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THE INFINITENESS OF THE SAGBI BASES FOR CERTAIN INVARIANT RINGS

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Introduction

The concept of initial ideals for ideals of a polynomial ring in Gröbner basis theory is generalized in a natural way for subalgebras of a polynomial ring, and they are called initial algebras. A set of generators of a subalgebra is called a SAGBI (Subalgebra Analogue to Gröbner Bases for Ideals) basis [6] if their initial monomials generate the initial algebra. The main difference between the initial ideal and the initial algebra is that the former always has finite generators by Hilbert's basis theorem while the latter does not. Hence it is an important problem to find a criterion for the finite generation of initial algebras.

Göbel [2] studied this problem for the subalgebras which are invariant rings of permutation groups G. He showed that, with respect to the lexicographic order, the initial algebra of $k[\mathbf{x}]^G$ is finitely generated if and only if G is a direct product of symmetric groups.

In this paper, we prove that a similar result holds for any multiplicative order, i.e. a monomial order which does not require the minimality of the unit 1. We introduce a topological structure to the set of multiplicative orders, and make use of it for the proof of our results.

In case of initial ideals, there exist only finite cardinality of distinct initial ideals for an ideal under a certain condition, although there exist infinite cardinality of orders in general. However, this is not always true in case of initial algebras. Our second result is about the cardinality of distinct initial algebras of invariant rings of permutation groups. We will show that there exist uncountable cardinality of distinct initial algebras for each invariant ring, when G is not a direct product of symmetric groups. If G is a product of symmetric groups, there exist finite cardinality of distinct initial algebras. The exact number is given in Proposition 3.3.

We prove similar results on initial algebras for $k[\mathbf{x}, \mathbf{x}^{-1}]^G$, i.e., for invariant subrings of the Laurent polynomial ring $k[\mathbf{x}, \mathbf{x}^{-1}]$.

In Section 1, we introduce a topology on the set of multiplicative orders. This section also contains our notation and the basic definitions. Section 2 presents our main results.

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1. The topological structure of multiplicative orders and standard bases for vector spaces

We fix a field k of an arbitrary characteristic. Let n be a positive integer, $k[\mathbf{x}] := k[x_1, \dots, x_n]$ the polynomial ring of n variables, and

$$k[\mathbf{x}, \mathbf{x}^{-1}] := k[x_1, \dots, x_n, x_1^{-1}, \dots, x_n^{-1}]$$

the Laurent polynomial ring of n variables. Throughout this paper, the monomials in $k[\mathbf{x}, \mathbf{x}^{-1}]$ are denoted $\mathbf{x}^{\mathbf{a}} = x_1^{a_1} \cdots x_n^{a_n}$ and identified with lattice points $\mathbf{a} = (a_1, \dots, a_n)$ in $\mathbf{Z}^n$. An algebra always means a k-algebra.

A total order $\prec$ on $\mathbf{Z}^n$ is said to be *multiplicative* if $\mathbf{a} \prec \mathbf{b}$ implies $\mathbf{a} + \mathbf{c} \prec \mathbf{b} + \mathbf{c}$ for all $\mathbf{a}$, $\mathbf{b}$, $\mathbf{c} \in \mathbf{Z}^n$. A *monomial order* is a total order which is a multiplicative order and the zero vector 0 is the minimum element among $\mathbf{Z}^n_{\geq 0}$. We denote by $\mathbf{S}'$ the set of vectors $\omega = (\omega^1, \dots, \omega^n)$ on the $(n-1)$-dimensional unit sphere $\mathbf{S}^{n-1} \subset \mathbf{R}^n$ whose components $\omega^1, \dots, \omega^n \in \mathbf{R}$ are linearly independent over $\mathbf{Q}$. For each $\omega \in \mathbf{S}'$, the multiplicative order $\prec = \iota(\omega)$ is defined by

$$\mathbf{a} \prec \mathbf{b} : \Leftrightarrow \omega \cdot \mathbf{a} \leq \omega \cdot \mathbf{b}.$$

Note that the inner products $\omega \cdot \mathbf{a}$ and $\omega \cdot \mathbf{b}$ are not equal for any distinct $\mathbf{a}$ and $\mathbf{b}$ in $\mathbf{Z}^n$ by the linear independence of $\omega^1, \dots, \omega^n$ over $\mathbf{Q}$.

For a convex polytope $P \subset \mathbf{R}^n$ and $\omega \in \mathbf{R}^n$, the *face* $\mathrm{face}_\omega(P)$ of P is defined by

$$\mathrm{face}_\omega(P) := \{\mathbf{a} \in \mathbf{R}^n \mid \omega \cdot \mathbf{a}' \leq \omega \cdot \mathbf{a} \text{ for all } \mathbf{a}' \in P\}.$$

We denote by Ω the set of multiplicative orders, by Ω_0 the set of monomial orders, and by $\mathcal{V}$ the set of k-vector spaces $V \subset k[\mathbf{x}, \mathbf{x}^{-1}]$ spanned by monomials.

We introduce topologies on Ω and $\mathcal{V}$ as follows. We take a map ρ from $\mathbf{Z}^n$ to $\mathbf{Z}_{>0}$ such that $\rho^{-1}(l)$ is a finite set for every $l \in \mathbf{Z}_{>0}$. Let $d_\rho \colon \Omega \times \Omega \to \mathbf{R}$ and $\delta_\rho \colon \mathcal{V} \times \mathcal{V} \to \mathbf{R}$ be functions defined as follows. For all $\prec, \prec' \in \Omega$, we set

$$d_\rho(\prec, \prec') := \begin{cases} 0 & \text{if } \prec = \prec' \\ 1/e & \text{if } e = \max\{e \in \mathbf{Z}_{>0} \mid \mathbf{x}^{\mathbf{a}} \prec \mathbf{x}^{\mathbf{b}} \Leftrightarrow \mathbf{x}^{\mathbf{a}} \prec' \mathbf{x}^{\mathbf{b}} \\ & \qquad \text{for all } \mathbf{x}^{\mathbf{a}},\ \mathbf{x}^{\mathbf{b}} \in k[\mathbf{x}, \mathbf{x}^{-1}] \text{ such that } \rho(\mathbf{a}),\ \rho(\mathbf{b}) < e\}. \end{cases}$$

For all $V, V' \in \mathcal{V}$, we set

$$\delta_\rho(V, V') := \begin{cases} 0 & \text{if } V = V' \\ 1/e & \text{if } e = \max\{e \in \mathbf{Z}_{>0} \mid \mathbf{x}^{\mathbf{a}} \in V \Leftrightarrow \mathbf{x}^{\mathbf{a}} \in V' \\ & \qquad \text{for all } \mathbf{x}^{\mathbf{a}} \in k[\mathbf{x}, \mathbf{x}^{-1}] \text{ such that } \rho(\mathbf{a}) < e\}. \end{cases}$$

It is easy to see that d_ρ and δ_ρ define metrics of Ω and $\mathcal{V}$, respectively. For $\mathbf{S}'$, we consider the topology induced from $\mathbf{R}^n$.

Theorem 1.1. *The topological structures of the metric spaces* (Ω, d_ρ) *and* $(\mathcal{V}, \delta_\rho)$ *are independent of the choice of* ρ. *The set* Ω *of multiplicative orders is compact with respect to this topology. Furthermore, the injection* $\iota\colon \mathbf{S}' \to \Omega$ *is continuous. The image* $\iota(\mathbf{S}')$ *is a dense subset of* Ω.

