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ASYMPTOTIC DISTRIBUTION OF EIGENVALUES OF NON-SYMMETRIC OPERATORS ASSOCIATED WITH STRONGLY ELLIPTIC SESQUILINEAR FORMS

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1. Introduction

The main object of this paper is to extend the result of K. Maruo and H. Tanabe [4] on the eigenvalue distribution of symmetric elliptic operators to a non symmetric case. Some amelioration of the result of [4] on the remainder estimates in Weyl's formula as well as the formula under less restrictive smoothness assumptions is also obtained.

Let Ω be a bounded domain in R^n having the restricted cone property. We use the same notations as those of [4] to denote various norms and functional spaces. In this paper it is assumed that $2m>n$ as in the previous paper [4]. Let B be a sesquilinear form defined in $H_m(\Omega)\times H_m(\Omega)$ satisfying

$$\operatorname{Re} B[u, u]\geq\delta_0||u||^2_m \qquad \text{for any} \quad u\in V \qquad a-(1)$$

where V is a closed subspace of $H_m(\Omega)$ containing $\overset{\circ}{H}_m(\Omega)$ and δ_0 is some positive constant independent of u. We assume that B has the following form

$$B[u, v] = B_0[u, v]+B_1[u, v] \qquad (1.1)$$

where B_0 which is the principal part of B is a symmetric integro-differential sesquilinear form of order m with bounded coefficients

$$B_0[u, v] = \int_\Omega \sum_{|\alpha|=|\beta|=m} a_{\alpha\beta}(x)D^\alpha uD^\beta v dx$$

and B_1 is a not necessarily symmetric sesquilinear form satisfying

$$|B_1[u, v]|\leq K(||u||_m||v||_{m-1}+||u||_{m-1}||v||_m) \qquad a-(2)$$

for any $u, v\in V$ i.e. B_1 is the lower order part of B. Let A be the operator associated with the form B: an element u of V belongs to $D(A)$ and $Au=f\in L^2(\Omega)$ if $B[u, v]=(f, v)$ holds for any $v\in V$. A is a not necessarily symmetric operator in $L^2(\Omega)$ and all rays $\arg\lambda=\theta$ different from the positive real axis are rays of minimal growth of the resolvent of A. By $N(t)$ we denote the number of eigenvalues of A whose real part does not exceed t. The main conclusion of this paper is that the following asymptotic formula holds:

$$N(t) = C_0 t^{n/2m} + o(t^{n/2m}) \qquad \text{as} \quad t \to \infty , \tag{1.2}$$

if the coefficients of B_0 are Riemann integrable,
and

$$N(t) = C_0 t^{n/2m} + 0\ (t^{(n-\theta)/2m}) \qquad \text{as} \quad t \to \infty \tag{1.3}$$

for any $\theta < h/(h+2)$ if B_0 has uniformly Hoelder continuous coefficients of order h and for any $\theta < (h+1)/(h+3)$ if the coefficients of B_0 belong to the class C^{1+h} in some domain containing Ω. The formula (1. 3) is an improvement of the corresponding result obtained for symmetric operators in [4] where (1. 3) was established only for $\theta < h/(h+3)$ and $\theta < (h+1)/(h+4)$ respectively making some more restrictive assumptions and in order to prove (1. 3) for $(h+1)/(h+4) \leq \theta < 1/2$ still more hypotheses were required.

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2. Main theorem

As was stated in the introduction let Ω be a bounded domain in R^n having the restricted cone property (p. 11 of S. Agmon [1]) and it is assumed that $2m > n$. For $x \in \Omega$ we write $\delta(x) = \min\{1, \text{dist}(x, \partial\Omega)\}$. Suppose that

$$\int_\Omega \delta(x)^{-p} dx < \infty \tag*{$a-(3)$}$$

for some positive number $p < 1$ which will be specified later.

Since all coefficients of of B_0 are bounded it follows from $a-(2)$ that for any $u, v \in V$

$$|B[u, v]| \leq K \|u\|_m \|v\|_m$$

for some constant K.

We state various smoothness assumptions on the coefficients of B_0:

they are Riemann integrable, i.e. continuous almost everywhere in Ω: $s-(0)$

they are uniformly Hoelder continuous of order h in Ω: $s-(1)$

they belong to $C^{1+h}(\Omega_1)$ where Ω_1 is some domain containing Ω and $C^{1+h}(\Omega_1)$ is the subclass of functions in $C^1(\Omega_1)$ with derivatives Hoelder continuous of order h in Ω_1. $s-(2)$

Main Theorem. *The following asymptotic formulas for $N(t)$ hold as $t\to\infty$:*

$$N(t) = C_0 t^{n/2m} + o\,(t^{n/2m}) \qquad \textit{under } s-(0)$$
$$N(t) = C_0 t^{n/2m} + 0\,(t^{(n-\theta)/2m})$$

for any θ satisfying

$$0<\theta<h/(h+2) \qquad \textit{under } s-(1)$$
$$0<\theta<(h+1)/(h+3) \qquad \textit{under } s-(2)$$

where

$$C_0 = \frac{\sin(n/2m)}{n/2m}\int_{\Omega} C(x)\,dx$$

$$C(x) = (2\pi)^{-n}\int_{R^n}\{\sum_{|\alpha|=|\beta|=m} a_{\alpha\beta}(x)\,\xi^{\alpha+\beta}+1\}^{-1}d\xi\,.$$

Remark. As was mentioned in the Introduction the remainder estimates described in the main theorem is an improvement of those established in [4]. Furthermore applying the theorem to the sesquilinear form (Au, Av) where A is the elliptic operator satisfying the conditions of R. Beals [3] we may prove Theorem C of [3] with $0<\theta<h/(h+2)$ instead of $0<\theta<h/(h+3)$ if the order of A is greater than $n/2$.

