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THE LEFSCHETZ NUMBER FOR EQUIVARIANT MAPS

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1. Introduction and results

Let G be a compact Lie group. A G -ENR (Euclidean Neighborhood Retract) is a G -space which is a G -retract of some G -invariant open subspace in a Euclidean G -space. In this paper we will consider the Lefschetz number

$$\lambda(f) = \sum_i (-1)^i \text{trace } f_{*,i}: H_i(X; \mathbb{Z})/\text{Tor} \rightarrow H_i(X; \mathbb{Z})/\text{Tor}$$

of a self G -map $f: X \rightarrow X$ of a compact G -ENR X . f restricts to the self map $f^G: X^G \rightarrow X^G$ of the G -fixed point set X^G of X . Then we will show

Theorem 1. *Let $f: X \rightarrow X$ be a self G -map of a compact G -ENR X .*

- (i) *If X has only one isotropy type (H) , then $\lambda(f) \equiv 0 \pmod{\chi(G/H)}$ where $\chi(\)$ denotes the Euler characteristic.*
- (ii) *If the G -action on X is semifree, then $\lambda(f) \equiv \lambda(f^G) \pmod{\chi(G)}$.*
- (iii) *If G is finite and of prime power order p^k , then $\lambda(f) \equiv \lambda(f^G) \pmod{p}$.*
- (iv) *If G is connected and abelian (i.e., torus), then $\lambda(f) = \lambda(f^G)$.*

In section 4 we will prove this theorem by using the fixed point index defined by Dold [2]. (i) of the theorem is a special case of Dold [3; (8.18)]. If G is finite and the G -action is free, related results are in Nakaoka [9] and Gottlieb [5]. As a corollary of the theorem we obtain

Corollary 2. (i) *If the G -action on X is semifree and $\lambda(f) \not\equiv 0 \pmod{\chi(G)}$, then f has a fixed point in X^G .*

(ii) *If G is of prime power order p^k and $\lambda(f) \not\equiv 0 \pmod{p}$, then f has a fixed point in X^G .*

Proof. In either case it follows $\lambda(f^G) \not\equiv 0$ and by the Lefschetz fixed point theorem there exists a fixed point of $f^G: X^G \rightarrow X^G$. q.e.d.

If G is a compact monogenic Lie group (i.e., finite cyclic group, torus and product of these) and $f \in G$ is its generator, then we may regard f as a self G -map of a G -ENR X . In this case we can show, as in the proof of Theorem 1, that $\lambda(f) = \chi(X^G)$, although this has already appeared in the literature, tom

Dieck [1; (5.3.11)] and Huang [6; Corollary 1] for G a finite cyclic group, Kobayashi [7; p. 63] for X a Riemannian manifold. As applications of this we will show the following two results.

Proposition 3. *If X is a compact G -ENR and G is monogenic, then*

$$|\chi(X^G)| \leq \sum_i \text{rank } H_i(X; Z).$$

In connection with this we note that if G is finite and of prime power order p^k , Floyd [4] shows

$$|\chi(X^G)| \leq \sum_i \dim H_i(X^G; Z_p) \leq \sum_i \dim H_i(X; Z_p).$$

Proposition 4. *Let G be of order 2 and f be its generator. Let M be a $2n$ -dimensional closed smooth G -manifold and orientable over Z . If f is orientation preserving, then*

$$\chi(M^G) \equiv \text{trace } f_{*,n} \pmod{2}.$$

If f is orientation reversing, then

$$\chi(M^G) = \text{trace } f_{*,n} = 0.$$

Here $f_{,n}$ is the automorphism of $H_n(M; Z)$ induced from f .*

These two propositions will be proved in section 5.

2. A lemma

If M is a G -space and $x \in M$, then $G(x)$ denotes the orbit of x and G_x the isotropy subgroup at x . The conjugacy class (G_x) of an isotropy subgroup G_x is called an isotropy type. For a subgroup H of G let $M_{(H)} = \{x \in M \mid (G_x) = (H)\}$. If N is a G -invariant subspace of M and $h: N \rightarrow M$ a G -map, then the fixed point set $\text{Fix}(h)$ of h is a union of orbits. If N and M are smooth G -manifolds, then for any fixed orbit $G(x) \subset \text{Fix}(h)$ we may take G -invariant tubular neighborhoods T and T' of $G(x)$ such that $T \subset T'$ and $h(T) \subset T'$. We decompose T into $T = T_i \oplus T_n$, where $T_i = T \cap N_{(H)}$, $H = G_x$, is the component tangent to $N_{(H)}$, and T_n the component normal to $N_{(H)}$. Similarly we decompose T' into $T' = T'_i \oplus T'_n$. Then we see $h(T_i) \subset T'_i$. We may regard T and T' as G -vector bundles over $G(x) \approx G/H$.

