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QF-3 AND SEMI-PRIMARY PP-RINGS II

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In the previous paper [5] the author has studied semi-primary left (resp. right) QF-3 rings, which is a ring A with the following property: there exists a faithful, projective, injective left (resp. right) A-module. Especially, we have considered, in [5], a semi-primary left QF-3 and partially PP-ring[1]. We have shown in [5], Remark 4 that the basic ring of such a ring is characterized as a special subring of a semi-simple ring.

In §3 of this short note we shall study a similar problem to the above in a case of a semi-primary left and right QF-3 ring with the following properties: Let Ae is a unique minimal faithful, projective, injective left ideal and $e=\sum_{i=1}^{t} e_i$ a decomposition of e into a sum of mutually orthogonal primitive idempotents e_i. 1) The left socle of Ae (the sum of irreducible A-module of Ae) is A-projective 2) e_iAe_i is a division ring for all i and 3) eAe is a direct sum of division rings.

It is clear that 3) implies 2). We shall shown in §3 that 1) implies 2) and that 3) is equivalent to 1) if A is a left and right QF-3 ring. Furthermore, we shall show that the basic ring of left QF-3 ring is a partially PP-ring if and only if A satisfies condition 1) and a condition that the socle of every primitive left ideal is irreducible.

In §1 we shall show that if A satisfies left and right minimum conditions, then A is left QF-3 if and only if A is right QF-3. However in §2 we shall give a semi-primary ring which is left QF-3, but not right QF-3.

1. QF-3 rings with minimum conditions.

Let A be a ring with identity element 1 and N the radical of A. In this note we always consider a semi-primary ring A, namely A/N is a semi-simple ring with minimum conditions and N is nilpotent. We call A a *left QF-3 ring* if there exists a *faithful, projective, injective A-module*. Since A is semi-primary, we obtain a faithful, injective left ideal Ae if A is left QF-3, where e is an idempotent.

1) See [4] or [5].

We shall show in this section that a left QF–3 ring satisfying left and right minimum conditions is a right QF–3 ring. On the other hand in §2 we shall give an example which shows that the above fact is not true for semi-primary left QF–3 rings.

Theorem 1. *We assume that A satisfies left and right minimum conditions. Then A is a left QF–3 if and only if A is right QF–3.*

Proof. Let Q be the factor module of the ring of rationals modulo the ring Z of integers. We assume that A is left QF–3 and L a faithful, projective, injective A-module. Put $L^*=\mathrm{Hom}_Z(L, Q)$. Since Q is Z-injective by [2], p. 134, Proposition 5.1, L^* is a right A-faithful module. Furthermore, L^* is A-injective by [2], p. 166, Proposition 2.5a. Let M be a finitely generated left A-module. The $L^*\underset{A}{\otimes}M\approx\mathrm{Hom}_Z(\mathrm{Hom}_A(M, L), Q)$ by [2], p. 124, Proposition 5,3. Hence, L^* is A-flat, since L is A-injective and Q is Z-injective. Therefore, L^* is a faithful, injective, projective A-module by [3]. Hence, A is right QF–3. The converse is similar.

Corollary 1. *Let A be as above. Then the left A-injective envelope of A is A-projective if and only if the right A-injective envelope is A-projective.*

Proof. It is clear from [8], Theorems 3.2 and 3.1.

2. Generalized trianglar matrix rings.

We shall consider a *g.t.a.* matrix ring $T_n(\Delta_i; M_{i,j})$ over division ringe Δ_i which is left QF–3, (see [6] for the definition of $T_n(\Delta_i; M_{i,j})$).

