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Osaka University
ON A COMPLEXITY OF A SURFACE IN 3-SPHERE

Dedicated to Professor Ralph H. Fox for his 60th birthday

SHIN’ICHI SUZUKI

(Received June 29, 1973)

0. Introduction

Throughout this paper we shall only be concerned with the combinatorial category, consisting of simplicial complexes and piecewise-linear maps. It is the purpose of the paper to prove intuitively obvious topological theorems which are interesting in the Morse theory of 3-manifolds. The theorems concern "knot types" of embeddings of a closed (=compact, without boundary), connected and orientable surface $M_p$ of genus $p$ into the 3-dimensional sphere $S^3$.

As widely known, a surface $M_p$ in $S^3$, denoted by $(M_p \subset S^3)$, is obtained from some 2-spheres by adding handles, Fox [3] and Homma [5]. Using the fact, we shall define a complexity $(s, t)$, a pair of natural numbers, for the knot type of the $(M_p \subset S^3)$ in §1. After establishing a canonical representative for the knot type of $(M_p \subset S^3)$ in §2, we first consider some non-existence results in §3. In §4 and §5, we construct some pairs $(M_p \subset S^3)$'s for some complexities $(s, t)$'s.

In the paper, homeomorphism is denoted by $\cong$, while $\approx$ and $\sim$ refer to homotopy and homology, respectively. $\partial X$, $\text{cl}(X)$ and $\text{o}X$ denote, respectively, the boundary, the closure and the interior of a manifold $X$. By $D^n$ and $S^{n-1}$ we shall denote the standard $n$-cell and the standard $(n-1)$-sphere $\partial D^n$, respectively, and particularly, $D^1=[-1, 1]$.

1. Definitions and notation

First let us explain several definitions and notation, and formulate our main theorem.

In general, we shall denote by $M$ a compact orientable surface, and $\#(M)$ and $g(M)$ stand for the number of connected components of $M$ and the total genus of $M$, respectively.

We shall say that a submanifold $X$ of a manifold $Y$ is properly embedded (or simply proper) if $X \cap \partial Y = \partial X$.

By $(M \subset M')$ we denote a pair of manifolds such that a 3-manifold $M'$ and
a properly embedded surface $M$. Throughout §§1, 2, 3 and 4, we do not give any orientation on $M$ and $M^3$. Two pairs $(M \subseteq M^3)$ and $(M' \subseteq M^3)$ are said to be congruent, or of the same knot type, if there is a homeomorphism $\psi : M^3 \to M^3$ such that $\psi(M) = M'$. We denote the congruence class of a pair $(M \subseteq M^3)$ by $\langle M \subseteq M^3 \rangle$, so the $(M \subseteq M^3)$ is a representative of $\langle M \subseteq M^3 \rangle$.

For a pair $(M \subseteq M^3)$, a simple loop $\lambda$ on $M$ is said to be co-unknotted, if $\lambda$ bounds a 2-cell $D(\lambda)$ in $M^3$ with $D(\lambda) \cap M = \partial D(\lambda) = \lambda$, and $D(\lambda)$ will be called an associated disk. Especially, a co-unknotted loop $\lambda$ is said to be essential if $\lambda$ is not contractible in $M$, and otherwise, $\lambda$ is inessential. Note that $\lambda$ is contractible in $M$ if and only if $\lambda$ bounds a 2-cell on $M$, see Epstein [2].

We may say a 3-manifold $F^3$ has Fox's property, if for any pair $(M \subseteq F^3)$ with $g(M) > 0$ there exists an essential co-unknotted loop on $M$, and throughout the paper by $F^3$ we will denote a 3-manifold which has Fox's property.

1.1. Proposition. (Kinoshita [8], Fox [3], Homma [5]) Any orientable 3-manifold whose fundamental group is either finite or a finitely generated free group has Fox's property. (Refer to Haken [4]).

For a pair $(M \subseteq M^3)$, let $h : D^1 \times D^2 \to M^3$ and $d : D^2 \times D^1 \to M^3$ be embeddings of 3-cells such that

(i) $h(D^1 \times D^2) \cap M = h(\partial D^1 \times D^2)$,

(ii) $d(D^2 \times D^1) \cap M = d(\partial D^2 \times D^1)$.

Then we have another embedded surfaces

(i) $M(h) = M - h(\partial D^1 \times D^2) \cup h(D^1 \times \partial D^2)$,

(ii) $M(d) = M - d(\partial D^2 \times D^1) \cup d(D^2 \times \partial D^1)$.

We will say that "$M(h)$ is formed from $M$ by adding a handle $h$" and similarly, "$M(d)$ is formed from $M$ by adding a dome $d$". It will be noticed that:

1.2. (i) $\tau(M(h)) = \tau(M) - 1$ and $g(M(h)) = g(M)$ if $h(\{0\} \times \partial D^2) \sim 0$ in $M(h)$, and $\tau(M(h)) = \tau(M)$ and $g(M(h)) = g(M) + 1$ if $h(\{0\} \times \partial D^2) \not\sim 0$ in $M(h)$.

(ii) $\tau(M(d)) = \tau(M) + 1$ and $g(M(d)) = g(M)$ if $d(\partial D^2 \times \{0\}) \sim 0$ in $M$, and

Then, as an immediate consequence of 1.1 and 1.2, we have:

1.3. Proposition. For any $\langle M_p \subseteq F^3 \rangle$, there exists a representative $(M_p \subseteq F^3)$ such that $M_p$ is formed from $P = \Sigma_1 \cup \cdots \cup \Sigma_s$, a union of non-intersecting 2-spheres in $F^3$, by adding one by one $s+p-1$ handles $h_1, \ldots, h_{s+p-1}$.