Following the method of S. Agmon [5] or Dunford-Schwartz [6] it is possible to show that the generalized eigenfunctions of A are complete in $L^2(\Omega)$ under our assumptions.

3. Some lemmas

As in the previous paper [4] we extend the operator A to a mapping on V to V^* where V^* is the antidual of V. This extended operator which is again denoted by A is defined by

$$B[u, v] = (Au, v) \qquad \text{for any} \quad v\in V$$

where the bracket on the right stands for the duality between V^* and V in this case.

Identifying $L^2(\Omega)$ with its antidual we may consider $V\subset L^2(\Omega)\subset V^*$ algebraically and topologically, and as is easily seen V is a dense subspace of V^* under this convention. The resolvent of A thus extended is a bounded linear operator on V^* to V. We denote by $\rho(A)$ the resolvent set of A and $d(\lambda)$ the distance from the point λ to the positive real axis for a complex number λ.

Lemma 3.1. *The resolvent set $\rho(A)$ of A in either sense contains the set $\{\lambda: d(\lambda)\geqq C|\lambda|^{1-1/2m}, |\lambda|\geqq C\}$ for some constant C. The eigenvalues $\{\lambda_j\}_{j=0}^{\infty}$ of A have finite multiplicity and eigenvalues of A can have only ∞ as a limite point.*

Proof. We put $(A-\lambda)u=f$ for any $u\in D(A)$. We see that

$$B[u, u]-\lambda(u, u) = (f, u) \tag{3.1}$$

From (3. 1), (1. 1), $a-(2)$ and $\operatorname{Im} B_0[u, u]=0$, we get:

$$|\operatorname{Im}\lambda|\,||u||_0^2\leq||f||_0||u||_0+2K||u||_m||u||_{m-1}\,. \tag{3.2}$$

Applying to the last term $||u||_m||u||_{m-1}$ Young's inequality and then using the interpolation inequality, for any positive constant δ_1 and $\delta_2\leq 1$ we find that

$$\begin{aligned}||u||_m||u||_{m-1}&\leq\delta_1||u||_m^2+\delta_1^{-1}||u||_{m-1}^2\\ &\geq K_1\{\delta_1||u||_m^2+\delta_1^{-1}\delta_2||u||_m^2+\delta_1^{-1}\delta_2^{-m+1}||u||_0^2\}\,.\end{aligned} \tag{3.3}$$

From (3. 1) and $a-(1)$ we get

$$\delta||u||_m^2\leqq|\lambda|\,||u||_0^2+||u||_0||f||_0\,. \tag{3.4}$$

Putting $\delta_1=\delta_2^{1/2}=|\lambda|^{-1/2m}$ and combining (3. 2), (3. 3) and (3. 4) we find that

$$(|\operatorname{Im}\lambda|-K_2|\lambda|^{-1/2m}||u||_0^2\leq(1+K_2|\lambda|^{-1/2m})\,||f||_0||u||_0\,. \tag{3.5}$$

If $|\operatorname{Im}\lambda|>C|\lambda|^{1-1/2m}$ for large C, we know that

$$||u||_0\leq K_3/|\operatorname{Im}\lambda|\,||f||_0\,. \tag{3.6}$$

If $\operatorname{Re}\lambda<0$ we get

$$|\operatorname{Re}\lambda|\,||u||_0^2\leq||f||_0||u||_0 \tag{3.7}$$

from (3. 1).
Combining (3. 6) and (3. 7) we find that there is a constant K_4 independent of λ such that

$$||u||_0\leq K_4/d(\lambda)||f||_0 \tag{3.8}$$

On the other hand for an adjoint operator A^* we find the same estimate (3. 8). Thus the null space of the operator $(A^*-\bar{\lambda})$consists only of zero and we know

$$\{\lambda\colon d(\lambda)\geq C|\lambda|^{1-1/2m},\ |\lambda|\geq C\}\subset\rho(A)\,.$$

Next we put $(A-\lambda)u=f$ for any $u\in V$.
From (1. 1), $a-(1)$ and $a-(2)$ it follows that

$$||u||_0^2\leq K_5/d(\lambda)\{||f||_{V^*}||u||_m+||u||_m||u||_{m-1}\}\,. \tag{3.9}$$

For any number δ_3 such that $0<\delta_3\leqq 1$ we know

$$||u||_{m-1}\leq K_6\{\delta_3||u||_m+\delta_3^{-2m+1}||u||_{V^*}\}\,. \tag{3.10}$$

From the inequality

$$|\lambda|\,|(u, v)| \leq ||f||_{V^*}||v||_m + K||u||_m||v||_m \qquad \text{for any} \quad v \in V$$

it follows that

$$|\lambda|\,||u||_{V^*} \leq ||f||_{V^*} + K_7||u||_m \tag{3.11}$$

Combining $a-(1)$, (3. 9), (3. 10) and (3. 11) and putting $\delta_3 = |\lambda|^{-1/2m}$ we get the following estimate:

$$\begin{aligned}\delta||u||_m^2 &\leq ||f||_{V^*}||u||_m + |\lambda|\,||u||_0^2 \\ &\leq ||f||_{V^*}||u||_m + K_8|\lambda|/d(\lambda)\{||f||_{V^*}||u|| + {}_m||u||_m||u||_{m-1}\} \\ &\leq ||f||_{V^*}||u||_m + K_9|\lambda|/d(\lambda)\{(1+|\lambda|^{-1/2m})||u||_m||f||_{V^*} \\ &\quad + |\lambda|^{-1/2m}||u||_m^2\}\end{aligned}$$

