Tokunaga, H.
Osaka J. Math.
39 (2002), 621–645

GALOIS COVERS FOR $\mathfrak{S}_4$ AND $\mathfrak{A}_4$ AND THEIR APPLICATIONS

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(Received September 4, 2000)

Introduction

The main purpose of this article is to explain new methods in studying Galois covers of algebraic varieties for $\mathfrak{S}_4$, the symmetric group of degree 4 and $\mathfrak{A}_4$, the alternating group of degree 4, and to apply them to studying the topology of the complements to plane curves.

Branched Galois covers has been playing important roles in the study of algebraic varieties. Double covers have been intensively used to construct algebraic surfaces having a prescribed Chern invariants (for example, [18]), and cyclic covers have been used to investigate the topology of the complements to plane curves (for example, [11], [29]). In most cases, however, they are abelian covers, i.e., Galois covers with abelian Galois groups. This is because the systematic methods to study abelian covers have been established and it is, in fact, quite user-friendly. On the other hand, there seem to be few systematic methods for non-abelian covers which are as useful as those for abelian covers; and there do not seem to be many results by using non-abelian Galois covers. Therefore it seems worthwhile to make a study of non-abelian Galois covers even for elementary non-abelian groups.

The author has studied Galois covers having dihedral groups as their Galois groups in [21], and applied such covers to the study of the complements to plane algebraic curves ([22], [23], [24]). As it is well-known, dihedral groups are a class of so-called *regular polyhedral groups*. Thus, as a next step, it is natural to consider Galois covers having such groups as their Galois groups. In [27] Tsuchihashi has made a study of singularities which appear in Galois covers with Galois groups, D_{2n}, $\mathfrak{A}_4$ and $\mathfrak{S}_4$. In this article, being inspired Tsuchihashi's work, we consider Galois covers of algebraic varieties with $\mathfrak{A}_4$ and $\mathfrak{S}_4$ as their Galois groups.

One could say that the difference between Tsuchihashi's results and ours is the one between *local* and *global*. In [27], Tsuchihashi's condition for constructing Galois covering singularities are given by the germs of holomorphic functions and the group action over them. In this paper, in order to describe our conditions for constructing $\mathfrak{S}_4$ covers, we use rather global language: divisors and their linear equivalences.

Both Tsuchihashi's approach and ours are based on Galois theory for $\mathfrak{A}_4$ and $\mathfrak{S}_4$.

Research partly supported by the research grant 11640034 from JSPS.

More explicitly, it is based on Lagrange's method in solving quartic equations ([10]). We try to understand Lagrange's method by geometric language. This is our goal for the first half of this article (Part I).

In Part II, we apply the results for $\mathfrak{S}_4$ covers in Part I to studying the topology of the complements to plane sextic curves. In order to make our problem clear and to see the role of $\mathfrak{S}_4$ covers, let us review our fundamental question and previous known results about it.

The fundamental question throughout Part II is as follows:

Question 0.1. Let B be a reduced plane curve in $\mathbf{P}^2$. What can one say about $\mathbf{P}^2 \setminus B$ just from the data of local topological type of singularities? For example, can one determine whether the fundamental group $\pi_1(\mathbf{P}^2 \setminus B)$ is abelian or non-abelian just from such data?

In what follows, we simply say the configuration of singularities instead of the data of local topological types of singularities.

From the viewpoint of Question 0.1, there do not seem to be many results on the non-commutativity on $\pi_1(\mathbf{P}^2 \setminus B)$, while there are several results on the commutativity (see [3], [8], [12], [19]).

In [25], the author gave a statement on the non-commutativity. We need some notations to explain it.

Let B be as before and assume that B has at most simple singularities. We use the lower cases, a_n, d_n and e_n to describe the types of them. For $x \in \mathrm{Sing}(B)$, we denote its Milnor number by μ_x. We define the total Milnor number, μ_B, to be $\sum_{x\in \mathrm{Sing}(B)} \mu_x$. We next define a non-negative integer, l_p, for an odd prime p as follows:

if $p = 3$, l_3 = the number of singularities of types a_{3k-1} ($k \geq 1$) and e_6, and
if $p \geq 5$, l_p = the number of singularities of type a_{pk-1}.