Of course, this representative $(M_p \subseteq F^3)$ in 1.3 is not uniquely determined. If $r$ handles $h_{r1}, \ldots, h_{rr}$, $1 \leq i_1 < \cdots < i_r \leq s+p-1$, are mutually independent, that
is, \( h_{ir}(D^1 \times D^1), \ldots, h_{ir}(D^1 \times D^1) \) are mutually disjoint in \( F^3 \), we can add these \( r \) handles at a time. Therefore, the \((M_p \subset F^3)\) is formed from \( \mathcal{P}_0 \) by \( t \) times, \( 1 \leq t \leq s+p-1 \), as a process

\[
\begin{align*}
\mathcal{P}_0 &= \sum_1 \cup \cdots \cup \sum_s \to \mathcal{P}_1 = \mathcal{P}_0(h_{11}, \ldots, h_{s(t)}) \\
&\to \mathcal{P}_2 = \mathcal{P}_1(h_{12}, \ldots, h_{t(r(t)}) \\
&\to \cdots \\
&\to \mathcal{P}_t = \mathcal{P}_{t-1}(h_{11}, \ldots, h_{t(r(t)}) = M_p,
\end{align*}
\]

where \( r(1) + r(2) + \cdots + r(t) = s + p - 1 \). A handle \( h_{ij}, 1 \leq i \leq t, 1 \leq i \leq r(i) \), is said to belong to \( \mathcal{P}_i \), and we denote \( h_{ij} \in \mathcal{P}_i \).

Now, to the pair \((M_p \subset F^3)\) we associate a pair \((s, t) \in \mathbb{N} \times \mathbb{N}\) of natural numbers, and we define a total order \(<(or >)\) in \{(s, t) \in \mathbb{N} \times \mathbb{N}\} as follows:

\[
(1.5) \quad (s, t) < (s', t') \text{ if } s < s' \text{ or if } s = s' \text{ and } t < t'.
\]

Then, for every congruence class \(\langle(M_p \subset F^3)\rangle\) we can define an invariant \(\langle s, t \rangle\) as follows:

\[
1.6. \text{ Definition.} \quad \langle(M_p \subset F^3)\rangle \text{ is with complexity } \langle s, t \rangle \text{ if there exists a representative } (M_p \subset F^3) \text{ with } \langle s, t \rangle \text{ and for any representative } (M_p' \subset F^3) \text{ with } \langle s', t' \rangle \text{ of } \langle(M_p \subset F^3)\rangle, \langle s, t \rangle \leq \langle s', t' \rangle.
\]

It is clear that the complexity \(\langle s, t \rangle\) is an invariant of a congruence class \(\langle(M_p \subset F^3)\rangle\). Now we can state our version of a special case of Proposition 1.1.

\[
1.7. \text{ Proposition.} \quad \text{Every } \langle(M_p \subset F^3)\rangle \text{ is with complexity } \langle 1, 1 \rangle.
\]

In the notion of (1.4) and 1.3, if there is a handle \( h_{ij} \) such that \( h_{ij} \{0\} \times \partial D^2 \) is inessential on \( \mathcal{P}_{1-i} (h_{ij}) \), then the handle \( h_{ij} \) and one of the 2-spheres \( \Sigma_i, \ldots, \Sigma_s \) can be omitted from the definition of complexity. Consequently, we have:

\[
1.8. \text{ Proposition.} \quad \text{For every } p \geq 1 \text{ and for every } \langle(M_p \subset F^3)\rangle \text{ with complexity } \langle s, t \rangle, \langle 1, 1 \rangle \leq \langle s, t \rangle \leq \langle s, s+p-1 \rangle \leq \langle p, 2p \rangle.
\]

More sharp statements will be given later.

We call a disk-sum of \( p \) copies of \( D^2 \times S^1 \) a solid-torus of genus \( p \), and denote it by \( p(D^2 \times S^1) \). Since \( p(D^2 \times S^1) \) is embeddable in any 3-manifold, the following is obvious.

\[
1.9. \text{ Proposition.} \quad \text{For any } p \geq 1, \text{ there exists a } \langle(M_p \subset F^3)\rangle \text{ with complexity } \langle 1, 1 \rangle.
\]

In §4, we will prove the following:

\[
1.10. \text{ Theorem.} \quad \text{For any } p \geq 2, \text{ there exists a } \langle(M_p \subset F^3)\rangle \text{ with complexity } \langle s, t \rangle \text{ such that } \langle s, t \rangle > \langle 1, 1 \rangle. \quad \text{(Refer to Kneser [10]).}
\]
2. Handle-isotopy and canonical representative

Let \((M, F^3)\) be a pair and let \(h: D^3 \times D^2 \to F^3\) be a handle for \(M\). Let \(D^2_1 = h\{-1\} \times D^2\) and \(D^2_2 = h\{1\} \times D^2\), and let \(D^2\) be a 2-cell on \(M\) with \(D^2 = (D^2_1 \cup D^2_2) = \phi\), and let \(\gamma\) be a simple arc on \(M\) with \(\gamma \cap (D^2_1 \cup D^2_2) = \gamma \cap (\partial D^2 \cup \partial D^2) = \partial \gamma\).

Then, sliding the end \(D^2_2\) of the handle \(h\) along \(\gamma\) in a regular neighborhood \(N(D^2_1 \cup \gamma \cup D^2; F^3)\), we have a new handle \(h': D^3 \times D^2 \to F^3\) for \(M\) such that \(h'(\{1\} \times D^2) = D^2_1\) and \(h'(\{-1\} \times D^2) = D^2_2\). It is easily seen that this deformation of a handle can be extended to an ambient isotopy of \(F^3\); and so \(\langle (M(h) \subset F^3) \rangle = \langle (M(h') \subset F^3) \rangle\). We will call this deformation a handle-isotopy (along \(\gamma\)). Of course, the above remains valid if \(D^2_1\) is substituted for \(D^2_2\).

