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Osaka University
On Alternating Knots

By Kunio Murasugi

Introduction

Let $k$ be a knot in $S^3$. Let $k$ be an image of the regular projection of $k$ onto $S^2$. A knot projection $K$ is said to be alternating if and only if it is connected and, as one follows along this knot, undercrossings and overcrossings alternate. A knot is said to be alternating if it possesses an alternating projection. There will be, of course, non-alternating knots. In fact, the first proof of their existence was given by Bankwitz in 1930 [3]. We do not know the general method by which we can decide whether or not a given knot projection represents an alternating knot. But some good methods have been found up to present. Recently, R. H. Crowell proved the theorem (cf. Theorem (6.5) [5]) which much improved the Bankwitz's theorem (cf. Satz, p. 145 [3]). He showed by means of this theorem that seven of eleven non-alternating projections, in the Knot Table at the end of Reidemeister's Knotentheorie [11], represent non-alternating knots.

In a previous paper [9] we gave a necessary condition for a given knot to be alternating by means of its Alexander polynomial (cf. [4], [8]). In the present paper, in order to characterize the Alexander polynomial of the alternating knot, we shall assign a matrix, called the knot matrix, to an alternating knot. The relation between the knot matrix and its Alexander polynomial is expressed in Theorem 1.17 which is the fundamental theorem of the present paper. From this theorem simply follow the main theorems in [4], [7], [8], [9] (Theorem 3.8, Theorem 3.13). Theorem 3.12 is a simple application of Theorem 1.17 and it plays a particular rôle in §3.

---

1) A knot is an oriented polygonal simple closed curve. A link of multiplicity $\mu$ is the union of $\mu$ ordered, pairwise disjoint knots. In the present paper we do not distinct exactly between links and knots, except the cases 3.7 and 3.8. Thus, by a knot (of multiplicity $\mu$) is meant an ordinary knot or link according to $\mu=1$ or $\mu>1$.

2) For any knot we may select a “point at infinity” $\infty \in S^3$ and consider a Cartesian coordinate system $R \times R \times R = S^3$. The projection $p : S^3 \rightarrow S^2$ is defined by $p(\infty) = \infty$ and $p(x, y, z) = (x, y)$. For each double point $p(a) = p(b)$, one of $a$ and $b$ with the larger $z$-coordinate is called the overcrossing and the other the undercrossing.
As an application of these theorems, it will be shown, moreover, that eleven non-alternating projections in the Knot Table [11] really represent non-alternating knots.

§ 1. Knot Matrix

In this section we shall assign a matrix to an alternating knot.

1.1. Let $k$ be a knot and $K$ be a regular projection of $k$. Let $K$ be oriented by the orientation induced by that of $k$. Let $K$ have $n$ double points $D_1, D_2, \ldots, D_n$. Throughout the present paper, we may assume that

(1.1) $K$ has no trivial double point.

By a trivial double point is meant a double point $D$ as is shown in Fig. 1.

![Fig. 1.](image)

$K$ divides $S^2$ into $n+2$ regions $r_0, r_1, \ldots, r_{n+1}$. At each double point $D_i$, just four corners of four regions, say, $r_j, r_k, r_l$ and $r_m$, meet. Two corners among these four corners are marked with dots as is shown in Fig. 2.

![Fig. 2.](image)

The segments of $K$ connecting two consecutive double points are called sides of $K$.

Now, let us divide $K$ into some oriented loops, called the standard loops, as follows. Imagine an insect crawling along $K$ in the positive direction of $K$. This insect must always turn to the left or to the right at a double point in the positive direction of $K$. Then, it will traverse a loop $L$, called a standard loop. It is clear that

(1.2) Each side of $K$ is contained in one and only one standard loop.

If $L$ bounds a region $r_i$, we say $L$ is of the first kind and $r_i$ is bounded by $L$. Otherwise, $L$ is of the second kind. If $K$ has no standard loop of the second kind, the alternating knot $k$ is called a special alternating one. From the rule of the marking with dots, we have immediately,

(1.3) The corners of the regions bounded by the standard loops of

3) By a loop is meant a simple closed curve.
the first kind are either all dotted or all undotted. The converse is also true (cf. Lemma 6.3 [8]).

Let \( m \) be the number of the standard loops of the second kind of \( K \). We may assume that these \( m \) loops are disjoint. (We have only to deform slightly some of these loops, if necessary.) Then, it follows

(1.4) \( m \) standard loops of the second kind divide \( S^2 \) into \( m+1 \) domains\(^5\) (cf. Lemma 2.4 [8]).

By the genus of a domain is meant the number of the standard loops of the second kind bounding the domain minus 1. Then it is easy to show that

(1.5) The sum of the genera of \( m+1 \) domains is equal to \( m-1 \).

Now we can introduce the rules on the numbering of \( m \) standard loops of the second kind \( C_1, \ldots, C_m \) and \( m+1 \) domains, \( E_1, E_2, \ldots, E_{m+1} \) in such a way that the following conditions hold (cf. § 3 [8]):

(1.6) If \( \hat{E}_i= C_{i_0} \cup C_{i_1} \cup \cdots \cup C_{i_j} \), then \( i_0=i \) and \( i_1, \ldots, i_j < i \), for \( i=1, \ldots, m \).

\( C_i \) is called the outer boundary of \( E_i \). We denote the remaining domain by \( E_{m+1} \), and the outer boundary of it is defined as follows: Let \( \hat{E}_{m+1}= C_{\lambda_1} \cup \cdots \cup C_{\lambda_p} \). Denoting \( \lambda=\max(\lambda_1, \ldots, \lambda_p) \), \( C_\lambda \) is the outer boundary of \( E_{m+1} \).

Let us denote \( (E_i \cup \hat{E}_i) \cap K=K_i \).

A double point such that at least two of the four regions meeting at it are contained in \( E_i \) is called a double point contained in \( K_i \) (or simply in \( K_i \)).

(1.7) Each double point is necessarily contained in only one of these \( K_i \). By the sides of \( K_i \) are meant segments of \( K_i \) connecting two consecutive double points in \( K_i \). Then it is clear that

(1.8) Each \( K_i \) is a knot projection of a knot, say \( k_i \), and, in particular, each \( k_i \) is special alternating.

And from (1.1), we have immediately

(1.9) \( K_i \) have no trivial double points.

Moreover, we can prove easily

---

4) A domain is a connected open subset of \( S^2 \).
5) A dot over the symbol denotes the set of boundary points.
Lemma 1.10. The regions contained in $E_i$ can be classified into two classes, called "black" or "white", in such a way that every side of $K_i$ is the common boundary of a black and a white region and that the region having some sides in common with $E_i$ is a white region and that each black region is bounded by a standard loop of the first kind (cf. Lemma 2.6 [8]).

1.2. The Alexander matrix $A$ of a knot is defined as follows. Let four regions meet at a double point $D_i$ of $K$ as is shown in Fig. 2. Then we assign to $D_i$ a linear equation:

$$D_i(r) = tr_j - tr_k + r_l - r_m = 0.$$ 

From (1.1) we see $j, k, l$ and $m$ are different from one another. Then $A$ is defined to be the matrix constructed by all coefficients of these equations. $A$ has $n$ rows and $n+2$ columns. Each row and each column correspond to a double point and a region respectively. Then the determinant of the square matrix obtained from $A$ by striking out two columns corresponding to a pair of regions, which have a side in common, is uniquely determined, freed from the factor $\pm t^\lambda$. This is a knot invariant. We call it the $A$-polynomial and denote it by $\Delta(t)$. Hereafter we may assume without loss of generality that the constant term $\Delta(0)>0$. Here we should notice that

$$(1.11) \text{ If the multiplicity } \mu \text{ of a knot } k \text{ is one, then the } A \text{-polynomial and the ordinary Alexander polynomial are the same. If } \mu>1, \text{ then } \Delta(t) = (1-t)^\mu \Delta(t, \cdots, t), \text{ where } \Delta(t, \cdots, t_\mu) \text{ is the ordinary Alexander polynomial of } k.$$ 

Moreover, if the knot projection is separated into two disjoint parts, the $A$-polynomial is zero (cf. [1]). We have to exclude such a knot, because we treat only the knots whose projection are connected as stated in the introduction.

Let $k$ be a special alternating knot. We shall assign a matrix $M$ to $k$ by means of its projection $K$.

Definition 1.12. Let $W_1, W_2, \cdots, W_p$ be the white regions in $K$. Then a matrix $M = (a_{ij})_{i,j=1,2,\ldots,p}$, called the knot matrix, is defined as follows:

- $a_{ii}$ is one half of the number of the double points lying on $\hat{W}^i$.
- $-a_{ij}(i \neq j)$ is the number of the dotted corners of $W_i$ at the double points lying on $\hat{W}_i \cap \hat{W}_j$.