Proof. Let d_{ρ_1}, d_{ρ_2} be distance functions on Ω determined by maps ρ_1, ρ_2 from $\mathbf{Z}^n$ to $\mathbf{Z}_{>0}$ as above. We take an arbitrary $\prec \in \Omega$ and $e > 0$. Then, there exists $e' \gg 0$ such that $\{\mathbf{x}^{\mathbf{a}} \in k[\mathbf{x}, \mathbf{x}^{-1}] \mid \rho_1(\mathbf{a}) \leq e'\}$ and $\{\mathbf{x}^{\mathbf{a}} \in k[\mathbf{x}, \mathbf{x}^{-1}] \mid \rho_2(\mathbf{a}) \leq e'\}$ contain $\{\mathbf{x}^{\mathbf{a}} \in k[\mathbf{x}, \mathbf{x}^{-1}] \mid \rho_1(\mathbf{a}) \leq e$ or $\rho_2(\mathbf{a}) \leq e\}$. Now, it follows for every $\prec' \in \Omega$ that $d_{\rho_1}(\prec, \prec') < 1/e'$ implies $d_{\rho_2}(\prec, \prec') < 1/e$, and $d_{\rho_2}(\prec, \prec') < 1/e'$ implies $d_{\rho_1}(\prec, \prec') < 1/e$. Hence d_{ρ_1} and d_{ρ_2} define the same topology.

By a similar argument, we can prove that any two distance functions δ_{ρ_1} and δ_{ρ_2} define the same topology on $\mathcal{V}$.

We prove the totally boundedness of Ω. We take a positive number e. Then the cardinality of monomials $\mathbf{x}^{\mathbf{a}}$ with $\rho(\mathbf{a}) \leq e$ is finite. So, there exist only finite cardinality of distinct orders on the set of monomials $\mathbf{x}^{\mathbf{a}}$ with $\rho(\mathbf{a}) \leq e$. Hence we can take $\prec_1, \ldots, \prec_l \in \Omega$ such that, for every $\prec \in \Omega$, it follows that $d_\rho(\prec, \prec_i) < 1/e$ for some i. Then the $1/e$-neighborhoods of $\prec_i$'s is a finite $1/e$-covering of Ω.

Now we see the completeness of Ω as follows. Let $\{\prec_i\}_i \subset \Omega$ be a Cauchy sequence. Then, for every integer $e > 0$, there exists an integer $k_e > 0$ such that $d_\rho(\prec_i, \prec_j) < 1/e$ for all i, $j \geq k_e$. Now, $\{\prec_i\}_i$ tends to the order $\prec \in \Omega$ which is defined by

$$\mathbf{x}^{\mathbf{a}} \prec \mathbf{x}^{\mathbf{b}} :\Leftrightarrow \mathbf{x}^{\mathbf{a}} \prec_{k_e} \mathbf{x}^{\mathbf{b}},$$

where e is an integer greater than $\rho(\mathbf{a})$ and $\rho(\mathbf{b})$.

Finally, we prove the continuity of the injection $\iota\colon \mathbf{S}' \to \Omega$, and the density of its image. Let $\prec_0 = \iota(\omega_0)$ be the multiplicative order defined by $\omega_0 \in \mathbf{S}'$, and let e be a positive number. Then the following three conditions are equivalent for $\omega \in \mathbf{S}'$ and $\prec = \iota(\omega)$:

$d_\rho(\prec_0, \prec) < 1/e$,

$\omega_0 \cdot \mathbf{a} \leq \omega_0 \cdot \mathbf{b} \Leftrightarrow \omega \cdot \mathbf{a} \leq \omega \cdot \mathbf{b}$ for all $\mathbf{a}$, $\mathbf{b} \in \mathbf{Z}^n$ with $\rho(\mathbf{a})$, $\rho(\mathbf{b}) \leq e$,

$\mathrm{face}_\omega(\mathrm{conv}\{\mathbf{a}, \mathbf{b}\}) = \mathrm{face}_{\omega_0}(\mathrm{conv}\{\mathbf{a}, \mathbf{b}\})$ for all $\mathbf{a}$, $\mathbf{b} \in \mathbf{Z}^n$ with $\rho(\mathbf{a})$, $\rho(\mathbf{b}) \leq e$,

where $\mathrm{conv}\{\mathbf{a}, \mathbf{b}\}$ is the convex hull of $\{\mathbf{a}, \mathbf{b}\}$. In general, for a convex polytope $P \subset \mathbf{R}^n$ and a vertex $\{v_0\} = \mathrm{face}_{\eta_0}(P)$, the set $\{\eta \in \mathbf{R}^n \mid \mathrm{face}_\eta(P) = \{v_0\}\}$ of vectors is an open cone of $\mathbf{R}^n$. In particular,

$$U(\mathbf{a}, \mathbf{b}) := \{\omega \in \mathbf{S}' \mid \mathrm{face}_\omega\big(\mathrm{conv}\{\mathbf{a}, \mathbf{b}\}\big) = \mathrm{face}_{\omega_0}\big(\mathrm{conv}\{\mathbf{a}, \mathbf{b}\}\big)\}$$

is an open set of $\mathbf{S}'$. Since $\{\omega \in \mathbf{S}' \mid d_\rho(\prec_0, \iota(\omega)) < 1/e\}$ is the intersection of

$U(\mathbf{a},\mathbf{b})$'s for $\mathbf{a}$, $\mathbf{b} \in \mathbf{Z}^n$ with $\rho(\mathbf{a})$, $\rho(\mathbf{b}) \leq e$, it is an open set of $\mathbf{S}'$. Hence the map ι is continuous.

The density of $\iota(\mathbf{S}')$ in Ω follows from Robbiano's classification of multiplicative orders [5, Theorem 2.5]:

Let $\prec$ be a multiplicative order. Then there exist vectors $\omega_1, \dots, \omega_N \in \mathbf{R}^n$ such that $\mathbf{x}^{\mathbf{a}} \prec \mathbf{x}^{\mathbf{b}}$ if and only if $\omega_i \cdot \mathbf{a} < \omega_i \cdot \mathbf{b}$ for the first i such that $\omega_i \cdot \mathbf{a} \neq \omega_i \cdot \mathbf{b}$, for all $\mathbf{a}$, $\mathbf{b} \in \mathbf{Z}^n$.

Indeed, we set $\omega(T) := \sum_{i=1}^{N} \omega_i T^{N-i}$ and take $\{t_i\}_i \subset \mathbf{R}$ such that $t_i \to +\infty$ as $i \to +\infty$ and $|\omega(t_i)|^{-1}\omega(t_i) \in \mathbf{S}'$. Then the sequence $\{\iota(|\omega(t_i)|^{-1}\omega(t_i))\}_i$ tends to $\prec$. $\square$

The topology of Ω defined as above is the same as the topology which is defined as follows (cf. [4, Lecture 3], [7]): Let $\Omega \to \{1, -1\}^{\mathbf{Z}^n}$ be the inclusion map which is defined, for each $\prec \in \Omega$, by $\prec(\mathbf{a}) := 1$ if $0 \prec \mathbf{a}$, and -1 otherwise, for all $\mathbf{a} \in \mathbf{Z}^n$. The set $\{1, -1\}^{\mathbf{Z}^n}$ is considered to be the topological space which is the product of the discrete topological space $\{1, -1\}$. The topological structure of Ω is induced from this topology.

In what follows, by a vector space $V \subset k[\mathbf{x}, \mathbf{x}^{-1}]$, we mean a vector space over the field k.