An immediate consequence is:

2.1. Lemma. In the notation of (1.4), a handle \(h_{ij}\) can be deformable by a handle-isotopy along a simple arc \(\gamma\) on \(M_p\) if the associated loop \(h_{ij}(\{0\} \times \partial D^2)\) is co-unknotted and \(\gamma \subset M_p - h_{ij}(D^1 \times \partial D^2)\). Especially, every handle \(h_{ij}\) can be deformable by a handle-isotopy along \(\gamma\), provided that \(\gamma \subset M_p - h_{ij}(D^1 \times \partial D^2)\).

By successive application of 2.1, we deduce:

2.2. Theorem. In the notation of (1.4), every handle \(h_{ij} \in \mathcal{P}_i\) can be deformable by a handle-isotopy along \(\gamma\) by deforming handles belonging to \(\mathcal{P}_{i-1} \cup \cdots \cup \mathcal{P}_1\) suitably, provided that \(\gamma \subset M_p - h_{ij}(D^1 \times \partial D^2)\), for \(i = 1, \ldots, t\).

Remark. In 2.1 and 2.3, the handles belonging to \(\mathcal{P}_1 \cup \cdots \cup \mathcal{P}_t\) with \(h_{kl}(D^1 \times \partial D^2) \ni \gamma = \phi\) may be considered to be changed by the handle-isotopy, but there may be no confusion if we denote them by the same symbols.

2.3. Theorem. For any \(\langle (M_p \subset F^3) \rangle\) with complexity \(\langle s, t \rangle\), we can take a canonical representative \((M_p^* \subset F^3)\) as follows:

(0) \(M_p^*\) consists of \(s\) 2-spheres \(\Sigma_1 \cup \cdots \cup \Sigma_s\) and \(s + p - 1\) handles.

(i) In the \(s + p - 1\) handles, there are just \(p\) handles, say \(h_{r_1}, \ldots, h_{r_p}\), such that \(h_{r_i}(\partial D^1 \times D^2)\) is contained in one of \(\Sigma_1, \ldots, \Sigma_s\), \(i = 1, \ldots, p\). (So, \(h_{r_i}(\{0\} \times \partial D^2) \not\sim 0\) on \(M^*_p\).)

(ii) In the \(s + p - 1\) handles, there are just \(s - 1\) handles, say \(h_{r_1}, \ldots, h_{r_{s-1}}\), such that \(h_{r_i}(\{-1\} \times D^2)\) and \(h_{r_i}(\{1\} \times D^2)\) are contained in different 2-spheres of \(\Sigma_1, \ldots, \Sigma_s\), \(i = 1, \ldots, s - 1\). (So, \(h_{r_i}(\{0\} \times \partial D^2) \not\sim 0\) on \(M^*_p\).)

Proof. Let \((M_p \subset F^3)\) be a representative of \(\langle (M_p \subset F^3) \rangle\) which consists of \(s\) 2-spheres \(\Sigma_1 \cup \cdots \cup \Sigma_s\) and \(s + p - 1\) handles \(h_{r_1}, \ldots, h_{r_{s+p-1}}\). For brevity, we will call a handle \(h_{r_i} c\)-handle if \(h_{r_i}\) connects two of \(\Sigma_1, \ldots, \Sigma_s\), that is, \(h_{r_i}(\{-1\} \times D^2)\) and \(h_{r_i}(\{1\} \times D^2)\) are contained in different 2-spheres of \(\Sigma_1, \ldots, \Sigma_s\).

If in the \(s + p - 1\) handles, there are exactly \(s - 1\) \(c\)-handles, we are finished,
and so we assume that there are more than $s - 1$ $c$-handles. Suppose that there are at least two $c$-handles for $\Sigma_1$; and let $h_{\lambda_1}$ and $h_{\mu_1}$ be such $c$-handles which connect $\Sigma_{i_1}$ with $\Sigma_1$ and $\Sigma_j$ with $\Sigma_1$, respectively. We may assume that $h_{\lambda_1} \in \mathcal{P}_{u_1}$ and $h_{\mu_1} \in \mathcal{P}_{v_1}$, and $1 \leq u_1 \leq v_1 \leq t$. Then, we can take a simple arc $\gamma$ on $P_{v_1} - h_{\mu_1}$ such that $\gamma$ runs from $\Sigma_1$ to $\Sigma_i$ through $h_{\lambda_1}(D^1 \times \partial D^2)$. By 2.2, there is a handle-isotopy along $\gamma$ so that $h_{\mu_1}$ connects $\Sigma_{i_1}$ with $\Sigma_j$. Note that if $i_1 = j$, then $h_{\mu_1}$ must be now a $c$-handle, and if $i_1 \neq j$ then $h_{\mu_1}$ is not a $c$-handle now.

Repeating the procedure, we may assume that there is only one $c$-handle, say $h_{r_2}$ for $\Sigma_1$, and that $h_{r_2}$ connects $\Sigma_1$ with $\Sigma_2$.

Next we observe $\Sigma_2$. Since $M_p$ is connected, there are some $c$-handles for $\Sigma_2$ other than $h_{r_2}$ if $s > 2$. Let $h_{\lambda_2}$ and $h_{\mu_2}$ be $c$-handles such that $h_{\lambda_2}$ connects $\Sigma_{i_2}$ with $\Sigma_2$ and $h_{\mu_2}$ connects $\Sigma_{i_2}$ with $\Sigma_2$, and we may assume that $h_{\lambda_2} \in \mathcal{P}_{u_2}$, $h_{\mu_2} \in \mathcal{P}_{v_2}$, and $1 \leq u_2 \leq v_2 \leq t$. Then, we have a handle-isotopy so that $h_{\mu_2}$ connects $\Sigma_{i_2}$ with $\Sigma_2$. Repeating the procedure, we may assume that there is exactly one $c$-handle, say $h_{s_2}$ for $\Sigma_2$, other than $h_{r_2}$, and that $h_{s_2}$ connects $\Sigma_2$ with $\Sigma_3$.