If $d(\lambda) \geq C|\lambda|^{1-1/2m}$ with $|\lambda|$ sufficiently large there is a constant K_{10} independent of λ such that

$$||u||_m \leq K_{10}|\lambda|/d(\lambda)||f||_{V^*} \tag{3.12}$$

On the other hand we put $(A^* - \bar{\lambda})u = f$ for any $u \in V$. Then we find the same estimate (3. 12) for A^*. Thus we see that

$$\{\lambda\colon d(\lambda) \geq C|\lambda|^{1+1/2m}\colon |\lambda| \geq C\} \subset \rho(A)\,.$$

The last part of the lemma is a simple consequence of Rellich's theorem.

Q.E.D.

For a bounded operator S on V^* to V we use the notations $||S||_{V^* \to L^2}$ $||S||_{V^* \to L^2}$ etc, to denote the norms of S considered as an operator on V^* to V, V^* to $L^2(\Omega)$, etc.

Lemma 3. 2. *There exists a constant C_1 such that*

i) $||(A-\lambda)^{-1}||_{L^2 \to L^2} \leq C_1/d(\lambda)$ ii) $||(A-\lambda)^{-1}||_{L^2 \to V} \leq C_1|\lambda|^{1/2}/d(\lambda)$
iii) $||(A-\lambda)^{-1}||_{V^* \to V} \leq C_1|\lambda|/d(\lambda)$ iv) $||(A-\lambda)^{-1}||_{V^* \to L^2} \leq C_1|\lambda|^{1/2}/d(\lambda)$

if $d(\lambda) \geq C|\lambda|^{1-1/2m}$, $|\lambda| \geqq C$ where C is the constant in the statement of Lemma 3. 1.

Proof. The statement i) is clear from (3. 8).
If $u = (A-\lambda)^{-1}f$ for any $f \in L^2(\Omega)$ we get;

$$\begin{aligned}\delta||u||_m^2 &\leq ||f||_0||u||_0 + |\lambda|\,||u||_0^2 \\ &\leq K_{11}|\lambda|(||f||_0/d(\lambda))^2\end{aligned}$$

from $a-(1)$ and i).
The statement iii) is clear from (3. 12). Finally with the aid of (3. 12) and the following inequality

$$|\lambda|\,\|u\|_0^2 \leq K\|u\|_m^2 + \|f\|_{V^*}\|u\|_m$$

we can easily show iv). Q.E.D.

Lemma 3.3. *Let S be a bounded operator on V^* to V. Then S has a kernel M in the following sense:*

$$Sf(x) = \int_\Omega M(x, y)f(y)\,dy \qquad \text{for} \quad f \in L_2(\Omega)\,.$$

$M(x, y)$ is continuous in $\Omega \times \Omega$ and there exists a constant C_2 such that for any $x, y \in \Omega$.

$$|M(x, y)| \leq C_2\|S\|_{V^* \to V}^{n^2/4m^2}\|S\|_{V^* \to L^2}^{n/2m - n^2/4m^2}\|S\|_{V^2 \to V}^{n/2m - n^2/4m^2}\|S\|_{L^2 \to L^2}^{(1-n/2m)^2}$$

Proof. see [4]. Q.E.D.

Lemma 3.4. *There are positive constants C_3 and C_4 such that*

$$B_0[u, u] \geq C_2\|u\|_m^2 - C_4\|u\|_0^2 \qquad \textit{for any} \quad u \in V\,.$$

Proof. From $a-(1)$ and the interpolation inequality, we can easily show the statement. Q.E.D.

4. Estimates of the resolvent kernel

We shall estimate the difference between the resolvent kernel of A and that of the operator A_0 associated with $B_0 + C_4$, thus $B_0[u, v] + C_4(u, v) = (A_0 u, v)$ for any $u, v \in V$. Obviously for the operator A_0 the analogues of Lemma 3.2 hold.

LetS_λ be the operator defined by

$$S_\lambda f = (A-\lambda)^{-1}f - (A_0-\lambda)^{-1}f \qquad \text{for any} \quad f \in V^*\,.$$

Lemma 4.1. *There is a constant C_7 such that for $d(\lambda) \geq C|\lambda|^{1-1/m}$, $|\lambda| \geq C$,*

i) $\|S_\lambda\|_{V^* \to V} \leq C_5|\lambda|/d(\lambda)\ (|\lambda|^{1-1/2m}/d(\lambda))$

ii) $\|S_\lambda\|_{V^* \to L^2}$
iii) $\|S_\lambda\|_{L^2 \to V}$
$\Big\} \leq C_5|\lambda|^{1/2}/d(\lambda)\ (|\lambda|^{1-1/2m}/d(\lambda))$

iv) $\|S_\lambda\|_{L^2 \to L^2} \leq C_5/d(\lambda)\ (|\lambda|^{1-1/2m}/d(\lambda))$.