DEFINITION 1.2. Let $\prec$ be a multiplicative order, $f = \sum_i c_i \mathbf{x}^{\mathbf{a}_i} \in k[\mathbf{x}, \mathbf{x}^{-1}]$ a nonzero polynomial, and $V \subset k[\mathbf{x}, \mathbf{x}^{-1}]$ a vector space.
(1) The *initial monomial* of f with respect to $\prec$ is defined by

$$\mathrm{in}_{\prec}(f) := \max_{\prec}\{\mathbf{x}^{\mathbf{a}_i} \mid c_i \neq 0\}. \tag{1.1}$$

Then it follows that $\mathrm{in}_{\prec}(f \cdot g) = \mathrm{in}_{\prec}(f) \cdot \mathrm{in}_{\prec}(g)$ for f, $g \in k[\mathbf{x}, \mathbf{x}^{-1}] \setminus \{0\}$.
(2) The *initial vector space* of V with respect to $\prec$ is by definition the vector space spanned by $\{\mathrm{in}_{\prec}(f) \mid f \in V \setminus \{0\}\}$. If A is a subalgebra of $k[\mathbf{x}, \mathbf{x}^{-1}]$, then $\mathrm{in}_{\prec}(A)$ has an algebra structure, since $\mathrm{in}_{\prec}(f) \cdot \mathrm{in}_{\prec}(g) = \mathrm{in}_{\prec}(f \cdot g)$ for any f, $g \in A \setminus \{0\}$. We call it the *initial algebra* of A with respect to $\prec$.

A set S of generators of A is called a *SAGBI basis* with respect to $\prec \in \Omega$, if $\{\mathrm{in}_{\prec}(f) \mid 0 \neq f \in S\}$ generates $\mathrm{in}_{\prec}(A)$ as an algebra. Note that A has a finite SAGBI basis only if the initial algebra $\mathrm{in}_{\prec}(A)$ is finitely generated.

The correspondence $\prec \mapsto \mathrm{in}_{\prec}(V)$ is a map from the set Ω of multiplicative orders to the set $\mathcal{V}$ of vector spaces spanned by monomials. This map is denoted by F_V. It is not continuous in general. However, if the vector space V satisfies the following separation condition, then F_V is continuous.

For each monomial m, there exist subspaces H, $K \subset V$ such that $V = H + K$. Here, the number of monomials appearing in polynomials in H is finite, m does not appear in any polynomials in K, and a polynomial in H and a polynomial in

K have no common monomials.
Actually, if $F_V(\prec)$ does not contain m, then neither does $F_V(\prec')$ for $\prec'$ in a sufficiently small neighborhood of $\prec$, since $F_V(\prec'') = F_H(\prec'') + F_K(\prec'')$ holds for any $\prec'' \in \Omega$. We denote by $U_V(\prec)$ the inverse image of the initial vector space $\mathrm{in}_\prec(V) \in \mathcal{V}$. Namely,

$$U_V(\prec) := \{\prec' \in \Omega \mid \mathrm{in}_{\prec'}(V) = \mathrm{in}_\prec(V)\}. \tag{1.2}$$

If V satisfies the separation condition, then $U_V(\prec)$ is a closed subset of Ω, because $\mathcal{V}$ is Hausdorff and the map F_V is continuous.

DEFINITION 1.3. Let $V \subset k[\mathbf{x}, \mathbf{x}^{-1}]$ be a vector space, and $\prec$ a multiplicative order.
(1) A basis $\{f_i\}_i$ of the vector space V is said to be *standard* with respect to $\prec$, if $\{\mathrm{in}_\prec(f_i)\}_i$ is a basis of the vector space $\mathrm{in}_\prec(V)$.
(2) A polynomial $0 \neq f \in V$ is said to be *reduced*, if all monomials of f but $\mathrm{in}_\prec(f)$ are not contained in $\mathrm{in}_\prec(V)$.
(3) A standard basis $\{f_i\}_i$ is said to be *reduced* if every f_i is reduced.

We remark that the index set of a standard basis of a vector space V with respect to $\prec \in \Omega$ can be taken as the set of monomials in $\mathrm{in}_\prec(V)$. Namely, we denote a standard basis by $\{f_m\}_m$ with $m = \mathrm{in}_\prec(f_m)$ where m runs through the monomials of $\mathrm{in}_\prec(V)$.

The following lemma is well known.

Lemma 1.4. *Let $V \subset k[\mathbf{x}, \mathbf{x}^{-1}]$ be a vector space and $\prec$, $\prec'$ multiplicative orders. Assume that there exists a reduced standard basis of V with respect to $\prec$ and $\prec'$. Then, $\mathrm{in}_\prec(V) \subset \mathrm{in}_{\prec'}(V)$ implies $\mathrm{in}_\prec(V) = \mathrm{in}_{\prec'}(V)$.*

Proof. Let $\{f_m\}_m$ and $\{f'_{m'}\}_{m'}$ be reduced standard bases of V with respect to $\prec$ and $\prec'$ respectively. For each monomial m in $\mathrm{in}_\prec(V)$, it follows that $f'_m = c_m f_m$ for some $c_m \neq 0$. Actually, we choose c_m so that the coefficient of m in $f'_m - c_m f_m$ is zero. Since f_m and f'_m are reduced, none of the monomials of $f'_m - c_m f_m \in V$ lie in $\mathrm{in}_\prec(V)$. Therefore $f'_m - c_m f_m$ is equal to zero. Hence, by replacing f_m with $c_m f_m$, we may assume $f_m = f'_m$ for every monomial m in $\mathrm{in}_\prec(V)$.

Suppose there existed a proper inclusion of $\mathrm{in}_\prec(V)$ to $\mathrm{in}_{\prec'}(V)$. Then, there exists a proper inclusion $\{f_m\}_m \subset \{f'_{m'}\}_{m'}$ of the reduced standard bases. This is a contradiction, since both $\{f_m\}_m$ and $\{f'_{m'}\}_{m'}$ are bases of V. □

Let $\{f_m\}_m$ and $\{f'_m\}_m$ be reduced standard bases of V with respect to multiplicative orders $\prec$ and $\prec'$, respectively. If $\mathrm{in}_\prec(V) = \mathrm{in}_{\prec'}(V)$ then we have $f'_m = c_m f_m$ for some $c_m \in k \setminus \{0\}$ for each monomial $m \in \mathrm{in}_\prec(V)$, by the proof of Lemma 1.4.

Namely, the reduced standard basis of V is uniquely determined by the vector space $\mathrm{in}_{\prec}(V)$ up to multiplications of elements of $k \setminus \{0\}$. We sometimes say $\{f_m\}_m$ a reduced standard basis with respect to $\mathrm{in}_{\prec}(V)$.

Lemma 1.5. *Let $V \subset V' \subset k[\mathbf{x}, \mathbf{x}^{-1}]$ be vector spaces, and $\prec \in \Omega$. With respect to $\prec$, we suppose that a reduced standard basis of V is a subset of a reduced standard basis of V'. Then it follows that*

$$U_{V'}(\prec) \subset U_V(\prec).$$

Proof. Let $\{f_m\}_m$ and $\{f'_{m'}\}_{m'}$ be reduced standard bases of V and V' with respect to $\prec$, respectively. Then it follows that

$$U_V(\prec) = \{\prec'' \in \Omega \mid \mathrm{in}_{\prec''}(f_m) = m \text{ for every monomial } m \in \mathrm{in}_{\prec}(V)\}$$

and

$$U_{V'}(\prec) = \{\prec'' \in \Omega \mid \mathrm{in}_{\prec''}(f'_{m'}) = m' \text{ for every monomial } m' \in \mathrm{in}_{\prec}(V')\}.$$

Now we assume that $\{f_m\}_m \subset \{f'_{m'}\}_{m'}$. Then, for each monomial $m \in \mathrm{in}_{\prec}(V)$, $f_m = f'_{m'}$ implies $m = \mathrm{in}_{\prec}(f_m) = \mathrm{in}_{\prec}(f'_{m'}) = m'$. Hence we have $U_{V'}(\prec) \subset U_V(\prec)$. $\square$

For a vector space $V \subset k[\mathbf{x}, \mathbf{x}^{-1}]$, we denote by $\Delta(V)$ the set of multiplicative orders with respect to which reduced standard bases of V exist. Note that $U_V(\prec)$ is contained in $\Delta(V)$ if $\prec \in \Delta(V)$.