Proposition 1. *Let A be a g.t.a. matrix ring $T_n(\Delta_i; M_{i,j})$ over division rings. We assume Ae_i is A-injective and t is the maximal index among j such that $M_{j,i}\neq(0)$. Then $Ae_i\approx\mathrm{Hom}_{\Delta_t}(e_tA, \Delta_t)$, and this isomorphism is given by the multiplication of elements in Ae_i from the right side, where $e_i=T_n(o, \overset{i}{\vee}{1}, o; o)$.*

Proof. First, we show that $M_{j,i}\approx\mathrm{Hom}_{\Delta_t}(M_{t,j}, M_{t,i})$ by the multiplication of elements in $M_{j,i}$. Since $M_{k,i}=(0)$ for $k<t$ and Ae_i is an indecomposable injective ideal, $M_{t,i}$ is a unique minimal left ideal in Ae_i. Hence, $[M_{t,i}:\Delta_t]=1$ and $M_{t,i}\approx\Delta_t$ as a left Δ_t-module. Let $X=\mathrm{Hom}_{\Delta_t}(M_{t,j}, M_{t,i})$ and $f\in X$. We define $\bar{f}\in\mathrm{Hom}_A(\sum_{k=t}^{n}\oplus M_{k,j}, M_{t,i})$ by setting $\bar{f}(M_{k,j})=(0)$ for $k>t$ and $\bar{f}|M_{t,j}=f$. Since $M_{t,i}\subseteq Ae_i$, there exists an element $m_j\in M_{j,i}$ such that $f(m)=mm_j$ for any $m\in M_{t,j}$. Therefore, X

coincides with the set of right multiplication of elements in $M_{j,i}$. Furthermore, $Ae_i \supseteq Am_j \cap M_{t,i} = M_{t,j}m_j$. Hence, $M_{t,j}m_j \neq (0)$ whenever $m_j \neq 0$, since $M_{t,i}$ is the socle of Ae_i. Therefore, $X \approx M_{j,i}$. It is clear from this fact that $Ae_i \approx \mathrm{Hom}_{\Delta_t}(e_t A, M_{t,i}) \approx \mathrm{Hom}_{\Delta_t}(e_t A, \Delta_t)$ as a left A-module.

Corollary 2. *Let A be as above. We assume that A is a left and right QF-3 ring. Then* $[Ae_1 : \Delta_1] < \infty$.

Proof. Since A is left QF-3, Ae_1 must be A-injective. We assum $M_{t,1} \neq (0)$ and $M_{k,1} = (0)$ for $k > t$. Then $e_k AM_{t,1} \subseteq M_{k,t}M_{k,1} = M_{k,1} = (0)$ for $k \neq t$. Therefore, if A is right QF-3, e_tA must be A-injective. Furthermore, $Ae_1 \approx \mathrm{Hom}_{\Delta_t}(e_tA, M_{t,1})$, $e_tA \approx \mathrm{Hom}_{\Delta_1}(Ae_1, M_{t,1})$ and $M_{t,1} = \Delta_t x = x\Delta_1$ for some $x \in M_{t,1}$ by Poroposition 1. Therefore, $[Ae_1 : \Delta_1] < \infty$ by [7], p. 68, Theorem 1.

EXAMPLE. Let $\Delta = \Delta_1 = \Delta_3$ and Δ_2 be division rings and $M_{3,2}$ a $\Delta_3 - \Delta_2$ module such that $[M_{3,2} : \Delta_3] = \infty$. Put $M_{3,1} = \Delta$ and $M_{2,1} = \mathrm{Hom}_{\Delta}(M_{3,2}, M_{3,1})$. Let

$$A = \begin{pmatrix} \Delta_1 & 0 & 0 \\ M_{2,1} & \Delta_2 & 0 \\ M_{3,1} & M_{3,2} & \Delta_3 \end{pmatrix}.$$

Then Ae_1 is A-faithful. Furthermore, $Ae_1 \approx \mathrm{Hom}_{\Delta_3}(e_3A, \Delta_3)$ as an A-module. Therefore, A is left QF-3. However, A is not right QF-3 from Corollary 2.

We obtain immediately Lemma 5 in [5] from Poroposition 1.

3. PP-rings.

Let e be a primitive idempotent of a semi-primary ring A. In this section we shall study the ring A such that Ae is injective and its socle is A-projective. If A is a partially PP-ring[2] and QF-3 ring, then A satisfies the above condition, (cf. Theorem 2 below).

