  is a  $G$ -invariant subspace of  $M$  and  $h: N \rightarrow M$  a  $G$ -map, then the fixed point set  $\text{Fix}(h)$  of  $h$  is a union of orbits. If  $N$  and  $M$  are smooth  $G$ -manifolds, then for any fixed orbit  $G(x) \subset \text{Fix}(h)$  we may take  $G$ -invariant tubular neighborhoods  $T$  and  $T'$  of  $G(x)$  such that  $T \subset T'$  and  $h(T) \subset T'$ . We decompose  $T$  into  $T = T_t \oplus T_n$ , where  $T_t = T \cap N_{(H)}$ ,  $H = G_x$ , is the component tangent to  $N_{(H)}$ , and  $T_n$  the component normal to  $N_{(H)}$ . Similarly we decompose  $T'$  into  $T' = T'_t \oplus T'_n$ . Then we see  $h(T_t) \subset T'_t$ . We may regard  $T$  and  $T'$  as  $G$ -vector bundles over  $G(x) \approx G/H$ .

**Lemma 5.** *Let  $M$  be a smooth  $G$ -manifold and  $N$  a  $G$ -invariant codimension 0 submanifold of  $M$  with finite isotropy types. If  $f: N \rightarrow M$  is a  $G$ -map with  $\text{Fix}(f)$  compact, then there exists a  $G$ -map  $h: N \rightarrow M$  such that*

- (i)  $h$  is  $G$ -homotopic to  $f$  relative to the outside of some  $G$ -invariant compact neighborhood of  $\text{Fix}(f)$ ,
- (ii)  $\text{Fix}(h)$  consists of a finite number of orbits,

- (iii) if  $f(N_{(H)}) \cap M_{(H)} = \phi$  then  $h(N_{(H)}) \cap M_{(H)} = \phi$  and hence  $\text{Fix}(h) \cap N_{(H)} = \phi$ ,
- (iv) for any fixed orbit  $G(x) \subset \text{Fix}(h)$  if  $T = T_i \oplus T_n$  and  $T' = T'_i \oplus T'_n$  are  $G$ -invariant tubular neighborhoods of  $G(x)$  as above, then  $h|T: T \rightarrow T'$  is fibre preserving and decomposes into  $h|T = (h|T_i) \oplus 0$  where  $0: T_n \rightarrow T'_n$  maps any vector to 0.

Proof. (I) *The case in which the  $G$ -action on  $N$  is free.*  $N \times M$  is a  $G$ -manifold with diagonal  $G$ -action, and its action is also free. Thus the orbit spaces  $N/G$  and  $N \times_G M$  are smooth manifolds. Define a  $G$ -map  $\tilde{f}: N \rightarrow N \times M$  by  $\tilde{f}(x) = (x, f(x))$  for  $x \in N$ . Passing to the orbit spaces,  $\tilde{f}$  induces a map  $\tilde{f}/G: N/G \rightarrow N \times_G M$ . By the transversality theorem we obtain a smooth map  $h_1: N/G \rightarrow N \times_G M$  such that

- (i)  $h_1$  is transverse to  $\Delta/G$ , where  $\Delta$  is the diagonal set in  $N \times M$ , and
- (ii)  $h_1$  is close enough and homotopic to  $\tilde{f}/G$  relative to  $N - V/G$ , where  $V$  is some  $G$ -invariant compact neighborhood of  $\text{Fix}(h)$ .

By the dimension reason  $h_1^{-1}(\Delta/G)$  is a finite set, in particular it is empty if  $\dim G > 0$ . If  $f(N) \cap M_{(1)} = \phi$  where  $M_{(1)}$  is the points of  $M$  with the identity isotropy subgroup, then  $\text{Fix}(f) = \phi$ ,  $\tilde{f}/G(N/G) \cap \Delta/G = \phi$  and hence we may take  $h_1 = \tilde{f}/G$ . By the equivariant covering homotopy property we may lift the homotopy of (ii) and obtain a  $G$ -map  $h_2: N \rightarrow N \times M$   $G$ -homotopic to  $\tilde{f}$  relative to the outside of some  $G$ -invariant compact neighborhood of  $\text{Fix}(f)$ .  $h_2^{-1}(\Delta)$  consists of a finite number of orbits. Let  $p_1: N \times M \rightarrow N$  and  $p_2: N \times M \rightarrow M$  be the projections.  $p_1 h_2: N \rightarrow N$  is a diffeomorphism since it is close enough to  $p_1 \tilde{f}$  = identity. Let  $h_3 = h_2(p_1 h_2)^{-1}: N \rightarrow N \times M$  and  $h = p_2 h_3: N \rightarrow M$ . then  $h_3(x) = (x, h(x))$  and  $\text{Fix}(h) = h_3^{-1}(\Delta) \approx h_2^{-1}(\Delta)$ .  $h$  is a desired  $G$ -map.