Lemma 1.6. *Let $A \subset k[\mathbf{x}, \mathbf{x}^{-1}]$ be a subalgebra, and $\prec \in \Delta(A)$. If the algebra $\mathrm{in}_{\prec}(A)$ is finitely generated, then $U_A(\prec)$ is an open subset of $\Delta(A)$.*

Proof. Let $\{f_m\}_m$ be a reduced standard basis of A with respect to $\mathrm{in}_{\prec}(A)$. For $0 \neq f = \sum_i c_i \mathbf{x}^{\mathbf{a}_i} \in k[\mathbf{x}, \mathbf{x}^{-1}]$, we set $\rho(f) := \max\{\rho(\mathbf{a}_i) \mid c_i \neq 0\}$. Then there exists a positive integer e such that $\mathrm{in}_{\prec}(A)$ is generated by its monomials m with $\rho(f_m) \leq e$. We will show that $1/e$-neighborhood of every $\prec' \in U_A(\prec)$ is contained in $U_A(\prec)$. We fix an arbitrary $\prec' \in U_A(\prec)$ and take $\prec'' \in \Delta(A)$ such that $d_\rho(\prec', \prec'') < 1/e$. Note that $\{f_m\}_m$ is a reduced standard basis with respect to $\prec'$ as well. Then monomial $m \in \mathrm{in}_{\prec'}(A)$ is contained in $\mathrm{in}_{\prec''}(A)$ if $\rho(f_m) \leq e$, because $m = \mathrm{in}_{\prec'}(f_m) = \mathrm{in}_{\prec''}(f_m)$ for $\rho(f_m) \leq e$. Since $\mathrm{in}_{\prec}(A) = \mathrm{in}_{\prec'}(A)$ is generated by monomials m with $\rho(f_m) \leq e$, we have $\mathrm{in}_{\prec'}(A) \subset \mathrm{in}_{\prec''}(A)$. This implies $\mathrm{in}_{\prec'}(A) = \mathrm{in}_{\prec''}(A)$ by Lemma 1.4. Hence $\prec''$ is contained in $U_A(\prec)$. Therefore the $1/e$-neighborhood of $\prec'$ is contained in $U_A(\prec)$. $\square$

The converse of Lemma 1.6 is not true in general. Actually, there exists a subalgebra A of $k[\mathbf{x}, \mathbf{x}^{-1}]$ which is generated by monomials but is not finitely generated. In this case, $U_A(\prec) = \Delta(A) = \Omega$ for any $\prec \in \Omega$.

Let I be an ideal of $k[\mathbf{x}]$. By Hilbert's basis theorem, the ideal $\mathrm{in}_\prec(I)$ is always finitely generated. By the argument similar to Lemma 1.6, Schwartz [7, Theorems 13 and 30] showed that, for any subset G of I,

$$U_{I,G} := \{\prec \in \Omega_0 \mid G \text{ is a Gröbner basis of } I \text{ with respect to } \prec\} \tag{1.3}$$

is an open subset of Ω_0. Note that Ω_0 is a compact subset of Ω. In fact, we have the following lemma.

Lemma 1.7. *Let $S \subset k[\mathbf{x}, \mathbf{x}^{-1}]$ be an algebra which is generated by a finite subset of monomials in S. Then the set of multiplicative orders which are well-orderings on the set of monomials in S is compact* (*may be empty*).

Proof. We remark that $\prec \in \Omega$ is a well-ordering on the set of monomials in S, if and only if the unit 1 is the minimum element among the monomials in S. Indeed, if there exists a monomial $1 \neq \mathbf{x}^{\mathbf{a}} \in S$ with $\mathbf{x}^{\mathbf{a}} \prec 1$, then $\{\mathbf{x}^{l\mathbf{a}} \mid l = 1, 2, \dots\} \subset S$ does not have the minimum element. For the converse, suppose that every monomial of S is greater than 1. Since S is Noetherian, the ideal $(U) \subset S$ is finitely generated (say, by $\{\mathbf{x}^{\mathbf{a}_1}, \dots, \mathbf{x}^{\mathbf{a}_r}\} \subset U$) for any subset U of monomials in S. Then we have $\min_\prec U = \min_\prec\{\mathbf{x}^{\mathbf{a}_1}, \dots, \mathbf{x}^{\mathbf{a}_r}\}$.

We set W the set of multiplicative orders which are not well-orderings on the set of monomials in S. We will show that W is an open subset of Ω. For $\prec \in W$, there exists a monomial $1 \neq \mathbf{x}^{\mathbf{a}} \in S$ with $\mathbf{x}^{\mathbf{a}} \prec 1$. We take a positive number e which is greater than $\rho(0)$ and $\rho(\mathbf{a})$. For any multiplicative order $\prec'$ in the $1/e$-neighborhood of $\prec$, we have $\mathbf{x}^{\mathbf{a}} \prec' 1$. So $\prec'$ is not a well-ordering on the set of monomials in S as well. Hence the $1/e$-neighborhood of $\prec$ is contained in W. Therefore W is open. □

By using the compactness of Ω_0, Schwartz [7, Corollaries 16 and 31] showed the finiteness of the cardinality of distinct initial ideals for a fixed ideal of $k[\mathbf{x}]$ with respect to monomial orders. By a similar argument, we get the following proposition.

Proposition 1.8. *Let $A \subset k[\mathbf{x}, \mathbf{x}^{-1}]$ be a subalgebra, and Δ a compact subset of $\Delta(A)$. Assume that the initial algebras $\mathrm{in}_\prec(A)$ are finitely generated for all $\prec \in \Delta$. Then there exist only finite distinct $\mathrm{in}_\prec(A)$'s when $\prec$ runs over Δ.*

Proof. By Lemma 1.6, $U_A(\prec)$ is an open subset of $\Delta(A)$ for any $\prec \in \Delta$. Hence

$$\{U_A(\prec) \cap \Delta \mid \prec \in \Delta\}$$

is a disjoint open covering of Δ. Since Δ is compact, it is a finite covering. Therefore, the cardinality of distinct initial algebras for A with respect to $\prec \in \Delta$ is finite. $\square$

2. Main result

Throughout Sections 2 and 3, we fix a subgroup G of the symmetric group S_n of degree n. The action of G on $k[\mathbf{x}, \mathbf{x}^{-1}]$ is defined by $\sigma(f) := f(x_{\sigma(1)}, \ldots, x_{\sigma(n)})$ for $\sigma \in G$ and $f = f(x_1, \ldots, x_n) \in k[\mathbf{x}, \mathbf{x}^{-1}]$. Let $k[\mathbf{x}, \mathbf{x}^{-1}]^G$ and $k[\mathbf{x}]^G$ be the invariant subrings of $k[\mathbf{x}, \mathbf{x}^{-1}]$ and $k[\mathbf{x}]$, respectively, by the action of G.

Recall the following result by Göbel.

Theorem 2.1 (Göbel [2]). *Let $\prec_{\text{lex}} \in \Omega$ be a lexicographic order. Then $\text{in}_{\prec_{\text{lex}}}(k[\mathbf{x}]^G)$ is finitely generated if and only if G is a direct product of symmetric groups.*

Here, by symmetric groups, we mean those of subsets of $\{1, \ldots, n\}$. Note that G is a direct product of symmetric groups if and only if G is generated by the set of transpositions in G. We will show that similar results hold for any multiplicative orders.

Theorem 2.2. *Assume that G is not a direct product of symmetric groups. Then the initial algebra $\text{in}_{\prec}(k[\mathbf{x}]^G)$ is not finitely generated for any multiplicative order $\prec \in \Omega$. There are uncountable cardinality of distinct initial algebras for $k[\mathbf{x}]^G$.*

We get a similar result for $k[\mathbf{x}, \mathbf{x}^{-1}]^G$ as follows.

