Maruo, K.
Osaka J. Math.
9 (1972), 547-560

ASYMPTOTIC DISTRIBUTION OF EIGENVALUES OF NON-SYMMETRIC OPERATORS ASSOCIATED WITH STRONGLY ELLIPTIC SESQUILINEAR FORMS

Kenji MARUO

(Received January 31, 1972)

1. Introduction

The main object of this paper is to extend the result of K. Maruo and H. Tanabe [4] on the eigenvalue distribution of symmetric elliptic operators to a non symmetric case. Some amelioration of the result of [4] on the remainder estimates in Weyl's formula as well as the formula under less restrictive smoothness assumptions is also obtained.

Let Ω be a bounded domain in R^n having the restricted cone property. We use the same notations as those of [4] to denote various norms and functional spaces. In this paper it is assumed that $2m>n$ as in the previous paper [4]. Let B be a sesquilinear form defined in $H_m(\Omega)\times H_m(\Omega)$ satisfying

$$\operatorname{Re} B[u, u]\geq\delta_0||u||^2_m \qquad \text{for any} \quad u\in V \qquad a-(1)$$

where V is a closed subspace of $H_m(\Omega)$ containing $\mathring{H}_m(\Omega)$ and δ_0 is some positive constant independent of u. We assume that B has the following form

$$B[u, v] = B_0[u, v]+B_1[u, v] \tag{1.1}$$

where B_0 which is the principal part of B is a symmetric integro-differential sesquilinear form of order m with bounded coefficients

$$B_0[u, v] = \int_\Omega \sum_{|\alpha|=|\beta|=m} a_{\alpha\beta}(x) D^\alpha u D^\beta v dx$$

and B_1 is a not necessarily symmetric sesquilinear form satisfying

$$|B_1[u, v]|\leq K(||u||_m||v||_{m-1}+||u||_{m-1}||v||_m) \qquad a-(2)$$

for any $u, v\in V$ i.e. B_1 is the lower order part of B. Let A be the operator associated with the form B: an element u of V belongs to $D(A)$ and $Au=f\in L^2(\Omega)$ if $B[u, v]=(f, v)$ holds for any $v\in V$. A is a not necessarily symmetric operator in $L^2(\Omega)$ and all rays $\arg\lambda=\theta$ different from the positive real axis are rays of minimal growth of the resolvent of A. By $N(t)$ we denote the number of eigenvalues of A whose real part does not exceed t. The main conclusion of this paper is that the following asymptotic formula holds:

$$N(t) = C_0 t^{n/2m} + o(t^{n/2m}) \qquad \text{as} \quad t \to \infty\,, \tag{1. 2}$$

if the coefficients of B_0 are Riemann integrable,
and

$$N(t) = C_0 t^{n/2m} + 0\,(t^{(n-\theta)/2m}) \qquad \text{as} \quad t \to \infty \tag{1. 3}$$

for any $\theta < h/(h+2)$ if B_0 has uniformly Hoelder continuous coefficients of order h and for any $\theta < (h+1)/(h+3)$ if the coefficients of B_0 belong to the class C^{1+h} in some domain containing Ω. The formula (1. 3) is an improvement of the corresponding result obtained for symmetric operators in [4] where (1. 3) was established only for $\theta < h/(h+3)$ and $\theta < (h+1)/(h+4)$ respectively making some more restrictive assumptions and in order to prove (1. 3) for $(h+1)/(h+4) \leq \theta < 1/2$ still more hypotheses were required.

The author wishes to thank Professor H. Tanabe and Mr. M. Nagase for suggesting this problem and helpful advices.

2. Main theorem

As was stated in the introduction let Ω be a bounded domain in R^n having the restricted cone property (p. 11 of S. Agmon [1]) and it is assumed that $2m > n$. For $x \in \Omega$ we write $\delta(x) = \min\{1, \text{dist}(x, \partial\Omega)\}$. Suppose that

$$\int_\Omega \delta(x)^{-p}\, dx < \infty \tag{$a-(3)$}$$

for some positive number $p < 1$ which will be specified later.

Since all coefficients of of B_0 are bounded it follows from $a-(2)$ that for any $u, v \in V$

$$|B[u, v]| \leq K \|u\|_m \|v\|_m$$

for some constant K.

We state various smoothness assumptions on the coefficients of B_0:

they are Riemann integrable, i.e. continuous almost everywhere in Ω: $s-(0)$

they are uniformly Hoelder continuous of order h in Ω: $s-(1)$

they belong to $C^{1+h}(\Omega_1)$ where Ω_1 is some domain containing Ω and $C^{1+h}(\Omega_1)$ is the subclass of functions in $C^1(\Omega_1)$ with derivatives Hoelder continuous of order h in Ω_1. $s-(2)$

Main Theorem. *The following asymptotic formulas for $N(t)$ hold as $t \to \infty$:*

$$N(t) = C_0 t^{n/2m} + o\,(t^{n/2m}) \qquad \textit{under } s-(0)$$
$$N(t) = C_0 t^{n/2m} + 0\,(t^{(n-\theta)/2m})$$

for any θ satisfying

$$0 < \theta < h/(h+2) \qquad \textit{under } s-(1)$$
$$0 < \theta < (h+1)/(h+3) \qquad \textit{under } s-(2)$$

where

$$C_0 = \frac{\sin(n/2m)}{n/2m} \int_\Omega C(x)\,dx$$

$$C(x) = (2\pi)^{-n} \int_{R^n} \{ \sum_{|\alpha|=|\beta|=m} a_{\alpha\beta}(x)\, \xi^{\alpha+\beta} + 1 \}^{-1} d\xi \,.$$

REMARK. As was mentioned in the Introduction the remainder estimates described in the main theorem is an improvement of those established in [4]. Furthermore applying the theorem to the sesquilinear form (Au, Av) where A is the elliptic operator satisfying the conditions of R. Beals [3] we may prove Theorem C of [3] with $0 < \theta < h/(h+2)$ instead of $0 < \theta < h/(h+3)$ if the order of A is greater than $n/2$.