By the repetition of the procedure, we can assume that there is only one $c$-handle for each of $\Sigma_i$ and $\Sigma_s$, and there are exactly two $c$-handles for $\Sigma_i$, for $i = 2, \ldots, s - 1$. Thus, we have a required representative $(M_p^{\#} \subset F^3)$ which satisfies (0) and (ii), and so (i).

On a surface $M_p$, we can choose a system of $2p$ simple loops $\{a_1, \ldots, a_p\} \cup \{b_1, \ldots, b_p\}$ such that $a_i \cap b_i$ consists of one crossing point and $a_i \cap a_j = \phi$, $b_i \cap b_j = \phi$ for $i \neq j$. We will call such a system canonical.

2.4. Corollary. (Homma [5]) For any $(M_p \subset F^3)$, there exists a canonical system $\{a_1, \ldots, a_p\} \cup \{b_1, \ldots, b_p\}$ on $M_p$ such that $a_1 \cup \cdots \cup a_p$ are the boundaries of mutually disjoint 2-cells $D_1^3 \cup \cdots \cup D_p^3$.

Of course, this canonical system is not uniquely determined, and $a_i$ is not always co-unknotted.

3. Non-existence results

In this section, we will give some non-existence theorems by contrast to Theorem 1.10.

3.1. Theorem. For any $p \geq 2$ and $s \geq 2$, there is no $(M_p \subset F^3)$ with complexity $\langle s, 1 \rangle$.

Proof. Suppose that there exists a $(M_p \subset F^3)$ with complexity $\langle s, 1 \rangle$ for $p \geq 2$ and $s \geq 2$. Then, by Theorem 2.3 there exists a canonical representative $(M_p^{\#} \subset F^3)$ of the $(M_p \subset F^3)$, and let $h_{\lambda_1}, \ldots, h_{s-1}$ be handles of type (ii) in 2.3. From the definition of complexity, all handles of $M_p^{\#}$ belong to $\mathcal{P}_s$, i.e. all handles are mutually independent. So, $\mathcal{P}_s(h_{r_1}, \ldots, h_{r_{s-1}})$ must be a 2-sphere, hence the $(M_p \subset F^3)$ is with complexity $\langle 1, 1 \rangle$, which contradicts our hypothesis.
In general, we claim:

3.2. Proposition. For any \( p \geq 2 \) and \( s \geq 2 \), if \( \langle (M_p \subset S^3) \rangle \) is with complexity \( \langle s, t \rangle \), then for a canonical representative \( (M_p^* \subset F^3) \) of the \( \langle (M_p \subset F^3) \rangle \) in Theorem 2.3, every handle \( h_i \in \mathcal{P}_i \) is of type (i) in 2.3.

Remainder of the paper, we consider only pairs \( (M \subset S^3) \)'s. For a pair \( (M \subset S^3) \), the residual space \( S^3 - M \) consists of \( (M) + 1 = q \) non-intersecting 3-manifolds. We denote the closures of these manifolds in \( S^3 \) by \( W_1(M \subset S^3) \cup \cdots \cup W_q(M \subset S^3) \) or simply by \( W_1 \cup \cdots \cup W_q \), and call the disjoint union of them the closed complement of \( (M \subset S^3) \). We record the following well-known theorem due to J.W. Alexander.

3.3. Proposition. (Alexander [1]) For any pair \( (M_0 \subset S^3) \), \( W_1(M_0 \subset S^3) \cong S^3 \). (Remember that \( M_0 \cong S^2 \).)

3.4. Theorem. For any \( p \geq 2 \), there is no \( \langle (M_p \subset S^3) \rangle \) with complexity \( \langle p, 2p-1 \rangle \).

Proof. Assume the contrary, then there is a \( \langle (M_p \subset S^3) \rangle \) with complexity \( \langle p, 2p-1 \rangle \) for \( p \geq 2 \), and let \( (M_p^* \subset S^3) \) be a canonical representative of it in Theorem 2.3.

Let \( \mathcal{P}_o = \Sigma_1 \cup \cdots \cup \Sigma_p \) be the 0-th step of \( M_p^* \), and let \( W_1(\mathcal{P}_o \subset S^3) \cup \cdots \cup W_{p+1}(\mathcal{P}_o \subset S^3) \) be the closed complement. It will be noticed that for every \( \Sigma_k \) there exists only one handle of type (i) in 2.3. From the definition of complexity, to each step \( \mathcal{P}_i, i = 1, \cdots, 2p-1 \), only one handle, say \( h_i \), belongs.

For clarity, the proof will be divided into five steps.

Step 1: From Proposition 3.2, \( h_i \) is of type (i) in 2.3. Without loss of generality, we may assume that \( h_i(\partial D^1 \times D^2) \) is contained in \( \Sigma_i \) and \( h_i(D^1 \times D^2) \) is contained in \( W_1(\mathcal{P}_i \subset S^3) \). Moreover, we may assume that \( W_1(\mathcal{P}_i \subset S^3) \cap W_2(\mathcal{P}_o \subset S^3) = \Sigma_i \) by 3.3. Then, the complement of \( (\mathcal{P}_1 \subset S^3) = (\mathcal{P}_o(h_i) \subset S^3) \) consists of

\[
\begin{align*}
W_1(\mathcal{P}_1 \subset S^3) &= \text{cl}(W_1(\mathcal{P}_o \subset S^3) - h_i(D^1 \times D^2)), \\
W_2(\mathcal{P}_1 \subset S^3) &= W_2(\mathcal{P}_o \subset S^3) \cup h_i(D^1 \times D^2), \\
W_k(\mathcal{P}_1 \subset S^3) &= W_k(\mathcal{P}_o \subset S^3) \text{ for } k = 3, \cdots, p+1.
\end{align*}
\]