Using these notations, we have

Theorem 0.2 ([25]). *Suppose that* $\deg B$ *is even. If there exists an odd prime* p *such that*

$$l_p + \mu_B > d^2 - 3d + 3,$$

then there exists a surjective homomorphism

$$\pi_1(\mathbf{P}^2 \setminus B) \to \mathcal{D}_{2p} = \langle \sigma, \tau \mid \sigma^2 = \tau^p = (\sigma\tau)^2 = 1 \rangle.$$

In particular, $\pi_1(\mathbf{P}^2 \setminus B)$ *is non-abelian.*

Corollary 0.3. *The notations are the same as in* Theorem 0.2. *Suppose that* B *has only nodes and cusps and let* a *and* b *be the number of nodes and cusps, respec-*

tively. If $a + 3b > d^2 - 3d + 3$, *then* $\pi_1(\mathbf{P}^2 \setminus B)$ *is non-abelian.*

The proof of Theorem 0.2 is based on an existence theorem on $\mathcal{D}_{2p}$ covers branched along B. Hence the inequality in Theorem 0.2 seems to give a very rough estimate. For sextic curves and $p = 3$, however, the inequality is sharp from the following result:

Theorem 0.4 ([5], [16], [23], [24]). *There exists a pair of irreducible sextic curves* (B_1, B_2) *as follows*:
(i) *Both* B_1 *and* B_2 *have the same configuration of singularities*; *and it is one of the following*:

$$3a_5 + 3a_1, \quad 6a_2 + 3a_1, \quad 3e_6, \quad e_6 + 4a_2 + 2a_1.$$

(ii) *There exists a surjective homomorphism* $\pi_1(\mathbf{P}^2 \setminus B_1) \to \mathfrak{S}_3$ *for* B_1, *while there is no such homomorphism for* B_2.

Also the inequality in Theorem 0.2 is sharp for $p = 5$ ([1]). On the other hand, it is known that there exist sextic curves, B_3, having the configurations of singularities: $3a_5 + 4a_1$, $6a_2 + 4a_1$, $e_6 + 4a_2 + 3a_1$, $3e_6 + a_1$. For B_3, the inequality in Theorem 0.2 is satisfied for $p = 3$. Hence there exists a surjective morphism $\pi_1(\mathbf{P}^2 \setminus B_3) \to \mathfrak{S}_3$. In particular, $\pi_1(\mathbf{P}^2 \setminus B_3)$ is non-abelian.

These examples seem to be rather interesting, since the difference of the configurations of singularities between B_1 in Theorem 0.2 and B_3 is just the number of nodes. From observation from the commutativity statements as in [3], [8], [12], [19], the number of nodes does not seem to give much effect on the non-commutativity on the fundamental group of the residual space. In fact, Oka posed the following conjecture in [15]:

Conjecture 0.5 ([15], p. 402). The fundamental group of the complement to a curve does not change by a degeneration which puts only nodes.

Moreover, by [5], the Alexander polynomials for B_1 in Theorem 0.4 and those for B_3 are $t^2 - t + 1$. This shows that one can not measure the difference of the topology between $\mathbf{P}^2 \setminus B_1$ and $\mathbf{P}^2 \setminus B_3$ by the Alexander polynomials, while they are likely to be different.

Now $\mathfrak{S}_4$-covers come in to our picture. We need them to see that the topology of $\mathbf{P}^2 \setminus B_1$ is different from $\mathbf{P}^2 \setminus B_3$; and it is the goal of Part II.