Following the method of S. Agmon [5] or Dunford-Schwartz [6] it is possible to show that the generalized eigenfunctions of A are complete in $L^2(\Omega)$ under our assumptions.

3. Some lemmas

As in the previous paper [4] we extend the operator A to a mapping on V to V^* where V^* is the antidual of V. This extended operator which is again denoted by A is defined by

$$B[u, v] = (Au, v) \qquad \text{for any} \quad v \in V$$

where the bracket on the right stands for the duality between V^* and V in this case.

Identifying $L^2(\Omega)$ with its antidual we may consider $V \subset L^2(\Omega) \subset V^*$ algebraically and topologically, and as is easily seen V is a dense subspace of V^* under this convention. The resolvent of A thus extended is a bounded linear operator on V^* to V. We denote by $\rho(A)$ the resolvent set of A and $d(\lambda)$ the distance from the point λ to the positive real axis for a complex number λ.

Lemma 3.1. *The resolvent set $\rho(A)$ of A in either sense contains the set $\{\lambda: d(\lambda) \geqq C|\lambda|^{1-1/2m}, |\lambda| \geqq C\}$ for some constant C. The eigenvalues $\{\lambda_j\}_{j=0}^\infty$ of A have finite multiplicity and eigenvalues of A can have only ∞ as a limite point.*

Proof. We put $(A-\lambda)u=f$ for any $u\in D(A)$. We see that

$$B[u, u]-\lambda(u, u) = (f, u) \tag{3.1}$$

From (3. 1), (1. 1), $a-(2)$ and $\mathrm{Im}\, B_0[u, u]=0$, we get:

$$|\mathrm{Im}\,\lambda|\,||u||_0^2\leq||f||_0||u||_0+2K||u||_m||u||_{m-1}\,. \tag{3.2}$$

Applying to the last term $||u||_m||u||_{m-1}$ Young's inequality and then using the interpolation inequality, for any positive constant δ_1 and $\delta_2\leq 1$ we find that

$$\begin{aligned}||u||_m||u||_{m-1}&\leq\delta_1||u||_m^2+\delta_1^{-1}||u||_{m-1}^2\\&\geq K_1\{\delta_1||u||_m^2+\delta_1^{-1}\delta_2||u||_m^2+\delta_1^{-1}\delta_2^{-m+1}||u||_0^2\}\,.\end{aligned} \tag{3.3}$$

From (3. 1) and $a-(1)$ we get

$$\delta||u||_m^2\leqq|\lambda|\,||u||_0^2+||u||_0||f||_0\,. \tag{3.4}$$

Putting $\delta_1=\delta_2^{1/2}=|\lambda|^{-1/2m}$ and combining (3. 2), (3. 3) and (3. 4) we find that

$$(|\mathrm{Im}\,\lambda|-K_2|\lambda|^{-1/2m}||u||_0^2\leq(1+K_2|\lambda|^{-1/2m})\,||f||_0||u||_0\,. \tag{3.5}$$

If $|\mathrm{Im}\,\lambda|>C|\lambda|^{1-1/2m}$ for large C, we know that

$$||u||_0\leq K_3/|\mathrm{Im}\,\lambda|\,||f||_0\,. \tag{3.6}$$

If $\mathrm{Re}\,\lambda<0$ we get

$$|\mathrm{Re}\,\lambda|\,||u||_0^2\leq||f||_0||u||_0 \tag{3.7}$$

from (3. 1).

Combining (3. 6) and (3. 7) we find that there is a constant K_4 independent of λ such that

$$||u||_0\leq K_4/d(\lambda)||f||_0 \tag{3.8}$$

On the other hand for an adjoint operator A^* we find the same estimate (3. 8). Thus the null space of the operator $(A^*-\bar{\lambda})$consists only of zero and we know

$$\{\lambda\colon d(\lambda)\geq C|\lambda|^{1-1/2m},\ |\lambda|\geq C\}\subset\rho(A)\,.$$

Next we put $(A-\lambda)u=f$ for any $u\in V$.

From (1. 1), $a-(1)$ and $a-(2)$ it follows that

$$||u||_0^2\leq K_5/d(\lambda)\{||f||_{V^*}||u||_m+||u||_m||u||_{m-1}\}\,. \tag{3.9}$$

For any number δ_3 such that $0<\delta_3\leqq 1$ we know

$$||u||_{m-1}\leq K_6\{\delta_3||u||_m+\delta_3^{-2m+1}||u||_{V^*}\}\,. \tag{3.10}$$

From the inequality

$$|\lambda||(u, v)| \leq ||f||_{V^*}||v||_m + K||u||_m||v||_m \qquad \text{for any} \quad v \in V$$

it follows that

$$|\lambda| ||u||_{V^*} \leq ||f||_{V^*} + K_7||u||_m \tag{3. 11}$$

Combining $a-(1)$, (3. 9), (3. 10) and (3. 11) and putting $\delta_3 = |\lambda|^{-1/2m}$ we get the following estimate:

$$\begin{aligned}\delta||u||_m^2 &\leq ||f||_{V^*}||u||_m + |\lambda| ||u||_0^2 \\ &\leq ||f||_{V^*}||u||_m + K_8|\lambda|/d(\lambda)\{||f||_{V^*}||u|| + {}_m||u||_m||u||_{m-1}\} \\ &\leq ||f||_{V^*}||u||_m + K_9|\lambda|/d(\lambda)\{(1+|\lambda|^{-1/2m})||u||_m||f||_{V^*} \\ &\quad + |\lambda|^{-1/2m}||u||_m^2\}\end{aligned}$$

