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ON MODULES THAT COMPLEMENT DIRECT SUMMANDS

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A module M is said to *complement direct summands* if every direct summand of M has the exchange property with respect to completely indecomposable modules, or in other words if for each direct summand B of M and for each decomposition $M=\bigoplus_I A_i$, where every A_i is completely indecomposable (i.e. has local endomorphism ring), there exists a subset K of I with $M=B\oplus\bigoplus_K A_k$.

There are several characterisations by a theorem of Harada [3, 3.1.2].

Theorem. *Let $M=\bigoplus_I A_i$ be a c. indec. decomposition. Equivalent are*

(1) *M satisfies the take-out property.*
(2) *Every direct summand of M has the exchange property in M.*
(3) *M complements direct summands.*
(4) *$(A_i: I)$ is a locally-semi-T-nilpotent family.*
(5) *$J'\cap End(M)$ is equal to the Jacobson radical of $End(M)$.*

One step of the proof, "(4)$\Rightarrow$(5)", does merit a certain attention. In an earlier version of the theorem by Harada and Sai [2, Thm 9], the proof of that step uses assumptions stronger than at hand [2, Lemma 12]. We would like to present an alternative and elementary proof of that step. In particular one does not need transfinite induction as in [3, Lemma 2.2.3]. All notation may be found in [3]. For the proofs let perpetually be $M=\bigoplus_I A_i$ a completely indec. decomposition and let $(e_i: I)$ be a related set of orthogonal idempotents (i.e. $e_i(M)=A_i$).

By definition, for an element f of End(M) not contained in J', there exist some elements $i, j\in I$ and $g\in$End(M) with $ge_jfe_i=e_i$. Thus the Jacobson radical of End(M) is always contained in $J'\cap$End(M), otherwise it would contain a nonzero idempotent.

Lemma 1. *For all $t\in J'\cap End(M)$ and for all $i\in I$, e_it and te_i are elements of the Jacobson radical.*

Proof. Write $e_i=vp$, where v is the inclusion of A_i in M and p is the projection onto A_i induced by e_i. $J'\cap\mathrm{End}(M)$ is an ideal, thus the composition $pstv$ is not an isomorphism for all endomorphisms s. As $\mathrm{End}(A_i)$ is local, 1_A-pstv is an isomorphism. By Beck [1, Lemma 1.1], 1_M-ste_i is also an isomorphism and so te_i is an element of the radical. The other case works similarly.

Corollaries.

(a) *For all* $t\in J'\cap End(M)$, $1-t$ *is a monomorphism.*

(b) $J'\cap End(M)$ *does not contain nonzero idempotents.*

(c) *Lemma* 1 *is also true for arbitrary local idempotents and for finite sums of orthogonal local idempotents.*

(d) *Suppose* J *is a finite subset of* I, *take* $x\in\bigoplus_J A_j$ *and* $d:=\sum_J e_j$. *Then* $x=(1-dt)(1-dtd)^{-1}(x)$. (*Condition* (§)).

Proofs. The definition of J' does not depend on a particular decomposition of M and this implies the first statement of (c). The second statement of (c) and (d) are obtained by a straightforward calculation. For (a), take $0\neq x\in M$. There exists a finite subset J of I with $x\in\bigoplus_J A_j$. By (c), td is in the radical and $1-td$ is an isomorphism, where $d=\sum_J e_j$. Thus $(1-t)(x)=(1-td)(x)\neq 0$. (b) follows from that, as $1-e$ is not monic for each nonzero idempotent e.

Having (a) in mind, in order to complete the proof of "(4)⇒(5)" it is enough to show $1-t$ is an epimorphism for all $t\in J'\cap\mathrm{End}(M)$. This is where (4) turns up. The idea is to apply the König-Graph-Lemma somehow.

Lemma 2. *Let* $(A_i\colon I)$ *be a locally-semi-T-nilpotent family and take* $t\in J'\cap End(M)$. *Then* $1-t$ *is an epimorphism.*

Proof. For an arbitrary $j\in I$ and $x\in e_jM$ there is constructed a $f_x\in\mathrm{End}(M)$ with $(1-t)f_x(x)=x$. Then $e_iM\subset(1-t)M$ for all $i\in I$ and $1-t$ is onto. Let $j\in I$ and $x\in e_jM$ be as above. Sequences $(f_n\colon \boldsymbol{N})$, $(g_n\colon \boldsymbol{N})$, $(h_n\colon \boldsymbol{N})$, $(d_n\colon \boldsymbol{N})$ with elements in $\mathrm{End}(M)$ and $(K_n\colon \boldsymbol{N})$, $(I_n\colon \boldsymbol{N})$ with subsets of I are constructed by induction, having the following properties:

(A) d_n is an idempotent

(B) $K_n\cap I_{n-1}=\emptyset$ and $\{j\}\cup K_1\cup\cdots\cup K_n=I_n$

(C) $1-g_n=(1-t)f_n$

(D) $g_n(x)=\prod_{1\leq i\leq n}\sum_{k_i\in K_i}e_{k_i}(1-d_i)th_i(x)$

$n=1$ Define $d_1:=e_j$, $I_0:=\{j\}$, $h_1:=(1-d_1td_1)^{-1}$, $f_1:=h_1$, $g_1:=1-(1-t)f_1$. (A) and (C) are valid per def. Now, $(1-g_1)(x)=(1-t)h_1(x)=(1-d_1t)h_1(x)-(1-d_1)th_1(x)$. As by Condition (§), $(1-d_1t)h_1(x)=x$, follows $g_1(x)=$

$(1-d_1)th_1(x)$ and so $j\notin\operatorname{supp}(g_1(x))=:K_1$ (for $y\in M$, $\operatorname{supp}(y)$ is the finite set of $i\in I$ with $e_i(y)\neq 0$). $g_1(x)=\sum_{k_1\in K_1} e_{k_1}(1-d_1)th_1(x)$. Take $I_1:=K_1\cup I_0$ and get (D) and (B).

$n\rightsquigarrow n+1$ Define $d_{n+1}:=\sum_{i\in I_n} e_i$, $h_{n+1}:=(1-d_{n+1}td_{n+1})^{-1}$, $f_{n+1}:=f_n+h_{n+1}g_n$, $g_{n+1}:=1-(1-t)f_{n+1}$, $K_{n+1}:=\operatorname{supp}(g_{n+1}(x))$, $I_{n+1}:=K_{n+1}\cup I_n$. Again, (A) and (C) are valid per def. For the rest:

$$\begin{aligned}(1-g_{n+1})(x) &= (1-t)(f_n+h_{n+1}g_n)(x)\\ &= \underbrace{x-g_n(x)+\underbrace{(1-d_{n+1}t)h_{n+1}g_n(x)}_{=\,g_n(x)\text{ by Condition }(\S)}}_{=\,0}-(1-d_{n+1})th_{n+1}g_n(x)\end{aligned}$$

From $g_{n+1}(x)=(1-d_{n+1})th_{n+1}g_n$ one gets (D) by insertion. It is easy to see that $K_{n+1}\cap I_n=\emptyset$, which gives (B).

The construction is now complete. All summands in (D) are nonisomorphisms (as compositions with t) between certain A_i, and none of these A_i occur twice. Now by locally-semi-T-nilpotency and the König-Graph-Lemma there exists a natural number m with $g_m(x)=0$. (C) implies $x=(1-t)f_m(x)$.

References

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