Step 2: If \( h_2 \) is of type (ii) in 2.3, then \( h_2(\{0\} \times \partial D^2) \) is contractible on \( \mathcal{P}_1 \) (\( h_2 \)) because \( \mathcal{P}_1 \) consists of only one closed orientable surface of genus 1 and \( p-1 \) 2-spheres. So, \( \langle (M_p \subset S^3) \rangle \) must be with complexity smaller than or equal to \( \langle p-1, 2p-2 \rangle \), which contradicts our hypothesis. We know that \( h_2 \) is of type (i) in 2.3, and \( h_2(\partial D^1 \times D^2) \) is contained in one of \( \Sigma_z, \cdots, \Sigma_p \), say \( \Sigma_z \). Since \( h_2(D^1 \times \phi D^2) = \phi_2, h_2(D^1 \times D^2) \subset W_3(\mathcal{P}_1 \subset S^3) \), and we may assume that \( W_3(\mathcal{P}_1 \subset S^3) \cap W_4(\mathcal{P}_1 \subset S^3) = \Sigma_z \). Then, the closed complement of \( (\mathcal{P}_1 \subset S^3) \) =
(\mathcal{P}(h_2) \subset S^3) \text{ consists of}
\begin{align*}
W_i(\mathcal{P} \subset S^3) &= W_i(\mathcal{P}_1 \subset S^3), \\
W_i(\mathcal{P} \subset S^3) &= \text{cl}(W_{i+1}(\mathcal{P}_1 \subset S^3) - h_i(D^1 \times D^2)), \\
W_i(\mathcal{P} \subset S^3) &= W_i(\mathcal{P}_1 \subset S^3) \cup h_i(D^1 \times D^2), \\
W_k(\mathcal{P} \subset S^3) &= W_k(\mathcal{P}_1 \subset S^3) \text{ for } k = 4, \ldots, p+1.
\end{align*}

**Step 3:** If \( h_3 \) is of type (ii) 2.3, there are four cases to be considered: \( h_3 \) connects i) \( \Sigma_i(h_1) \) with \( \Sigma_i(h_2) \), ii) \( \Sigma_i(h_2) \) with \( \Sigma_k, k=3, \ldots, p, \), iii) \( \Sigma_i(h_3) \) with \( \Sigma_k, k=3, \ldots, p, \), and iv) \( \Sigma_k \) with \( \Sigma_j, k+j = 3, \ldots, p. \) Since \( h_2(D^1 \times D^2) \cap h_3(D^1 \times D^2) \neq \phi \), \( h_2(D^1 \times D^2) \subset W_4(\mathcal{P} \subset S^3). \) For \( \Sigma_i(h_1) \cap W_4(\mathcal{P} \subset S^3) = \phi, \) the case i) cannot occur actually. Moreover, every case of ii), iii) and iv) cannot occur by the same reason as that of Step 2.

Now, we know that \( h_3 \) is of type (i) in 2.3, and \( h_3(\partial D^1 \times D^2) \) is contained in one of \( \Sigma_3, \ldots, \Sigma_p, \) say \( \Sigma_j. \) Note that \( \Sigma_3 \cup \Sigma_4 \cup \Sigma_3 \) are considered to be concentric. We may assume that \( W_4(\mathcal{P} \subset S^3) \cap W_5(\mathcal{P} \subset S^3) = \Sigma_3. \) Then, the closed complement of \( (\mathcal{P} \subset S^3) \cap (\mathcal{P} \subset S^3) \) consists of
\begin{align*}
W_i(\mathcal{P} \subset S^3) &= W_i(\mathcal{P}_1 \subset S^3), \\
W_i(\mathcal{P} \subset S^3) &= \text{cl}(W_{i+1}(\mathcal{P}_1 \subset S^3) - h_i(D^1 \times D^2)), \\
W_i(\mathcal{P} \subset S^3) &= W_i(\mathcal{P}_1 \subset S^3) \cup h_i(D^1 \times D^2), \\
W_k(\mathcal{P} \subset S^3) &= W_k(\mathcal{P}_1 \subset S^3) \text{ for } k = 5, \ldots, p+1.
\end{align*}

**Step 4:** Repeating the same arguments in Step 3, we may assume that
i) \( h_{i+1}, \ldots, h_p \) are of type (i) in 2.3, and \( \mathcal{P}_p \) consists of closed orientable surfaces \( \Sigma_i(h_1), \ldots, \Sigma_p(h_p) \) of genus 1,
ii) the closed complement of \( (\mathcal{P} \subset S^3) \) consists of
\begin{align*}
W_i(\mathcal{P} \subset S^3) &= \text{cl}(W_{i+1}(\mathcal{P}_0 \subset S^3) - h_i(D^1 \times D^2)), \\
W_k(\mathcal{P} \subset S^3) &= \text{cl}(W_{k+1}(\mathcal{P}_0 \subset S^3) - h_{k-1}(D^1 \times D^2) - h_k(D^1 \times D^2))
\end{align*}
for \( k = 2, \ldots, p, \)
\begin{align*}
W_{p+1}(\mathcal{P} \subset S^3) &= W_{p+1}(\mathcal{P}_0 \subset S^3) \cup h_p(D^1 \times D^2).
\end{align*}

In particular, it will be noticed that \( \Sigma_3 \cup \cdots \cup \Sigma_p \) are concentric.