Theorem 2.3. *Assume that G is not a direct product of symmetric groups. Then the initial algebra $\text{in}_{\prec}(k[\mathbf{x}, \mathbf{x}^{-1}]^G)$ is not finitely generated for any multiplicative order $\prec \in \Omega$. There are uncountable cardinality of distinct initial algebras for $k[\mathbf{x}, \mathbf{x}^{-1}]^G$.*

For a subgroup G of a symmetric group and a monomial $\mathbf{x}^{\mathbf{a}} \in k[\mathbf{x}, \mathbf{x}^{-1}]$, we define

$$f_G(\mathbf{x}^{\mathbf{a}}) := \sum_{\sigma \in G/G(\mathbf{x}^{\mathbf{a}})} \sigma(\mathbf{x}^{\mathbf{a}}), \tag{2.1}$$

where $G(\mathbf{x}^{\mathbf{a}})$ is the stabilizer $\{\tau \in G \mid \tau(\mathbf{x}^{\mathbf{a}}) = \mathbf{x}^{\mathbf{a}}\}$. We set

$$B := \{f_G(\mathbf{x}^{\mathbf{a}}) \mid \mathbf{a} \in \mathbf{Z}^n\} \tag{2.2}$$

and

$$B_0 := \{f_G(\mathbf{x}^{\mathbf{a}}) \mid \mathbf{a} \in \mathbf{Z}^n_{\geq 0}\}. \tag{2.3}$$

Lemma 2.4. *For any multiplicative order, the sets B and B_0 are reduced standard bases of $k[\mathbf{x}, \mathbf{x}^{-1}]^G$ and $k[\mathbf{x}]^G$, respectively.*

Proof. We fix an arbitrary multiplicative order $\prec$. We first remark that if $f_G(\mathbf{x}^{\mathbf{a}})$ and $f_G(\mathbf{x}^{\mathbf{b}})$ have common terms then $f_G(\mathbf{x}^{\mathbf{a}}) = f_G(\mathbf{x}^{\mathbf{b}})$. This implies that B is linearly independent over k, and every $f_G(\mathbf{x}^{\mathbf{a}}) \in B$ is reduced.

We show that B spans $k[\mathbf{x}, \mathbf{x}^{-1}]^G$ over k. Let $0 \neq f = \sum_i c_i \mathbf{x}^{\mathbf{a}_i} \in k[\mathbf{x}, \mathbf{x}^{-1}]^G$ be an invariant. Then G acts on the set $\{c_i \mathbf{x}^{\mathbf{a}_i} \mid c_i \neq 0\}$ of terms of f. We decompose it into orbits as

$$\{c_i \mathbf{x}^{\mathbf{a}_i} \mid c_i \neq 0\} = \coprod_l \{c_{i_l} \sigma(\mathbf{x}^{\mathbf{a}_{i_l}}) \mid \sigma \in G\}.$$

The sum of the elements of $\{c_i \mathbf{x}^{\mathbf{a}_i} \mid c_i \neq 0\}$ is equal to f, and the sum of the elements of $\{\sigma(\mathbf{x}^{\mathbf{a}_{i_l}}) \mid \sigma \in G\}$ is equal to $f_G(\mathbf{x}^{\mathbf{a}_{i_l}})$. Hence we have

$$f = \sum_l c_{i_l} f_G(\mathbf{x}^{\mathbf{a}_{i_l}}).$$

Now, we show that B is a standard basis of $k[\mathbf{x}, \mathbf{x}^{-1}]^G$ with respect to $\prec$. Since B spans $k[\mathbf{x}, \mathbf{x}^{-1}]^G$, a G-invariant of $k[\mathbf{x}, \mathbf{x}^{-1}] \setminus \{0\}$ has an expression $f = \sum_i c_i f_G(\mathbf{x}^{\mathbf{a}_i})$. By the remark, the monomial $\mathrm{in}_\prec(f_G(\mathbf{x}^{\mathbf{a}_i}))$ appears in f with nonzero coefficient if $c_i \neq 0$. Hence we have

$$\mathrm{in}_\prec(f) = \max_\prec \{\mathrm{in}_\prec\left(f_G(\mathbf{x}^{\mathbf{a}_i})\right) \mid c_i \neq 0\} \in \{\mathrm{in}_\prec(g) \mid g \in B\}. \tag{2.4}$$

Thus, B is a standard basis of $k[\mathbf{x}, \mathbf{x}^{-1}]^G$ with respect to $\prec$.

We will prove that B_0 is a standard basis of $k[\mathbf{x}]^G$ with respect to $\prec$. Let $f = \sum_i c_i f_G(\mathbf{x}^{\mathbf{a}_i}) \in k[\mathbf{x}]^G$ be a nonzero invariant. By the remark, any term which appears in $c_i f_G(\mathbf{x}^{\mathbf{a}_i})$ appears in f as well. So, each $c_i f_G(\mathbf{x}^{\mathbf{a}_i})$ must be an element of $k[\mathbf{x}]$. Hence B_0 spans $k[\mathbf{x}]^G$. As (2.4), we have $\mathrm{in}_\prec(f) \in \{\mathrm{in}_\prec(g) \mid g \in B_0\}$. Thus B_0 is a standard basis of $k[\mathbf{x}]^G$. □

By this lemma, we have

$$\Delta\left(k[\mathbf{x}]^G\right) = \Delta\left(k[\mathbf{x}, \mathbf{x}^{-1}]^G\right) = \Omega.$$

Furthermore, it is easy to see that $k[\mathbf{x}]^G$ and $k[\mathbf{x}, \mathbf{x}^{-1}]^G$ satisfy the separation condition. Hence $U_{k[\mathbf{x}]^G}(\prec)$ and $U_{k[\mathbf{x},\mathbf{x}^{-1}]^G}(\prec)$ are closed for any $\prec \in \Omega$.

The following is the key lemma.

Lemma 2.5. *Assume that G is not a direct product of symmetric groups. Then every $\omega \in \mathbf{S}'$ is not an interior point of*

$$\iota^{-1}\big(U_{k[\mathbf{x}]^G}(\prec)\big) = \{\omega' \in \mathbf{S}' \mid \mathrm{in}_{\prec}\big(k[\mathbf{x}]^G\big) = \mathrm{in}_{\prec'}\big(k[\mathbf{x}]^G\big) \ \textit{for} \ \ \prec' = \iota(\omega')\}$$

for $\prec = \iota(\omega)$, with respect to the Euclidean topology.

Before we prove this lemma, we will prove Theorems 2.2 and 2.3 by assuming this lemma.

Let $\prec$ be a multiplicative order. Suppose that $\mathrm{in}_{\prec}(k[\mathbf{x}]^G)$ was finitely generated. Then by Lemma 1.6, $U_{k[\mathbf{x}]^G}(\prec)$ is a nonempty open subset of Ω. The inverse image $\iota^{-1}(U_{k[\mathbf{x}]^G}(\prec))$ is a nonempty open subset of $\mathbf{S}'$ by Theorem 1.1. For $\omega' \in \iota^{-1}(U_{k[\mathbf{x}]^G}(\prec))$, we set $\prec' = \iota(\omega')$. Then it follows that $\iota^{-1}(U_{k[\mathbf{x}]^G}(\prec')) = \iota^{-1}(U_{k[\mathbf{x}]^G}(\prec))$, which implies that ω' is an interior point of $\iota^{-1}(U_{k[\mathbf{x}]^G}(\prec'))$. This contradicts Lemma 2.5. Therefore $\mathrm{in}_{\prec}(k[\mathbf{x}]^G)$ is not finitely generated.