Proof. Let $(A-\lambda)^{-1}f - (A_0-\lambda)^{-1}f = S_\lambda f = u$. Now we know that

$$(A-\lambda)^{-1} - (A_0-\lambda)^{-1} = (A_0-\lambda)^{-1}(A_0-A)\,(A-\lambda)^{-1}\,.$$

On the other hand, since the operator A_0 is self-adjoint we know

$$\begin{aligned}(S_\lambda f, \phi) &= ((A_0-A)(A-\lambda)^{-1}f, (A_0-\lambda)^{-1}\phi)\\ &= (B_0-B)[(A-\lambda)^{-1}f, (A_0-\bar{\lambda})^{-1}\phi]+C_4((A-\lambda)^{-1}f, (A_0-\bar{\lambda})^{-1}\phi)\\ &= -B_1[(A-\lambda)^{-1}f, (A_0-\bar{\lambda})^{-1}\phi]+C_4((A-\lambda)^{-1}f, (A_0-\bar{\lambda})^{-1}\phi) \quad (4.1)\end{aligned}$$

for any $\phi \in V^*$.

Combining (4. 1), Lemma 3. 2 and the interpolation inequality we find that there are constants K_1 and K_2 such that

$$\begin{aligned}|(S_\lambda f, \phi)| &\leq K_1\{\|(A-\lambda)^{-1}f\|_m \|(A_0-\bar{\lambda})^{-1}\phi\|_{m-1}\\ &\quad + \|(A-\lambda)^{-1}f\|_{m-1}\|(A_0-\bar{\lambda})^{-1}\phi\|_m\}\\ &\leq K_2(|\lambda|/d(\lambda))^2|\lambda|^{-1/2m}\|f\|_{V^*}\|\phi\|_{V^*}\} .\end{aligned}$$

Then we get

$$\|S_\lambda\|_{V\to V^*} \leq C_5|\lambda|/d(\lambda)(|\lambda|^{1-1/2m}/d(\lambda)) .$$

The remaining inequalities can be proved in a similar manner. Q.E.D.

Since $m>n/2$ there exist the resolvent kernels $K_\lambda(x, y)$ and $K_\lambda^0(x, y)$ of the operator A and A_0 such that

$$(A-\lambda)^{-1}f(x) = \int_\Omega K_\lambda(x, y)f(y)\,dy$$

$$(A_0-\lambda)^{-1}f(x) = \int_\Omega K_\lambda^0(x, y)f(y)\,dy \qquad \text{for any} \quad f\in L^2(\Omega) .$$

Theorem 4. 2. *For any given positive numbers p, ε and any non-negative integer j, the following inequality holds*:

$$\begin{aligned}&|K_\lambda(x, x)-C(x)(-\lambda)^{-1+n/2m}| \leq C_6[|\lambda|^{n/2m}/d(\lambda)\{\gamma^{h+i}|\lambda|/d(\lambda)\\ &+(\gamma^{-1}|\lambda|^{1-1/2m}/d(\lambda))^j+|\lambda|^{1-1/2m}/d(\lambda)+(|\lambda|^{1-1/2m}/\delta(x)\,d(\lambda))^p\}] \qquad (4.2)\end{aligned}$$

for $d(\lambda)\geq|\lambda|^{1-1/4m}+\varepsilon$, $\gamma>0$, $\gamma^{-1}|\lambda|^{1-1/2m}/d(\lambda)\leq 1$, and $|\lambda|$ sufficiently large, where $i=0$ under $s-(1)$ and $i=1$ under $S-(2)$. C_6 is a constant depending on p, ε, j but not on λ, γ or x, and $C(x)$ is the function defined in the main theorem.

Proof. Combining Lemma 4. 2, 6. 2, 7.2 and 7.3 of [4] we get

$$\begin{aligned}&|K_\lambda^0(x, x)-C(x)(-\lambda)^{-1+n/2m}| \leq K_3[|\lambda|^{n/2m}/d(\lambda)\{\gamma^{h+i}/d(\lambda)\\ &+(\gamma^{-1}|\lambda|^{1-1/2m}/d(\lambda))^j+(|\lambda|^{1-1/2m}/\delta(x)\,d(\lambda))^p\}+|\lambda|^{(n-1)/2m-1}] \qquad (4.3)\end{aligned}$$

where i=0 or 1 according as we assume $s-(1)$ or $s-(2)$.

Formally we replaced $d(\lambda)$ by some power of $|\lambda|$ at this point (Theorem 7. 1 of [4]); however, in this paper we postpone this replacement for a little while to obtain better remainder estimates as was stated in the introduction.

On the other hand applying Lemma 3. 3 and Lemma 4. 1 to S_λ we get

$$|K_\lambda(x, y) - K_\lambda^0(x, y)| \leq K_4(|\lambda|/d(\lambda))^2|\lambda|^{(n-1)/2m-1} \tag{4. 4}$$

Combining (4. 3) and (4. 4) the desired estimate (4. 2) is obtained. Q.E.D.

Next we shall consider the case of the assumption $s-(0)$. We denote $P_{\alpha\beta}$ the set of points where $a_{\alpha\beta}$ is continuous and put $P = \bigcap_{|\alpha|=|\beta|=m} P_{\alpha\beta}$. We fix a point $x_0 \in P$ and set

$$B_2'[u, v] = \int_\Omega \sum_{|\alpha|=|\beta|=m} a_{\alpha\beta}(x_0) D^\alpha u \overline{D^\beta v} dx \qquad \text{for} \quad u, v \in H_m(\Omega).$$

Lemma 4. 3. *There exist positive constants C_7 and C_8 independent of u and x_0 such that*

$$B_2'[u, u] \geq C_7 \|u\|_m^2 - C_8 \|u\|_0^2 \qquad \textit{for} \quad u \in \mathring{H}_m(\Omega).$$

Proof. There is a constant K_5 such that

$$\sum_{|\alpha|=|\beta|=m} a_{\alpha\beta}(x_0)\, \xi^{\alpha+\beta} \geq K_5 |\xi|^{2m}$$

for any $\xi \in R^n$. That the desired inequality holds for any $u \in \mathring{H}_m(\Omega)$ is a well known fact. Q.E.D.

