Title	Stability of surfaces with constant mean curvature in 3-dimensional Riemannian manifolds
Author(s)	Sakaki, Makoto
Citation	Osaka Journal of Mathematics. 1990, 27(4), p. 893-898
Version Type	VoR
URL	https://doi.org/10.18910/9433
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Sakaki, M.
Osaka J. Math.
27 (1990), 893-898

STABILITY OF SURFACES WITH CONSTANT MEAN CURVATURE IN 3-DIMENSIONAL RIEMANNIAN MANIFOLDS

MAKOTO SAKAKI

(Received October 18, 1989)

0. Introduction

In [1] Barbosa and do Carmo adopted a new approach to the stability of minimal surfaces. In particular they discussed the stability of simply connected compact domains with boundary on minimal surfaces in space forms (cf. [2], [8] and [10]). Their method was applied also to the stability of surfaces with constant mean curvature in 3-dimensional space forms (see [5], [9] and [14]). It is natural to ask if these arguments can be generalized for general ambient spaces. In the case of minimal surfaces, a positive answer to this question was given in our previous paper [11]. In this paper we give a positive answer in the case of surfaces with constnat mean curvature. Namely we prove:

Theorem. *Let $f: M \rightarrow N$ be an immersion of a 2-dimensional orientable manifold M into a 3-dimensional orientable Riemannian manifold N. Assume that the mean curvature of the immersion f is constant. Let D be a simply connected compact domain on M with piecewise smooth boundary. We denote by A and dM the second fundamental form and the area element of M induced by f, respectively. Suppose that the sectional curvature of N and the norm of the covariant derivative of the curvature tensor of N are bounded. Then there is a positive constant c_1 depending only on N such that if $\int_D (1+|A|^2/2)dM<c_1$, then D is stable.*

REMARK. (i) The method in [11] is not available to the statility of surfaces with constant mean curvature.

(ii) If we omit the hypothesis that D is simply connected. it is not known whether the theorem is ture or not (cf. [11, Theorem 0.3]).

The author wishes to thank Professor S. Tanno for his constant encouragement and advice, and the referee for useful comments.

1. Preliminaries

Let $f: M \rightarrow N$ be an immersion of an m-dimensional orientable manifold

M into an $(m+1)$-dimensional orientable Riemannian manifold N. Assume that the mean curvature of the immersion f is constant. Let D be a compact domain on M with piecewise smooth boundary ∂D. We choose a unit normal vector field ν to f(M) We denote by $F(D)$ the space of smooth functions ψ on D such that $\psi=0$ on ∂D and $\int_D \psi dM=0$, where dM is the volume element of M with respect to the metric ds^2 induced by f.

For $\psi\in F(D)$, we consider a smooth map F: $[0, 1]\times D\rightarrow N$ such that F_t: $D\rightarrow N$ for $t\in[0, 1]$ defined by $F_t(p)=F(t, p)$ for $p\in D$ is an immersion, $F_0=f$, $F_t|_{\partial D}=f|_{\partial D}$, $(d/dt)F_t|_{t=0}=\psi\nu$ and $(d/dt)\int_{[0,t]\times D}F^*dN=0$, where dN is the volume element of N. The second variation $I(\psi)$ of the volume functional of D for the variational vector field $\psi\nu$ is defined by $I(\psi)=(d^2/dt^2)\mathrm{vol}(D, t)|_{t=0}$, where vol$(D, t)$ is the volume of D with respect to the metric indeced by F_t. Let ∇, A and Ric denote the Riemannian connection of (M, ds^2), the second fundamental form of f and the Ricci tensor of N, respectively. Then by the second variation formula of the volume for hypersurfaces with constant mean curvature (see [3] and [4]),

$$I(\psi)=\int_D\{|\nabla\psi|^2-(\mathrm{Ric}(\nu, \nu)+|A|^2)\psi^2\}dM. \tag{1}$$

The domain D is stable if $I(\psi)>0$ for any $\psi\in F(D)$ which is not identically zero, and D is unstable if $I(\psi)<0$ for some $\psi\in F(D)$.