If $d(\lambda) \geq C|\lambda|^{1-1/2m}$ with $|\lambda|$ sufficiently large there is a constant K_{10} independent of λ such that

$$||u||_m \leq K_{10}|\lambda|/d(\lambda)||f||_{V^*} \tag{3. 12}$$

On the other hand we put $(A^* - \bar{\lambda})u = f$ for any $u \in V$. Then we find the same estimate (3. 12) for A^*. Thus we see that

$$\{\lambda \colon d(\lambda) \geq C|\lambda|^{1+1/2m} \colon |\lambda| \geq C\} \subset \rho(A).$$

The last part of the lemma is a simple consequence of Rellich's theorem.

Q.E.D.

For a bounded operator S on V^* to V we use the notations $||S||_{V^* \to L^2}$ $||S||_{V^* \to L^2}$ etc, to denote the norms of S considered as an operator on V^* to V, V^* to $L^2(\Omega)$, etc.

Lemma 3. 2. *There exists a constant C_1 such that*

i) $||(A-\lambda)^{-1}||_{L^2 \to L^2} \leq C_1/d(\lambda)$ ii) $||(A-\lambda)^{-1}||_{L^2 \to V} \leq C_1|\lambda|^{1/2}/d(\lambda)$

iii) $||(A-\lambda)^{-1}||_{V^* \to V} \leq C_1|\lambda|/d(\lambda)$ iv) $||(A-\lambda)^{-1}||_{V^* \to L^2} \leq C_1|\lambda|^{1/2}/d(\lambda)$

if $d(\lambda) \geq C|\lambda|^{1-1/2m}$, $|\lambda| \geqq C$ where C is the constant in the statement of Lemma 3. 1.

Proof. The statement i) is clear from (3. 8).
If $u = (A-\lambda)^{-1}f$ for any $f \in L^2(\Omega)$ we get;

$$\begin{aligned}\delta||u||_m^2 &\leq ||f||_0||u||_0 + |\lambda| ||u||_0^2 \\ &\leq K_{11}|\lambda|(||f||_0/d(\lambda))^2\end{aligned}$$

from $a-(1)$ and i).
The statement iii) is clear from (3. 12). Finally with the aid of (3. 12) and the following inequality

$$|\lambda|\,||u||_0^2 \le K||u||_m^2 + ||f||_{V^*}||u||_m$$

we can easily show iv). Q.E.D.

Lemma 3.3. *Let S be a bounded operator on V^* to V. Then S has a kernel M in the following sense:*

$$Sf(x) = \int_\Omega M(x, y)f(y)\,dy \qquad \text{for} \quad f \in L_2(\Omega)\,.$$

$M(x, y)$ is continuous in $\Omega \times \Omega$ and there exists a constant C_2 such that for any $x, y \in \Omega$.

$$|M(x, y)|$$
$$\le C_2||S||_{V_*\to V}^{n^2/4m^2}||S||_{V^*\to L^2}^{n/2m-n^2/4m^2}||S||_{V^2\to V}^{n/2m-n^2/4m^2}||S||_{L^2\to L^2}^{(1-n/2m)^2}$$

Proof. see [4]. Q.E.D.

Lemma 3.4. *There are positive constants C_3 and C_4 such that*

$$B_0[u, u] \ge C_2||u||_m^2 - C_4||u||_0^2 \qquad \textit{for any} \quad u \in V\,.$$

Proof. From $a-(1)$ and the interpolation inequality, we can easily show the statement. Q.E.D.

4. Estimates of the resolvent kernel

We shall estimate the difference between the resolvent kernel of A and that of the operator A_0 associated with $B_0 + C_4$, thus $B_0[u, v] + C_4(u, v) = (A_0u, v)$ for any $u, v \in V$. Obviously for the operator A_0 the analogues of Lemma 3.2 hold.

LetS_λ be the operator defined by

$$S_\lambda f = (A-\lambda)^{-1}f - (A_0-\lambda)^{-1}f \qquad \text{for any} \quad f \in V^*\,.$$

Lemma 4.1. *There is a constant C_7 such that for $d(\lambda) \ge C|\lambda|^{1-1/m}$, $|\lambda| \ge C$,*

i) $||S_\lambda||_{V^*\to V} \le C_5|\lambda|/d(\lambda)\ (|\lambda|^{1-1/2m}/d(\lambda))$

ii) $||S_\lambda||_{V^*\to L^2}$
iii) $||S_\lambda||_{L^2\to V}$
$\Big\}\le C_5|\lambda|^{1/2}/d(\lambda)\ (|\lambda|^{1-1/2m}/d(\lambda))$

iv) $||S_\lambda||_{L^2\to L^2} \le C_5/d(\lambda)\ (|\lambda|^{1-1/2m}/d(\lambda))$.