**Step 5:** After Step 4, we know that all handles \( h_{i+1}, \ldots, h_{p-1} \) belonging to \( \mathcal{P}_{i+1} \cup \cdots \cup \mathcal{P}_{p-1} \) are of type (ii) in 2.3. Since \( h_{p+1}(D^1 \times D^2) \cap h_{p+1}(D^1 \times D^2) \neq \phi, \) \( h_{p+1}(D^1 \times D^2) \subset W_{p+1}(\mathcal{P} \subset S^3). \) But, since \( \partial W_{p+1}(\mathcal{P} \subset S^3) = \Sigma_p(h_p), \) \( h_{p+1} \) cannot be of type (ii) in 2.3, so the \( \langle (M_p \subset S^3) \rangle \) must be with complexity smaller than \( \langle p, 2p-1 \rangle. \)

After all, we obtain a desired contradiction, and completes the proof of Theorem 3.4.
We note the following, which is easily derived from the same argument as above.

3.5. Proposition. For any \( p \geq 2 \) and \( t \geq 2 \), if \( \langle (M_p \subset S^3) \rangle \) is with complexity \( \langle p, t \rangle \), then for a canonical representative \( (M^* \subset S^3) \) of the \( \langle (M_p \subset S^3) \rangle \) in 2.3, every handle \( h_{ij} \in \mathcal{P}_i \) is of type (ii) in 2.3.

We summarize our results Proposition 1.8 and Theorems 3.1 and 3.4 as follows:

3.6. Proposition. For every \( p \geq 1 \) and for every \( \langle (M_p \subset S^3) \rangle \) with complexity \( \langle s, t \rangle \), the positive integers \( p, s \) and \( t \) satisfy one of the followings:

1. \( s = 1, 1 \leq t \leq p \),
2. \( 2 \leq s \leq p - 1, 2 \leq t \leq s + p - 1 \),
3. \( s = p, 2 \leq t \leq 2p - 2 \).

4. Some existence results

In this section, we will give some existence theorems, and Theorem 1.10 is a direct consequence of these results.

4.1. Theorem. For any \( p \geq 2 \), there exists a \( \langle (M_p \subset S^3) \rangle \) with complexity \( \langle s, t \rangle \) such that \( s \geq 2 \) and \( t \geq 2 \).

Proof. The following Fig. 1 shows the case \( p = 2 \), which is due to Homma [5]. First, we will show that the \( \langle (M_2 \subset S^3) \rangle \) in Fig. 1 is with complexity \( \langle 2, 2 \rangle \). From the construction, \( W_1 \) is homeomorphic to \( V_{p^*} \) of Suzuki [12, Fig. 2]. So, we conclude that \( \pi_1(W_1) \) is indecomposable with respect to free products and

![Fig. 1: \( (M_2 \subset S^3) \)]
not free, that is, there is no essential co-unknotted loop $J$ on $M_z$ with $J \sim 1$ in $W_i$ by the bounded Kneser's theorem, Jaco [6], see [12, §2]. On the other hand, we know that $W_z$ is a disk-sum of two copies of a closed complement $V_K$ of the so-called clover-leaf knot. Since $\pi_1(V_K)$ is indecomposable with respect to free products and not free, we know that the essential co-unknotted loop $J_{z1}$ on $M_z$ is unique up to isotopy by [12, Cor. 3.5]. Now, we can easily conclude that the $\langle(M_z \subset S^3)\rangle$ is with complexity $\langle 2, 2 \rangle$.

By the same way as that of the proof [12, Th. 5.2], using the pair $(M_z \subset S^3)$ we can construct required pair $(M_p \subset S^3)$ for any $p > 2$. The following Fig. 2 illustrates the case $p = 3$.

![Fig. 2: (M₃⊂S³)](image)

From the construction, the closed complement $W_1$ in Fig. 2 is homeomorphic to $V_{F_3}$ of [12, Fig. 3]. Thus, $\pi_1(W_1)$ is also indecomposable and not free, that is, there is no essential co-unknotted loop $J$ on $M_z$ with $J \sim 1$ in $W_i$. On the other hand, $W_2$ in Fig. 2 is a disk-sum of three copies of $V_K$, the closed complement of the clover-leaf knot. So, for any essential co-unknotted loop $J$ on $M_z$, we conclude that $J \sim 0$ on $M_z$, Jaco [6], see [12, Prop. 2.15]. Now, we can easily conclude that the $\langle(M_z \subset S^3)\rangle$ satisfies the required condition.

The proof of the case $p > 3$, which is omitted here, is the same as that of the case $p = 3$.

**REMARK.** It can be shown by a long geometric proof that the $\langle(M_z \subset S^3)\rangle$ in Fig. 2 is with complexity $\langle 3, 2 \rangle$. In fact, the author suspects, but cannot prove, that the every class $\langle(M_p \subset S^3)\rangle$ obtained in the proof of 4.1 is with complexity $\langle p, 2 \rangle$.

As shown in the proof of 4.1, for every essential co-unknotted loop $J$ on $M_z$ in Fig. 1, $J \sim 0$ on $M_z$. With reference to Proposition 1.1, we record the
4.2. Proposition. For any pair \((M_2 \subset S^3)\), there exists an essential co-unknotted loop \(J\) on \(M_2\) with \(J \sim 0\) on \(M_2\), provided that \(p \geq 2\).

4.3. Theorem. For any \(p \geq 2\), there exists an \(\langle (M_2 \subset S^3) \rangle\) with complexity \(\langle 1, t \rangle\) such that \(t \geq 2\).

Proof. The following \((M_2' \subset S^3)\) in Fig. 3 shows the case \(p = 2\). From the construction, it is easy to check that \(W'\) is homeomorphic to the \(W_1\) of the

\[\text{Fig. 3: } (M_2' \subset S^3)\]

\((M_2 \subset S^3)\) in Fig. 1; so there is no essential co-unknotted loop \(J\) on \(M_2'\) with \(J \sim 1\) in \(W'\). On the other hand, \(W_2'\) is a disk-sum of \(D^2 \times S^1\) and \(V_K\). By [12, Cor. 3.6], the essential co-unknotted loop \(J_{21}\) on \(M_2'\) with \(J_{21} \sim 0\) on \(M_2'\), is unique up to isotopy, and now we can conclude that the \(\langle (M_2' \subset S^3) \rangle\) is with complexity \(\langle 1, 2 \rangle\).