Then it is easy to prove the following

6) It is clear that $a_{ii}$ is an integer. (cf. Lemma 3.6 [7])
Theorem 1.13. Two special alternating knots of s-equivalent knot matrices are equivalent.

Two matrices $M, M'$ are said to be s-equivalent if one can be transformed into the other by applying finitely many times the following operations: To exchange $i$-th row and $j$-th row and, at the same time, to exchange $i$-th column and $j$-th column ($i, j=1, \cdots, p$).

Example 1. The knot matrix $M$ of the knot as is shown in Fig. 3 is the following:

$$M = \begin{pmatrix} 1 & -1 & 0 \\ 0 & 1 & -1 \\ -1 & 0 & 1 \end{pmatrix}$$

(1.14) $M$ is of the following properties:

(i) $a_{ii} > 0$, $a_{ij} \leq 0$, $(i, j=1, \cdots, p, i \neq j)$,

(ii) $\sum_{i=1}^{p} a_{ij} = 0$, $\sum_{j=1}^{p} a_{ij} = 0$,

(iii) $|a_{ij} - a_{ji}| \leq 1$.

Next, we shall assign a matrix to an alternating knot. Let $K$ have $m$ standard loops of the second kind $C_1, C_2, \cdots, C_m$. Let $E_i, K_i$ be the same as in 1.1. Then

Definition 1.15. The knot matrix $M$ of $k$ is defined as follows:

$$M = \begin{pmatrix} M_{11} & M_{12} & \cdots & M_{1,m+1} \\ M_{21} & M_{22} & \cdots & M_{2,m+1} \\ \vdots & \vdots & \ddots & \vdots \\ M_{m+1,1} & M_{m+1,2} & \cdots & M_{m+1,m+1} \end{pmatrix}$$

(i) $M_{ij}$ are the special knot matrices assigned to the special alternating knots $k_i$.

(ii) $M_{ij} = (b_{ab})$ are defined as follows:

(a) If $E_i \cap \bar{E}_j = \emptyset$, $M_{ij} = M_{ji} = 0$.

(b) Let $E_i \cap \bar{E}_j = C_i$, $l = \min(i, j)$.

(i) If $E_i$ stands on the right hand side in the positive direction of $C_i$, then $M_{ij} = 0$.

(ii) Otherwise, $W_a \subset E_i$, $W_b \subset E_j$, $b_{ab} = \lambda - \mu$, where $\lambda, \mu$ denote the numbers of the dotted and undotted corners of $W_a$ at all double points in $K_i$ lying on $W_a \cap W_b$ respectively.

Example 2. The knot matrix $M$ of the knot as is shown in Fig. 4 is the following:
$M = \begin{pmatrix}
1 & -1 & 0 \\
-1 & 1 & 0 \\
1 & -1 & 0 \\
0 & 0 & 1 -1 & 0 \\
0 & 0 & -1 & 0 & 2 -1 \\
-1 & 1 & 0 & 0 & -1 & 1
\end{pmatrix}$

(1.16) $M_{ij}$ (i $\neq$ j) have the following properties:

1. At least one of $M_{ij}$, or $M_{ji}$ is equal to 0.

2. Each row and each column of $M_{ij}$ contain only two elements different from 0, if these row and column are different from 0. And one is 1, the other is -1.

3. $\sum b_{ab} = 0$, $\sum b_{ab} = 0$.

1.3. We shall introduce the notations on matrices and determinants used in the present paper from now on.

Let $M = (a_{ij})_{i=1, \ldots, n, j=1, \ldots, m}$ be a matrix. By $M(i_1, i_2, \ldots, i_p)$ is denoted the matrix consisting of $i_1$ row, $i_2$ row, $\ldots$, $i_p$ row, and $j_1$ column, $j_2$ column, $\ldots$, $j_q$ column, of $M$:

$$M(i_1, i_2, \ldots, i_p) = \begin{pmatrix}
a_{i_1i_1} & a_{i_1i_2} & \cdots & a_{i_1i_q} \\
a_{i_2i_1} & a_{i_2i_2} & \cdots & a_{i_2i_q} \\
\vdots & \vdots & \ddots & \vdots \\
a_{ip_{i_1}} & a_{ip_{i_2}} & \cdots & a_{ip_{i_q}}
\end{pmatrix}$$

In particular, we may denote $M(i, j)$ by $M(i, j)$. By $M(i_1, i_2, \ldots, i_p)$ is denoted the matrix obtained from $M$ by striking out $i_1$, row, $\ldots$, $i_p$ row, and $j_1$ column, $\ldots$, $j_q$ column:

$$M(i_1, i_2, \ldots, i_p) = M(r_1, \ldots, r_{n-p})$$

where $r_1 < \cdots < r_{n-p}$, $s_1 < \cdots < s_{m-q}$, and $(r_1, \cdots, r_{n-p})$, $(s_1, \cdots, s_{m-q})$ denote the sets of $(n-p)$, $(m-q)$ integers obtained from the sets of $n$, $m$ integers $(1, 2, \cdots, n)$, $(1, 2, \cdots, m)$ by striking out the sets of $p$, $q$ integers $(i_1, i_2, \ldots, i_p)$, $(j_1, j_2, \ldots, j_q)$ respectively. By $M'$ is denoted the transposed matrix of $M$:

$$M' = (b_{ij})_{i=1, \ldots, m, j=1, \ldots, n} \text{ where } b_{ij} = a_{ji}.$$
Using these notations, the fundamental theorem of the present paper can be stated as follows:

**Theorem 1.17.** Let $\Delta(t)$ denote the $A$-polynomial of an alternating knot. Then,

$$\pm t^k \Delta(t) = \det \left[ M(\mathbf{i}) - t M^T(\mathbf{i}) \right],$$

where (i) $\lambda$ is a suitable integer, (ii) each $i_p$ is chosen arbitrarily such that $M(\mathbf{i}_p)$ is contained in $M_{p\mathbf{i}}$ for $p = 1, \ldots, m+1$, and (iii) $M^T$ denotes the following matrix:

$$M^T = \begin{pmatrix}
M_{11} & -M_{12} & \cdots & -M_{1, m+1, 1} \\
-M_{21} & M_{22} & \cdots & -M_{2, m+1, 2} \\
\vdots & \vdots & \ddots & \vdots \\
-M_{m+1, m+1} & -M_{m+1, m+2} & \cdots & M_{m+1, m+1}
\end{pmatrix}.$$  

Proof will be given in 1.4, 1.5.

**1.4.** Let $W_{i, 1}, \ldots, W_{i, p_i}$ and $B_{i, 1}, \ldots, B_{i, q_i}$ denote the white and black regions in $K_i$ respectively. Let $D_{i, 1}, \ldots, D_{i, q_i}$ denote the double points in $K_i$ and let $s_i$ denote the genera of $E_i(i = 1, \ldots, m+1)$. Then we see immediately,

\begin{equation}
(1.18) \quad p_i + q_i + s_i - 1 = n_i.
\end{equation}

We may assume without loss of generality that

\begin{equation}
(1.19) \quad \begin{array}{l}
(i) \quad (p_i + q_i) + \cdots + (p_i - 1 + q_i - 1) + j^\text{th} \text{ column corresponds to the white region } W_{i, j}, \\
(ii) \quad (p_i + q_i) + \cdots + (p_i - 1 + q_i - 1) + p_i + l^\text{th} \text{ column corresponds to the black region } B_{i, l}, \\
(iii) \quad n_i + \cdots + n_i - 1 + r^\text{th} \text{ row corresponds to } D_{i, r}, \text{ for } i = 1, \ldots, m+1.
\end{array}
\end{equation}

(We have only to change the permutation of columns and rows, if necessary.) For the sake of brevity, we say, for example, the column corresponding to $W_{i, j}$ is $W_{i, j}$-column and say the row corresponding to $D_{i, j}$ is $D_{i, j}$-row. And we denote, for example, $M_{W_{i, j}}^{D_{i, j}}$ by $M(\frac{1}{W_{i, j}})$. Now, let us take a point, called the center, from each white region and a point from each black region, and fix them. Moreover, we take a point from each component of the complementary domain of $E_i(i = 1, \ldots, m+1)$, and fix them. We call it the center of the complementary
domain. The subset $G_i$ (or $G^*_i$) of $S^2$ obtained by connecting the centers of all the white regions (or all the black regions) with the double points lying on their boundaries will be called the graph (or the dual graph) of $K_i$. The center of each white region (or each black region) is called the vertex of $G_i$ (or $G^*_i$) and the segments of $G_i$ (or $G^*_i$) connecting two consecutive vertices are called the sides of $G_i$ (or $G^*_i$).