Let B be a reduced sextic curve with at most simple singularities, and let $f: Z' \to \mathbf{P}^2$ be the double cover branched along B and let $\mu: Z \to Z'$ be the canonical resolution (see [9] for the canonical resolution). By the assumption μ is a minimal resolution, and Z is a $K3$ surface. Let NS(Z) be the Néron-Severi group of Z and let R be the subgroup of NS(Z) generated by all the irreducible components of the exceptional divisor of μ. Both NS(Z) and R are lattices with respect to the intersection product. Note that R has a natural orthogonal decomposition $R = \bigoplus_{x \in \mathrm{Sing}(Z')} R_x$, where R_x is the subgroup of NS(Z) generated by the exceptional divisor arising from x. As we assume that B has only simple singularities, R_x is isomorphic to one of the so-called A-D-E lattices. The graph of R_x means the dual graph of the exceptional set for x. We denote it by $G(R_x)$ and the graph, $G(R)$, of R is $\sum_{x \in \mathrm{Sing}(Z')} G(R_x)$. Note that the involution induced by the double cover $f \circ \mu$ canonically acts $G(R)$. By our assumption, $G(R_x)$ is one of the Dynkin graphs, which we denote by the bold characters $\mathbf{A}_n$, $\mathbf{D}_n$ and $\mathbf{E}_n$. (Note that these types correspond to those of lattices.) Let G_1 be a subgraph of $G(R)$. We denote the subgroup (or lattice) of NS(Z) generated by the vertices of G_1 by $\mathcal{L}(G_1)$.

Now we are in position to state our main result (for the terminology of Galois covers, see §1 and §3):

Theorem 0.6. *Let B be a reduced sextic curve with at most simple singularities, and let $f\colon Z' \to \mathbf{P}^2$ be the double cover branched along B. If there exists an $\mathfrak{S}_4$-cover $\pi\colon S \to \mathbf{P}^2$ of $\mathbf{P}^2$ such that* (i) *π is branched at $2B$ and* (ii) *π factors $f\colon Z' \to \mathbf{P}^2$. Then $G(R)$ contains a subgraph either $\mathbf{A}_2^{\oplus 9}$ or $\mathbf{A}_2^{\oplus 6} \oplus \mathbf{A}_1^{\oplus 4}$.*

Theorem 0.7. *Suppose that $G(R)$ contains $\mathbf{A}_2^{\oplus 6} \oplus \mathbf{A}_1^{\oplus 4}$ such that $\mathbf{A}_1^{\oplus 4}$ is a invariant block under the involution induced by the covering transformation. Then there exists an $\mathfrak{S}_4$-cover of $\mathbf{P}^2$ such that* (i) *π is branched at $2B$ and* (ii) *π factors $f\colon Z' \to \mathbf{P}^2$.*

By Theorem 0.6, we can infer that there is no $\mathfrak{S}_4$-cover for B_1 in Theorem 0.4, while there exists an $\mathfrak{S}_4$-cover for B_3 as above.

REMARK 0.8. By Theorems 0.6 and 0.7, we know that Conjecture 0.5 is false in general. In fact, there is a family of sextic curves $\{C_t\}_{t \in \Delta}$, $\Delta = \{t \in \mathbf{C} \mid |t| < 1\}$ such that C_0 is a sextic curve having $3e_6 + a_1$ as its singularities, while C_t ($t \neq 0$) is a sextic curve having $3e_6$ as its singularities. By Theorems 0.6 and 0.7, we know that there exists an $\mathfrak{S}_4$-cover branched at $2C_0$, while there exist no such covers for C_t ($t \neq 0$). This implies $\pi_1(\mathbf{P}^2 \setminus C_0) \not\cong \pi_1(\mathbf{P}^2 \setminus C_t)$ ($t \neq 0$).

Recently, Oka and Pho have figured out $\pi_1(\mathbf{P}^2 \setminus C_t)$ explicitly ([17]):

$$\pi_1(\mathbf{P}^2 \setminus C_t) \cong \mathbf{Z}/2\mathbf{Z} * \mathbf{Z}/3\mathbf{Z} \quad (t \neq 0),$$

while

$$\pi_1(\mathbf{P}^2 \setminus C_0) \cong B_4(\mathbf{P}^1),$$

where $B_4(\mathbf{P}^1)$ is the braid group of 4 strings for $\mathbf{P}^1$.