Let A^* be a basic ring[3] of A. Then A is isomorphic to the endomorphism rings of a finitely generated projective right A^*-module (see [6]). We note from this fact that primitive left ideals in A and A^* enjoy many similar properties.

Lemma. *Let e, f be primitive idempotents. We assume that the left socle of Ae is A-irreducible and A-projective. If $fAe \neq (0)$, then either*

2) See [4].
3) See [5].

Ae contains an isomorphic image of Af or the left socle of Af is not irreducible.

Proof. If $x \neq 0 \in fAe$. Then $Afx = Ax \neq (0)$ in Ae. Let φ be an A-homomorphism of Af to Ax by setting $\varphi(yf) = yfx$; $y \in A$. Since $Ax \neq (0)$, Ax contains the left socles S of Ae. Then $o \to \varphi^{-1}(o) \to \varphi^{-1}(S) \to S \to o$ is exact. Hence, $\varphi^{-1}(S) \approx S \oplus \varphi^{-1}(o)$. If the left socle of Af is irreducible, then $\varphi^{-1}(o) = (0)$. Therefore, φ is isomorphic.

Proposition 2. 1) *Let A be semi-primary and e a primitive idempotent in A. If Ae is A-injective and its left socle is A-projective, then $Ae \approx \mathrm{Hom}_{fAf}(fA, fAe)$ as a left A-module and $eAe \approx fAf$ is a division ring.* 2) *Furthermore, we assume that A is a left QF-3 ring with faithful injective ideal AE. If the left socle of AE is A-projective, then EAE is a semi-simple ring, where f is a prinitive idempotent.*

Proof. We may assume that A coincides with its basic ring. Let S be the socle of Ae. Since S is A-projective, $S \approx Af$ and $Nf = (0)$ for some primitive idempotent f. Let $1 = \sum_{j=1}^{n} e_j$ a decomposition of 1 into a sum of mutually orthogonal primitive idempotents e_j (assume $e = e_i$, $f = e_k$). Then $e_j A e_k = e_j N e_k = (0)$ for $j \neq k$. It is clear that $\Delta \equiv e_k A e_k = e_k A e_k / e_k N e_k$ is a division ring. Since $e_l A e_k A e_j = (0) = e_l AS$ for $l \neq k$, $\mathrm{Hom}_\Delta(e_k A, S) = \sum_{j=1}^{n} \mathrm{Hom}_A(e_k A e_j, S)$. Furthermore, $S = e_k A e_i$, since $e_k A e_i$ is a left ideal in Ae_i. Then we can prove similarly to Proposition 1 that $\mathrm{Hom}_\Delta(e_k A, S) \approx Ae_i$ as a left A-module. We have the last part of 1) from this isomorphism (cf. the proof of Theorem 3 below).
2) Let $E = \sum_{i=1}^{t} e_i$ be the usual decomposition. From the assumption we know that the left socle of Ae_i is A-projective. Then we obtain $e_i A e_j = (0)$ for $i \neq j$ by Lemma. Therefore, we have proved 2) from 1).

Corollary 3. *Let A, Ae and fA be as above. Then the following facts are equivalent.*
1) $eN = (0)$, 2) *The right socle of fA is A-projective.*

Proof. We assume that A is basic. Let T be the right socle of fA, then $\Delta T = T$, where $\Delta = fAf$. Since $Ae \approx \mathrm{Hom}_\Delta(fA, \Delta)$, $\Delta \approx eAe \approx Ae/Ne \approx \mathrm{Hom}_\Delta(T, \Delta)$. Hence, $T \approx eA/eN$. Therefore, 1) and 2) are equivalent.

We note that if A is a semi-primary left QF-3 ring with a faithful injective Ae, then every irreducible left ideal in a primitive left ideal is isomorphic to one of Ae, (cf. [8]).