Proof. Let $(A-\lambda)^{-1}f - (A_0-\lambda)^{-1}f = S_\lambda f = u$. Now we know that

$$(A-\lambda)^{-1} - (A_0-\lambda)^{-1} = (A_0-\lambda)^{-1}(A_0-A)\,(A-\lambda)^{-1}\,.$$

On the other hand, since the operator A_0 is self-adjoint we know

$$\begin{aligned}(S_\lambda f, \phi) &= ((A_0-A)(A-\lambda)^{-1}f, (A_0-\lambda)^{-1}\phi)\\ &= (B_0-B)[(A-\lambda)^{-1}f, (A_0-\bar{\lambda})^{-1}\phi]+C_4((A-\lambda)^{-1}f, (A_0-\bar{\lambda})^{-1}\phi)\\ &= -B_1[(A-\lambda)^{-1}f, (A_0-\bar{\lambda})^{-1}\phi]+C_4((A-\lambda)^{-1}f, (A_0-\bar{\lambda})^{-1}\phi) \quad (4.1)\end{aligned}$$

for any $\phi\in V^*$.
Combining (4. 1), Lemma 3. 2 and the interpolation inequality we find that there are constants K_1 and K_2 such that

$$\begin{aligned}|(S_\lambda f, \phi)| &\le K_1\{||(A-\lambda)^{-1}f||_m||(A_0-\bar{\lambda})^{-1}\phi||_{m-1}\\ &\quad +||(A-\lambda)^{-1}f||_{m-1}||(A_0-\bar{\lambda})^{-1}\phi||_m\}\\ &\le K_2(|\lambda|/d(\lambda))^2|\lambda|^{-1/2m}||f||_{V^*}||\phi||_{V^*}\} .\end{aligned}$$

Then we get

$$||S_\lambda||_{V\to V^*}\le C_5|\lambda|/d(\lambda)(|\lambda|^{1-1/2m}/d(\lambda)) .$$

The remaining inequalities can be proved in a similar manner. Q.E.D.

Since $m>n/2$ there exist the resolvent kernels $K_\lambda(x, y)$ and $K^0_\lambda(x, y)$ of the operator A and A_0 such that

$$(A-\lambda)^{-1}f(x) = \int_\Omega K_\lambda(x, y)f(y)\,dy$$

$$(A_0-\lambda)^{-1}f(x) = \int_\Omega K^0_\lambda(x, y)f(y)\,dy \qquad \text{for any} \quad f\in L^2(\Omega) .$$

Theorem 4. 2. *For any given positive numbers p, ε and any non-negative integer j, the following inequality holds*:

$$\begin{aligned}&|K_\lambda(x, x)-C(x)(-\lambda)^{-1+n/2m}|\le C_6[|\lambda|^{n/2m}/d(\lambda)\{\gamma^{h+i}|\lambda|/d(\lambda)\\ &+(\gamma^{-1}|\lambda|^{1-1/2m}/d(\lambda))^j+|\lambda|^{1-1/2m}/d(\lambda)+(|\lambda|^{1-1/2m}/\delta(x)\,d(\lambda))^p\}] \qquad (4.2)\end{aligned}$$

for $d(\lambda)\ge|\lambda|^{1-1/4m}+\varepsilon$, $\gamma>0$, $\gamma^{-1}|\lambda|^{1-1/2m}/d(\lambda)\le 1$, *and* $|\lambda|$ *sufficiently large, where* $i=0$ *under* $s-(1)$ *and* $i=1$ *under* $S-(2)$. C_6 *is a constant depending on* p, ε, j *but not on* λ, γ *or* x, *and* $C(x)$ *is the function defined in the main theorem.*

Proof. Combining Lemma 4. 2, 6. 2, 7.2 and 7.3 of [4] we get

$$\begin{aligned}&|K^0_\lambda(x, x)-C(x)(-\lambda)^{-1+n/2m}|\le K_3[|\lambda|^{n/2m}/d(\lambda)\{\gamma^{h+i}/d(\lambda)\\ &+(\gamma^{-1}|\lambda|^{1-1/2m}/d(\lambda))^j+(|\lambda|^{1-1/2m}/\delta(x)\,d(\lambda))^p\}+|\lambda|^{(n-1)/2m-1}] \qquad (4.3)\end{aligned}$$

where i=0 or 1 according as we assume $s-(1)$ or $s-(2)$.
Formally we replaced $d(\lambda)$ by some power of $|\lambda|$ at this point (Theorem 7. 1 of [4]); however, in this paper we postpone this replacement for a little while to obtain better remainder estimates as was stated in the introduction.
On the other hand applying Lemma 3. 3 and Lemma 4. 1 to S_λ we get

$$|K_\lambda(x, y)-K_\lambda^0(x, y)| \leq K_4(|\lambda|/d(\lambda))^2|\lambda|^{(n-1)/2m-1} \tag{4. 4}$$

Combining (4. 3) and (4. 4) the desired estimate (4. 2) is obtained. Q.E.D.

Next we shall consider the case of the assumption $s-(0)$. We denote $P_{\alpha\beta}$ the set of points where $a_{\alpha\beta}$ is continuous and put $P=\bigcap_{|\alpha|=|\beta|=m} P_{\alpha\beta}$. We fix a point $x_0 \in P$ and set

$$B_2'[u, v] = \int_\Omega \sum_{|\alpha|=|\beta|=m} a_{\alpha\beta}(x_0) D^\alpha u \overline{D^\beta v} dx \quad \text{for} \quad u, v \in H_m(\Omega).$$

Lemma 4. 3. *There exist positive constants C_7 and C_8 independent of u and x_0 such that*

$$B_2'[u, u] \geq C_7||u||_m^2 - C_8||u||_0^2 \quad \textit{for} \quad u \in \mathring{H}_m(\Omega).$$

Proof. There is a constant K_5 such that

$$\sum_{|\alpha|=|\beta|=m} a_{\alpha\beta}(x_0)\xi^{\alpha+\beta} \geq K_5|\xi|^{2m}$$

for any $\xi \in R^n$. That the desired inequality holds for any $u \in \mathring{H}_m(\Omega)$ is a well known fact. Q.E.D.