Lemma 5. *Let M be a smooth G -manifold and N a G -invariant codimension 0 submanifold of M with finite isotropy types. If $f: N \rightarrow M$ is a G -map with $\text{Fix}(f)$ compact, then there exists a G -map $h: N \rightarrow M$ such that*

- (i) *h is G -homotopic to f relative to the outside of some G -invariant compact neighborhood of $\text{Fix}(f)$,*
- (ii) *$\text{Fix}(h)$ consists of a finite number of orbits,*

- (iii) if $f(N_{(H)}) \cap M_{(H)} = \phi$ then $h(N_{(H)}) \cap M_{(H)} = \phi$ and hence $\text{Fix}(h) \cap N_{(H)} = \phi$,
 (iv) for any fixed orbit $G(x) \subset \text{Fix}(h)$ if $T = T_i \oplus T_n$ and $T' = T'_i \oplus T'_n$ are G -invariant tubular neighborhoods of $G(x)$ as above, then $h|T: T \rightarrow T'$ is fibre preserving and decomposes into $h|T = (h|T_i) \oplus 0$ where $0: T_n \rightarrow T'_n$ maps any vector to 0.

Proof. (I) The case in which the G -action on N is free. $N \times M$ is a G -manifold with diagonal G -action, and its action is also free. Thus the orbit spaces N/G and $N \times_G M$ are smooth manifolds. Define a G -map $\tilde{f}: N \rightarrow N \times M$ by $\tilde{f}(x) = (x, f(x))$ for $x \in N$. Passing to the orbit spaces, \tilde{f} induces a map $\tilde{f}/G: N/G \rightarrow N \times_G M$. By the transversality theorem we obtain a smooth map $h_1: N/G \rightarrow N \times_G M$ such that

- (i) h_1 is transverse to Δ/G , where Δ is the diagonal set in $N \times M$, and
 (ii) h_1 is close enough and homotopic to \tilde{f}/G relative to $N - V/G$, where V is some G -invariant compact neighborhood of $\text{Fix}(h)$.

By the dimension reason $h_1^{-1}(\Delta/G)$ is a finite set, in particular it is empty if $\dim G > 0$. If $f(N) \cap M_{(1)} = \phi$ where $M_{(1)}$ is the points of M with the identity isotropy subgroup, then $\text{Fix}(f) = \phi$, $\tilde{f}/G(N/G) \cap \Delta/G = \phi$ and hence we may take $h_1 = \tilde{f}/G$. By the equivariant covering homotopy property we may lift the homotopy of (ii) and obtain a G -map $h_2: N \rightarrow N \times M$ G -homotopic to \tilde{f} relative to the outside of some G -invariant compact neighborhood of $\text{Fix}(f)$. $h_2^{-1}(\Delta)$ consists of a finite number of orbits. Let $p_1: N \times M \rightarrow N$ and $p_2: N \times M \rightarrow M$ be the projections. $p_1 h_2: N \rightarrow N$ is a diffeomorphism since it is close enough to $p_1 \tilde{f} = \text{identity}$. Let $h_3 = h_2(p_1 h_2)^{-1}: N \rightarrow N \times M$ and $h = p_2 h_3: N \rightarrow M$. then $h_3(x) = (x, h(x))$ and $\text{Fix}(h) = h_3^{-1}(\Delta) \approx h_2^{-1}(\Delta)$. h is a desired G -map.

(II) The general case. Let $\{(H_1), (H_2), \dots, (H_a)\}$ be the set of isotropy types on N ordered in such a way that if H_i is conjugate to a subgroup of H_j then $j \leq i$. Consider the following assertion $A(i)$ for $0 \leq i \leq a$:

$A(i)$. There exist a G -map $h_i: N \rightarrow M$ and a G -invariant neighborhood U_i of $X_i = N_{(H_1)} \cup \dots \cup N_{(H_i)}$ such that

- (i) h_i is G -homotopic to f relative to the outside of some G -invariant compact neighborhood of $\text{Fix}(f|X_i)$,
 (ii) $\text{Fix}(h_i) \cap (U_i - X_i) = \phi$,
 (iii) $h_i|U_i: U_i \rightarrow M$ satisfies the conditions (ii), (iii) and (iv) of the lemma.

