



Title	On modules that complement direct summands
Author(s)	Zöllner, Andreas
Citation	Osaka Journal of Mathematics. 1986, 23(2), p. 457-459
Version Type	VoR
URL	https://doi.org/10.18910/9470
rights	
Note	

The University of Osaka Institutional Knowledge Archive : OUKA

<https://ir.library.osaka-u.ac.jp/>

The University of Osaka

ON MODULES THAT COMPLEMENT DIRECT SUMMANDS

ANDREAS ZÖLLNER

(Received October 5, 1984)

A module M is said to *complement direct summands* if every direct summand of M has the exchange property with respect to completely indecomposable modules, or in other words if for each direct summand B of M and for each decomposition $M = \bigoplus_I A_i$, where every A_i is completely indecomposable (i.e. has local endomorphism ring), there exists a subset K of I with $M = B \oplus \bigoplus_K A_k$.

There are several characterisations by a theorem of Harada [3, 3.1.2].

Theorem. *Let $M = \bigoplus_I A_i$ be a c. indec. decomposition. Equivalent are*

- (1) *M satisfies the take-out property.*
- (2) *Every direct summand of M has the exchange property in M .*
- (3) *M complements direct summands.*
- (4) *$(A_i: I)$ is a locally-semi- T -nilpotent family.*
- (5) *$J' \cap \text{End}(M)$ is equal to the Jacobson radical of $\text{End}(M)$.*

One step of the proof, “(4) \Rightarrow (5)”, does merit a certain attention. In an earlier version of the theorem by Harada and Sai [2, Thm 9], the proof of that step uses assumptions stronger than at hand [2, Lemma 12]. We would like to present an alternative and elementary proof of that step. In particular one does not need transfinite induction as in [3, Lemma 2.2.3]. All notation may be found in [3]. For the proofs let perpetually be $M = \bigoplus_I A_i$ a completely indec. decomposition and let $(e_i: I)$ be a related set of orthogonal idempotents (i.e. $e_i(M) = A_i$).

By definition, for an element f of $\text{End}(M)$ not contained in J' , there exist some elements $i, j \in I$ and $g \in \text{End}(M)$ with $ge_jfe_i = e_i$. Thus the Jacobson radical of $\text{End}(M)$ is always contained in $J' \cap \text{End}(M)$, otherwise it would contain a nonzero idempotent.

Lemma 1. *For all $t \in J' \cap \text{End}(M)$ and for all $i \in I$, e_it and te_i are elements of the Jacobson radical.*

Proof. Write $e_i = vp$, where v is the inclusion of A_i in M and p is the projection onto A_i induced by e_i . $J' \cap \text{End}(M)$ is an ideal, thus the composition $pstv$ is not an isomorphism for all endomorphisms s . As $\text{End}(A_i)$ is local, $1_A - pstv$ is an isomorphism. By Beck [1, Lemma 1.1], $1_M - ste_i$ is also an isomorphism and so te_i is an element of the radical. The other case works similarly.

Corollaries.

- (a) For all $t \in J' \cap \text{End}(M)$, $1 - t$ is a monomorphism.
- (b) $J' \cap \text{End}(M)$ does not contain nonzero idempotents.
- (c) Lemma 1 is also true for arbitrary local idempotents and for finite sums of orthogonal local idempotents.
- (d) Suppose J is a finite subset of I , take $x \in \bigoplus_J A_j$ and $d := \sum_J e_j$. Then $x = (1 - dt)(1 - dtd)^{-1}(x)$. (Condition (§)).

Proofs. The definition of J' does not depend on a particular decomposition of M and this implies the first statement of (c). The second statement of (c) and (d) are obtained by a straightforward calculation. For (a), take $0 \neq x \in M$. There exists a finite subset J of I with $x \in \bigoplus_J A_j$. By (c), td is in the radical and $1 - td$ is an isomorphism, where $d = \sum_J e_j$. Thus $(1 - t)(x) = (1 - td)(x) \neq 0$. (b) follows from that, as $1 - e$ is not monic for each nonzero idempotent e .

Having (a) in mind, in order to complete the proof of “(4) \Rightarrow (5)” it is enough to show $1 - t$ is an epimorphism for all $t \in J' \cap \text{End}(M)$. This is where (4) turns up. The idea is to apply the König-Graph-Lemma somehow.