Theorem 2. *Let A be a semi-primary left QF-3 ring with faithful injective left ideal Ae. Then the basic ring of A is a partially PP-ring[1] if and only if the left socle of Ae is A-projective and the socle of any primitive left ideal is irreducible. In this case A is also right QF-3.*

Proof. Let A be a basic and partially PP-ring. Then we may assume by [5], Theorem 3 that $A=T_n(\Delta_i\,;\,M_{i,j})$ and $[M_{n,i}:\Delta_n]=1$ for all i, where the Δ_i's are division rings. Let $e'=T_n(1, o, \cdots, o\,;\,o)$, then Ae' is faithful and injective. Therefore, the irreducible left ideal in a primitive left ideal Af is isomorphic to the socle $T_n(o\,;M_{n,1}, o, \cdots, o)$, which is A-projective. Since $[M_{n,i}:\Delta_n]=1$, we obtain that the socle of Af is irreducible. Conversely, we assume that A is basic and Ae is a faithful injective and its socle is A-projective, and that the socle of any primitive left ideal is irreducible. Let $1=\sum_{i=1,j=1}^{t,\rho(i)} e_{i,j}$ be a decomposition of 1 into a sum of mutually orthogonal primitive idempotent $e_{i,j}$ such that $e=\sum_{i=1}^{t} e_{i,1}$ and the left socle of $Ae_{i,j}$ is isomorphic to one of $Ae_{i,1}$ and not isomorphic to one of $Ae_{k,1}$ for $k\neq i$. Then $e_{i,j}Ae_{k,l}=(0)$ for $i\neq k$ by Lemma. Therefore, $A=\sum_i\oplus E_iAE_i$ as a ring, where $E_i=\sum_{j=1}^{\rho(i)} e_{i,j}$. Hence, we may assume $t=1$, $e=e_1$, and $1=e_1+\cdots+e_s$. By $n(e_i)$[4] we shall mean the largest integer p such that $N^pe_i\neq(0)$. If $e_iAe_j\neq(0)$ for $i\neq j$, then there exists an isomorphism φ of Ae_i into Ae_j. Since Ne_j is a unique maximal left ideal in Ae_j, $\varphi(Ae_i)\subseteq Ne_j$. Hence, $n(e_j)>n(e_i)$. After rearranging $\{e_k\}$, we may assume that $1=f_1+\cdots+f_s$, $n(f_i)\geqslant n(f_j)$ if $i\leqslant j$ and $\{f_i, \cdots, f_s\}\equiv\{e_1, \cdots, e_s\}$. Since Ae_1 is faithful, $e_1=f_1$. Furthermore, $f_iAf_j=(0)$ if $i<j$ from the above. Hence, $A=T_s(f_iAf_i\,;\,f_kAf_p)$[5]. It is clear that the f_iAf_i are division rings[6]. Furthermore, Af_1 is injective and $f_kAf_1\neq(0)$ for all k and hence, $Af_1\approx\mathrm{Hom}_{f_sAf_s}(f_sA, f_sAf_1)$ by Proposition 1. Therefore, $f_sAf_i\neq(0)$ for all i. Since Af_i has the irreducible socle, $[f_sAf_i:f_sAf_s]=1$. Hence $[f_iAf_1:f_1Af_1]=1$. Therefore, A is a partially PP-ring by [5], Proposition 5.

REMARK. Using the similar argument as above and Corollary 3 we can prove directly [5], Theorem 1.

Theorem 3. *Let A be a left and right QF-3 ring and semi-primary.*

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4) cf. [6].

5) See [6].

6) Since $f_sAf_1\neq(0)$ is left ideal in Af_1, the socle S of Af_1 is contained in f_sAf_1. Hence $S\approx Af_k$. We know $f_sAf_s\equiv\Delta_s$ is a division ring by Prop. 2. In the proof. of Prop. 1 we have used a fact that Δ_s is a division ring, and hence we obtain $[f_iAf_1:f_1Af_1]=1$ for all i as in the proof. Therefore, $f_iNf_iAf_i=(0)$ and hence $f_iNf_i=(0)$