We put $B_2[u, v]=B_2'[u, v]+C_2(u, v)$ for $u, v \in \mathring{H}_m(\Omega)$. We know that

$$B_2[u, u] \geq K_6||u||_m^2 \quad \text{for} \quad u \in \mathring{H}_m(\Omega) \tag{4. 5}$$

from Lemma 4. 3.

We denote by A_2 the operator associated with B_2 under the Dirichlet boundary condition. By definition for any $u, v \in \mathring{H}_m(\Omega)$ we have

$$B_2[u, v] = (A_2 u, v)$$

where the bracket on the right denotes the pairing between the antidual $H_{-m}(\Omega)$ of $\mathring{H}_m(\Omega)$ and $\mathring{H}_m(\Omega)$ this case. Obviously for the operator A_2 the analogues of Lemma 3. 1 and Lemma 3. 2 hold.

We denote by $\xi(x)$ a function in $C_0^\infty(R^n)$ the support of which is contained in the set $\{x \in R^n: |x|<1\}$ and which takes the valued 1 at the origin. We write $\xi_\delta(x)=\xi((x-x_0)/\delta)$ where δ is any positive number $<\delta(x_0)$.

Let $S_{\lambda\delta}$ be the operator defined by

$$S_{\lambda\delta}f = \xi_\delta\{(A-\lambda)^{-1}f-(A_2-\lambda)^{-1}(rf)\} \quad \text{for} \quad f \in V^*$$

where rf is the restriction of $f \in V^*$ to $\mathring{H}_m(\Omega)$.

Obviously $S_{\lambda\varepsilon}$ is a bounded operator on V^* to $\mathring{H}_m(\Omega)$ and hence a fortiori to V. Since $a_{\alpha\beta}$ is continuous at x_0 for any α and β with $|\alpha|=|\beta|=m$ there is a positive number θ_δ such that

$\theta_\delta \to 0$ as $\delta \to 0$ and

$$|a_{\alpha\beta}(x) - a_{\alpha\beta}(x_0)| < \theta_\delta \qquad \text{for} \quad |x - x_0| < \delta \tag{4.6}$$

Lemma 4.4. *If λ is real <0 and $\delta^{-1}|\lambda|^{-1/2m} \le 1$ we get*

i) $||S_{\lambda\delta}||_{V^* \to V} \le C_9\{\theta_\delta + \delta^{-1}|\lambda|^{-1/2m}\}$

ii) $||S_{\lambda\delta}||_{V^* \to L^2} \le C_9\{\theta_\delta + \delta^{-1}|\lambda|^{-1/2m}\}|\lambda|^{-1/2}$

iii) $||S_{\lambda\delta}||_{L^2 \to V} \le C_9\{\theta_\delta + \delta^{-1}|\lambda|^{-1/2m}\}|\lambda|^{-1/2}$

iv) $||S_{\lambda\delta}||_{L^2 \to L^2} \le C_9\{\theta_\delta + \delta^{-1}|\lambda|^{-1}\}|\lambda|^{-1}$

Proof. Let $u = (A - \lambda)^{-1} f - (A_2 - \lambda)^{-1}(rf)$ and $v = \xi_\delta u = S_{\lambda\delta} f$. Noting that $v \in \overset{\circ}{H}_m(\Omega)$ we have

$$\begin{aligned} & B_2[v, v] - \lambda(v, v) \\ &= B_2[v, v] - B_2[u, \xi_\delta v] + B_2[u, \xi_\delta v] - \lambda(u, \xi_\delta v) \\ &= B_2[v, v] - B_2[u, \xi_\delta v] + (B_2 - B)[(A - \lambda)^{-1} f, \xi_\delta v]\,. \end{aligned} \tag{4.7}$$

In view of (4.5) we get

$$|B_2[v, v] - \lambda(v, v)| \ge K_7\{||v||_m + |\lambda|^{1/2}||v||_0\}^2\,. \tag{4.8}$$

Next from (4.7)

$$\begin{aligned} & |B_2[v, v] - \lambda(v, v)| \\ & \le |B_2[v, v] - B_2[u, \xi_\delta v]| + |(B_2 - B)[(A - \lambda)^{-1} f, \xi_\delta v]| \\ & \leqq \Big|\int_\Omega \sum_{|\alpha| = |\beta| = m} a_{\alpha\beta}(x_0) \sum_{\alpha > \gamma} \binom{\alpha}{\gamma} D^{\alpha-\gamma}\xi_\delta D^\gamma u \overline{D^\beta v}\, dx\Big| \\ & + \Big|\int_\Omega \sum_{|\alpha| = |\beta| = m} a_{\alpha\beta}(x_0) \sum_{\beta > \gamma} \binom{\beta}{\gamma} D^\alpha u D^{\beta-\gamma}\xi_\delta \overline{D^\gamma u}\, dx\Big| \\ & + \Big|\int_\Omega \sum_{|\alpha| = |\beta| = m} \{a_{\alpha\beta}(x) - a_{\alpha\beta}(x_0)\} D^\alpha (A - \lambda)^{-1} f \sum_{\beta \ge \gamma} D^{\beta-\gamma}\xi_\delta \overline{D^\gamma v}\, dx\Big| \\ & + |B_1[(A - \lambda)^{-1} f, \xi_\delta v] + C_8((A - \lambda)^{-1} f, \xi_\delta v)| \\ & = I_1 + I_2 + I_3 + I_4\,. \end{aligned} \tag{4.9}$$