Acknowledgement. Part of this work was done during the author's visit to Professor Alan Huckleberry under the support from SFB 237. The author thanks Professor Huckleberry for his hospitality. He also thanks the referee for his/her comments on the first version of this article.

General theory for $\mathfrak{S}_4$- and $\mathfrak{A}_4$-covers

1. Galois covers of algebraic varieties

In this section, we introduce some notations and terminologies, which we use throughout this article to describe Galois covers. Let Y be a normal projective variety. Let X be a normal variety with a finite surjective morphism $\pi\colon X \to Y$. The field of rational functions, $\mathbf{C}(X)$, of X is a finite extension of that of Y, $\mathbf{C}(Y)$, with $[\mathbf{C}(X) : \mathbf{C}(Y)] = \deg\pi$. We call X a Galois cover of Y if $\mathbf{C}(X)$ is a Galois extension of $\mathbf{C}(Y)$. Let G be a finite group. If $\mathbf{C}(X)$ is a Galois extension with $\mathrm{Gal}(\mathbf{C}(X)/\mathbf{C}(Y)) \cong G$, we simply call X a G-cover. Let H be a subgroup of G, and let $\mathbf{C}(X)^H$ be the H-invariant subfield of $\mathbf{C}(X)$. We denote the $\mathbf{C}(X)^H$-normalization of Y by $D(X/Y, H)$. Note that there are canonical morphisms:

$$\beta_1(\pi, H)\colon D(X/Y, H) \to Y, \quad \beta_2(\pi, H)\colon X \to D(X/Y, H)$$

such that (i) $\beta_2(\pi, H)$ is a H-cover, and (ii) $\pi = \beta_1(\pi, H) \circ \beta_2(\pi, H)$. We call $D(X/Y, H)$ the *intermediate cover with respect to* H. Note that if H is a normal subgroup of G, then $\beta_1(\pi, H)\colon D(X/Y, H) \to Y$ is a G/H-cover.

We define the branch locus of π to be the subset given by

$$\{y \in Y \mid \pi \text{ is not locally isomorphic over } y\}$$

We denote it by $\Delta(X/Y)$ or Δ_π. In what follows, we assume that Y is smooth. By the purity of the branch locus ([29]), Δ_π is an algebraic subset of codimension 1. Let $\Delta_\pi = B_1 + \cdots + B_r$ be the decomposition into its irreducible components. The ramification index of π along B_i is the one along the smooth part of B_i. If we say that a G-cover $\pi\colon X \to Y$ is branched at $e_1B_1 + \cdots + e_rB_r$ (resp. at most $e_1B_1 + \cdots + e_rB_r$), it means that (i) $\Delta_\pi = B_1 + \cdots + B_r$ (resp. $\Delta_\pi \subset B_1 + \cdots + B_r$), and (ii) the ramification index along B_i is e_i (resp. $\le e_i$).

Now we formulate our basic problem on Galois covers.

Problem 1.1. Let G be a finite group and let H be a normal subgroup of G. Put $G_1 = G/H$.
(i) *Function field version* Let $f\colon Z \to Y$ be a G_1 cover of Y. Find a condition for the existence of an H extension, K, of $\mathbf{C}(Z)$ such that (a) K is a Galois extension of $\mathbf{C}(Y)$ with Galois group G, and (b) $K^H = \mathbf{C}(Z)$. Note that the K-normalization of Y gives a G cover of Y.

(ii) *Geometric version* Let $f: Z \to Y$ be a smooth G_1-cover of Y, and let D be an effective divisor on Z. Find a condition on D for the existence of an H-cover, $g: X \to Z$, satisfying (a) $f \circ g$ gives a G-cover of Y such that $D(X/Y, H) = Z$ and (b) $\Delta(X/Z) \subset \mathrm{Supp}(D)$.
(ii)$'$ Let $f: Z \to Y$ be a smooth G_1-cover of Y, and let D be an effective divisor on Z. We denote its irreducible decomposition by $D = e_1 D_1 + \cdots + e_r D_r$. Find a condition on D for the existence of an H-cover, $g: X \to Z$, satisfying (a) $f \circ g$ gives a G-cover of Y such that $D(X/Y, H) = Z$ and (b) g is branched at at most D.