Let $T$ be a tree\(^7\) in $G_i$. The subset of $G^*_i$ consisting of sides disjoint to $T$ will be called the dual of $T$, denoted by $T^*$. Then, we see

\[(1.20) \text{ If } T \text{ is a maximal tree in } G_i, \text{ then } T^* \text{ is also a maximal tree in } G^*_i.\]

Let $T_i$ be a maximal tree in $G_i$ and let us fix it. Since $T_i$ contains $p_i$ vertices, it contains $p_i-1$ sides. Then it follows

**Lemma 1.21.** There exists 1-1 correspondence $\varphi_i$ between $p_i$ vertices except one vertex, the center of $W_{i,1}$, say, and $p_i-1$ sides of $T_i$ in such a way that each side of $T_i$ corresponds to one of vertices lying on its side by $\varphi_i$. Moreover a correspondence $\varphi_i$ is unique. In the same way, there exists one and only one 1-1 correspondence $\varphi^*_i$ between $q_i+s_i+1$ vertices except one vertex, one of the centers of components of the complementary domain $S^2-E_i$, and $q_i+s_i$ sides of $T^*_i$.

Since this lemma will be proved by induction on the number of the sides in $T_i$, we omit the details of proof.

$\varphi_i$ naturally gives rise to 1-1 correspondence between $W_{i,1}, \ldots, W_{i,p_i-1}$, and $D_{i,1}, \ldots, D_{i,p_i-1}$, and $\varphi^*_i$ gives rise to 1-1 correspondence between $B_{i,1}, \ldots, B_{i,q_i}$ and $s_i$ components of complementary domain $F_{i,1}, \ldots, F_{i,s_i}$, and $q_i+s_i$ double points $D_{i,p_i}, \ldots, D_{i,n_i}$ ($n_i=p_i+q_i+s_i-1$). We may assume that $\varphi_i(W_{i,\lambda})=D_{i,\lambda}$, $\varphi^*_i(B_{i,\mu})=D_{i,p_i+\mu-1}$, $\varphi^*_i(F_{i,\nu})=D_{i,p_i+s_i+\nu-1}$. (We may have only to change the numbering of the double points, if necessary.) We should note that there is no double point corresponding to $m+1$ white regions $W_{1,\rho_1}, \ldots, W_{m+1,\rho_{m+1}}$. We may assume, moreover, that $W_{r,\rho_r}$ and $W_{m+1,\rho_{m+1}}$ have a side in common, because the outer boundary $C_r$ of $E_r$ for some $r$ is also the outer boundary of $E_{m+1}$. See 1.1.

**1.5.** We say the transformation from a matrix $M=(a_{ij})_{i=1, \ldots, n; \; j=1, \ldots, m}$ to a matrix $M'$:\n
\[7) \text{ By a tree is meant the connected subset of } G \text{ which contains no loop. A tree is called maximal if it contains all vertices of } G.\]
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\[ M' = \begin{pmatrix}
  a_{11} & a_{12} & \cdots & a_{1m} \\
  \sum_{i=1}^{n} \alpha_{i1}a_{i1} & \sum_{i=1}^{n} \alpha_{i2}a_{i2} & \cdots & \sum_{i=1}^{n} \alpha_{im}a_{im} \\
  \sum_{i=1}^{n} \alpha_{11}a_{1i} & \sum_{i=1}^{n} \alpha_{12}a_{1i} & \cdots & \sum_{i=1}^{n} \alpha_{1m}a_{1i} \\
  \sum_{i=1}^{n} \alpha_{21}a_{2i} & \sum_{i=1}^{n} \alpha_{22}a_{2i} & \cdots & \sum_{i=1}^{n} \alpha_{2m}a_{2i} \\
  \cdots & \cdots & \cdots & \cdots \\
  a_{n1} & a_{n2} & \cdots & a_{nm}
\end{pmatrix}
\]

is obtained by adding essentially \( \alpha_{11} \) times of the first row, \( \ldots \), \( \alpha_{1m} \) times of the \( n^{th} \) row to \( j_1^{th} \) row and by adding essentially \( \alpha_{21} \) times of the first row, \( \ldots \), \( \alpha_{2m} \) times of the \( n^{th} \) row to \( j_2^{th} \) row, \( \ldots \). Then it will be easily proved

(1.22) The essential addition is the elementary transformation, if \( \alpha_{1,j_1}, \ldots, \alpha_{i,j_1} \) are all \( \pm 1 \).

By the elementary transformation is meant the transformation obtained by applying a finite number of times the following three operations \( E_i \) (or \( E_i^* \)):

\( E_i \) (or \( E_i^* \)): To exchange two rows (or two columns).

\( E_2 \) (or \( E_2^* \)): To add \( \lambda \) times of a row (or a column) to the other row (or column), \( \lambda \) being integer.

\( E_3 \) (or \( E_3^* \)): To multiply a row (or a column) by \( \pm 1 \).

Two matrices are called equivalent if one can be transformed into the other by elementary transformations.

We shall transform the Alexander matrix \( A \).

Let \( D_{i,j_1}, \ldots, D_{i,j_2} \), in which \( D_{i,j} \) is contained, be double points in \( K_i \) lying on \( W_{i,j} \) for \( j=1, \ldots, p_i-1 \). Then we see

(1.23) \( \lambda \) is even, and just one half of all corners of \( W_{i,j} \) at these double points are dotted. (cf. Lemma 6.3 [8])

Moreover, we have

(1.24) The signs of all elements in \( A \left( \begin{array}{cccc}
  D_{i_1,j_1} & \cdots & D_{i_1,j_{p_i-1}} \\
  W_{i_1,j_1} & \cdots & W_{i_1,j_{p_i}}
\end{array} \right) \) are constant. (cf. Lemma 3.1 [7])

Hence we can denote these elements different from 0 by \( \varepsilon_it \) or \( \varepsilon_i \cdot 1 \), \( \varepsilon_i = \pm 1 \). Let us assume, then, \( A(D_{i,j_1}, W_{i,j_1}) = A(D_{i,j_2}, W_{i,j_2}) = \cdots = A(D_{i,j_\mu}, W_{i,j_\mu}) = \varepsilon_i \cdot 1 \) and \( A(D_{i,j_\mu+1}, W_{i,j_\mu+1}) = A(D_{i,j_\mu+2}, W_{i,j_\mu+2}) = \cdots = A(D_{i,j_\lambda}, W_{i,j_\lambda}) = \varepsilon_it \), where \( \mu = \lambda/2 \). Now we essentially add \( \varepsilon_i \) times of \( D_{i,j_1} \)-row, \( \cdots \), \( D_{i,j_\mu} \)-row to \( D_{i,j} \)-row, and essentially add \( -\varepsilon_i \) times of \( D_{i,j_\mu+1} \)-row, \( \cdots \), \( D_{i,j_\lambda} \)-row to \( D_{i,j} \)-row. Thus \( A \) is transformed into a matrix \( A' \). Then it follows
(1.25) $A'$ is equivalent to $A$.
Moreover, we have

$$A'(D_{i,j}, W_{i,j}) = a_{i,j} (1-t), \quad A'(D_{i,j}, W_{i,j}) = a_{i,j} - ta_{i,j},$$

where $a_{i,j}$ denotes the number of the undotted corners of $W_{i,j}$ at all double points in $K_i$ lying on $W_{i,j}$, and where $a_{i,j}$ denotes the number of the dotted corners of $W_{i,j}$ at all double points in $K_i$ lying on $W_{i,j}$.

Thus it follows

$$A'(D_{i,1}, ... D_{i,P_{i-1}}) = \tilde{M}^{-1}_{i,j} (p_i) - t \tilde{M}^{-1}_{i,j} (p_i).$$

Moreover, we can prove

$$A'(D_{i,1}, ... D_{i,P_{i-1}}) = 0, \quad \text{for } i = 1, ..., m+1.$$ 

Proof. Because the common part of a black region and a white region has only to contain the consecutive two double points, q.e.d.

Since double points lying on the boundary of a black region of $E_i$ are contained in $K_i$, we have obviously

$$A'(D_{1,1} ... D_{i,1} ... D_{i,P_{i-1}}) = 0,$$

for $i = 1, ..., m+1.$

Next, we shall decide the form of the matrix $P'_{ij}$.