If $i=0$, then $X_i = \phi$ and hence we may take $U_i = \phi$, $h_i = f$. Thus $A(0)$ is valid. $A(a)$ is equivalent to the lemma since $X_a = N$. Thus, to prove the lemma it suffices to prove that $A(i)$ implies $A(i+1)$.

Now suppose $A(i)$. As in the author [8; Lemma 3.1] there exists a G -invariant codimension 0 submanifold P (with boundary) of N such that $X_i \subset \text{Int } P \subset P \subset \text{Int } U_i$. Let $Q = N - \text{Int } P$ and $K = H_{i+1}$. Consider an $N(K)$ -map

$h_i|Q^K: Q^K \rightarrow M^K$, where $N(K)$ is the normalizer of K in G . $h_i|Q^K$ may also be considered as an $N(K)/K$ -map. Since K is the maximal isotropy subgroup on Q , then the action of $N(K)/K$ on Q^K is free. Thus we may apply the preceding argument (I) to the $N(K)/K$ -map $h_i|Q^K$, and obtain a resulting $N(K)/K$ -map $Q^K \rightarrow M^K$. By G -equivariancy it extends to a G -map $f_1: Q_{(K)} = G(Q^K) \rightarrow M$, which satisfies the conditions (i)~(iv) of the lemma. To be precise for the condition (i) it says that f_1 is G -homotopic to $h_i|Q_{(K)}$ relative to the outside of some G -invariant compact neighborhood (in $Q_{(K)}$) of $\text{Fix}(h_i|Q_{(K)})$. Moreover its G -homotopy may be so taken as to be relative to a neighborhood of $\partial Q_{(K)}$, since h_i has no fixed point in a neighborhood of $\partial Q_{(K)}$. Let T be a G -invariant tubular neighborhood of $Q_{(K)}$ in Q and $\pi: T \rightarrow Q_{(K)}$ be the projection. Then we may extend f_1 to a G -map $f_2: T \rightarrow M$ such that

(i) for some two neighborhoods $U \subset U'$ (U' compact) of $\text{Fix}(f_1)$ in $Q_{(K)}$, $f_2 = f_1 \circ \pi$ on $T|U$ and $f_2 = h_i$ on $T|Q_{(K)} - U'$,

(ii) $\text{Fix}(f_2) \cap (T - Q_{(K)}) = \emptyset$.

From $h_i|Q$ and f_2 , as in the author [8; Lemma 3.2], we obtain a G -map $f_3: Q \rightarrow M$ such that

(i) $f_3 = h_i$ on a neighborhood A of ∂Q , $f_3 = f_2$ on a neighborhood of $Q_{(K)}$, $f_3 = h_i = f$ on the outside of a G -invariant compact neighborhood B of $\text{Fix}(f_1)$ ($= \text{Fix}(f_2)$),

(ii) f_3 is G -homotopic to $h_i|Q$ relative to $A \cup (Q - B)$.

Define $h_{i+1}: N \rightarrow M$ as $h_{i+1} = h_i$ on P and $h_{i+1} = f_3$ on Q . Then h_{i+1} is a G -map required in $A(i+1)$. q.e.d.

3. Fixed point index

We first recall the definition of the fixed point index from Dold [2]. Let $F \subset N \subset R^n \subset R^n \cup \{\infty\} = S^n$, where F is compact and N is open. The *fundamental class* $\alpha_F \in H_n(N, N - F; Z)$ is the image of 1 under the composite homomorphism

$$Z = H_n(S^n; Z) \rightarrow H_n(S^n, S^n - F; Z) \cong H_n(N, N - F; Z).$$

Let $h: N \rightarrow R^n$ be a map with $\text{Fix}(h)$ compact. Define the map $1 - h: (N, N - F) \rightarrow (R^n, R^n - 0)$ by $(1 - h)(x) = x - h(x)$ for $x \in N$. Then the *fixed point index* $\text{ind}(h)$ of h is defined as $\text{ind}(h) = (1 - h)_* \alpha_F \in H_n(R^n, R^n - 0; Z) = Z$. Dold uses the symbol I_h for the index, but we use the symbol $\text{ind}(h)$ to facilitate the printing.