2. A curvature estimate

The purpose of this section is to prove the following:

Lemma. *Let f: $M\rightarrow N$ be an immersion of a 2-dimensional orientable manifold M into a 3-dimensional orientable Reimannian manifold N. Assume that the mean curvature of the immersion f is constant. We denote by A and ds^2 the second fundamental form and the metric of M indeced by f, respectively. Suppose that the sectional curvature of N and the norm of the covariant derivative of the curvature tensor of N are bounded. Then the Gaussian curvature $\tilde{K}$ of M with respect to the metric $d\tilde{s}^2=(1+|A|^2/2)ds^2$ satisfies $\tilde{K}\leqq c_2$ for a positive constant c_2 depending only on N.*

Proof. By the hypothesis we may assume that the sectional curvature of N is bounded from above by a and below by b, and the norm of the covariant derivative of the curvature tensor of N is not greater than ξ.

Let K, ∇ and Δ denote the Gaussian curvature, the Riemannian connection and the Laplacian of (M, ds^2), respectively. Then we have

$$(2)\quad \tilde{K}=\frac{K}{1+|A|^2/2}-\frac{1}{2(1+|A|^2/2)}\Delta\log\left(1+\frac{1}{2}|A|^2\right)$$
$$=\frac{K}{1+|A|^2/2}-\frac{\langle A,\Delta A\rangle}{2(1+|A|^2/2)^2}$$
$$+\frac{1}{2(1+|A|^2/2)^3}\left\{-|\nabla A|^2-\frac{1}{2}|A|^2|\nabla A|^2+\left|\frac{1}{2}\nabla(|A|^2)\right|^2\right\}.$$

We shall make a pointwise argument at a point p on M. Let $\{e_1, e_2, e_3\}$ be an orthonormal basis for the tangent space of N at $f(p)$ such that e_1, e_2 are tangent to $f(M)$. In what follows, we use the following convention on the ranges of indices: $1\leqq i, j, k, \cdots\leqq 2$, $1\leqq B, C, D, \cdots\leqq 3$. We denote by h_{ij}, h_{ijk}, R^B_{CDE} and R^B_{CDEF} the components of A, ∇A, the curvature tensor of N and the covariant derivative of the curvature tensor of N with respect to the basis, respectively. We may choose $\{e_1, e_2\}$ so that $h_{11}=\lambda$, $h_{12}=h_{21}=0$ and $h_{22}=\mu$ for some λ and μ. By the Gauss equation we have

$$(3)\qquad K=R^1_{212}+\lambda\mu\leqq a+\frac{1}{2}(\lambda^2+\mu^2)=a+\frac{1}{2}|A|^2.$$

Using the equations (2.7), (2.9), (2.17) and (2.21) in [7], we find that

$$(4)\quad -\langle A,\Delta A\rangle=(R^3_{2121}+R^3_{1122})\lambda+(R^3_{1212}+R^3_{2211})\mu+(R^2_{323}-2R^1_{212})\lambda^2$$
$$+(R^1_{313}-2R^1_{212})\mu^2+2\left(2R^1_{212}-\frac{1}{2}R^1_{313}-\frac{1}{2}R^2_{323}\right)\lambda\mu-\lambda\mu(\lambda-\mu)^2$$
$$\leqq\sqrt{2}\,\{(R^3_{2121})^2+(R^3_{1122})^2+(R^3_{1212})^2+(R^3_{2211})^2\}^{1/2}(\lambda^2+\mu^2)^{1/2}$$
$$+(a-2b)(\lambda^2+\mu^2)+\max\{|2a-b|,\ |a-2b|\}(\lambda^2+\mu^2)+(\lambda^2+\mu^2)^2$$
$$\leqq\sqrt{2}\,\xi|A|+c_3|A|^2+|A|^4,$$

where

$$c_3=a-2b+\max\{|2a-b|,\ |a-2b|\}\ .$$

Noting that the components h_{ijk} satisfy $h_{11i}+h_{22i}=0$, $h_{12i}=h_{21i}$ and $h_{ijk}-h_{ikj}=R^3_{ikj}$ by the Codazzi equation, we have