Since $P'_{ij}$ is transformed from $P_{ij}$:

$$P'_{ij} = A'(D_{j,1} ... D_{j,P_{j-1}})(i \neq j).$$

we shall first decide the form of $P_{ij}$. From the definition, we have

$$A(D_{j,1} ... D_{j,P_{j-1}}) = 0.$$ 

Let $E_i \cap E_j = C_i$, $l = \min(i, j)$. Then elements of $P_{ij}$ are given as follows:

(1.31) (i) If $D_{i,\lambda}$ does not lie on $W_{j,\mu}$, then

$$A(D_{i,\lambda}, W_{j,\mu}) = 0.$$
(ii) **The case where** $D_{i,\lambda}$ **lies on** $\hat{W}_{j,\mu}$.

(a) **If** $E_i$ **stands on the left hand side in the positive direction on** $C_i$,

$$A(D_{i,\lambda}, W_{j,\mu}) = \varepsilon_j \cdot t.$$  

(b) **Otherwise,** $A(D_{i,\lambda}, W_{j,\mu}) = \varepsilon_j \cdot t$.

Hence, we can decide the elements of $P_{ij}$ as follows:

\[ (1.32) \]

Let $D_{i,t}, \ldots$ be the double points except $D_{i,\lambda}$ in $K_i$ lying on $\hat{W}_{i,n} \cap \hat{W}_{j,\mu}$.

(i) **If** $E_i$ **stands on the left hand side in the positive direction on** $C_i$,

$$A'(D_{i,\lambda}, W_{j,\mu}) = A(D_{i,\lambda}, W_{j,\mu}) + \varepsilon_i A(D_{i,t}, W_{j,\mu}) + \cdots.$$  

(ii) **Otherwise,** $A'(D_{i,\lambda}, W_{j,\mu}) = A(D_{i,\lambda}, W_{j,\mu}) - \varepsilon_i A(D_{i,t}, W_{j,\mu}) - \cdots$.

Thus, we have from (1.32),

\[ (1.33) \]

(i) **If** $E_i$ **stands on the left hand side in the positive direction on** $C_i$, **then** $A'(D_{i,\lambda}, W_{j,\mu}) = \alpha - \beta$,

(ii) **Otherwise,** $A'(D_{i,\lambda}, W_{j,\mu}) = (\alpha - \beta)t$,

where $\alpha, \beta$ denote the numbers of the dotted and undotted corners of $\hat{W}_{i,\lambda}$ at all double points in $K_i$ lying on $\hat{W}_{i,\lambda} \cap \hat{W}_{j,\mu}$ respectively and $\mu = p_1$.

Thus it follows

\[ (1.34) \]

Let $A''$ be a matrix obtained from $A'$ by adding all $W_{i,1}$-column, $\ldots$, $W_{i,p_{i-1}}$-column to $W_{i,1}$-column for $i = 1, \ldots, m+1$. Then it is clear that

\[ (1.35) \]

Moreover, we have

\[ (1.36) \]

Because $A''(D_{i,\lambda}, W_{j,p_j}) = \sum_{i=1}^{p_j} A'(D_{i,\lambda}, W_{j,\mu}) = 0$.

Let $E_i$ consist of the outer boundary and $s_i$ standard loops of the second kind $C_{i,1}, \ldots, C_{i,s_i}$. $C_{i,t}$ are the outer boundaries of $E_{i,t}$. Then let $A'''$ be a matrix obtained from $A''$ by arranging $s_i$ columns $W_{i,1}p_i$-column, $\ldots$, $W_{i,s_i}p_i$-column, after $B_{i,q_i}$-column. We should note that $W_{r,p_r}$-column and $W_{m+1,p_{m+1}}$-column do not move. Let us denote

$$A'''(W_{r,p_r}, W_{m+1,p_{m+1}}) = A_1,$$
Then it follows by noting that $\varphi^k_i$ in Lemma 1.21 is unique,

\begin{equation}
\det A_0 \left( \frac{D_i,p_i \cdots D_i,p_i}{W_{i_1,p_i} \cdots W_{i_2,p_i} \cdots W_{i_3,p_i}} \right) = \pm t^v,
\end{equation}

where $v$ is a suitable integer.

Moreover, if $g(E_i) = 0$, $E_i$ is bounded only by its outer boundary. Hence we have

\begin{equation}
\text{If the genus } g(E_i) \text{ of } E_i \text{ is zero, then}
\end{equation}

\begin{equation}
A_0 \left( \frac{D_{i-1} \cdots D_{i-1}}{W_{i_1,p_i} \cdots W_{i_2,p_i} \cdots W_{i_3,p_i}} \right) = 0.
\end{equation}

If $E_i(i \neq m + 1)$ are bounded by $C_1, \cdots, C_i$, and hence $A_0 \left( \frac{D_{i,p_i} \cdots D_{i,p_i}}{W_{i_1,p_i} \cdots W_{i_2,p_i} \cdots W_{i_3,p_i}} \right) = 0$. Thus we have at last the following:

\begin{equation}
\det A = \det \begin{pmatrix}
A_i(t) & B_{i_1}(t) & \cdots & B_{i_m}(t) \\
B_{i_1}(t) & A_i(t) & \cdots & B_{i_m}(t) \\
\vdots & \vdots & \ddots & \vdots \\
B_{m+1}(t) & B_{m+1}(t) & \cdots & A_{m+1}(t)
\end{pmatrix}
\end{equation}

where

\begin{align*}
A_i(t) &= A_0 \left( \frac{D_{i-1} \cdots D_{i-1}}{W_{i_1} \cdots W_{i_2,p_i}} \right) = A_i \left( \frac{D_{i-1} \cdots D_{i-1}}{W_{i_1} \cdots W_{i_2,p_i}} \right), \\
B_{i_1}(t) &= A_0 \left( \frac{D_{i-1} \cdots D_{i-1}}{W_{i_1} \cdots W_{i_2,p_i}} \right) = A_i \left( \frac{D_{i-1} \cdots D_{i-1}}{W_{i_1} \cdots W_{i_2,p_i}} \right).
\end{align*}

Thus the proof of Theorem 1.17 is complete.

**1.6.**

**Corollary 1.40.** If we set

\begin{equation}
D(t) = \det \left[ M \left( i_1 \cdots i_{m+1} \right) - tM^T \left( i_1 \cdots i_{m+1} \right) \right],
\end{equation}

then

\begin{equation}
D(0) = \prod_{j=1}^{m+1} \det \bar{M}_{ij} \left( i_j \right).
\end{equation}

Proof. It is sufficient to prove that $D(0) = \prod_{j=1}^{m+1} A_j(0)$. Since one of $B_{ij}(0)$ or $B_{ji}(0)$ is always 0, noting the numbering of $E_j$, we obtain the required result.

This corollary expresses that the special alternating knots play an important rôle in the studying of $D(0)$.
§ 2. Determinant

2.1. The square matrix $M=(a_{ij})_{i,j=1,...,n}$ is called the matrix of special type on the rows (or the columns) if it satisfies the following conditions (i), (ii) (or (ii)*):

(i) $a_{ii} > 0, \quad a_{ij} \leq 0, \quad \text{for } i, j=1, \ldots, n, \quad i \not= j.$

(ii) $\sum_{j=1}^{n} a_{ij} \geq 0, \quad \text{for } i=1, \ldots, n,$

((ii)*) $\sum_{i=1}^{n} a_{ij} \geq 0, \quad \text{for } j=1, \ldots, n$).

Moreover, $M$ is said to be of $(P)$-property on the row (or the column) if it satisfies the following condition:

(iii) There exists an $i$ such that $\sum_{j=1}^{n} a_{ij} \neq 0.$

((iii)*) There exists a $j$ such that $\sum_{i=1}^{n} a_{ij} \neq 0.$

$M$ is called the matrix of strongly special type on the row (or the column), if it is the matrix of special type on the row (or the column) and $M(i_1, \ldots, i_p)$ $(1 \leq i_1 < \cdots < i_p \leq n)$ are of $(P)$-property on the row (or the column). We first state a well-known result about matrices of this type.

Lemma 2.1. If $M$ is a matrix of special type on the row (or the column) and if $M$ is of $(P)$-property on all rows (or all columns):

\begin{equation}
\sum_{i=1}^{n} a_{ij} > 0 \quad \text{for all } i,
\end{equation}

then $\det M > 0.$

From this Lemma, it follows

Lemma 2.3. If $M=(a_{ij})_{i,j=1,\ldots,n}$ is a matrix of special type on the row (or the column), then $\det M \geq 0.$

Proof. Let $N(t)=(b_{ij})_{i,j=1,\ldots,n},$ where $b_{ii}=a_{ii}, \quad b_{ij}=ta_{ij}.$ Since $N(t)$ satisfies the condition (2.2) for $0 \leq t < 1,$ it follows $\det N(t) > 0.$ Moreover, since $\det N(t)$ is a continuous function of $t,$ it follows $\det M = \det N(1) = \lim_{t \to 1} N(t) \geq 0,$ q.e.d.