We put $B_2[u, v] = B_2'[u, v] + C_2(u, v)$ for $u, v \in \mathring{H}_m(\Omega)$. We know that

$$B_2[u, u] \geq K_6 \|u\|_m^2 \qquad \text{for} \quad u \in \mathring{H}_m(\Omega) \tag{4. 5}$$

from Lemma 4. 3.

We denote by A_2 the operator associated with B_2 under the Dirichlet boundary condition. By definition for any $u, v \in \mathring{H}_m(\Omega)$ we have

$$B_2[u, v] = (A_2 u, v)$$

where the bracket on the right denotes the pairing between the antidual $H_{-m}(\Omega)$ of $\mathring{H}_m(\Omega)$ and $\mathring{H}_m(\Omega)$ this case. Obviously for the operator A_2 the analogues of Lemma 3. 1 and Lemma 3. 2 hold.

We denote by $\xi(x)$ a function in $C_0^\infty(R^n)$ the support of which is contained in the set $\{x \in R^n : |x| < 1\}$ and which takes the valued 1 at the origin. We write $\xi_\delta(x) = \xi((x - x_0)/\delta)$ where δ is any positive number $< \delta(x_0)$.

Let $S_{\lambda\delta}$ be the operator defined by

$$S_{\lambda\delta} f = \xi_\delta \{(A - \lambda)^{-1} f - (A_2 - \lambda)^{-1}(rf)\} \qquad \text{for} \quad f \in V^*$$

where rf is the restriction of $f \in V^*$ to $\mathring{H}_m(\Omega)$.

Obviously $S_{\lambda\varepsilon}$ is a bounded operator on V^* to $\mathring{H}_m(\Omega)$ and hence a fortiori to V. Since $a_{\alpha\beta}$ is continuous at x_0 for any α and β with $|\alpha| = |\beta| = m$ there is a positive number θ_δ such that

$\theta_\delta \to 0$ as $\delta \to 0$ and

$$|a_{\alpha\beta}(x)-a_{\alpha\beta}(x_0)|<\theta_\delta \qquad \text{for} \quad |x-x_0|<\delta \tag{4.6}$$

Lemma 4.4. *If* λ *is real* <0 *and* $\delta^{-1}|\lambda|^{-1/2m}\leq 1$ *we get*

i) $||S_{\lambda\delta}||_{V^*\to V}\leq C_9\{\theta_\delta+\delta^{-1}|\lambda|^{-1/2m}\}$

ii) $||S_{\lambda\delta}||_{V^*\to L^2}\leq C_9\{\theta_\delta+\delta^{-1}|\lambda|^{-1/2m}\}|\lambda|^{-1/2}$

iii) $||S_{\lambda\delta}||_{L^2\to V}\leq C_9\{\theta_\delta+\delta^{-1}|\lambda|^{-1/2m}\}|\lambda|^{-1/2}$

iv) $||S_{\lambda\delta}||_{L^2\to L^2}\leq C_9\{\theta_\delta+\delta^{-1}|\lambda|^{-1}\}|\lambda|^{-1}$

Proof. Let $u=(A-\lambda)^{-1}f-(A_2-\lambda)^{-1}(rf)$ and $v=\xi_\delta u=S_{\lambda\delta}f$. Noting that $v\in \mathring{H}_m(\Omega)$ we have

$$\begin{aligned} &B_2[v, v]-\lambda(v, v)\\ &= B_2[v, v]-B_2[u, \xi_\delta v]+B_2[u, \xi_\delta v]-\lambda(u, \xi_\delta v)\\ &= B_2[v, v]-B_2[u, \xi_\delta v]+(B_2-B)[(A-\lambda)^{-1}f, \xi_\delta v]\,. \end{aligned} \tag{4.7}$$

In view of (4. 5) we get

$$|B_2[v, v]-\lambda(v, v)|\geq K_7\{||v||_m+|\lambda|^{1/2}||v||_0\}^2\,. \tag{4.8}$$

Next from (4. 7)

$$\begin{aligned} &|B_2[v, v]-\lambda(v, v)|\\ &\leq |B_2[v, v]-B_2[u, \xi_\delta v]|+|(B_2-B)[(A-\lambda)^{-1}f, \xi_\delta v]|\\ &\leqq \Big|\int_\Omega \sum_{|\alpha|=|\beta|=m} a_{\alpha\beta}(x_0)\sum_{\alpha>\gamma}\binom{\alpha}{\gamma}D^{\alpha-\gamma}\xi_\delta D^\gamma u\overline{D^\beta v}\,dx\Big|\\ &+\Big|\int_\Omega \sum_{|\alpha|=|\beta|=m} a_{\alpha\beta}(x_0)\sum_{\beta>\gamma}\binom{\beta}{\gamma}D^\alpha u D^{\beta-\gamma}\xi_\delta\overline{D^\gamma u}\,dx\Big|\\ &+\Big|\int_\Omega \sum_{|\alpha|=|\beta|=m}\{a_{\alpha\beta}(x)-a_{\alpha\beta}(x_0)\}D^\alpha(A-\lambda)^{-1}f\sum_{\beta\geq\gamma}D^{\beta-\gamma}\xi_\delta\overline{D^\gamma v}\,dx\Big|\\ &+|B_1[(A-\lambda)^{-1}f, \xi_\delta v]+C_8((A-\lambda)^{-1}f, \xi_\delta v)|\\ &=I_1+I_2+I_3+I_4\,. \end{aligned} \tag{4.9}$$