$$(5)\quad -\frac{1}{2}|A|^2|\nabla A|^2+\left|\frac{1}{2}\nabla(|A|^2)\right|^2$$
$$=-\frac{1}{2}\sum_{i,j}(h_{ij})^2\sum_{i,j,k}(h_{ijk})^2+\sum_k\Big(\sum_{i,j}h_{ij}h_{ijk}\Big)^2$$
$$=-(\lambda^2+\mu^2)\{(h_{111})^2+(h_{121})^2+(h_{112})^2+(h_{122})^2\}$$
$$+(\lambda h_{111}+\mu h_{221})^2+(\lambda h_{112}+\mu h_{222})^2$$
$$=-(\lambda^2+\mu^2)\{(h_{111})^2+(h_{112}+R^3_{112})^2+(h_{112})^2+(-h_{111}+R^3_{221})^2\}$$
$$+(\lambda-\mu)^2\{(h_{111})^2+(h_{112})^2\}$$

$$
\begin{aligned}
&= -(\lambda+\mu)^2\{(h_{111})^2+(h_{112})^2\}-(\lambda^2+\mu^2)\{(R^3_{112})^2+(R^3_{221})^2\} \\
&\quad +2(\lambda^2+\mu^2)(h_{111}R^3_{221}-h_{112}R^3_{112}) \\
&\leqq 2|A|^2(h_{111}R^3_{221}-h_{112}R^3_{112}) \\
&\leqq 2\{(h_{111})^2+(h_{112})^2\}+\frac{1}{2}\{(R^3_{221})^2+(R^3_{112})^2\}|A|^4 \\
&\leqq |\nabla A|^2+\frac{1}{4}(a-b)^2|A|^4,
\end{aligned}
$$

where for the last inequality we use the following (see [6]):

$$|R^3_{221}|\leqq\frac{1}{2}(a-b),\ |R^3_{112}|\leqq\frac{1}{2}(a-b).$$

By (2), (3), (4) and (5) we get

$$
\begin{aligned}
\tilde{K} &\leqq \frac{a+|A|^2/2}{1+|A|^2/2}+\frac{\sqrt{2}\,\xi|A|+c_3|A|^2+|A|^4}{2(1+|A|^2/2)^2}+\frac{(a-b)^2|A|^4}{8(1+|A|^2/2)^3} \\
&\leqq \sup_{t\geqq 0}\left\{\frac{a+t^2/2}{1+t^2/2}+\frac{\sqrt{2}\,\xi t+c_3t^2+t^4}{2(1+t^2/2)^2}+\frac{(a-b)^2t^4}{8(1+t^2/2)^3}\right\}=c_2\,.
\end{aligned}
$$

It is easy to see that $0<c_2<\infty$ and c_2 depends only on N. Thus the proof is complete.

REMARK. For example, let us consider the case where N is the 3-dimensional unit sphere. Then $a=b=1$, $c_3=0$ and $\xi=0$. Hence we have

$$c_2=\sup_{t\geqq 0}\left\{1+\frac{t^4}{2(1+t^2/2)^2}\right\}=3\,,$$

which is worse than Proposition 1 of [5].

3. Proof of Theorem

Proof of Theorem. Let $F(D)$, $I(\ )$, ∇ and ds^2 be as in Section 1. By the hypothesis we may assume that the sectional curvature of N is bounded from above by a. Then by (1) we have

$$I(\psi)\geq\int_D\{|\nabla\psi|^2-(2a+|A|^2)\psi^2\}dM \tag{6}$$

for $\psi\in F(D)$. Set $d\tilde{s}^2=(1+|A|^2/2)ds^2$. Let $\tilde{\nabla}$ and $d\tilde{M}$ denote the Riemannian connection and the area element of $(M, d\tilde{s}^2)$, respectively. Then we see that