We shall prove the following

Lemma 2.4. If $M$ is of strongly special type on the row (or the column), then

(i) $\det M > 0,$

(ii) $(-1)^{i+j} \det M(i_j) \geq 0, \quad (i, j=1, \ldots, n)$

(iii) $\det M(i_i) \geq \det M(i_j), \quad (i, j=1, \ldots, n).$
Proof. Since (ii), (iii) are proved in the same way as used in the proofs of Satz 2, Satz 3 in [3], we omit the details. We shall prove (i) by induction on $n$. The case $n=1$ is clear. Suppose that Lemma is true for the case $n-1$. We may assume without loss of generality

\[ a_{21} + a_{32} + \cdots + a_{1n} > 0. \]  

We may assume that at least one of $a_{21}, a_{32}, \ldots, a_{1n}$ is different from 0. For otherwise, it follows $\det M = a_{11} \det \begin{pmatrix} 1 \\ 1 \end{pmatrix} > 0$. Let $a_{21} = 0$.

Now, let us denote

\[
\det M = \det \begin{pmatrix} a_{11} & 0 & 0 & \cdots & 0 \\ a_{21} & b_{22} & b_{23} & \cdots & b_{2n} \\ a_{31} & b_{32} & b_{33} & \cdots & b_{3n} \\ \vdots \\ a_{n1} & b_{n2} & b_{n3} & \cdots & b_{nn} \\ \end{pmatrix} = a_{11} \det M' = a_{11} \det \begin{pmatrix} b_{ij} \end{pmatrix},
\]

where

\[
b_{ii} = a_{ii} - \frac{a_{i1}a_{ii}}{a_{11}} \quad (i = 1),
\]

\[
b_{ij} = a_{ij} - \frac{a_{i1}a_{ij}}{a_{11}} \quad (i \neq j).
\]

To prove Lemma, it is sufficient to show that $M'$ is a matrix of strongly special type on the row. From (2.6), it follows immediately that $b_{ii} > 0$, $b_{ij} \leq 0$. Moreover, it follows $\sum_{j=2}^n b_{ij} = \sum_{j=2}^n \left( a_{ij} - \frac{a_{i1}a_{ij}}{a_{11}} \right) = \sum_{j=2}^n a_{ij} - \frac{a_{i1} \sum_{j=1}^n a_{ij} - a_{i1} a_{ij}}{a_{11}}$. In particular, $b_{21} + \cdots + b_{2n} > 0$. Thus we see $M'$ is of $(P)$-property on the row. Next, we shall prove that $M'(i_1, \ldots, i_p)$ are of $(P)$-property on the row. Suppose $M'(i_1, \ldots, i_p)$ be not of $(P)$-property on the row. For the sake of brevity, we shall prove the case on $M'(23 \cdots p)$. Then we see

\[
(2.7) \quad \begin{cases} b_{33} + b_{34} + \cdots + b_{3, p+1} = 0, \\ b_{43} + b_{44} + \cdots + b_{4, p+1} = 0, \\ \vdots \\ b_{p+1, 3} + b_{p+1, 4} + \cdots + b_{p+1, p+1} = 0. \end{cases}
\]

Since $b_{j3} + \cdots + b_{j, p+1} = a_{j3} + \cdots + a_{j, p+1} - \frac{a_{j1}(a_{i3} + \cdots + a_{1, p+1})}{a_{11}} = 0$, for $j = 3, \ldots, p+1$, and since $a_{i1} > - (a_{i2} + \cdots + a_{1n})$, we have

\[
(2.8) \quad a_{j3} + a_{j4} + \cdots + a_{j, p+1} = 0, \quad \text{for } j = 3, \ldots, p+1.
\]
Thus, it follows $M' \oplus (34 \cdots p+1)$ is not of $(P)$-property, which contradicts the assumption. Thus, $M'$ is a matrix of strongly special type, q.e.d.

2.2. A matrix $M=(a_{ij})$ of strongly special type is said to be $k$-strongly special type if it satisfies the conditions

$$|a_{ij} - a_{ji}| \leq 1,$$

for all $i, j$.

We can first prove

**Lemma 2.10.** Let $M=\begin{pmatrix} a & b \\ c & d \end{pmatrix}$ be a matrix of $k$-strongly special type on the row and the column, $a, b, c, d$ being integers. If $\det M=p$, then $a$ or $d \leq p$. If, in particular, $p=3$, then matrices without $s$-equivalent admit only the following 7 matrices:

$$\begin{pmatrix} 1 & -1 \\ 0 & 3 \end{pmatrix}, \begin{pmatrix} 1 & -1 \\ -1 & 4 \end{pmatrix}, \begin{pmatrix} 3 & -1 \\ 0 & 1 \end{pmatrix}, \begin{pmatrix} 3 & 0 \\ 1 & -1 \end{pmatrix}, \begin{pmatrix} 2 & -1 \\ -1 & 2 \end{pmatrix},$$

$$\begin{pmatrix} 3 & -3 \\ -2 & 3 \end{pmatrix}, \begin{pmatrix} -3 & 4 \end{pmatrix}.$$      

Proof. If $a=|b|$ or $d=|c|$, then it is clear that $a=|b| \leq p$ or $|c|=d \leq p$. Let $a>|b|$, $d>|c|$. Set $a=-b+\varepsilon$, $d=-c+\eta$, where $\varepsilon, \eta \geq 1$. Since $p=ad-bc=an-c\varepsilon \geq an$, it follows $a \leq p$. The latter half of this Lemma will be easily shown.

Our main theorem in this section, which is an extension of Lemma 2.4, is the following

**Theorem 2.11.** If $M=(a_{ij})_{i, j=1, \ldots, n}$, $a_{ij}$ being integers, is a matrix of $k$-strongly special type on the row and the column, then

$$\det M \geq \min \{a_{11}, a_{22}, \ldots, a_{nn}\}.$$ 

Proof. Let $\det M=p$. Theorem will be proved by induction on $p$ and $n$. The case $n=2$, the theorem is the same as Lemma 2.10.

Suppose that the theorem holds for the matrix $N$ such that degree of $N=n-1$ and that $\det N \leq p$. We may assume without loss of generality that $a_{i1}+a_{i2}+\cdots+a_{in}>0$. Let us set $p_i=(-1)^{i+1} \det M_i^{(1)}$, for $i=1, \ldots, n$. Then, it follows from Lemma 2.4 that $p_i \geq p_i$ and $p_i \geq 0$. Now, if $p_i \leq p$, then the theorem is proved by applying the assumption to $M_i^{(1)}$.

Otherwise, it follows $a_{ii} \leq p$. For, let $a_{ii} \geq p$. Then since $p_i \geq p_i$ and $a_{ii} \leq 0$ ($i=1$), we have $a_{ii}p_i \leq a_{ii}p_i$. Thus it follows $p=a_{i1}p_1+a_{i2}p_2+\cdots+a_{in}p_n \geq (a_{i1}+a_{i2}+\cdots+a_{in})p_i \geq p_i > p_i$. This is a contradiction. Thus we have $a_{ii} \leq p$.

Moreover, it follows
Lemma 2.12. If $M=(a_{ij})_{i,j=1,...,n}$ is a matrix of strongly special type on the row and the column, then

$$\sum_{1 \leq i_1 < \cdots < i_p \leq n} \det N_{i_1,\ldots,i_p} > 0,$$

for $p = 1, \ldots, n$,

where

$$N_{i_1,\ldots,i_p} = \begin{pmatrix}
  a_{i_1} & \cdots & a_{i_{p-1}} & a_{i_p} & \cdots & a_{in} \\
  a_{i_2} & \cdots & a_{i_{p-2}} & a_{i_{p-1}} & \cdots & a_{in} \\
  \vdots & & & & & \\
  a_{i_n} & a_{i_{n-p}} & a_{i_{n-p+1}} & \cdots & a_{in}
\end{pmatrix}.$$

Proof. We may assume without loss of generality that

$$\sum_{1 \leq i_1 < \cdots < i_p \leq n} \det N_{i_1,\ldots,i_p} > 0,$$

for $p = 1, \ldots, n$.