Noting that $||rf||_{-m}\leq||f||_{V^*}$ we get, by Lemma 3. 2

$$||u||_l\leq K_7|\lambda|^{-1/2-l/2m}||f||_{V^*} \qquad \text{for} \quad f\in V^* \tag{4.10}$$

$$||u||_l\leq K_8|\lambda|^{-1-l/2m}||f||_0 \qquad \text{for} \quad f\in L^2(\Omega) \tag{4.11}$$

if $0\leq l\leq m$.
We have

$$|D^\gamma\xi_\delta(x)|\leq K_9\delta^{-|\gamma|}\,. \tag{4.12}$$

From (4. 10) and (4. 12) it follows that

$$|I_1| \leq K_9 \sum_{k=0}^{m-1} \delta^{k-m} ||u||_k ||v||_m$$
$$\leq K_{10} \delta^{-1} |\lambda|^{-1/2m} ||f||_{V^*} ||v||_m \qquad \text{for any} \quad f \in V^* \tag{4.13}$$

and

$$|I_2| \leq K_{11} ||u||_m \sum_{k=0}^{m-1} \delta^{k-m} ||v||_k$$
$$\leq K_{12} \delta^{-1} |\lambda|^{-1/2m} ||f||_{V^*} (||v||_m + |\lambda|^{1/2} ||v||_0) . \tag{4.14}$$

for any $f \in V^*$.
From (4. 6) it follows that

$$|I_3| \leq K_{13} \theta_\delta ||(A-\lambda)^{-1} f||_m \sum_{k=0}^{m} \delta^{k-m} ||v||_k$$
$$\leq K_{14} \theta_\delta ||f||_{V^*} ||v||_m + |\lambda|^{1/2} ||v||_0) . \tag{4.15}$$

From $a-(2)$, (4. 12) and the interpolation we know

$$|I_4| \leq K_{15} \{||(A-\lambda)^{-1} f||_m ||\xi_\delta v||_{m-1} + ||(A-\lambda)^{-1} f||_{m-1} ||\xi_\delta v||_m\}$$
$$\leq K_{16} |\lambda|^{-1/2m} ||f||_{V^*} (||v||_m + |\lambda|^{1/2} ||v||_0) . \tag{4.16}$$

Combining (4. 8), (4. 13), (4. 14), (4. 15) and (4. 16) we find that

$$(||v||_m + |\lambda|^{1/2} ||v||_0) \leq K_{17} \{\theta_\delta + \delta^{-1} |\lambda|^{-1/2m}\} ||f||_{V^*}$$

where K_{17} is a positive constant independent of λ and δ.
Thus the statements i) and ii) are clear. The inequalities iii) and iv) can be proved similarly. Q.E.D.

Lemma 4. 5. *For any $x \in P$ we have*

$$\lim_{\lambda \to -\infty} (-\lambda)^{1-n/2m} K_\lambda(x, x) = C(x) .$$

Proof. From Lemma 3. 3 and Lemma 4. 4, it follows that if $\lambda < 0$ and $\delta^{-1} |\lambda|^{-1/2m} \leq 1$.

$$|K_\lambda(x_0, x_0) - K_\lambda^0(x_0, x_0)| \leq K_{18} (\theta_\delta + \delta^{-1/2m}) |\lambda|^{-1+n/2m} \tag{4.17}$$

where $K_\lambda^0(x, y)$ is the kernel of the operator $(A_2 - \lambda)^{-1}$.
On the other hand, from Agmon [2], we get

$$|K_\lambda^0(x_0, x_0) - C(x_0)(-\lambda)^{-1+n/2m}| \leq K_{19} (|\lambda|^{-1+(n-1)/2m}$$
$$+ |\lambda|^{-1+(n-p)/2m} / \delta^p(x_0)) \tag{4.18}$$

where p is the any positive constant.

In view of (4. 17) and (4. 18) with $p=1/2$ we find

$$|K_\lambda(x_0, x_0)-(-\lambda)^{-1+n/2m}C(x_0)|$$
$$\leq K_{20}(\theta_\delta+\delta^{-1}|\lambda|^{-1/2m}+\delta(x_0)^{-1/2}|\lambda|^{-1/4m})|\lambda|^{-1+n/2m}.$$

Thus we know

$$\lim_{\lambda\to-\infty}(-\lambda)^{1-n/2m}K_\lambda(x_0\ x_0,) = C(x_0) \qquad \text{Q.E.D.}$$

5. Proof of the main theorem

First we shall consider the relation between the resolvent kernel and eigenvalues.