The following \((M_3' \subset S^3)\) in Fig. 4 shows the case \(p = 3\), which is obtained from the \((M_3' \subset S^3)\) by adding a handle \(h_{31}\), where \(h_{31}(D^1 \times D^2)\) is shown by an arc in the figure. In the other cases \(p > 3\), we can construct required pairs inductively using this \((M_3' \subset S^3)\), and so on.

From the construction, the \(W_3'\) in Fig. 4 is a disk-sum of the \(W_1\) of the \((M_3 \subset S^3)\) in Fig. 1 and \(D^2 \times S^1\). We have the essential co-unknotted loop \(J_{31} = h_{31}(\{0\} \times \partial D^3)\) on \(M_3'\) with \(J_{31} \sim 0\) on \(M_3'\), and \(J_{31}\) is unique up to isotopy by [12, Cor. 3.6]. On the other hand, the \(W_4'\) in Fig. 4 is a disk-sum of \(V_K\) and the \(W_1\) of the \((M_2 \subset S^3)\) in Fig. 1. So, there is no essential co-unknotted loop \(J\) on \(M_3'\) with \(J \sim 1\) in \(W_2'\) and \(J \sim 0\) on \(M_3'\). Since the \((M_3' \subset S^3)\) is obtained from the \((M_3' \subset S^3)\) by adding a dome along the 2-cell \(h_{31}(\{0\} \times D^3)\), we can conclude that the \(\langle (M_3' \subset S^3) \rangle\) is with complexity \(\langle 1, 3 \rangle\).
In general, if $p=2n$, then $W'$ of $(M'_p \subset S^3)$ is a disk-sum of $n$ copies of the $W$ of the $(M_2 \subset S^3)$ in Fig. 1, and $W'$ of $(M'_p \subset S^3)$ is a disk-sum of $D^2 \times S^1$, $V_K$ and $n-1$ copies of the $W$ of the $(M_2 \subset S^3)$ in Fig. 1. If $p=2n+1$, then $W'_1$ of $(M'_p \subset S^3)$ is a disk-sum of $D^2 \times S^1$ and $n$ copies of the $W_1$ of the $(M_2 \subset S^3)$ in Fig. 1, and $W'_2$ of $(M'_p \subset S^3)$ is a disk-sum of $V_K$ and $n$ copies of the $W$ of the $(M_2 \subset S^3)$ in Fig. 1. So, in every case, a system of mutually disjoint and homologically independent essential co-unknotted loops on $M'_p$, we can take a system which consists of exactly one loop, and completing the proof.

REMARK. In the above, an essential co-unknotted loop $J$ on $M'_p$ with $J \not\sim 0$ on $M'_p$ is not unique up to isotopy for $p>3$, but the every $(M'_p \subset S^3)$ may be with complexity $<1, p>$.

REMARK. In the proof of Theorems 4.1 and 4.3, we based on the Homma's example $(M_2 \subset S^3)$ in Fig. 1. To construct another examples, we refer the reader to Jaco [7], Kinoshita [9] and Suzuki [11], etc..

5. Remarks and questions

In the preceding section, we have constructed some pairs and actually determined its complexity in some of the simplest cases. In more complicated cases we will need much information on 3-manifolds in $S^3$. While, the author suspects, but cannot prove, that:

5.1. Question. For positive integers $p$, $s$ and $t$ satisfying one of the (1), (2) and (3) in Proposition 3.6, does there exist a $(M_p \subset S^3)$ with complexity $<s, t>$?
In fact, Theorems 4.1 and 4.3 and Proposition 1.9 imply that in the case $p=2$ Question 5.1 is affirmative. In the case $p=3$ we can easily give some $\langle (M_3 \subset S^3) \rangle$'s with complexity $\langle s, t \rangle = \langle 1, 1 \rangle, \langle 1, 3 \rangle$ and $\langle 3, 2 \rangle$. Generally, using $\langle (M_3 \subset S^3) \rangle$'s whose complexities have been known, we can construct some kind of $\langle (M_q \subset S^3) \rangle$'s. For example:

**5.2. Example.** (Fig. 5) There exists a $\langle (M_3 \subset S^3) \rangle$ with complexity $\langle 2, 2 \rangle$.

Proof. Using the pair $(M_3 \subset S^3)$ in Fig. 1, we give the following $(M_3'' \subset S^3)$ in Fig. 5. From the construction, it is easy to check that $W''_1$ is a disk-sum of $D^2 \times S^1$ and the $W_i$ in Fig. 1, and $W''_2$ is a disk-sum of three copies of $V_K$. As homologically non-trivial co-unknotted loops, we have a unique loop $J_3$ on $M_3''$. From the definition of complexity, we can easily conclude that the $\langle (M_3'' \subset S^3) \rangle$ is with complexity $\langle 2, 2 \rangle$.

**5.3. Example.** (Fig. 6) For any $p \geq 2$, there exists a $\langle (M_p \subset S^3) \rangle$ with complexity $\langle 1, 2 \rangle$.

Proof. The case $p=2$ is Theorem 4.3 (Fig. 3). Using the $(M_3 \subset S^3)$ in Fig. 3, we give the following pair $(M_p'' \subset S^3)$ in Fig. 6 for $p \geq 3$. It is easy to check that $W''_1$ is a disk-sum of $(p-2) (D^2 \times S^1)$ and the $W_i$ in Fig. 1, and $W''_2$ is a disk-sum of $D^2 \times S^1$ and $p-1$ copies of $V_K$. So, we can choose at most $p-1$ mutually disjoint and homologically independent essential co-unknotted loops on $M_p''$. Now, we can easily conclude that the $\langle (M_p'' \subset S^3) \rangle$ satisfies the required condition.