First, $N_{i_1,\ldots,i_p,\ldots}$ is a matrix of special type on the column, because $M$ is a matrix of special type on the row and the column. Thus it follows from Lemma 2.3 that

$$\det N_{i_1,\ldots,i_p,\ldots} \geq 0.$$

(We should note that $N_{i_1,\ldots,i_p,\ldots}$ is not necessarily a matrix of strongly special type.) We have to prove that one of the terms in summation has a positive value, that is, it is of strongly special type. To do this, we shall prove that $N_{i_1,\ldots,i_p}$ (for $p = 1, \ldots, n$) is a matrix of strongly special type. Set $N = N_{i_1,\ldots,i_p}$. Then $N$ is of $(P)$-property on the column from the assumption (2.14). Next, $\tilde{N}_{1,1}$ is of $(P)$-property on the column from (2.14). In the same way, we can generally prove that $\tilde{N}_{1,1}$ is of $(P)$-property. Moreover, $\tilde{N}_{12,\ldots,12}$ is of $(P)$-property, because it is the same as $\tilde{M}_{12,\ldots,12}$. The same way, $N_{r,\ldots,s}$, $(r > 1)$ is of $(P)$-property from (2.14). Thus it follows that $N$ is a matrix of strongly special type.

§ 3. Applications to knot theory

In this section, we shall apply the results obtained in §§ 1, 2 to the knot theory.
3.1. When an alternating knot $k$ is divided into $m+1$ special alternating knots $k_1, \cdots, k_{m+1}$ by $m$ standard loops of the second kind, we say $k$ is the *-product of $k_1, \cdots, k_{m+1}$, and denote by

$$
(3.1) \quad k = k_1 \ast k_2 \ast \cdots \ast k_{m+1}.
$$

We have first to prove the following

**Theorem 3.2.** Let $M=(a_{ij})_{i,j=1,\ldots,n}$ be the knot matrix of a special alternating knot $k$. Then $\tilde{M}(i,j)$ $(i=1, \cdots, n)$ are matrices of $k$-strongly special type on the row and the column.

Proof. It will be shown from (1.14) that $\tilde{M}(i,j)$ are matrices of special type and that these are of $k$–property. Hence we have only to show that $\tilde{M}(i,j)$ are matrices of strongly special type. We shall show that $\tilde{M}(i_1, \ldots, i_{\lambda-1}, i_{\lambda+1})$ is of $(P)$-property on the row and the column. Now suppose the contrary. For the sake of brevity, we assume that $i=1$, $j_1=2, \cdots, j_{\lambda-1}=\lambda$. Then it follows

$$
(3.2) \quad \begin{cases}
    a_{\lambda+1, \lambda+1} + a_{\lambda+2, \lambda+2} + \cdots + a_{\lambda+1, n} = 0, \\
    a_{\lambda+2, \lambda+1} + a_{\lambda+2, \lambda+2} + \cdots + a_{\lambda+2, n} = 0, \\
    \vdots \\
    a_{n, \lambda+1} + a_{n, \lambda+2} + \cdots + a_{n, n} = 0,
\end{cases}
$$

or

$$
(3.3) \quad \begin{cases}
    a_{\lambda+1, \lambda+1} + a_{\lambda+2, \lambda+2} + \cdots + a_{n, \lambda+1} = 0, \\
    a_{\lambda+1, \lambda+2} + a_{\lambda+2, \lambda+2} + \cdots + a_{n, \lambda+2} = 0, \\
    \vdots \\
    a_{\lambda+1, n} + a_{\lambda+2, n} + \cdots + a_{n, n} = 0.
\end{cases}
$$

And it follows, moreover,

$$
(3.4) \quad \text{If one of (3.2) and (3.3) holds, then it follows the other.}
$$

Because, let (3.2) hold. Then, it follows $(a_{\lambda+1, \lambda+1} + \cdots + a_{n, \lambda+1}) + \cdots + (a_{\lambda+1, n} + \cdots + a_{n, n}) = 0$. Since $\tilde{M}(1,1)$ is a matrix of special type, the value of the sum in each bracket is $\geq 0$. Hence we have (3.3).

Thus, we have from (3.2), (3.3)

$$
\begin{cases}
    a_{\lambda+1, 1} = a_{\lambda+1, 2} = \cdots = a_{\lambda+1, \lambda} = 0, \\
    \vdots \\
    a_{n, 1} = a_{n, 2} = \cdots = a_{n, \lambda} = 0,
\end{cases}
$$

and
\[
\begin{align*}
\begin{cases}
a_{1, \lambda+1} = a_{2, \lambda+1} = \cdots = a_{\lambda, \lambda+1} = 0, \\
\cdots \\
a_{1, n} = a_{2, n} = \cdots = a_{\lambda, n} = 0.
\end{cases}
\]
\]

Hence \( M \) is of the form \( M = \begin{pmatrix} * & 0 \\ 0 & * \end{pmatrix} \). This shows that the knot projection is separated into two parts, which contradicts the assumption in 1.2. The proof of Theorem 3.2 is thus complete.

From (1.14), it follows, moreover,

\[ \det \tilde{M} \left( \begin{array}{c} 1 \\ 1 \end{array} \right) = \det \tilde{M} \left( \begin{array}{c} 2 \\ 2 \end{array} \right) = \cdots = \det \tilde{M} \left( \begin{array}{c} n \\ n \end{array} \right). \]

3.2. Let \( k = k_1 \cdots k_{m+1} \), and denote the \( A \)-polynomial of \( k \) by \( \Delta(t) \). Then, it follows in the same notations as used in corollary 1.40,

**Lemma 3.6.** \( D(0) > 0 \).

Proof. Since \( D(0) = \Pi_{i=1}^{m+1} A_i(0) \), it is sufficient to prove \( A_i(0) = \det \tilde{M}_i \left( \begin{array}{c} p_i \\ p_i \end{array} \right) > 0 \), where \( M_i \) denote the special knot matrices of \( k_i \).

From Theorem 3.2, it follows \( \tilde{M}_i \left( \begin{array}{c} p_i \\ p_i \end{array} \right) \) are matrices of \( k \)-strongly special type on the row and the column. Thus, we have \( \det \tilde{M}_i \left( \begin{array}{c} p_i \\ p_i \end{array} \right) > 0 \), q.e.d.

On account of this Lemma, the fundamental theorem 1.17 can be stated as follows:

\[ \Delta(t) = D(t) = \det \left\{ \tilde{M} \left( \begin{array}{c} i_1 \cdots i_{m+1} \\ i_1 \cdots i_{m+1} \end{array} \right) - t \tilde{M}^T \left( \begin{array}{c} i_1 \cdots i_{m+1} \\ i_1 \cdots i_{m+1} \end{array} \right) \right\}. \]

Now, Lemma 3.6 is equivalent to the following theorem which is obtained in [4, p. 265] [7], [8].

**Theorem 3.8.** The degree of the \( A \)-polynomial of an alternating knot \( k \) plus one equals twice its genus plus its multiplicity \( \mu \).

Proof. Applying the same notations as used in §1, Lemma 3.6 is equivalent to the following.

\[ \Delta(t) \text{ is equal to } (p_1 + 1) + \cdots + (p_{m+1} - 1) = (p_1 + \cdots + p_{m+1}) - (m + 1). \]

On the other hand, Seifert showed that \( k \) is spanned by an orientable surface without singularity with genus \( h \), where \( 2h = n_1 + \cdots + n_{m+1} - (q_1 + \cdots + q_{m+1} + m) - \mu + 2. \) (See [12].) Hence it follows,
Alternating Knots

\[2h = (n_1 - q_1) + \cdots + (n_{m+1} - q_{m+1}) - m - \mu + 2\]
\[= (p_1 + s_1 - 1) + \cdots + (p_{m+1} + s_{m+1} - 1) - m - \mu + 2\]
\[= (p_1 + \cdots + p_{m+1}) + (s_1 + \cdots + s_{m+1}) - (m + 1) - m - \mu + 2\]
\[= p_1 + \cdots + p_{m+1} + m - 1 - m - 1 - m - \mu + 2\]
\[= p_1 + \cdots + p_{m+1} - m - \mu.\]

Thus we have

\[(3.10) \quad 2h + \mu - 1 = p_1 + \cdots + p_{m+1} - m - 1.\]

Denoting the genus of \(k\) by \(g\), it follows (cf. [12])

\[(3.11) \quad \text{The degree of } \Delta(t) \leq 2g + \mu - 1.\]

Hence, it follows from (3.9) (3.10) (3.11), \(2g + \mu - 1 \geq \text{the degree of } \Delta(t) = (p_1 + \cdots + p_{m+1}) - m - 1 = 2h + \mu - 1 \geq 2g + \mu - 1\), which is the required result, q.e.d.