Lemma 5. 1. *We get the following equality and estimates*:

i) $\int_\Omega K_\lambda(x, x)\,dx = \sum_{j=1}^{\infty}(\lambda_j-\lambda)^{-1}$

ii) $\sum_{j=0}^{\infty}(\lambda_j-\lambda)^{-1} = C_{40}(-\lambda)^{-1+n/2m}+o(|\lambda|^{-1+n/2m})$

under $s-(0)$ *as* $\lambda\to-\infty$.

iii) *If* $d(\lambda)\geq|\lambda|^{1-1/4m+\varepsilon}$
$$\sum(\lambda_j-\lambda)^{-1} = C_{10}(-\lambda)^{-1+n/2m}$$
$$+0\,[|\lambda|^{(i+1+h)+(n-i-h)/2m+\delta}/d(\lambda)^{2+h+\varepsilon}$$
$$+|\lambda|^{p+(n-p)/2m}/d(\lambda)^{1+p}] \qquad as \quad |\lambda|\to\infty.$$

where $i=0$ *or* 1 *under* $s=(1)$ *or* $s-(2)$ *respectively* p *is the any positive number such that* $0<p<1$ *and* $C_{10}=\int_\Omega C(x)\,dx$.

Proof. For the statement i) see § 13 of Agmon [1].
From Lemma 3. 2 and Lemma 3. 3 we see that

$$|K_\lambda(x, x)|\leq K_1|\lambda|^{n/2m-1}. \tag{5. 1}$$

Since $a_{\alpha\beta}(x)$ are Riemann-integrable functions we find that the measure of $(\Omega-P)$ is zero. Using Lemma 4. 5, (5. 1) and Lebesgue theorem we know that

$$\lim_{\lambda\to-\infty}\int_\Omega(-\lambda)^{1-n/2m}K_\lambda(x, x)\,dx = \int_\Omega\lim_{\lambda\to-\infty}(-\lambda)^{1-n/2m}K_\lambda(x, x)\,dx.$$

Thus ii) is proved.
Putting $\gamma=|\gamma|^{1-1/2m+\varepsilon}/d(\lambda)$ in (4. 2) and integrating both sides over Ω we get the desired estimate since the second term is smaller than the first if j is

sufficiently large and the third term is dominated by the integral of the last.

Q.E.D.

Lemma 5.2. *Under* $s-(0)$ *it follows that*

$$N(t) = C_0 t^{n/2m} + o\,(t^{n/2m})\,.$$

Proof. Using Lemma 5.1 (ii) and arguing as in § 14 of Agmon [1] we get the desired statement. Q.E.D.

Lemma 5.3. *There is a constant* C_{11} *such that*

$$\operatorname{Re} \lambda_j \geq C_{11} j^{2m/n} \qquad \textit{for large} \quad j\,.$$

Proof. From $j \leq N(\operatorname{Re} \lambda_j)$ and Lemma 5.2 we can easily show the estimate.

Q.E.D.

Lemma 5.4. *If* $d(\lambda) \geq C|\lambda|^{1-1/2m+\varepsilon}$ *and* $|\lambda|$ *is sufficiently large then we have the following estimate*

$$\Big|\sum_{j=0}^{\infty} (\lambda_j-\lambda)^{-1} - \sum_{j=0}^{\infty} (\operatorname{Re} \lambda_j - \lambda)^{-1}\Big| \leq C_{12} |\lambda|^{1+(n-1)/2m+\varepsilon} / d(\lambda)^2\,.$$

Proof. We have the following equality

$$\sum_{j=0}^{\infty} (\lambda_j-\lambda)^{-1} - \sum_{j=0}^{\infty} (\operatorname{Re} \lambda_j-\lambda)^{-1} = -\sum_{j=0}^{\infty} \operatorname{Im} \lambda_j (\lambda_j-\lambda)^{-1} (\operatorname{Re} \lambda_j-\lambda)^{-1}$$
$$= -\sum_{\operatorname{Re}\lambda_j \leq 2|\lambda|} - \sum_{\operatorname{Re}\lambda_j > 2|\lambda|} = I_1 + I_2\,.$$

If $\operatorname{Re} \lambda_j \leq 2|\lambda|$ there is a constant K_2 such that

$$|\operatorname{Im} \lambda_j| \leq K_2 |\lambda|^{1-1/2m} \tag{5.2}$$

from Lemma 3.1.

On the other hand, if $d(\lambda) \geq C|\lambda|^{1-1/2m+\varepsilon}$ and $|\lambda|$ is sufficiently large, then an elementary geometrical observation shows that there is a positive constant K_3 such that

$$|\lambda_j - \lambda| \geq K_3 d(\lambda) \tag{5.3}$$

for any j.

In view of Lemma 5.2, (5.2) and (5.3) we get

$$|I_1| \leq \sum_{\operatorname{Re}\lambda_j \leq 2|\lambda|} |\operatorname{Im} \lambda_j|\, |\lambda_j-\lambda|^{-1} |\operatorname{Re} \lambda_j - \lambda|^{-1}$$
$$\leq K_4 |\lambda|^{1+(n-1)/2m} / d(\lambda)^2\,.$$

Next from Lemma 5.3 and $\operatorname{Re} \lambda_j > 2|\lambda|$ we see

$$|\lambda_j-\lambda| = |\lambda_j-\lambda|^{1-n(1+\varepsilon)/2m} |\lambda_j-\lambda|^{n(1+\varepsilon)/2m}$$
$$\geq K_5 |\lambda|^{1-n/2m-\varepsilon}\, j^{(1+\varepsilon)}\,.$$