Let R^n be a Euclidean G -space, N be a G -invariant open subspace of R^n , and $h: N \rightarrow R^n$ be a G -map satisfying the conditions (ii) and (iv) of Lemma 5. Let $\text{Fix}(h) = G(x_1) \cup G(x_2) \cup \cdots \cup G(x_a)$ with $G_{x_i} = H_i$ ($1 \leq i \leq a$). If T_i is a small G -invariant open tubular neighborhood of $G(x_i)$ in N , then by the additivity of the index [2; (1.5)] it follows that

$$\text{ind}(h) = \sum_{i=1}^a \text{ind}(h|T_i).$$

For a while let $x=x_i$, $H=H_i$, $T=T_i$. We may consider that a fibre in T over $g(x) \in G(x)$ is a subspace in R^n which is a parallel translation to $g(x)$ of (a small open disc in) a linear subspace through the origin. Let $\pi: T \rightarrow G(x) \subset R^n$ be the projection, and T' be the other G -invariant open tubular neighborhood of $G(x)$ as in (iv) of Lemma 5. Define $1-h+\pi: T \rightarrow T'$ as $(1-h+\pi)(v) = v - h(v) + \pi(v)$. This map is fibre preserving, and the following diagram is commutative for any $g \in G$.

$$\begin{array}{ccc} H_n(T, T-G(x)) & \xrightarrow{j_*} & H_n(T, T-g(x)) = Z \\ (1-h+\pi)_* \downarrow & & \downarrow (1-h+\pi)_* \\ H_n(T', T'-G(x)) & \xrightarrow{j_*} & H_n(T', T'-g(x)) = Z, \end{array}$$

where $j: (T, T-G(x)) \rightarrow (T, T-g(x))$ is the inclusion. Let $\alpha = \alpha_{G(x)} \in H_n(T, T-G(x))$ be the fundamental class. Let $\alpha_g = j_*(1-h+\pi)_*\alpha \in Z$. By the commutativity of the diagram, $\alpha_g = (1-h+\pi)_*j_*\alpha$ and $j_*\alpha = 1$ in $H_n(T, T-g(x)) = Z$. Since $1-h+\pi$ is G -equivariant, α_g are all equal for every $g \in G$. So, if α is its the same value, then we see that $(1-h+\pi)_*\alpha = \alpha \cdot \alpha$ in $H_n(T', T'-G(x))$.

$$(1-h)_*: H_n(T, T-G(x)) \rightarrow H_n(R^n, R^n-0)$$

factors as

$$H_n(T, T-G(x)) \xrightarrow{(1-h+\pi)_*} H_n(T', T'-G(x)) \xrightarrow{(1-\pi)_*} H_n(R^n, R^n-0).$$

Thus we see that in $H_n(R^n, R^n-0)$

$$(1-h)_*\alpha = (1-\pi)_*(1-h+\pi)_*\alpha = \alpha \cdot (1-\pi)_*\alpha,$$

and hence $\text{ind}(h|T) = \alpha \cdot \text{ind}(\pi)$. Since $\text{ind}(\pi) = \chi(G/H)$ by [2; (4.1)], it follows that $\text{ind}(h|T_i)$ is a multiple of $\chi(G/H_i)$ for $i=1, 2, \dots, a$.

Let $\text{Fix}(h) \cap N^G = \{x_1, x_2, \dots, x_b\}$ ($1 \leq b \leq a$). For $1 \leq i \leq b$ the tubular neighborhood T_i is a disc with x_i as its center. As before T_i decomposes into the direct sum $T_i = T_{i,t} \oplus T_{i,n}$ where $T_{i,t} = T_i^G$ is the component tangent to N^G and $T_{i,n}$ is the component normal to N^G . Then, from the condition (iv) of Lemma 5 we see that h on T_i decomposes into $h(u, v) = (h(u), 0)$. Thus, by [2; (1.4), (1.6)], $\text{ind}(h|T_i) = \text{ind}(h|T_i^G)$ and hence

$$\sum_{i=1}^b \text{ind}(h|T_i) = \text{ind}(h^G).$$

From the above argument it follows the following.

- (i) If $\text{Fix}(h) \subset N^G \cup N_{(H)}$, then $\text{ind}(h) \equiv \text{ind}(h^G) \pmod{\chi(G/H)}$.
- (ii) If G is finite and of prime power order p^k , then $\text{ind}(h) \equiv \text{ind}(h^G) \pmod{p}$. For $\chi(G/H) \equiv 0 \pmod{p}$ for any proper subgroup H of G .

(iii) If G is connected and abelian, then $\text{ind}(h) = \text{ind}(h^G)$. For $\chi(G/H) = 0$ for any proper subgroup H of G .