**Theorem 3.12.** If \(k\) is a special alternating knot of multiplicity 1, then \(\Delta(1) = 1\).

Proof. As is well-known, \(\Delta(1) = \pm 1\). Thus it is sufficient to prove \(\Delta(1) \geq 0\). It follows from (3.7),

\[\Delta(1) = \det \left\{ \begin{bmatrix} p & p \\ \bar{M} & \bar{M}' \end{bmatrix} \right\} = \det \begin{pmatrix} 0 & \epsilon_{12} \cdots \epsilon_{1, p-1} \\ \epsilon_{21} & 0 \cdots \epsilon_{2, p-1} \\ \vdots & & \ddots & \epsilon_{p-1, 1} \\ \epsilon_{p-1, 2} & \cdots & 0 \end{pmatrix} \geq 0,\]

where \(\epsilon_{ij} = a_{ij} - a_{ji}, \quad \epsilon_{ij} = -\epsilon_{ji}, \quad \text{q.e.d.}\)

We see from the above Lemma that there are knots which cannot be transformed into the special alternating. The knot shown in Fig. 4 is one of these knots, for \(\Delta(t) = 1 - 5t + 7t^2 - 5t^3 + t^4\) and \(\Delta(1) = -1\).

**Theorem 3.13.** Let \(\Delta(t) = c_0 + c_1 t + \cdots + c_{p-1} t^{p-1}\) be the \(A\)-polynomial of a special alternating knot, then it follows \((-1)^j c_j > 0\), for \(j = 0, 1, \cdots, p - 1\).

Proof. From (3.7), we have

\[\Delta(t) = \det \left\{ \bar{M} \left( \begin{array}{c} p \\ p \end{array} \right) - t \bar{M}' \left( \begin{array}{c} p \\ p \end{array} \right) \right\}.\]

Then since \((-1)^j c_j = \frac{1}{j!} \left[ \frac{d^j}{dt^j} \Delta(t) \right]_{t=0}\), it follows from Lemma 2.12 that

\[(3.14) \quad (-1)^j c_j = \frac{1}{j!} \sum_{1 \leq i_1 \leq \cdots \leq i_j \leq p-1} N_{i_1, \cdots, i_j} > 0, \quad \text{q.e.d.}\]
3.3. Let $\mathcal{R}$ be a ring of all polynomials $f(t)=a_0+a_1t+\cdots+a_nt^n$, $a_i$ being integers, satisfying the following conditions:

\[ a_0 > 0, \quad a_i = (-1)^na_{n-i}, \quad \text{for } i = 0, \ldots, n. \]

We introduce a semi-order, denoted by $\geq$, in $\mathcal{R}$. Let $f(t)=a_0+a_1t+\cdots+a_nt^n$, $g(t)=b_0+b_1t+\cdots+b_mt^m$. By $f(t)\geq g(t)$ is meant that

\[ n \geq m \quad \text{and} \quad |a_i| \geq |b_i| \quad \text{for } i = 0, \ldots, m. \]

Then, as is shown in [6], [12], it follows

**Lemma 3.17.** $\mathcal{R}$ coincides with a ring of all $A$-polynomials of knots.

Proof has been given in [12] in the case that the multiplicity $\mu$ of knots is equal to 1, and given in [6] in the case $\mu > 1$.

Let us denote, moreover, the subring of all polynomials satisfying the conditions

\[ (-1)^ia_i > 0, \quad \text{for all } i, \]

by $\mathcal{S}$.

Now, let an alternating knot $k = k_1 \cdots k_{m+1}$. Let the $A$-polynomials of $k, k_1, \ldots, k_{m+1}$ be denoted by $\Delta(t), \Delta_1(t), \ldots, \Delta_{m+1}(t)$. Then it follows,

**Lemma 3.19.** $\Delta(t) \geq \Delta_1(t) \cdots \Delta_{m+1}(t)$.

Since the proof of this Lemma has been given in [8, pp. 247-248], [9, pp. 181-185], we omit the details.

Hereafter we shall symbolize these as follows:

\[ \Delta(t) = \Delta_1(t) \cdots \Delta_{m+1}(t). \]

From Theorem 3.13, Lemma 3.19, we have immediately,

\[ \Delta(-1) \geq \Delta_1(-1) \cdots \Delta_{m+1}(-1). \]

Moreover, we have,

**Lemma 3.22.** The totality $\mathcal{B}$ of all $A$-polynomials of alternating knots is contained in $\mathcal{S}$. [9]

In the following, it will be shown

\[ \mathcal{A} \subseteq \mathcal{S}. \]

3.4. The main theorem of this section is the following

**Theorem 3.24.** The special alternating knots with $\Delta(0)=1$ admit only elementary torus knots or their products.
By an \textit{elementary torus knot} is meant a special alternating knot whose graph or dual graph of its projection is a polygon. (See Fig. 5)

Proof. Let \(M=(a_{ij})_{i,j=1\ldots,n}\) be the knot matrix of a knot satisfying the condition in this theorem. Then, we see from Theorem 2.11 that at least two of \(a_{11}, \ldots, a_{nn}\) are equal to 1. We shall prove the theorem by induction on \(n\). If \(n=2\), it follows \(M=\begin{pmatrix} 1 & -1 \\ -1 & 1 \end{pmatrix}\), from which the theorem follows. Now suppose that the theorem is proved in the case \(n-1\). We may assume without loss of generality that

\[
(3.25) \quad a_{11} = 1, \quad a_{12} = -1, \quad a_{13} = \cdots = a_{1n} = 0.
\]

Then it follows that \(\Delta(0)=\det \bar{M} \begin{pmatrix} 2 \\ -2 \end{pmatrix} = \det \bar{M} \begin{pmatrix} 12 \\ 12 \end{pmatrix} = 1\).

Let

\[
N = \begin{pmatrix} a_{11} + a_{22} & a_{23} \cdots a_{2n} \\ a_{11} + a_{32} & a_{33} \cdots a_{3n} \\ \vdots \\ a_{11} + a_{n2} & a_{n3} \cdots a_{nn} \end{pmatrix}.
\]

\(N\) is the knot matrix of a special alternating knot \(k_1\), where \(k_1\) is transformed from \(k\) by applying in its projection the following operation as is shown in Fig. 6:

\[
\begin{array}{c}
\begin{array}{c}
\begin{array}{c}
\begin{array}{c}
\begin{array}{c}
 w_1 \\
 w_2 \\
 w_3
\end{array}
\end{array}
\end{array}
\end{array}
\end{array}\quad \Rightarrow \quad \begin{array}{c}
\begin{array}{c}
\begin{array}{c}
\begin{array}{c}
\begin{array}{c}
 w_i \\
 w_1 \\
 w_2
\end{array}
\end{array}
\end{array}
\end{array}
\end{array}
\]

Fig. 6.

Denoting the \(A\)-polynomial of \(k_1\) by \(\Delta_1(t)\), it follows

\[
\Delta_1(0) = \det \bar{N} \begin{pmatrix} 1 \\ 1 \end{pmatrix} = \det \bar{M} \begin{pmatrix} 12 \\ 12 \end{pmatrix} = 1.
\]

Thus we see from the assumption of the induction that \(k_1\) is an elementary torus knot or their product. Hence \(k\) must be also an elementary torus knot or their product, q.e.d.

\[
(3.26) \quad \text{The } A\text{-polynomial of an elementary torus knot is of the form: } 1 - t + \cdots + (-1)^i t^i + \cdots + (-1)^n t^n.
\]
The converse of (3.26) is also true. That is,

**Lemma 3.27.** An alternating knot whose $A$-polynomial is of the form, $1-t+\cdots+(-1)^nt^n$, is an elementary torus knot.

Proof. Let $k=k_1*\cdots*k_{m+1}$, and denote the $A$-polynomials of $k$, $k_1, \cdots, k_{m+1}$ by $\Delta(t)$, $\Delta_1(t), \cdots, \Delta_{m+1}(t)$. Then, we see from Lemma 3.19 and (1.40) that $\Delta_1(0) = \cdots = \Delta_{m+1}(0) = 1$. Hence it follows from Theorem 3.24 that $k_i$ are elementary torus knots or their products. Then,

$$\Delta(t) \leq \Delta_1(t) \cdots \Delta_{m+1}(t),$$

which implies $m=0$. Hence $k$ is an elementary torus knot or their product. It is clear, however, that $k$ is not a product knot, q.e.d.

3.5. From Theorem 3.24, (3.26), Lemma 3.27, we have the following

**Theorem 3.28.** An alternating knot with $\Delta(0)=1$ is represented as the $*$-product of some elementary torus knots.