(II) *The general case.* Let  $\{(H_1), (H_2), \dots, (H_a)\}$  be the set of isotropy types on  $N$  ordered in such a way that if  $H_i$  is conjugate to a subgroup of  $H_j$  then  $j \leq i$ . Consider the following assertion A( $i$ ) for  $0 \leq i \leq a$ :

A( $i$ ). *There exist a  $G$ -map  $h_i: N \rightarrow M$  and a  $G$ -invariant neighborhood  $U_i$  of  $X_i = N_{(H_i)} \cup \dots \cup N_{(H_1)}$  such that*

- (i)  $h_i$  is  $G$ -homotopic to  $f$  relative to the outside of some  $G$ -invariant compact neighborhood of  $\text{Fix}(f|X_i)$ ,
- (ii)  $\text{Fix}(h_i) \cap (U_i - X_i) = \phi$ ,
- (iii)  $h_i|U_i: U_i \rightarrow M$  satisfies the conditions (ii), (iii) and (iv) of the lemma.

If  $i = 0$ , then  $X_i = \phi$  and hence we may take  $U_i = \phi$ ,  $h_i = f$ . Thus A(0) is valid. A( $a$ ) is equivalent to the lemma since  $X_a = N$ . Thus, to prove the lemma it suffices to prove that A( $i$ ) implies A( $i+1$ ).

Now suppose A( $i$ ). As in the author [8; Lemma 3.1] there exists a  $G$ -invariant codimension 0 submanifold  $P$  (with boundary) of  $N$  such that  $X_i \subset \text{Int } P \subset P \subset \text{Int } U_i$ . Let  $Q = N - \text{Int } P$  and  $K = H_{i+1}$ . Consider an  $N(K)$ -map

$h_i|Q^K: Q^K \rightarrow M^K$ , where  $N(K)$  is the normalizer of  $K$  in  $G$ .  $h_i|Q^K$  may also be considered as an  $N(K)/K$ -map. Since  $K$  is the maximal isotropy subgroup on  $Q$ , then the action of  $N(K)/K$  on  $Q^K$  is free. Thus we may apply the preceding argument (I) to the  $N(K)/K$ -map  $h_i|Q^K$ , and obtain a resulting  $N(K)/K$ -map  $Q^K \rightarrow M^K$ . By  $G$ -equivariancy it extends to a  $G$ -map  $f_1: Q_{(K)} = G(Q^K) \rightarrow M$ , which satisfies the conditions (i)~(iv) of the lemma. To be precise for the condition (i) it says that  $f_1$  is  $G$ -homotopic to  $h_i|Q_{(K)}$  relative to the outside of some  $G$ -invariant compact neighborhood (in  $Q_{(K)}$ ) of  $\text{Fix}(h_i|Q_{(K)})$ . Moreover its  $G$ -homotopy may be so taken as to be relative to a neighborhood of  $\partial Q_{(K)}$ , since  $h_i$  has no fixed point in a neighborhood of  $\partial Q_{(K)}$ . Let  $T$  be a  $G$ -invariant tubular neighborhood of  $Q_{(K)}$  in  $Q$  and  $\pi: T \rightarrow Q_{(K)}$  be the projection. Then we may extend  $f_1$  to a  $G$ -map  $f_2: T \rightarrow M$  such that

- (i) for some two neighborhoods  $U \subset U'$  ( $U'$  compact) of  $\text{Fix}(f_1)$  in  $Q_{(K)}$ ,  $f_2 = f_1 \circ \pi$  on  $T \setminus U$  and  $f_2 = h_i$  on  $T \setminus Q_{(K)} - U'$ ,
- (ii)  $\text{Fix}(f_2) \cap (T - Q_{(K)}) = \phi$ .