The set $U_{k[\mathbf{x}]^G}(\prec)$ can not contain interior points by Lemma 2.5, and also it is closed. Hence it is a nowhere dense subset of Ω. Suppose that there were only countable cardinality of distinct initial algebras for $k[\mathbf{x}]^G$. Then Ω is covered by countable cardinality of $U_{k[\mathbf{x}]^G}(\prec)$'s. Since Ω is a compact metric space, this contradicts the Baire theorem which says that the complement of the union of countable cardinality of nowhere dense subsets of a complete metric space is dense.

By Lemma 2.4, we see that a reduced standard basis of $k[\mathbf{x}]^G$ is a subset of that of $k[\mathbf{x}, \mathbf{x}^{-1}]^G$. Hence we have

$$U_{k[\mathbf{x},\mathbf{x}^{-1}]^G}(\prec) \subset U_{k[\mathbf{x}]^G}(\prec)$$

by Lemma 1.5. Since $U_{k[\mathbf{x}]^G}(\prec)$ is nowhere dense, the subset $U_{k[\mathbf{x},\mathbf{x}^{-1}]^G}(\prec)$ is also nowhere dense and is not open. Hence $\mathrm{in}_{\prec}(k[\mathbf{x}, \mathbf{x}^{-1}]^G)$ is not finitely generated by Lemma 1.6.

The assertion of Theorem 2.3 for the cardinality of distinct initial algebras follows, since the disjoint covering $\{U_{k[\mathbf{x},\mathbf{x}^{-1}]^G}(\prec) \mid \prec \in \Omega\}$ of Ω is a refinement of $\{U_{k[\mathbf{x}]^G}(\prec) \mid \prec \in \Omega\}$.

The rest of this section is devoted to the proof of Lemma 2.5. Our strategy is to translate polynomial informations into the geometry of convex polytopes. Let

$$\overline{\mathcal{M}} := \left\{(a_1, \ldots, a_n) \in \mathbf{R}^n_{\geq 0} \;\middle|\; \sum_{i=1}^{n} a_i = 1\right\} \tag{2.5}$$

and $\mathcal{M} := \overline{\mathcal{M}} \cap \mathbf{Q}^n$. We define the surjection

$$\pi : \{\mathbf{x}^{\mathbf{a}} \mid \mathbf{a} \in \mathbf{Z}^n_{\geq 0} \setminus \{0\}\} \to \mathcal{M} \tag{2.6}$$

by $\mathbf{x}^{\mathbf{a}} \mapsto (\sum_{i=1}^{n} a_i)^{-1}\mathbf{a}$ for $\mathbf{a} = (a_1, \ldots, a_n) \in \mathbf{Z}^n_{\geq 0}$. The action of G on $\overline{\mathcal{M}}$ is by def-

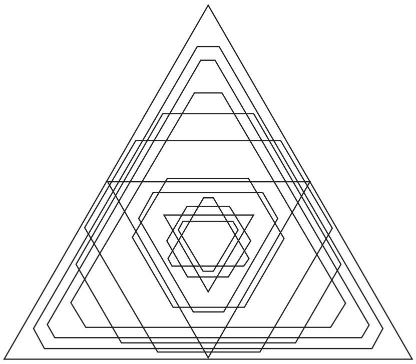

Fig. 1: Some $P_{S_3}(\mathbf{a})$'s in $\overline{\mathcal{M}}$.

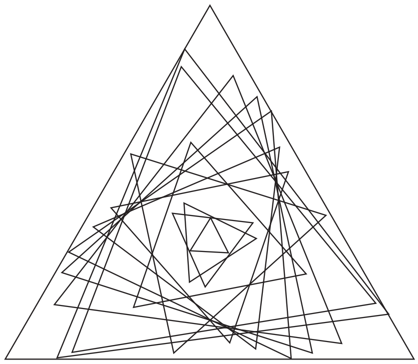

Fig. 2: Some $P_{A_3}(\mathbf{a})$'s in $\overline{\mathcal{M}}$.

inition $\sigma(\mathbf{a}) := (a_{\sigma(1)}, \ldots, a_{\sigma(n)})$ for $\mathbf{a} = (a_1, \ldots, a_n) \in \overline{\mathcal{M}}$ and $\sigma \in G$. For each point $\mathbf{a} \in \overline{\mathcal{M}}$, we denote by $P_G(\mathbf{a})$ the convex hull of the G-orbit $\{\sigma(\mathbf{a}) \mid \sigma \in G\}$. Note that the set of vertices of $P_G(\mathbf{a})$ is $\{\sigma(\mathbf{a}) \mid \sigma \in G\}$, for each point in $\{\sigma(\mathbf{a}) \mid \sigma \in G\}$ lies on the sphere $\{\mathbf{a}' \in \overline{\mathcal{M}} \mid |\mathbf{a}'| = |\mathbf{a}|\}$.

Let $\prec$ be a multiplicative order defined by $\omega \in \mathbf{S}'$. Then, for each element $\mathbf{a} \in \mathbf{Z}^n_{\geq 0}$, we have $\mathrm{face}_\omega(P_G(\pi(\mathbf{a}))) = \{\pi(\mathbf{a})\}$ if and only if $\mathrm{in}_\prec(f_G(\mathbf{x}^{\mathbf{a}})) = \mathbf{x}^{\mathbf{a}}$. By Lemma 2.4, we get the following lemma.

Lemma 2.6. *Assume that $\prec \in \Omega$ is defined by $\omega \in \mathbf{S}'$. Then*

$$\bigcup_{\mathbf{a} \in \mathcal{M}} \pi^{-1}\big(\mathrm{face}_\omega(P_G(\mathbf{a}))\big) \cup \{1\}$$

is a basis of the vector space $\mathrm{in}_\prec(k[\mathbf{x}]^G)$. For ω, $\omega' \in \mathbf{S}'$, set $\prec = \iota(\omega)$ and $\prec' = \iota(\omega')$. If there exists $\mathbf{a} \in \mathcal{M}$ with $\mathrm{face}_\omega(P_G(\mathbf{a})) \neq \mathrm{face}_{\omega'}(P_G(\mathbf{a}))$, then we have $\mathrm{in}_\prec(k[\mathbf{x}]^G) \neq \mathrm{in}_{\prec'}(k[\mathbf{x}]^G)$.

Figs. 1 and 2 show the examples of $P_G(\mathbf{a})$'s for $n = 3$. Fig. 1 is for $G = S_3$ and Fig. 2 is for $G = A_3$.

We will construct a "deformation" of a polytope $P_G(\mathbf{a})$, when G is not a direct product of symmetric groups.

We set $I_\sigma := \{\mathbf{a} \in \overline{\mathcal{M}} \mid \sigma(\mathbf{a}) = \mathbf{a}\}$ for each $\sigma \in G$, and let I be the union of I_σ's for $\sigma \in G \setminus \{1\}$. Then $\overline{\mathcal{M}} \setminus I$ consists of finite number of connected components. For $1 \neq \sigma \in G$ the condition that I_σ has codimension one is equivalent to that σ is a transposition. Since $\overline{\mathcal{M}}$ is a convex set of dimension $n-1$, it is connected even if we remove finite number of linear subspaces of codimension greater than one from it.

Lemma 2.7. *Assume that G is not a direct product of symmetric groups. Then for all $\mathbf{a} \in \overline{\mathcal{M}} \setminus I$, every connected component of $\overline{\mathcal{M}} \setminus I$ contains at least two points* *of* $\{\sigma(\mathbf{a}) \mid \sigma \in G\}$.

Proof. Let $\tau \in G$ be a transposition. Then, the action of τ is the reflection of $\overline{\mathcal{M}}$ with respect to the hyperplane I_τ. For every $1 \neq \sigma \in G$, the subset I_σ of $\overline{\mathcal{M}}$ is the reflection of $I_{\tau\sigma\tau}$ in the hyperplane I_τ. So, the union I of them is symmetric with respect to I_τ. The complement $\overline{\mathcal{M}} \setminus I$ is also symmetric with respect to I_τ.