Lemma 2. Let $(A_i: I)$ be a locally-semi- T -nilpotent family and take $t \in J' \cap \text{End}(M)$. Then $1 - t$ is an epimorphism.

Proof. For an arbitrary $j \in I$ and $x \in e_j M$ there is constructed a $f_x \in \text{End}(M)$ with $(1 - t)f_x(x) = x$. Then $e_i M \subset (1 - t)M$ for all $i \in I$ and $1 - t$ is onto. Let $j \in I$ and $x \in e_j M$ be as above. Sequences $(f_n: N)$, $(g_n: N)$, $(h_n: N)$, $(d_n: N)$ with elements in $\text{End}(M)$ and $(K_n: N)$, $(I_n: N)$ with subsets of I are constructed by induction, having the following properties:

- (A) d_n is an idempotent
- (B) $K_n \cap I_{n-1} = \emptyset$ and $\{j\} \cup K_1 \cup \dots \cup K_n = I_n$
- (C) $1 - g_n = (1 - t)f_n$
- (D) $g_n(x) = \prod_{1 \leq i \leq n} \sum_{k_i \in K_i} e_{k_i} (1 - d_i) t h_i(x)$

$n=1$ Define $d_1 := e_j$, $I_0 := \{j\}$, $h_1 := (1 - d_1 t d_1)^{-1}$, $f_1 := h_1$, $g_1 := 1 - (1 - t)f_1$. (A) and (C) are valid per def. Now, $(1 - g_1)(x) = (1 - t)h_1(x) = (1 - d_1 t)h_1(x) - (1 - d_1)t h_1(x)$. As by Condition (§), $(1 - d_1 t)h_1(x) = x$, follows $g_1(x) =$

$(1-d_1)th_1(x)$ and so $j \notin \text{supp}(g_1(x)) =: K_1$ (for $y \in M$, $\text{supp}(y)$ is the finite set of $i \in I$ with $e_i(y) \neq 0$). $g_1(x) = \sum_{k_1 \in K_1} e_{k_1}(1-d_1)th_1(x)$. Take $I_1 := K_1 \cup I_0$ and get (D) and (B).

$n \rightsquigarrow n+1$ Define $d_{n+1} := \sum_{i \in I_n} e_i$, $h_{n+1} := (1-d_{n+1}td_{n+1})^{-1}$, $f_{n+1} := f_n + h_{n+1}g_n$, $g_{n+1} := 1 - (1-t)f_{n+1}$, $K_{n+1} := \text{supp}(g_{n+1}(x))$, $I_{n+1} := K_{n+1} \cup I_n$. Again, (A) and (C) are valid per def. For the rest:

$$\begin{aligned} (1-g_{n+1})(x) &= (1-t)(f_n + h_{n+1}g_n)(x) \\ &= x - g_n(x) + \underbrace{(1-d_{n+1}t)h_{n+1}g_n(x)}_{= g_n(x) \text{ by Condition (S)}} - (1-d_{n+1})th_{n+1}g_n(x) \\ &\quad \underbrace{\hspace{10em}}_{= 0} \end{aligned}$$

From $g_{n+1}(x) = (1-d_{n+1})th_{n+1}g_n$ one gets (D) by insertion. It is easy to see that $K_{n+1} \cap I_n = \emptyset$, which gives (B).

The construction is now complete. All summands in (D) are nonisomorphisms (as compositions with t) between certain A_i , and none of these A_i occur twice. Now by locally-semi-T-nilpotency and the König-Graph-Lemma there exists a natural number m with $g_m(x) = 0$. (C) implies $x = (1-t)f_m(x)$.

References

- [1] I. Beck: *On modules whose endomorphism ring is local*, Israel J. Math. **29** (1978), 393–407.
- [2] M. Harada and Y. Sai: *On categories of indecomposable modules I*, Osaka J. Math. **7** (1970), 323–344.
- [3] M. Harada: *Applications of factor categories to completely indecomposable modules*, Publ. Dép. Math. Lyon **11** (1974).

Mathematisches Institut
der Universität
Theresienstrasse 39
8000 München 2
West Germany