This theorem will follow Lemma 3.30 about the number of the double points in a knot projection. Before Lemma 3.30 is stated, we state the following

**Lemma 3.29.** Let $k=k_1*\cdots*k_{m+1}$, and denote the numbers of the double points in $K_i$ by $n_i$. Then the number of the double points in $K$ equals $n_1+\cdots+n_{m+1}$.

Thus $k$ possesses an alternating projection, in which there are at most $n_1+\cdots+n_{m+1}$ double points. From Lemma 3.29, it follows

**Lemma 3.30.** An alternating knot $k$ with $\Delta(0)=1$ possesses a projection, in which there are at most $2n$ double points, $n$ denoting the degree of $\Delta(t)$.

Proof. $k$ can be represented as the $*$-product of at most $n$ special alternating knots. Then, every $*$-component is an elementary torus knot whose $A$-polynomial is $1-t$. Since the number of the double points of it is 2, it follows the number of double points in $K$ is $2n$. If $k$ can be represented as the $*$-product of $m(<n)$ special elementary torus knots, then the number of double points in $K$ is less than $2n$, q.e.d.

**Corollary 3.31.** Alternating prime knots whose $A$-polynomials are of the form: $\Delta(t)=1-c_1t+c_2t^2-c_3t^3+t^4$, admit only the following knots: $5_1$, $6_2$, $6_3$, $7_6$, $7_7$, $8_{12}$. 
Hence it follows:

(3.32) Projection 8_{20}, 8_{21}, 9_{42}, 9_{43}, 9_{44}, 9_{45}, 9_{48}, really represent non-alternating knots.

Last of all, we shall show that 3 projections 9_{43}, 9_{47}, 9_{49} really represent non-alternating knots.

3.6. In the following, we denote knots 9_n and their A-polynomials by k_n and Δ_n(t) respectively. Let us denote the elementary torus knot whose A-polynomial is of the degree n, by k_{(n)} and denote its A-polynomial by Δ_{(n)}(t).

Now, since Δ_{(3)}(t) = 1 - 3t^2 + 2t^4 - t^5 + 2t^6 + t^8 and Δ_{(7)}(t) = 1 - 4t + 6t^2 - 5t^3 + 6t^4 - 4t^5 + t^6, it follows that Δ_{(3)}(1) = -1, Δ_{(7)}(-1) = 13 and Δ_{(1)}(1) = 1, Δ_{(1)}(-1) = 27.

Suppose k_{43}, k_{47} be alternating. Then it follows from Lemma 3.29, that each of k_{43}, k_{47} has to be represented as one of the following "-products:

1) k_{(3)}*k_{(3)}*k_{(1)}*k_{(1)},
2) k_{(3)}*k_{(3)}*k_{(1)}*k_{(1)},
3) k_{(1)}*k_{(3)}*k_{(1)}*k_{(1)}*k_{(1)},
4) k_{(1)}*k_{(1)}*k_{(1)}*k_{(1)}*k_{(1)}.

In each case we set t = -1. Then we have the following

1) Δ_{(3)}(-1)\{Δ_{(3)}(-1)\}^3 = 32 > 13, 27 ,
2) \{Δ_{(3)}(-1)\}^2\{Δ_{(3)}(-1)\}^2 = 36 > 13, 27 ,
3) Δ_{(1)}(-1)\{Δ_{(1)}(-1)\}^4 = 48 > 13, 27 ,
4) \{Δ_{(1)}(-1)\}^6 = 64 > 13, 27 .

All cases contradict (3.21). Thus we have

(3.33) k_{43}, k_{47} represent non-alternating knots.

To show that 9_{49} represents a non-alternating knot, we require some preparations.

3.7. In this and next paragraphs, we make an exact distinction between knots and links.

Let M = (a_{ij})_{i,j=1,...,n}, be the knot matrix of a special alternating knot k. Let N = (b_{ij})_{i,j=1,...,n}, be a matrix obtained from M as follows:

8) It will be easily shown from Theorem 3.13, Lemma 3.19 and by simple computations that two knot projections 8_{23}, 9_{46} represent non-alternating knots.
\[(3.34) \quad b_{ij} = a_{ij} + \min (|a_{ij}|, |a_{ji}|), \quad (i \neq j)\]
\[b_{ii} = a_{ii} - \sum_{j \neq i} \min (|a_{ij}|, |a_{ji}|).\]

Then, if follows

**Lemma 3.35.** *N is the knot matrix of a special alternating knot \(k_0\).*

Proof. Let \(W_i, W_j\) be two regions in \(K\) such that \(\overline{W_i} \cap \overline{W_j} = \emptyset\). Then, by applying, as much as possible, the operation as is shown in Fig. 7, we have a knot. It is clear that such a knot is \(k_0\).

![Fig. 7.](image)

We shall call \(k_0\) the *frame knot* of \(k\). We can naturally extend this concept to links, but we should note that link projection may be separated into some parts. (See Fig. 8)

![Fig. 8.](image)

In all cases, it follows

\[(3.36) \quad \mu = \mu_0,\]

where \(\mu, \mu_0\) denote the multiplicities of \(k, k_0\).

**3.8.** Since \(\Delta_{\nu}(t) = 3 - 6t + 7t^2 - 6t^3 + 3t^4\), it follows \(\Delta_{\nu}(1) = 1, \Delta_{\nu}(-1) = 25\). We shall first prove

\[(3.37) \quad \text{If } k_\nu \text{ is alternating, it must be a special alternating knot.}\]

Proof. By noting \(\Delta(1) = 1\), it follows that if \(k_\nu\) is alternating and if it is not a special alternating knot, then \(\Delta_{\nu}(t)\) must be represented as one of the following \(*\)-products:
(1) $\Delta_{\varphi_2}(t)* (3-5t+3t^3)$,
(2) $\Delta_{\varphi_2}(t)* (1-t)^2$,
(3) $\Delta_{\varphi_2}(t)* \Delta_{\varphi_1}(t)* (1-t)$,
(4) $3(1-t)^3* \Delta_{\varphi_1}(t)* \Delta_{\varphi_1}(t)$,
(5) $(3-5t+3t^3)* \Delta_{\varphi_1}(t)* \Delta_{\varphi_1}(t)$,
(6) $3(1-t)* \Delta_{\varphi_1}(t)* \Delta_{\varphi_1}(t)* \Delta_{\varphi_1}(t)$.

In all cases we set $t=-1$. Then every value is larger than $\Delta_{\varphi_1}(-1) = 25$. This contradicts Lemma 3.21. Thus $k_{4g}$ must be a special alternating knot.

Now, for the frame knot of a special alternating knot, we can prove the following

**Lemma 3.38.** The frame knot $k_0$ of a special alternating knot whose $A$-polynomial is of degree 4, admits only an elementary torus knot $k(t)$ or a product of two knots $k_{(a)}$.

This will be proved by simple computation, if we note that at least one of $b_{ij}$ in the knot matrix of $k_0 N=(b_{ij})$ is equal to 0.

From Lemma 3.38, we have, moreover, by computation,

(3.39) In the knot matrix $M$ of a special alternating knot whose $A$-polynomial $\Delta(t)$ is of the degree 4 and $\Delta(0)=3$, at least two of $a_{11}, \cdots, a_{55}$ are equal to 1.

Thus, it follows from Lemma 2.10, (3.39) and from the fact that $M$ is of $k$-strongly special type, without $s$-equivalent, that $M$ admits only the following:

(3.40) (1) \[
\begin{pmatrix}
1 & 0 & -1 & 0 & 0 \\
0 & 1 & -1 & 0 & 0 \\
0 & 0 & 2 & -1 & -1 \\
-1 & 0 & 0 & 2 & -1 \\
0 & -1 & 0 & -1 & 2 \\
\end{pmatrix}
\]
(2) \[
\begin{pmatrix}
1 & -1 & 0 & 0 & 0 \\
0 & 1 & 0 & -1 & 0 \\
-1 & 0 & 3 & 0 & -2 \\
0 & 0 & 0 & 1 & -1 \\
0 & 0 & -3 & 0 & 3 \\
\end{pmatrix}
\]
These matrices correspond to knot matrices of $8_{15}$, $9_4$, $9_7$ and of a product of $3_1$ and $7_2$, respectively. Thus we have the following

\[(3.41) \text{Special alternating knots whose } A\text{-polynomials } \Delta(t) \text{ are of degree 4 and } \Delta(0) = 3, \text{ admit only the knots: } 8_{15}, 9_4, 9_7, \text{ and a product of } 3_1 \text{ and } 7_2.\]

This naturally follows

\[(3.42) 9_{49} \text{ represents a non-alternating knot.}\]

Thus, it has been really shown that all non-alternating projections at the end of [11] represent non-alternating knots.

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References

Alternating Knots