Noting that $||rf||_{-m} \le ||f||_{V^*}$ we get, by Lemma 3.2

$$||u||_l \le K_7|\lambda|^{-1/2 - l/2m}||f||_{V^*} \qquad \text{for} \quad f \in V^* \tag{4.10}$$

$$||u||_l \le K_8|\lambda|^{-1 - l/2m}||f||_0 \qquad \text{for} \quad f \in L^2(\Omega) \tag{4.11}$$

if $0 \le l \le m$.
We have

$$|D^\gamma \xi_\delta(x)| \le K_9 \delta^{-|\gamma|}\,. \tag{4.12}$$

From (4. 10) and (4. 12) it follows that

$$|I_1| \leq K_9 \sum_{k=0}^{m-1} \delta^{k-m} ||u||_k ||v||_m$$
$$\leq K_{10} \delta^{-1} |\lambda|^{-1/2m} ||f||_{V^*} ||v||_m \qquad \text{for any} \quad f \in V^* \tag{4.13}$$

and

$$|I_2| \leq K_{11} ||u||_m \sum_{k=0}^{m-1} \delta^{k-m} ||v||_k$$
$$\leq K_{12} \delta^{-1} |\lambda|^{-1/2m} ||f||_{V^*} (||v||_m + |\lambda|^{1/2} ||v||_0) . \tag{4.14}$$

for any $f \in V^*$.
From (4. 6) it follows that

$$|I_3| \leq K_{13} \theta_\delta ||(A-\lambda)^{-1} f||_m \sum_{k=0}^{m} \delta^{k-m} ||v||_k$$
$$\leq K_{14} \theta_\delta ||f||_{V^*} ||v||_m + |\lambda|^{1/2} ||v||_0) . \tag{4.15}$$

From $a-(2)$, (4. 12) and the interpolation we know

$$|I_4| \leq K_{15} \{ ||(A-\lambda)^{-1} f||_m ||\xi_\delta |v||_{m-1} + ||(A-\lambda)^{-1} f||_{m-1} ||\xi_\delta v||_m \}$$
$$\leq K_{16} |\lambda|^{-1/2m} ||f||_{V^*} (||v||_m + |\lambda|^{1/2} ||v||_0) . \tag{4.16}$$

Combining (4. 8), (4. 13), (4. 14), (4. 15) and (4. 16) we find that

$$(||v||_m + |\lambda|^{1/2} ||v||_0) \leq K_{17} \{ \theta_\delta + \delta^{-1} |\lambda|^{-1/2m} \} ||f||_{V^*}$$

where K_{17} is a positive constant independent of λ and δ.
Thus the statements i) and ii) are clear. The inequalities iii) and iv) can be proved similarly. Q.E.D.

Lemma 4. 5. *For any $x \in P$ we have*

$$\lim_{\lambda \to -\infty} (-\lambda)^{1-n/2m} K_\lambda(x, x) = C(x) .$$

Proof. From Lemma 3. 3 and Lemma 4. 4, it follows that if $\lambda < 0$ and $\delta^{-1} |\lambda|^{-1/2m} \leq 1$.

$$|K_\lambda(x_0, x_0) - K_\lambda^0(x_0, x_0)| \leq K_{18} (\theta_\delta + \delta^{-1/2m}) |\lambda|^{-1+n/2m} \tag{4.17}$$

where $K_\lambda^0(x, y)$ is the kernel of the operator $(A_2 - \lambda)^{-1}$.
On the other hand, from Agmon [2], we get

$$|K_\lambda^0(x_0, x_0) - C(x_0)(-\lambda)^{-1+n/2m}| \leq K_{19} (|\lambda|^{-1+(n-1)/2m}$$
$$+ |\lambda|^{-1+(n-p)/2m} / \delta^p(x_0)) \tag{4.18}$$

where p is the any positive constant.

In view of (4. 17) and (4. 18) with $p=1/2$ we find

$$|K_\lambda(x_0, x_0)-(-\lambda)^{-1+n/2m}C(x_0)|$$
$$\leq K_{20}(\theta_\delta+\delta^{-1}|\lambda|^{-1/2m}+\delta(x_0)^{-1/2}|\lambda|^{-1/4m})|\lambda|^{-1+n/2m}.$$

Thus we know

$$\lim_{\lambda\to-\infty}(-\lambda)^{1-n/2m}K_\lambda(x_0\ x_0,) = C(x_0) \qquad \text{Q.E.D.}$$

5. Proof of the main theorem

First we shall consider the relation between the resolvent kernel and eigenvalues.

Lemma 5. 1. *We get the following equality and estimates:*

i) $\int_\Omega K_\lambda(x, x)\,dx = \sum_{j=1}^{\infty}(\lambda_j-\lambda)^{-1}$

ii) $\sum_{j=0}^{\infty}(\lambda_j-\lambda)^{-1} = C_{40}(-\lambda)^{-1+n/2m}+o(|\lambda|^{-1+n/2m})$

under $s-(0)$ *as* $\lambda\to-\infty$.

iii) *If* $d(\lambda)\geq|\lambda|^{1-1/4m+\varepsilon}$
$$\sum(\lambda_j-\lambda)^{-1} = C_{10}(-\lambda)^{-1+n/2m}$$
$$+0\,[|\lambda|^{(i+1+h)+(n-i-h)/2m+\delta}/d(\lambda)^{2+h+\varepsilon}$$
$$+|\lambda|^{p+(n-p)/2m}/d(\lambda)^{1+p}] \qquad as \quad |\lambda|\to\infty.$$

where $i=0$ *or* 1 *under* $s=(1)$ *or* $s-(2)$ *respectively* p *is the any positive number such that* $0<p<1$ *and* $C_{10}=\int_\Omega C(x)\,dx$.