Now, let $C \neq C'$ be connected components of $\overline{\mathcal{M}} \setminus I$. We will show that $C' = \tau_l \circ \cdots \circ \tau_1(C)$ for some transpositions $\tau_1, \ldots, \tau_l \in G$. Let $\phi : [0,1] \to \overline{\mathcal{M}}$ be a path from a point in C to a point in C'. We assume that ϕ does not intersect $I_\tau \cap I_{\tau'}$ for any transpositions $\tau \neq \tau'$ in S_n, and

$$\{t \in [0,1] \mid \phi(t) \in I_\tau \text{ for some transposition } \tau \in G\}$$

is a finite set, say $\{t_1, \ldots, t_l\}$ with $t_i < t_{i+1}$. We set τ_i the transposition in G with $\phi(t_i) \in I_{\tau_i}$. Then we have $C' = \tau_l \circ \cdots \circ \tau_1(C)$.

We remark that every connected component contains the same cardinality of points of $\{\sigma(\mathbf{a}') \mid \sigma \in G\}$ for each $\mathbf{a}' \in \overline{\mathcal{M}}$. Suppose that there existed a point $\mathbf{a} \in \overline{\mathcal{M}} \setminus I$ and a connected component of $\overline{\mathcal{M}} \setminus I$ which contains only one point of $\{\sigma(\mathbf{a}) \mid \sigma \in G\}$. Then every connected component contains only one point of $\{\sigma(\mathbf{a}) \mid \sigma \in G\}$. Assume that $\mathbf{a}$ is contained in a connected component C of $\overline{\mathcal{M}} \setminus I$. For each $1 \neq \sigma \in G$, we have $\sigma(\mathbf{a}) \neq \mathbf{a}$ because $\mathbf{a}$ is not an element of I. Hence there exists a connected component $C' \neq C$ of $\overline{\mathcal{M}} \setminus I$ such that $\sigma(\mathbf{a}) \in C'$. If $\tau_1, \ldots, \tau_l \in G$ are transpositions such that $C' = \tau_l \circ \cdots \circ \tau_1(C)$, then $\sigma(\mathbf{a}) = \tau_l \circ \cdots \circ \tau_1(\mathbf{a})$ since C' contains exactly one point of $\{\sigma(\mathbf{a}) \mid \sigma \in G\}$. Because $\mathbf{a}$ is not fixed by any element of $G \setminus \{1\}$, we see that $\sigma = \tau_l \circ \cdots \circ \tau_1$. Therefore G can be generated by transpositions in G. This contradicts the assumption. □

Proof of Lemma 2.5. We fix an arbitrary $\omega \in \mathbf{S}'$ and set $\prec = \iota(\omega)$. We will prove that ω is not an interior point of $\iota^{-1}(U_{k[\mathbf{x}]^G}(\prec))$.

Let $\mathbf{a} \in \mathcal{M} \setminus I$ such that $\{\mathbf{a}\} = \mathrm{face}_\omega(\mathbf{a})$. Then, by Lemma 2.7, there exists another point $\sigma(\mathbf{a}) \neq \mathbf{a}$, for some $\sigma \in G$, in the connected component of $\overline{\mathcal{M}} \setminus I$ which contains $\mathbf{a}$. We define a path γ with

$$\gamma\colon [0,1] \to \overline{\mathcal{M}} \setminus I,\ \ \gamma(0) = \mathbf{a},\ \ \gamma(1) = \sigma(\mathbf{a})$$

by combining rational points of $\mathcal{M} \setminus I$ with line segments. Then $\gamma([a,b])$ contains rational points densely for any $0 \leq a < b \leq 1$. Now,

$$T = \bigl\{t \in [0,1] \mid \omega \cdot \gamma(t) = \omega \cdot \sigma'\bigl(\gamma(t)\bigr) \quad \text{for some } \sigma' \in G\bigr\}$$

is not an empty set. Indeed, since

$$\omega \cdot \bigl(\gamma(0) - \sigma^{-1}(\gamma(0))\bigr) = \omega \cdot \bigl(\mathbf{a} - \sigma^{-1}(\mathbf{a})\bigr) > 0$$

and

$$\omega \cdot \left(\gamma(1) - \sigma^{-1}(\gamma(1))\right) = \omega \cdot \left(\sigma(\mathbf{a}) - \mathbf{a}\right) < 0,$$

there exists $t \in (0,1)$ such that $\omega \cdot (\gamma(t) - \sigma^{-1}(\gamma(t))) = 0$ by the intermediate value theorem. We set $t_0 := \inf(T)$, and $\mathbf{b} := \gamma(t_0)$. Then we have

$$\omega \cdot \mathbf{b} = \omega \cdot \sigma_0(\mathbf{b})$$

for some $1 \neq \sigma_0 \in G$, and

$$\omega \cdot \gamma(t) > \omega \cdot \sigma'\left(\gamma(t)\right) \tag{2.7}$$

for all $t \in [0, t_0)$ and $1 \neq \sigma' \in G$. Note that $\mathbf{b} \neq \sigma_0(\mathbf{b})$, since the path γ does not intersect I. For each $\delta \in \mathbf{R}_{>0}$, we set

$$\omega_\delta := \omega - \delta\left(\mathbf{b} - \sigma_0(\mathbf{b})\right).$$

Let $\{t_i\}_i \subset [0, t_0)$ be a sequence such that $\lim_{i\to\infty} t_i = t_0$ and $\mathbf{a}_i := \gamma(t_i) \in \mathcal{M}$. Then, for each $\varepsilon' > 0$, there exists a positive integer $N_{\varepsilon'}$ such that

$$\left|\left(\mathbf{b} - \sigma_0(\mathbf{b})\right) \cdot \left((\mathbf{b} - \sigma_0(\mathbf{b})) - (\mathbf{a}_i - \sigma_0(\mathbf{a}_i))\right)\right| < \varepsilon'$$

and

$$0 < \omega \cdot \left(\mathbf{a}_i - \sigma_0(\mathbf{a}_i)\right) < \varepsilon'$$

for every integer $i > N_{\varepsilon'}$.

Now, let ε be any positive number. Then there exists $\delta > 0$ such that

$$\left|\omega - \frac{\omega_\delta}{|\omega_\delta|}\right| < \varepsilon$$

and $|\omega_\delta|^{-1}\omega_\delta \in \mathbf{S}'$. We set $\varepsilon' = (1+\delta)^{-1}\delta|\mathbf{b} - \sigma_0(\mathbf{b})|^2$. Then, for any integer $i > N_{\varepsilon'}$, we have

$$\begin{aligned}
\omega_\delta \cdot \left(\sigma_0(\mathbf{a}_i) - \mathbf{a}_i\right) &= \left(\omega - \delta(\mathbf{b} - \sigma_0(\mathbf{b}))\right) \cdot \left(\sigma_0(\mathbf{a}_i) - \mathbf{a}_i\right) \\
&= \omega \cdot \left(\sigma_0(\mathbf{a}_i) - \mathbf{a}_i\right) \\
&\quad -\delta\left(\mathbf{b} - \sigma_0(\mathbf{b})\right) \cdot \left\{\left((\mathbf{b} - \sigma_0(\mathbf{b})) - (\mathbf{a}_i - \sigma_0(\mathbf{a}_i))\right) - \left(\mathbf{b} - \sigma_0(\mathbf{b})\right)\right\} \\
&> -\varepsilon' - \delta\varepsilon' + \delta|\mathbf{b} - \sigma_0(\mathbf{b})|^2 = 0.
\end{aligned}$$

Hence

$$\mathrm{face}_{\omega_\delta}\left(P_G(\mathbf{a}_i)\right) \neq \{\mathbf{a}_i\}$$

for $i > N_{\varepsilon'}$. On the other hand, $\{\mathbf{a}_i\} = \mathrm{face}_\omega(P_G(\mathbf{a}_i))$ for all i by (2.7). So, we have $\mathrm{in}_{\prec_\delta}(k[\mathbf{x}]^G) \neq \mathrm{in}_{\prec}(k[\mathbf{x}]^G)$ for $\prec_\delta = \iota(|\omega_\delta|^{-1}\omega_\delta)$ by Lemma 2.6. Thus, $|\omega_\delta|^{-1}\omega_\delta \notin \iota^{-1}(U_{k[\mathbf{x}]^G}(\prec))$. Therefore ω is not an interior point of $\iota^{-1}(U_{k[\mathbf{x}]^G}(\prec))$. □

3. Finite SAGBI bases

Now we will observe the case where G is a direct product of symmetric groups.