$$|\tilde{\nabla}\psi|^2=\frac{|\nabla\psi|^2}{1+|A|^2/2} \tag{7}$$

and

$$d\tilde{M} = \left(1+\frac{1}{2}|A|^2\right)dM. \tag{8}$$

By (6), (7) and (8) we have

$$\begin{aligned} I(\psi) &\geqq \int_D \left\{ |\tilde{\nabla}\psi|^2 - \frac{2a+|A|^2}{1+|A|^2/2}\psi^2 \right\} d\tilde{M} \\ &\geqq \int_D (|\tilde{\nabla}\psi|^2 - 2\eta\psi^2) d\tilde{M}, \end{aligned} \tag{9}$$

where $\eta=\max\{a, 1\}$. We denote by $\tilde{\lambda}_1(D)$ the first eigenvalue on D of the Laplacian of $(M, d\tilde{s}^2)$ with Dirichlet boundary condition. The inequality (9) says that D is stable if $\tilde{\lambda}_1(D)>2\eta$.

Let c_2 be as in Section 2. Set

$$c_1 = \frac{4\pi}{c_2+\eta},$$

which is positive and depends only on N. We note that $c_2 \geqq \eta$ by the difinitions of c_2 and η. Using Proposition 3.3 and 3.10 of [1] with Lemma, we can see that $\tilde{\lambda}_1(D)>2\eta$ if $\tilde{a}(D)<c_1$, where $\tilde{a}(D)$ denotes the area of D with respect to the metric $d\tilde{s}^2$. By (8) we have $\tilde{a}(D)=\int_D (1+|A|^2/2)dM$. Thus the proof is complete.

REMARK. In fact we prove that D is strongly stable in the sense of [14] if $\int_D (1+|A|^2/2)dM<c_1$.

References

[1] J.L. Barbosa and M. do Carmo: *Stability of minimal surfaces and eigenvalues of the Laplacian,* Math. Z. **173** (1980), 13–28.

[2] J.L. Barbosa and M. do Carmo: *Stability of minimal surfaces in spaces of constant curvature,* Bol. Soc. Brasil Mat. **11** (1980), 1–10.

[3] J.L. Barbosa and M. do Carmo: *Stability of hypersurfaces with constant mean curvature,* Math. Z. **185** (1984), 339–353.

[4] J.L. Barbosa, M. do Carmo and J. Eschenburg: *Stability of hypersurfaces of constant mean curvature in Riemannian manifolds,* Math. Z. **197** (1988), 123–138.

[5] J.L. Barbosa and H. Mori: *Stability of constant mean curvature sufaces in Riemannian 3-space form,* Yokohama Math. J. **30** (1982), 73–79.

[6] M. Berger: *Sur quelques varietes Riemanniennes suffisamment pincees,* Bull. Soc. Math. France **88** (1960), 57–71.

[7] S.S. Chern, M. do Carmo and S. Kobayashi: *Minimal submanifolds of a sphere*

with second fundamental form of constant length, Functional Analysis and Related Fields, ed. by F.E. Browder, Springer-Varlag, Berlin-Heidelberg-New York, 1970, 59–75.

[8] D. Hoffman and R. Osserman: *The area of the generalized Gaussian image and the stability of minimal surfaces in S^n and $\boldsymbol{R}^n$*, Math. Ann. **260** (1982), 437–452.

[9] H. Ruchert: *Ein Eindeutigkeitssatz für Flächen konstanter mittlerer Krümmung,* Arch. Math. **33** (1979), 91–104.

[10] M. Sakaki: *Remarks on the rigidity and stability of minimal submanifolds,* Proc. Amer. Math. Soc. **106** (1989), 793–795.

[11] M. Sakaki: *Estimates on the stability of minimal surfaces and harmonic maps,* J. Math. Soc. Japan **41** (1989), 641–650.

[12] M. Sakaki: *On the index of surfaces with constant mean curvature,* preprint.

[13] M. Sakaki: *Upper bounds for the index of minimal surfaces,* Tohoku Math. J. 42 (1990), 339–349.

[14] A.M. da Silveira: *Stability of complete noncompact surfaces with constant mean curvature,* Math. Ann. **277** (1987), 629–638.

Department of Mathematics
Faculty of Science
Hirosaki University
Hirosaki, Aomori-ken 036
Japan