Proof. For the statement i) see § 13 of Agmon [1].
From Lemma 3. 2 and Lemma 3. 3 we see that

$$|K_\lambda(x, x)|\leq K_1|\lambda|^{n/2m-1}. \tag{5. 1}$$

Since $a_{\alpha\beta}(x)$ are Riemann-integrable functions we find that the measure of $(\Omega-P)$ is zero. Using Lemma 4. 5, (5. 1) and Lebesgue theorem we know that

$$\lim_{\lambda\to-\infty}\int_\Omega(-\lambda)^{1-n/2m}K_\lambda(x, x)\,dx = \int_\Omega\lim_{\lambda\to-\infty}(-\lambda)^{1-n/2m}K_\lambda(x, x)\,dx.$$

Thus ii) is proved.
Putting $\gamma=|\gamma|^{1-1/2m+\varepsilon}/d(\lambda)$ in (4. 2) and integrating both sides over Ω we get the desired estimate since the second term is smaller than the first if j is

sufficiently large and the third term is dominated by the integral of the last.
Q.E.D.

Lemma 5. 2. *Under* $s-(0)$ *it follows that*

$$N(t) = C_0 t^{n/2m} + o\,(t^{n/2m})\,.$$

Proof. Using Lemma 5. 1 (ii) and arguing as in § 14 of Agmon [1] we get the desired statement. Q.E.D.

Lemma 5. 3. *There is a constant* C_{11} *such that*

$$\operatorname{Re}\lambda_j \geq C_{11} j^{2m/n} \qquad \textit{for large} \quad j\,.$$

Proof. From $j \leq N(\operatorname{Re}\lambda_j)$ and Lemma 5. 2 we can easily show the estimate.
Q.E.D.

Lemma 5. 4. *If* $d(\lambda) \geq C|\lambda|^{1-1/2m+\varepsilon}$ *and* $|\lambda|$ *is sufficiently large then we have the following estimate*

$$\Big|\sum_{j=0}^{\infty}(\lambda_j-\lambda)^{-1} - \sum_{j=0}^{\infty}(\operatorname{Re}\lambda_j-\lambda)^{-1}\Big| \leq C_{12}|\lambda|^{1+(n-1)/2m+\varepsilon}/d(\lambda)^2\,.$$

Proof. We have the following equality

$$\sum_{j=0}^{\infty}(\lambda_j-\lambda)^{-1} - \sum_{j=0}^{\infty}(\operatorname{Re}\lambda_j-\lambda)^{-1} = -\sum_{j=0}^{\infty}\operatorname{Im}\lambda_j(\lambda_j-\lambda)^{-1}(\operatorname{Re}\lambda_j-\lambda)^{-1}$$
$$= -\sum_{\operatorname{Re}\lambda_j\leq 2|\lambda|} - \sum_{\operatorname{Re}\lambda_j>2|\lambda|} = I_1+I_2\,.$$

If $\operatorname{Re}\lambda_j \leq 2|\lambda|$ there is a constant K_2 such that

$$|\operatorname{Im}\lambda_j| \leq K_2|\lambda|^{1-1/2m} \tag{5. 2}$$

from Lemma 3. 1.
On the other hand, if $d(\lambda) \geq C|\lambda|^{1-1/2m+\varepsilon}$ and $|\lambda|$ is sufficiently large, then an elementary geometrical observation shows that there is a positive constant K_3 such that

$$|\lambda_j-\lambda| \geq K_3 d(\lambda) \tag{5. 3}$$

for any j.
In view of Lemma 5. 2, (5. 2) and (5. 3) we get

$$|I_1| \leq \sum_{\operatorname{Re}\lambda_j\leq 2|\lambda|}|\operatorname{Im}\lambda_j|\,|\lambda_j-\lambda|^{-1}|\operatorname{Re}\lambda_j-\lambda|^{-1}$$
$$\leq K_4|\lambda|^{1+(n-1)/2m}/d(\lambda)^2\,.$$

Next from Lemma 5. 3 and $\operatorname{Re}\lambda_j > 2|\lambda|$ we see

$$|\lambda_j-\lambda| = |\lambda_j-\lambda|^{1-n(1+\varepsilon)/2m}|\lambda_j-\lambda|^{n(1+\varepsilon)/2m}$$
$$\geq K_5|\lambda|^{1-n/2m-\varepsilon} j^{(1+\varepsilon)}\,.$$

Thus we find

$$\sum_{\mathrm{Re}\,\lambda_j>2|\lambda|}|\lambda_j-\lambda|^{-1}\le K_6|\lambda|^{-1+n/2m+\varepsilon}\sum_{j=0}^{\infty}j^{-(1+\varepsilon)}$$
$$\le K_7|\lambda|^{-1+n/2m+\varepsilon}\,. \tag{5.4}$$

On the other hand, from Lemma 3. 1 and $\mathrm{Re}\,\lambda_j>2|\lambda|$, we get

$$|\mathrm{Im}\,\lambda_j|\,|\mathrm{Re}\,\lambda_j-\lambda|^{-1}\le K_8|\lambda|^{-1/2m}\,. \tag{5.5}$$

From (5. 4) and (5. 5) we know that

$$|I_2|\le\sum_{\mathrm{Re}\,\lambda_j>2|\lambda|}|\mathrm{Im}\,\lambda_j|\,|\lambda_j-\lambda|^{-1}|\mathrm{Re}\,\lambda_j-\lambda_j-\lambda|^{-1}$$
$$\le K_9|\lambda|^{-1+(n-1)/2m+\varepsilon}\le K_{10}|\lambda|^{1+(n-1)/2m+\varepsilon}/d(\lambda)^2\,.$$

Q.E.D.