Lemma 3.1. *Let A be $k[\mathbf{x}]^{S_n}$ or $k[\mathbf{x},\mathbf{x}^{-1}]^{S_n}$. We consider the initial algebras $\mathrm{in}_\prec(A)$ for all multiplicative orders $\prec$. Then the cardinality of distinct initial algebras for A is $n!$.*

Proof. It suffices to show that, if $\prec$ and $\prec'$ are multiplicative orders with $x_n \prec \cdots \prec x_1$ and $x_n \prec' \cdots \prec' x_1$, then $\mathrm{in}_\prec(A) = \mathrm{in}_{\prec'}(A)$. By Lemma 2.4, we see that a reduced standard basis of A is equal to

$$\begin{cases} \{f_{S_n}(x_1^{a_1}\cdots x_n^{a_n}) \mid 0 \le a_n \le \cdots \le a_1\} & \text{if } A = k[\mathbf{x}]^{S_n} \\ \{f_{S_n}(x_1^{a_1}\cdots x_n^{a_n}) \mid a_n \le \cdots \le a_1\} & \text{if } A = k[\mathbf{x},\mathbf{x}^{-1}]^{S_n}. \end{cases}$$

For every $\mathbf{a} = (a_1,\dots,a_n) \in \mathbf{Z}^n$ with $a_n \le \cdots \le a_1$, it follows that $\mathrm{in}_\prec(f_{S_n}(\mathbf{x}^\mathbf{a})) = \mathrm{in}_{\prec'}(f_{S_n}(\mathbf{x}^\mathbf{a})) = \mathbf{x}^\mathbf{a}$. This implies that $\mathrm{in}_\prec(A) = \mathrm{in}_{\prec'}(A)$. □

By the proof of Lemma 3.1, the initial algebras $\mathrm{in}_\prec(k[\mathbf{x}]^{S_n})$ and $\mathrm{in}_\prec(k[\mathbf{x},\mathbf{x}^{-1}]^{S_n})$ are spanned by the sets of monomials

$$\{x_1^{a_1}\cdots x_n^{a_n} \mid 0 \le a_n \le \cdots \le a_1\} \text{ and } \{x_1^{a_1}\cdots x_n^{a_n} \mid a_n \le \cdots \le a_1\},$$

respectively, if the multiplicative order $\prec$ satisfies $x_n \prec \cdots \prec x_1$. In these case, they are generated as algebras by

$$\{x_1, x_1x_2, \dots, x_1x_2\cdots x_n\} \text{ and } \{x_1, x_1x_2, \dots, x_1x_2\cdots x_n, x_1^{-1}x_2^{-1}\cdots x_n^{-1}\},$$

respectively. Therefore, the initial algebras $\mathrm{in}_\prec(k[\mathbf{x}]^{S_n})$ and $\mathrm{in}_\prec(k[\mathbf{x},\mathbf{x}^{-1}]^{S_n})$ are finitely generated for any multiplicative order $\prec$ (cf. Robbiano, Sweedler [6, Theorem 1.14]).

Lemma 3.2 (cf. [2, Lemma 3.8]). *Let G_1 and G_2 be subgroups of S_n which acts on $\mathbf{x}_1 := (x_1,\dots,x_l)$ and $\mathbf{x}_2 := (x_{l+1},\dots,x_n)$, respectively. We set $G = G_1 \times G_2$ the direct product of G_1 and G_2. If A is $k[\mathbf{x}]^G$ or $k[\mathbf{x},\mathbf{x}^{-1}]^G$, and A_i is $k[\mathbf{x}_i]^{G_i}$ or $k[\mathbf{x}_i,\mathbf{x}_i^{-1}]^{G_i}$ for $i = 1, 2$, respectively, then we have*

$$\mathrm{in}_\prec(A) = \mathrm{in}_\prec(A_1) \otimes_k \mathrm{in}_\prec(A_2).$$

Proof. By Lemma 2.4, the assertion follows from the equality

$$\begin{aligned} f_G(\mathbf{x}_1^{\mathbf{a}_1}\cdot\mathbf{x}_2^{\mathbf{a}_2}) &= \sum_{(\sigma_1,\sigma_2)\in G_1/G_1(\mathbf{x}_1^{\mathbf{a}_1})\times G_2/G_2(\mathbf{x}_2^{\mathbf{a}_2})} \sigma_1(\mathbf{x}_1^{\mathbf{a}_1})\cdot\sigma_2(\mathbf{x}_2^{\mathbf{a}_2}) \\ &= \left(\sum_{\sigma_1\in G_1/G_1(\mathbf{x}_1^{\mathbf{a}_1})}\sigma_1(\mathbf{x}_1^{\mathbf{a}_1})\right)\cdot\left(\sum_{\sigma_2\in G_2/G_2(\mathbf{x}_2^{\mathbf{a}_2})}\sigma_2(\mathbf{x}_2^{\mathbf{a}_2})\right) \\ &= f_{G_1}(\mathbf{x}_1^{\mathbf{a}_1})\cdot f_{G_2}(\mathbf{x}_2^{\mathbf{a}_2}) \end{aligned}$$

for every monomial $\mathbf{x}_1^{\mathbf{a}_1}\in k[\mathbf{x}_1,\mathbf{x}_1^{-1}]$ and $\mathbf{x}_2^{\mathbf{a}_2}\in k[\mathbf{x}_2,\mathbf{x}_2^{-1}]$. $\square$

Proposition 3.3. *Let A be $k[\mathbf{x}]^G$ or $k[\mathbf{x},\mathbf{x}^{-1}]^G$. Assume that G is a direct product of symmetric groups. Then the initial algebra* $\mathrm{in}_\prec(A)$ *is finitely generated for any multiplicative order $\prec$. The cardinality of distinct initial algebras for A is $|G|$.*

Proof. Assume that $n=n_1+\cdots+n_r$ and $G=S_{n_1}\times\cdots\times S_{n_r}$, and that S_{n_i} acts on the set of variables $\mathbf{x}_i=(x_{i,1},\dots,x_{i,n_i})$ for each i. Let A_i be $k[\mathbf{x}_i]^{S_{n_i}}$ if $A=k[\mathbf{x}]^G$, and $k[\mathbf{x}_i,\mathbf{x}_i^{-1}]^{S_{n_i}}$ if $A=k[\mathbf{x},\mathbf{x}^{-1}]^G$. Then there exist $n_i!$ distinct initial algebras for each A_i by Lemma 3.1. Since we can define the order in $\mathbf{x}_i$ independently for each i, there exist $n_1!\cdots n_r!$ distinct initial algebras for A. Clearly, this number is equal to the order of the group G.

Since each A_i is finitely generated for any $\prec\in\Omega$, the tensor product of them is also finitely generated. Hence the initial algebra $\mathrm{in}_\prec(A)$ is finitely generated for any $\prec\in\Omega$. $\square$

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