We assume Ae (resp. fA) is a unique minimal faithful, projective, injective left (resp. right) A-module, and $e=e_1+\cdots+e_t$ ($f=f_1+\cdots+f_s$) is a decomposition of e (resp. f) into a sum of mutually orthogonal primitive idempotents e_i (resp. f_i). If 1) *$e_iAe_i=\Delta_i$ is a division ring for all i, then* 2) *$s=t$ and $e_iAe_i\approx f_{\pi(i)}Af_{\pi(i)}$. And furthermore, A is contained in a semi-simple ring $B=\sum_{i=1}^{t}\oplus(\Delta_i)_{n_i}$ such that* 3) *Ae_i (resp. $f_{\pi(i)}A$) is isomorphic to an irreducible left B-ideals in $(\Delta_i)_{n_i}$ as $A-\Delta_i$ (resp. Δ_i-A) module for $i=1,\cdots,t$. Conversely, if A is a subring in a semi-simple ring B satisfying* 1) 2) *and* 3), *then A is a left and right QF-3 ring, (cf.* [5], *Remark* 4).

Proof. If A satisfies 1), 2) and 3), then $Ae_i\approx Be_{ii}\approx\mathrm{Hom}_{\Delta_i}(e_{ii}B, \Delta_i)\approx\mathrm{Hom}_{\Delta_i}(f_{\pi(i)}A, \Delta_i)$, where e_{ii} is a matrix unit in $(\Delta_i)_{n_i}$. Hence, Ae_i is A-injective by [2], p. 166, Proposition 2.5a. Similarly we obtain that f_iA is A-injective. Since $\sum_{i=1}^{t}Be_{ii}$ is B-faithful, $\sum Ae_i$ is A-faithful. We assume that A is semi-primary left and right QF-3 ring satisfying 1). First we assume that A is basic. We put $e_i=g$, $\Delta_i=e_iAe_i=\Delta$ and $\mathrm{Hom}_{\Delta}(Ag,\Delta)=(Ag)^*$. $T=\{t\in(Ag)^*\mid, t(Ng)=(0)\}$ is a right A-submodule of $(Ag)^*$ and $TN=(0)$. Conversely, if $y\in(Ag)^*$ satisfies $yN=(0)$, $(0)=(yN)(Ag)$. Hence, $y\in T$. Therefore, T coincides with right socle of $(Ag)^*$. It is clear that $T\approx\mathrm{Hom}_{\Delta}(Ag/Ng, \Delta)$ as an A-module and since $Ag/Ng\approx\Delta$ (A is basic), T is an irreducible right A-submodule of $(Ag)^*$. Hence, $(Ag)^*$ is an indecomposable A-module (A is semi-primary). Put $M=\sum\oplus(Ae_i)^*$. Since $\sum\oplus Ae_i$ is A-faithful, M is A-faithful as right A-module. Hence, M contains fA as a direct summand. Since $(Ae)^*$ is an indecomposable injective A-module, $f_iA\approx(Ae_{\pi(t)})^*$ for $i=1,\cdots,s$ by the generalized Krull-Schmidt's Theorem in [1], where π is a one-to-one mapping of $\{1,\cdots,s\}$ to $\{1,\cdots,t\}$. By φ we denote the isomprphism $f_{\pi^{-1}(t)}A\approx\mathrm{Hom}_{\Delta_i}(Ae_i, \Delta_i)$. The $\varphi(T)\approx\mathrm{Hom}_{\Delta_i}(Ae_i/Ne_i, \Delta_i)$. Since $e_i(Ae_i/Ne_i)=Ae_i/Ne_i$, $T=f_kTe_i$, $(k=\pi^{-1}(\mathrm{t}))$. $\varphi(f_kAe_i)=\mathrm{Hom}_{\Delta_i}(\Delta_i, \Delta_i)\approx\Delta_i$. Therefore, $T=f_kTe_i=f_kAe_i$. Let S be the left socle Ae_i. Then $S=S\Delta_i$, since Ae_i is A-injective. Hence, we obtain $\varphi(f_kA/f_kN)=\mathrm{Hom}_{\Delta_i}(S, \Delta_i)$. Therefore, $f_kS=S=f_kSe_i=f_kAe_i$. Furthermore, $\varphi(f_kNf_k)(Ae_i)=\varphi(f_k)(NS)=(0)$, $f_kAf_k/f_kNf_k=f_kAf_k$ is a division ring. Therefore, if we exchange "left" and "right" in the above argument, we obtain $(f_kA)^*\approx Ae_i$ and $s=t$. Similarly we obtain $[f_kAe_i:\Delta_k]=1$. Hence, $f_kAe_i=\Delta_kx'=x'\Delta_i$ for some $x'\neq 0$. Therefore, we have an isomorphism ψ of Δ_i to Δ_k such that $x'\delta=\psi(\delta)x'$ for $\delta\in\Delta_i$. Let $\varphi(f_k)=g$, then $gf_k=g$ and $g(ye_i)=g(f_kye_i)=g(x')\delta$, where $f_kye_i=x\delta$ and $y\in A$. We may assume $g(x)=1\in\Delta_i$ for some $x\in f_kAe_i$. Then we can easily check that φ is given by a multiplication of element of f_kA from the left side and φ is a right A and left (Δ_k, ψ)-semilinear mapping. Therefore, the facts $(Ae_i)^* \approx f_k A$ and $(f_k A)^* \approx Ae_i$ imply that $[Ae_i : \Delta_i] = [f_k A : \Delta_k] = n_i < \infty$ by [7], p. 68, Theorm 1. Let $C = \sum \oplus \Delta_i$ and $B = \mathrm{End}_C(Ae) = \sum \oplus \mathrm{End}_{\Delta_i}(Ae_i)$. Since Ae is A-faithful, we may regard A as a subring of B. Then it is clear from the fact $(Ae_i)^* \approx f_k A$ that $f_k A$ is isomorphic to an irreducible right ideal in B as $\Delta_i - A$ module. Furthermore, $(Ae_i)(Ae_i) = Ae_i$, hence Ae_i is isomorphic to an irreducible left ideal in B as an $A - \Delta_i$ module. If A is not basic, then we can use the same argument as the above after enlarging the degree n_i of the simple rings $B_i = \mathrm{End}_{\Delta_i}(Ae_i)$.