4. Proof of Theorem 1

Let $f: X \rightarrow X$ be as in the theorem. Let N be a G -invariant open subspace in a Euclidean G -space R^n , and $i: X \rightarrow N$, $r: N \rightarrow X$ be G -maps such that $ri = \text{identity}$. We easily see that X^H is also an ENR for any $H < G$. We apply Lemma 5 to the G -map $ifr: N \rightarrow R^n$ and obtain a G -map $h: N \rightarrow R^n$ satisfying the conditions (i)~(iv). By [2; (1.7), (4.1)] it follows that $\lambda(f^H) = \text{ind}((ifr)^H) = \text{ind}(h^H)$. From this and (ii), (iii) in the preceding section, (iii) and (iv) of the theorem immediately follow. If X has only one isotropy type (H) , then $(ifr)(N_{(K)}) \subset R^n_{(H)}$ for any $K < G$. Thus, from (iii) of Lemma 5 it follows $\text{Fix}(h) \subset N_{(H)}$ and from (i) in the preceding section it follows $\lambda(f) = \text{ind}(h) \equiv 0 \pmod{\chi(G/H)}$. This proves (i) of the theorem. If the G -action on X is semifree, from (iii) of the lemma it follows $\text{Fix}(h) \subset N^G \cup N_{(1)}$. Thus (ii) of the theorem follows from (i) in the preceding section.

5. Proof of Proposition 3 and 4

Let X be a compact G -ENR and N, i, r be as in section 4. If G is a compact monogenic Lie group, we regard a generator f of G as a G -map $f: X \rightarrow X$. Then $\text{Fix}(f) = X^G$. Let $h: N \rightarrow R^n$ be a G -map obtained by Lemma 5 from the G -map $ifr: N \rightarrow R^n$. In this case we may construct h satisfying the additional condition $\text{Fix}(h) \subset N^G$. This is ensured by the fact $\text{Fix}(ifr) = i(X^G) \subset N^G$. Then we see that $\lambda(f) = \text{ind}(h) = \text{ind}(h^G) = \lambda(f^G)$. Since f^G is the identity map of X^G , it follows $\lambda(f) = \chi(X^G)$. As noticed in Introduction this has already appeared in the literature. Using this result, Proposition 3 and 4 are proved as follows.

(1) Proof of Proposition 3. Let $f_{*,i}: H_i(X; C) \rightarrow H_i(X; C)$ be the automorphism induced from f , where C is the complex numbers. Let $z_1, z_2, \dots, z_r \in C$ be the eigenvalues of $f_{*,i}$ where $r = \dim H_i(X; C) = \text{rank } H_i(X; Z)$. Since the $\chi(G)$ times composition of $f_{*,i}$ is the identity, then $z_j^{\chi(G)} = 1$ and thus $|z_j| = 1$ for $1 \leq j \leq r$. We see that

$$|\text{trace } f_{*,i}| = |\sum_{j=1}^r z_j| \leq \sum_{j=1}^r |z_j| = \text{rank } H_i(X; Z),$$

and

$$|\chi(X^G)| = |\lambda(f)| \leq \sum_i \text{rank } H_i(X; Z). \quad \text{q.e.d.}$$

(2) Proof of Proposition 4. Let G, f and M be as in the proposition. Note that a smooth G -manifold with a finite number of isotropy types is a G -ENR. Let $z \in H_{2n}(M; Z)$ be the fundamental class defined from an orientation of M for which f is either orientation preserving or reversing. Consider the following commutative diagram.

$$\begin{array}{ccc}
 H_i(M; \mathbb{Z}) & \xrightarrow{\cap z} & H^{2n-i}(M; \mathbb{Z}) \\
 f_{*,i} \downarrow & & \uparrow f^{*,2n-1} \\
 H_i(M; \mathbb{Z}) & \xrightarrow{\cap f_{*,2n}(z)} & H^{2n-1}(M; \mathbb{Z}),
 \end{array}$$

where \cap denotes the cap product and the horizontal homomorphisms are the isomorphisms of the Poincaré duality. Note that the inverse of the isomorphism $f^{*,2n-i}$ is itself since f is an involution. It follows that if f is orientation preserving then $\text{trace } f_{*,i} = \text{trace } f^{*,2n-i}$, and if f is orientation reversing then $\text{trace } f_{*,i} = -\text{trace } f^{*,2n-i}$. From the universal coefficient theorem it follows that $\text{trace } f^{*,i} = \text{trace } f_{*,i}$. It thus follows that if f is orientation preserving then $\lambda(f) \equiv \text{trace } f_{*,n} \pmod{2}$, and if f is orientation reversing then $\lambda(f) = \text{trace } f_{*,n} = 0$. Thus $\lambda(f) = \chi(M^c)$ implies the proposition. q.e.d.

In case M is odd dimensional in Proposition 4, we then easily see that $\lambda(f) = \chi(M^c) = 0$ if f is orientation preserving, and $\lambda(f) = \chi(M^c) \equiv 0 \pmod{2}$ if f is orientation reversing.

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