In Problem 1.1, we divide a construction problem for G-covers into two parts: G/H-covers and H-covers. By this approach, we could reduce our difficulty of G-covers to that of rather elementary ones. This method, however, does not work at all for simple groups. Thus we need a new strategy to attack $\mathfrak{A}_5$-covers. Now we go on to two specific cases: $\mathfrak{S}_4$ and $\mathfrak{A}_4$. To this purpose, we first review Lagrange's method in solving quartic equations.

2. Lagrange's method

Let us recall Lagrange's idea to solve a quartic equation ([10], [20]). Let k be a field of $ch(k) = 0$ containing the fourth and third primitive root of unity. Consider an algebraic equation of degree 4 over k:

$$x^4 + a_1 x^2 + a_2 x + a_3 = 0.$$

We denote its four roots by α_i $i = 1, 2, 3, 4$. Suppose that $\mathrm{Gal}(k(\alpha_1, \alpha_2, \alpha_3, \alpha_4)/k) \cong \mathfrak{S}_4$. We fix an action of S_4 on the set $\{\alpha_1, \alpha_2, \alpha_3, \alpha_4\}$ in a canonical way. Namely, $\sigma: \alpha_i \mapsto \alpha_{\sigma(i)}$ Let V_4 be the subgroup of $\mathfrak{S}_4$ given by

$$V_4 = \{id, (12)(34), (13)(24), (14)(23)\},$$

i.e., the Klein group. Put

$$\begin{aligned} \gamma_1 &= (\alpha_1 + \alpha_2) - (\alpha_3 + \alpha_4) \\ \gamma_2 &= (\alpha_1 + \alpha_3) - (\alpha_2 + \alpha_4) \\ \gamma_3 &= (\alpha_1 + \alpha_4) - (\alpha_2 + \alpha_3) \end{aligned}$$

Then one can easily check

Lemma 2.1. $\gamma_1^2, \gamma_2^2, \gamma_3^2$ *are* V_4*-invariant.*

Put $\varphi_i = \gamma_i^2$, and let $A = \varphi_1 + \varphi_2 + \varphi_3$, $B = \varphi_1\varphi_2 + \varphi_2\varphi_3 + \varphi_3\varphi_1$ and $C = \gamma_1\gamma_2\gamma_3$. Then one can also check:

Lemma 2.2. *A, B and C are $\mathfrak{S}_4$-invariant. In particular, $A, B, C \in k$.*

Consider the cubic equation given by

$$x^3 - Ax^2 + Bx - C^2 = 0.$$

The left hand side is in $k[x]$, and φ_1, φ_2 and φ_3 are the solutions of the above cubic equation. Then we have

Lemma 2.3. *$k(\varphi_1, \varphi_2, \varphi_3)/k$ is a Galois extension with Galois group $\mathfrak{S}_3$, and $k(\varphi_1, \varphi_2, \varphi_3) = k(\alpha_1, \alpha_2, \alpha_3, \alpha_4)^{V_4}$.*

Proof. By Lemma 2.1, $k(\varphi_1, \varphi_2, \varphi_3) \subset k(\alpha_1, \alpha_2, \alpha_3, \alpha_4)^{V_4}$. Consider a bi-quadratic extension: $L = K(\varphi_1, \varphi_2, \varphi_3)(\gamma_1, \gamma_2)$. Since $\gamma_3 = C/\gamma_1\gamma_2$, $\gamma_3 \in L$. Hence, with $\sum_i \alpha_i = -a_1 \in K$, one can check all $\alpha_i \in L$. This implies that $K(\alpha_1, \alpha_2, \alpha_3, \alpha_4) = L$ and $[k(\alpha_1, \alpha_2, \alpha_3, \alpha_4) : k(\varphi_1, \varphi_2, \varphi_3)] = 4$. Thus we have $k(\varphi_1, \varphi_2, \varphi_3) = k(\alpha_1, \alpha_2, \alpha_3, \alpha_4)^{V_4}$. Since V_4 is a normal subgroup of $\mathfrak{S}_4$ such that $\mathfrak{S}_4/V_4 \cong \mathfrak{S}_3$, we have the assertion on the Galois group by the fundamental theorem of the Galois theory. □

Thus one can obtain an $\mathfrak{S}_3$ extension from the given $\mathfrak{S}_4$ extension canonically.