We shall consider the converse of Proposition 2.2)

Theorem 4. *In Theorem* 3 *we assume furthermore that* $e_i A e_j = (0)$ *for* $i \neq j$. *Then* Ae_i *and* $f_i A$ *coincide with irreducible left ideal and right ideals in* B, *respectively, and the socle of* Ae_i *and* $f_i A$ *ard projective.*

Proof. We may assume $(Ae_i)^* \approx f_i A$ and $(f_i A)^* \approx Ae_i$ and A is basic. Since $e_i A e_j = (0)$ for $i \neq j$, $Ae_i \subseteq \mathrm{Hom}_{\Delta_i}(Ae_i, Ae_i) = B_i$ coincides to an irreducible left ideal in B_i. Since $f_i A \approx \mathrm{Hom}_{\Delta_i}(Ae_i, \Delta_i)$, $f_i A e_j \approx \mathrm{Hom}_{\Delta_i}(e_j A e_i, \Delta_i) = (0)$ for $i \neq j$. Hence, $f_i A = \mathrm{Hom}_{\Delta_i}(Ae_i, \Delta_i) \subseteq B_i$. Hence, we can assume that $e_i = e_{ll}^{(i)}$, $f_i = e_{mm}^{(i)}$, where the $e_{jj}^{(j)}$ are matrix units in B_i. Let E_i be the identity element of B_i. If $E_i A$ contains $x = \sum_j \delta_j e_j m$ such that $\delta_j \neq 0$ for some $j \neq m$, then $x f_i A e_i \not\subset f_i A e_i$. However, $f_i A e_i$ is the left socle of Ae_i and $E_i A e_i = Ae_i$. Therefore, we obtain $E_i A \cap B_i f_i = \Delta_i f_i$. Hence $Af_i = f_i A f_i$ is a A-projective. Since $f_i A e_i \approx f_i A f_i$ as a left A-module, $f_i A e_i$ is A-projective.

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