From  $h_i|Q$  and  $f_2$ , as in the author [8; Lemma 3.2], we obtain a  $G$ -map  $f_3: Q \rightarrow M$  such that

- (i)  $f_3 = h_i$  on a neighborhood  $A$  of  $\partial Q$ ,  $f_3 = f_2$  on a neighborhood of  $Q_{(K)}$ ,  $f_3 = h_i = f$  on the outside of a  $G$ -invariant compact neighborhood  $B$  of  $\text{Fix}(f_1)$  ( $= \text{Fix}(f_2)$ ),
- (ii)  $f_3$  is  $G$ -homotopic to  $h_i|Q$  relative to  $A \cup (Q - B)$ .

Define  $h_{i+1}: N \rightarrow M$  as  $h_{i+1} = h_i$  on  $P$  and  $h_{i+1} = f_3$  on  $Q$ . Then  $h_{i+1}$  is a  $G$ -map required in A(i+1). q.e.d.

### 3. Fixed point index

We first recall the definition of the fixed point index from Dold [2]. Let  $F \subset N \subset R'' \subset R'' \cup \{\infty\} = S''$ , where  $F$  is compact and  $N$  is open. The *fundamental class*  $\alpha_F \in H_n(N, N - F; \mathbb{Z})$  is the image of 1 under the composite homomorphism

$$Z = H_n(S''; \mathbb{Z}) \rightarrow H_n(S'', S'' - F; \mathbb{Z}) \cong H_n(N, N - F; \mathbb{Z}).$$

Let  $h: N \rightarrow R''$  be a map with  $\text{Fix}(h)$  compact. Define the map  $1 - h: (N, N - F) \rightarrow (R'', R'' - 0)$  by  $(1 - h)(x) = x - h(x)$  for  $x \in N$ . Then the *fixed point index*  $\text{ind}(h)$  of  $h$  is defined as  $\text{ind}(h) = (1 - h)_* \alpha_F \in H_n(R'', R'' - 0; \mathbb{Z}) = \mathbb{Z}$ . Dold uses the symbol  $I_h$  for the index, but we use the symbol  $\text{ind}(h)$  to facilitate the printing.

Let  $R''$  be a Euclidean  $G$ -space,  $N$  be a  $G$ -invariant open subspace of  $R''$ , and  $h: N \rightarrow R''$  be a  $G$ -map satisfying the conditions (ii) and (iv) of Lemma 5. Let  $\text{Fix}(h) = G(x_1) \cup G(x_2) \cup \dots \cup G(x_a)$  with  $G_{x_i} = H_i$  ( $1 \leq i \leq a$ ). If  $T_i$  is a small  $G$ -invariant open tubular neighborhood of  $G(x_i)$  in  $N$ , then by the additivity of the index [2; (1.5)] it follows that

$$\text{ind}(h) = \sum_{i=1}^a \text{ind}(h|T_i).$$

For a while let  $x=x_i$ ,  $H=H_i$ ,  $T=T_i$ . We may consider that a fibre in  $T$  over  $g(x) \in G(x)$  is a subspace in  $R^n$  which is a parallel translation to  $g(x)$  of (a small open disc in) a linear subspace through the origin. Let  $\pi: T \rightarrow G(x) \subset R^n$  be the projection, and  $T'$  be the other  $G$ -invariant open tubular neighborhood of  $G(x)$  as in (iv) of Lemma 5. Define  $1-h+\pi: T \rightarrow T'$  as  $(1-h+\pi)(v)=v-h(v)+\pi(v)$ . This map is fibre preserving, and the following diagram is commutative for any  $g \in G$ .

$$\begin{array}{ccc} H_n(T, T-G(x)) & \xrightarrow{j_*} & H_n(T, T-g(x)) = Z \\ (1-h+\pi)^* \downarrow & & \downarrow (1-h+\pi)_* \\ H_n(T', T'-G(x)) & \xrightarrow{j_*} & H_n(T', T'-g(x)) = Z, \end{array}$$

where  $j: (T, T-G(x)) \rightarrow (T, T-g(x))$  is the inclusion. Let  $\alpha = \alpha_{G(x)} \in H_n(T, T-G(x))$  be the fundamental class. Let  $\alpha_g = j_*(1-h+\pi)_* \alpha \in Z$ . By the commutativity of the diagram,  $\alpha_g = (1-h+\pi)_* j_* \alpha$  and  $j_* \alpha = 1$  in  $H_n(T, T-g(x)) = Z$ . Since  $1-h+\pi$  is  $G$ -equivariant,  $\alpha_g$  are all equal for every  $g \in G$ . So, if  $\alpha$  is its the same value, then we see that  $(1-h+\pi)_* \alpha = \alpha \cdot \alpha$  in  $H_n(T', T'-G(x))$ .