Thus we find

$$\sum_{\mathrm{Re}\,\lambda_j>2|\lambda|}|\lambda_j-\lambda|^{-1}\leq K_6|\lambda|^{-1+n/2m+\varepsilon}\sum_{j=0}^{\infty}j^{-(1+\varepsilon)}$$
$$\leq K_7|\lambda|^{-1+n/2m+\varepsilon}\,. \tag{5.4}$$

On the other hand, from Lemma 3. 1 and $\mathrm{Re}\,\lambda_j>2|\lambda|$, we get

$$|\mathrm{Im}\,\lambda_j|\,|\mathrm{Re}\,\lambda_j-\lambda|^{-1}\leq K_8|\lambda|^{-1/2m}\,. \tag{5.5}$$

From (5. 4) and (5. 5) we know that

$$|I_2|\leq\sum_{\mathrm{Re}\,\lambda_j>2|\lambda|}|\mathrm{Im}\,\lambda_j|\,|\lambda_j-\lambda|^{-1}|\mathrm{Re}\,\lambda_j-\lambda_j-\lambda|^{-1}$$
$$\leq K_9|\lambda|^{-1+(n-1)/2m+\varepsilon}\leq K_{10}|\lambda|^{1+(n-1)/2m+\varepsilon}/d(\lambda)^2\,.$$

Q.E.D.

Now we follow the method of Agmon [2]. We put

$$f(\lambda)=\sum_{j=0}^{\infty}(\mathrm{Re}\,\lambda_j-\lambda)^{-1}\quad\text{and}\quad I(z)=(2\pi i)^{-1}\int_{L(z)}f(\lambda)\,d\lambda$$

where $L(z)$ is an oriented curve in the complex plane from $\bar{z}$ to $z=t+i\tau$ not intersecting $[0,\infty)$.

Thus for $t>0$, $\tau>0$

$$|I(z)-(\tau/\pi)\,\mathrm{Re}\,f(z)-N(t)+N(0)|\leq C_{12}\tau|\mathrm{Im}\,f(z)|\,. \tag{5.6}$$

First we consider the asymptotic formula for $N(t)$ under $s-(1)$. If $d(\lambda)\geq|\lambda|^{1-h/2m(h+2)+\varepsilon}$ and $|\lambda|$ is large then we get

$$|f(\lambda)|\leq K_{11}|\lambda|^{-1+n/2m} \tag{5.7}$$

from Lemma 5. 1 and Lemma 5. 4.

We put $z=t+it^{1-h/2m(h+2)+\varepsilon}$ and take

$$L(z)=\{\lambda=t+iu;\ t^{1-h/2m(h+2)+\varepsilon}\leq u\leq t\}$$
$$\cup\ \{\lambda;\ |\lambda|=\sqrt{2}\,t;\ \mathrm{Re}\,\lambda\leq t)$$

where t is a sufficiently large positive number.

From (5. 6), (5. 7) and $N(0)=0$ we find

$$|I(z)-N(t)|\leq K_{12}t^{n/2m-h/2m(h+2)+\varepsilon}\,. \tag{5.8}$$

On the other hand we know the following equality

$$I(z)=(2\pi i)^{-1}\int_{L(z)}f(\lambda)\,d\lambda=(2\pi i)^{-1}\int_{L(z)}\{f(\lambda)-C_{10}(-\lambda)^{-1+n/2m}\}\,d\lambda$$
$$+(2\pi i)^{-1}\int_{L(z)}C_{10}(-\lambda)\lambda^{-1+n/2m}d\lambda=I_1+I_2\,.$$

In view of Lemma 5. 1 and Lemma 5. 4, putting $1>p>h/2$ we get that

$$
\begin{aligned}
|I_1| &\leq K_{13}\Big\{\int_{L(z)} |\lambda|^{1+h+(n-h)/2m+\varepsilon}/d(\lambda)^{2+h}|d\lambda| \\
&+\int_{L(z)} |\lambda|^{p+(n-p)/2m}/d(\lambda)^{1+p}|d\lambda| \\
&+\int_{L(z)} |\lambda|^{1+(n-1)/2m+\varepsilon}/d(\lambda)^{2}|d\lambda| \\
&\leq K_{14}\Big\{t^{1+h+(n-h)2m+\varepsilon}\int_{t^{1-h/2m(h+2)+\varepsilon}}^{t} u^{-(2+h)}du \\
&+t^{1+h+(n-h)/2m+\varepsilon-(2+h)+1} \\
&+t^{pt(n-p)/2m}\int_{t^{1-h/2m(h+2)+\varepsilon}}^{t} u^{-(1+p)}du \\
&+t^{p+(n-p)/2m-(1+p)+1} \\
&+t^{1+(n-1)/2m+\varepsilon}\int_{t^{1+h/2m(h+2)+\varepsilon}}^{t} u^{-2}du \\
&+t^{1+(n-1)/2m+\varepsilon-2+1}\Big\} \\
&\leq K_{15}t^{n/2m-h/2m(h+2)+\varepsilon} \qquad (5.9)
\end{aligned}
$$

Noting that

$$
\left|\frac{1}{2\pi i}\int_{L(z)}(-\lambda)^{-1+n/2m}d\lambda - t^{n/2m}\frac{\sin(n\pi/2m)}{n\pi/2m}\right| \leq K_{16}t^{n/2m-h/2m(h2+\varepsilon}.
$$

from (5. 8) and (5. 9) we obtain the desired estimate.

In case of $s-(2)$ assuming that $a-(3)$ holds for some $p\geq(h+1)/2$ if $h<1$ and for any $p<1$ if $h=1$, we can prove the desired result in the same method as above.

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