Now we follow the method of Agmon [2]. We put

$$f(\lambda)=\sum_{j=0}^{\infty}(\mathrm{Re}\,\lambda_j-\lambda)^{-1}\quad\text{and}\quad I(z)=(2\pi i)^{-1}\int_{L(z)}f(\lambda)\,d\lambda$$

where $L(z)$ is an oriented curve in the complex plane from $\bar{z}$ to $z=t+i\tau$ not intersecting $[0, \infty)$.

Thus for $t>0$, $\tau>0$

$$|I(z)-(\tau/\pi)\,\mathrm{Re}\,f(z)-N(t)+N(0)|\le C_{12}\tau|\mathrm{Im}\,f(z)|\,. \tag{5.6}$$

First we consider the asymptotic formula for $N(t)$ under $s-(1)$. If $d(\lambda)\ge|\lambda|^{1-h/2m(h+2)+\varepsilon}$ and $|\lambda|$ is large then we get

$$|f(\lambda)|\le K_{11}|\lambda|^{-1+n/2m} \tag{5.7}$$

from Lemma 5. 1 and Lemma 5. 4.

We put $z=t+it^{1-h/2m(h+2)+\varepsilon}$ and take

$$L(z)=\{\lambda=t+iu;\ t^{1-h/2m(h+2)+\varepsilon}\le u\le t\}$$
$$\cup\ \{\lambda;\ |\lambda|=\sqrt{2}\,t;\ \mathrm{Re}\,\lambda\le t)$$

where t is a sufficiently large positive number.

From (5. 6), (5. 7) and $N(0)=0$ we find

$$|I(z)-N(t)|\le K_{12}t^{n/2m-h/2m(h+2)+\varepsilon}\,. \tag{5.8}$$

On the other hand we know the following equality

$$I(z)=(2\pi i)^{-1}\int_{L(z)}f(\lambda)\,d\lambda=(2\pi i)^{-1}\int_{L(z)}\{f(\lambda)-C_{10}(-\lambda)^{-1+n/2m}\}\,d\lambda$$
$$+(2\pi i)^{-1}\int_{L(z)}C_{10}(-\lambda)\lambda^{-1+n/2m}\,d\lambda=I_1+I_2\,.$$

In view of Lemma 5. 1 and Lemma 5. 4, putting $1>p>h/2$ we get that

$$
\begin{aligned}
|I_1| \leq & K_{13}\{\int_{L(z)} |\lambda|^{1+h+(n-h)/2m+\varepsilon}/d(\lambda)^{2+h}|d\lambda| \\
& +\int_{L(z)} |\lambda|^{p+(n-p)/2m}/d(\lambda)^{1+p}|d\lambda| \\
& +\int_{L(z)} |\lambda|^{1+(n-1)/2m+\varepsilon}/d(\lambda)^2|d\lambda| \\
\leq & K_{14}\{t^{1+h+(n-h)2m+\varepsilon}\int_{t^{1-h/2m(h+2)+\varepsilon}}^{t} u^{-(2+h)}du \\
& +t^{1+h+(n-h)/2m+\varepsilon-(2+h)+1} \\
& +t^{pt(n-p)/2m}\int_{t^{1-h/2m(h+2)+\varepsilon}}^{t} u^{-(1+p)}du \\
& +t^{p+(n-p)/2m-(1+p)+1} \\
& +t^{1+(n-1)/2m+\varepsilon}\int_{t^{1+h/2m(h+2)+\varepsilon}}^{t} u^{-2}du \\
& +t^{1+(n-1)/2m+\varepsilon-2+1}\} \\
\leq & K_{15}t^{n/2m-h/2m(h+2)+\varepsilon}
\end{aligned}
\tag{5. 9}
$$

Noting that

$$
\left|\frac{1}{2\pi i}\int_{L(z)}(-\lambda)^{-1+n/2m}d\lambda - t^{n/2m}\frac{\sin(n\pi/2m)}{n\pi/2m}\right| \leq K_{16}t^{n/2m-h/2m(h2+\varepsilon}.
$$

from (5. 8) and (5. 9) we obtain the desired estimate.

In case of $s-(2)$ assuming that $a-(3)$ holds for some $p\geq(h+1)/2$ if $h<1$ and for any $p<1$ if $h=1$, we can prove the desired result in the same method as above.

OSAKA UNIVERSITY

Bibliography

[1] S. Agmon: Lectures on Elliptic Boundary Value Problems, Van Nostrand Mathematical Studies, Princeton, 1965.

[2] S. Agmon: *Asymptotic formulas with remainder estimates for eigenvalues of elliptic operators,* Arch. Rational Mech. Anal. **28** (1968), 165–183.

[3] R. Beals: *Asymptotic behavior of the Green's function and spectral function of an elliptic operator,* J. Functional Analysis **5** (1970), 484–503.

[4] K. Maruo and H. Tanabe: *On the asymptotic distribution of eigenvalues of operators associated with strongly elliptic sesquilinear forms,* Osaka J. Math. **8** (1971), 323–345.

[5] S. Agmon: *On the eigenfunctions and on the eigenvalues of general elliptic boundary value problems,* Comm. Pure Appl. Math. **15** (1962), 119–147.

[6] N. Dunford and J.T. Schwartz: Linear Operator, II, Interscience Publishers, New York, 1963.