$$(1-h)_*: H_n(T, T-G(x)) \rightarrow H_n(R^n, R^n-0)$$

factors as

$$H_n(T, T-G(x)) \xrightarrow{(1-h+\pi)_*} H_n(T', T'-G(x)) \xrightarrow{(1-\pi)_*} H_n(R^n, R^n-0).$$

Thus we see that in  $H_n(R^n, R^n-0)$

$$(1-h)_* \alpha = (1-\pi)_* (1-h+\pi)_* \alpha = \alpha \cdot (1-\pi)_* \alpha,$$

and hence  $\text{ind}(h|T) = \alpha \cdot \text{ind}(\pi)$ . Since  $\text{ind}(\pi) = \chi(G/H)$  by [2; (4.1)], it follows that  $\text{ind}(h|T_i)$  is a multiple of  $\chi(G/H_i)$  for  $i=1, 2, \dots, a$ .

Let  $\text{Fix}(h) \cap N^G = \{x_1, x_2, \dots, x_b\}$  ( $1 \leq b \leq a$ ). For  $1 \leq i \leq b$  the tubular neighborhood  $T_i$  is a disc with  $x_i$  as its center. As before  $T_i$  decomposes into the direct sum  $T_i = T_{i,t} \oplus T_{i,n}$  where  $T_{i,t} = T_i^G$  is the component tangent to  $N^G$  and  $T_{i,n}$  is the component normal to  $N^G$ . Then, from the condition (iv) of Lemma 5 we see that  $h$  on  $T_i$  decomposes into  $h(u, v) = (h(u), 0)$ . Thus, by [2; (1.4), (1.6)],  $\text{ind}(h|T_i) = \text{ind}(h|T_i^G)$  and hence

$$\sum_{i=1}^b \text{ind}(h|T_i) = \text{ind}(h^G).$$

From the above argument it follows the following.

- (i) If  $\text{Fix}(h) \subset N^G \cup N_{(H)}$ , then  $\text{ind}(h) \equiv \text{ind}(h^G) \pmod{\chi(G/H)}$ .
- (ii) If  $G$  is finite and of prime power order  $p^k$ , then  $\text{ind}(h) \equiv \text{ind}(h^G) \pmod{p}$ . For  $\chi(G/H) \equiv 0 \pmod{p}$  for any proper subgroup  $H$  of  $G$ .

(iii) If  $G$  is connected and abelian, then  $\text{ind}(h) = \text{ind}(h^G)$ . For  $\chi(G/H) = 0$  for any proper subgroup  $H$  of  $G$ .

#### 4. Proof of Theorem 1

Let  $f: X \rightarrow X$  be as in the theorem. Let  $N$  be a  $G$ -invariant open subspace in a Euclidean  $G$ -space  $R^n$ , and  $i: X \rightarrow N$ ,  $r: N \rightarrow X$  be  $G$ -maps such that  $ri = \text{identity}$ . We easily see that  $X^H$  is also an ENR for any  $H < G$ . We apply Lemma 5 to the  $G$ -map  $ifr: N \rightarrow R^n$  and obtain a  $G$ -map  $h: N \rightarrow R^n$  satisfying the conditions (i)~(iv). By [2; (1.7), (4.1)] it follows that  $\lambda(f^H) = \text{ind}((ifr)^H) = \text{ind}(h^H)$ . From this and (ii), (iii) in the preceding section, (iii) and (iv) of the theorem immediately follow. If  $X$  has only one isotropy type ( $H$ ), then  $(ifr)(N_K) \subset R_{(H)}^n$  for any  $K < G$ . Thus, from (iii) of Lemma 5 it follows  $\text{Fix}(h) \subset N_{(H)}$  and from (i) in the preceding section it follows  $\lambda(f) = \text{ind}(h) \equiv 0 \pmod{\chi(G/H)}$ . This proves (i) of the theorem. If the  $G$ -action on  $X$  is semifree, from (iii) of the lemma it follows  $\text{Fix}(h) \subset N^G \cup N_{(1)}$ . Thus (ii) of the theorem follows from (i) in the preceding section.