Examples 5.2 and 5.3 also suggest an interesting point: The complexity of a $\langle (M_p \subset S^3) \rangle$ is connected with its prime decompositions, [12, Th. 1.6]. In
remainder of the paper, we consider only pairs $(M_p \subset S^3)$'s such that $M_p$ is oriented and $S^3$ has the right-handed orientation. In the sense of [12, Def. 1.11], the complexity $\langle s, t \rangle$ is also an invariant of the congruence class, denote it also by $\langle (M_p \subset S^3) \rangle$, of a pair $(M_p \subset S^3)$. For the other notation, see [12, §1]. From the definitions of complexity and composition of pairs [12], we have at once:

5.4. **Proposition.** Let $\langle s, t \rangle$, $\langle s_1, t_1 \rangle$ and $\langle s_2, t_2 \rangle$ be the complexities of $\langle (M_p \subset S^3) \rangle$, $\langle (M_{p_1} \subset S^3) \rangle$ and $\langle (M_{p_2} \subset S^3) \rangle$, respectively. Suppose that $(M_p \subset S^3) \cong (M_{p_1} \subset S^3) \# (M_{p_2} \subset S^3)$. Then, 

$$\langle s, t \rangle \leq \langle s_1 + s_2 - 1, \max \{t_1, t_2\} \rangle.$$ 

5.5. **Question.** Does it hold in the above 5.4 the equality 

$$\langle s, t \rangle = \langle s_1 + s_2 - 1, \max \{t_1, t_2\} \rangle?$$ 

In view of 5.4, we deduce the following:

5.6. **Theorem.** Every $\langle (M_p \subset S^3) \rangle$ with complexity $\langle p, t \rangle$ is prime.

Proof. The case $p = 1$ is obvious from 1.7 and [12, Prop. 1.5], so we assume that $p \geq 2$. Suppose that there exists a $\langle (M_p \subset S^3) \rangle$ with complexity $\langle p, t \rangle$ that is not prime. Let $(M_p \subset S^3) \cong (M_{p_1} \subset S^3) \# (M_{p_2} \subset S^3)$ be a non-trivial decomposition, and let $\langle s_1, t_1 \rangle$ and $\langle s_2, t_2 \rangle$ be the complexities of the $\langle (M_{p_1} \subset S^3) \rangle$ and $\langle (M_{p_2} \subset S^3) \rangle$, respectively. From Proposition 3.6 and [12, Prop. 1.3], $1 \leq s_i \leq p_i < p$, $(i = 1, 2)$, and $p_1 + p_2 = p$. Then, $s_1 + s_2 - 1 \leq p_1 + p_2 - 1 = p - 1 < p$, hence for any $t$
This contradicts to 5.4, and the proof completes.

5.7. **Theorem.** If Question 5.5 is affirmative, then every \( \langle (M_p \subset S^3) \rangle \) with complexity either \( \langle s, s+p-1 \rangle \) or \( \langle p-1, 2p-3 \rangle \) is prime.

**Proof.** By virtue of Proposition 3.6, we may assume that \( p > s \geq 1 \) and \( p \geq 2 \) for \( \langle s, s+p-1 \rangle \), and \( p \geq 3 \) for \( \langle p-1, 2p-3 \rangle \). As the same way as that of 5.6, we suppose that there exists a \( \langle (M_p \subset S^3) \rangle \) with complexity either \( \langle s, s+p-1 \rangle \) or \( \langle p-1, 2p-3 \rangle \) that is not prime. Let \( (M_p \subset S^3) \# (M_p \subset S^3) \) be a non-trivial decomposition, and let \( \langle s_1, t_1 \rangle \) and \( \langle s_2, t_2 \rangle \) be the complexities of \( \langle (M_p \subset S^3) \rangle \) and \( \langle (M_p \subset S^3) \rangle \), respectively. From Proposition 3.6 and [12, Prop. 1.3], we have \( 1 \leq s_i < p_i \), \( i = 1, 2 \), and \( p_1 + p_2 = p \).

**Case** (i) \( \langle s, s+p-1 \rangle \): By our assumption, we have \( s+1 = s_1 + s_2 \), and so \( s_1 \leq s \) and \( s_2 \leq s \). Hence, \( t_i \leq s_i + p_i - 1 < s + p - 1 \) for \( i = 1, 2 \) by Proposition 3.6 (or 1.8),

These contradict to our assumption.

**Case** (ii) \( \langle p-1, 2p-3 \rangle \): By our assumption, \( p-1 = s_1 + s_2 - 1 \). While, by Proposition 3.6 (or 1.8), if \( s_i = p_i \), then \( t_i \leq 2p_i - 1 \), and if \( s_i < p_i \) then \( t_i \leq s_i + p_i - 1 \). So, if \( s_i = p_i \) then \( t_i \leq s_i + p_i - 1 \leq 2p_i - 2 = 2p - 4 < 2p - 3 \), and if \( s_i < p_i \) then \( t_i \leq s_i + p_i - 1 \leq (p-2) + (p-1) - 1 < 2p - 3 \).

These contradictions complete the proof.

**Remark.** Theorems 5.6 and 5.7 are, of course, sufficient conditions for a \( \langle (M_p \subset S^3) \rangle \) to be prime, because the examples of prime pairs given in [12, Th. 5.2] are with complexity \( \langle 1,1 \rangle \). In fact, there may be a prime \( \langle (M_p \subset S^3) \rangle \) with every \( \langle s, t \rangle \).

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**References**