#### 5. Proof of Proposition 3 and 4

Let  $X$  be a compact  $G$ -ENR and  $N, i, r$  be as in section 4. If  $G$  is a compact monogenic Lie group, we regard a generator  $f$  of  $G$  as a  $G$ -map  $f: X \rightarrow X$ . Then  $\text{Fix}(f) = X^G$ . Let  $h: N \rightarrow R^n$  be a  $G$ -map obtained by Lemma 5 from the  $G$ -map  $ifr: N \rightarrow R^n$ . In this case we may construct  $h$  satisfying the additional condition  $\text{Fix}(h) \subset N^G$ . This is ensured by the fact  $\text{Fix}(ifr) = i(X^G) \subset N^G$ . Then we see that  $\lambda(f) = \text{ind}(h) = \text{ind}(h^G) = \lambda(f^G)$ . Since  $f^G$  is the identity map of  $X^G$ , it follows  $\lambda(f) = \chi(X^G)$ . As noticed in Introduction this has already appeared in the literature. Using this result, Proposition 3 and 4 are proved as follows.

(1) Proof of Proposition 3. Let  $f_{*,i}: H_i(X; C) \rightarrow H_i(X; C)$  be the automorphism induced from  $f$ , where  $C$  is the complex numbers. Let  $z_1, z_2, \dots, z_r \in C$  be the eigenvalues of  $f_{*,i}$ , where  $r = \dim H_i(X; C) = \text{rank } H_i(X; Z)$ . Since the  $\chi(G)$  times composition of  $f_{*,i}$  is the identity, then  $z_j^{\chi(G)} = 1$  and thus  $|z_j| = 1$  for  $1 \leq j \leq r$ . We see that

$$|\text{trace } f_{*,i}| = |\sum_{j=1}^r z_j| \leq \sum_{j=1}^r |z_j| = \text{rank } H_i(X; Z),$$

and

$$|\chi(X^G)| = |\lambda(f)| \leq \sum_i \text{rank } H_i(X; Z). \quad \text{q.e.d.}$$

(2) Proof of Proposition 4. Let  $G, f$  and  $M$  be as in the proposition. Note that a smooth  $G$ -manifold with a finite number of isotropy types is a  $G$ -ENR. Let  $[z] \in H_{2n}(M; Z)$  be the fundamental class defined from an orientation of  $M$  for which  $f$  is either orientation preserving or reversing. Consider the following commutative diagram.

$$\begin{array}{ccc}
 H_i(M; \mathbb{Z}) & \xrightarrow{\cap z} & H^{2n-i}(M; \mathbb{Z}) \\
 f_{*,i} \downarrow & & \uparrow f^{*,2n-1} \\
 H_i(M; \mathbb{Z}) & \xrightarrow{\cap f_{*,2n}(z)} & H^{2n-1}(M; \mathbb{Z}),
 \end{array}$$

where  $\cap$  denotes the cap product and the horizontal homomorphisms are the isomorphisms of the Poincaré duality. Note that the inverse of the isomorphism  $f^{*,2n-i}$  is itself since  $f$  is an involution. It follows that if  $f$  is orientation preserving then  $\text{trace } f_{*,i} = \text{trace } f^{*,2n-i}$ , and if  $f$  is orientation reversing then  $\text{trace } f_{*,i} = -\text{trace } f^{*,2n-i}$ . From the universal coefficient theorem it follows that  $\text{trace } f^{*,i} = \text{trace } f_{*,i}$ . It thus follows that if  $f$  is orientation preserving then  $\lambda(f) \equiv \text{trace } f_{*,n} \pmod{2}$ , and if  $f$  is orientation reversing then  $\lambda(f) = \text{trace } f_{*,n} = 0$ . Thus  $\lambda(f) = \chi(M^G)$  implies the proposition. q.e.d.

In case  $M$  is odd dimensional in Proposition 4, we then easily see that  $\lambda(f) = \chi(M^G) = 0$  if  $f$  is orientation preserving, and  $\lambda(f) = \chi(M^G) \equiv 0 \pmod{2}$  if  $f$  is orientation reversing.

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