We next consider the converse of this. Namely we show how we obtain an $\mathfrak{S}_4$ extension from a given $\mathfrak{S}_3$ extension.

Lemma 2.4. *Let F be an $\mathfrak{S}_3$ extension of k. Here $\mathfrak{S}_3 = \langle \sigma, \tau \mid \sigma^2 = \tau^3 = (\sigma\tau)^2 = 1\rangle$. Suppose that there exist three elements φ_1, φ_2 and φ_3 of F such that*
(i) $\varphi_i \notin (F^\times)^2$,
(ii) $\varphi_1^\sigma = \varphi_2$, $\varphi_3^\sigma = \varphi_3$; *and* $\varphi_1^\tau = \varphi_2$, $\varphi_2^\tau = \varphi_3$, $\varphi_3^\tau = \varphi_1$, *and*
(iii) $\varphi_1\varphi_2\varphi_3 = a^2$ *for some* $a \in k$.
Then the bi-quadratic extension $F(\sqrt{\varphi_1}, \sqrt{\varphi_2})$ is an $\mathfrak{S}_4$ extension of K.

Proof. By $\sqrt{\varphi_3} = a/\sqrt{\varphi_1}\sqrt{\varphi_2}$, we know that $F(\sqrt{\varphi_1}, \sqrt{\varphi_2})$ is a Galois extension of k. To see $\mathrm{Gal}\big(F(\sqrt{\varphi_1}, \sqrt{\varphi_2})/k\big)$, define α_1, α_2, α_3, α_4 as follows:

$$\begin{pmatrix} \alpha_1 \\ \alpha_2 \\ \alpha_3 \end{pmatrix} = \frac{1}{4} \begin{pmatrix} 1 & 1 & 1 \\ 1 & -1 & -1 \\ -1 & 1 & -1 \end{pmatrix} \begin{pmatrix} \sqrt{\varphi_1} \\ \sqrt{\varphi_2} \\ \sqrt{\varphi_3} \end{pmatrix},$$

and

$$\alpha_4 = -(\alpha_1 + \alpha_2 + \alpha_3).$$

Then $F(\sqrt{\varphi_1}, \sqrt{\varphi_2}) = K(\alpha_1, \ldots, \alpha_4)$; and by checking the action of the induced automorphisms explicitly, we have $\mathrm{Gal}\big(F(\sqrt{\varphi_1}, \sqrt{\varphi_2})/K\big) \cong \mathfrak{S}_4$. □

REMARK 2.5. As for the $\mathfrak{A}_4$-case, we replace $\mathfrak{S}_3$ by $\mathbf{Z}/3\mathbf{Z}$, and repeat the same argument. We omit its detail.

3. $\mathfrak{S}_4$-covers of algebraic varieties

We keep the same notation as before, and attack $\mathfrak{S}_4$-covers under the settings in §1. We choose V_4 as H in §1. Then from what we have seen in the previous section, we have the following proposition immediately:

Proposition 3.1. *Let $f: Z \to Y$ be an $\mathfrak{S}_3$-cover of Y. Suppose that there exist three distinct rational functions, φ_1, φ_2 and φ_3 with the following properties*:

(i) $\varphi_i \notin \left(\mathbf{C}(Z)^\times\right)^2$ *for each i.*

(ii) *If we denote* $\mathrm{Gal}(\mathbf{C}(Z)/\mathbf{C}(Y)) = \langle \sigma, \tau \mid \sigma^2 = \tau^3 = (\sigma\tau)^2 = 1\rangle$, *then*

(ii-a) $\varphi_1^\sigma = \varphi_2$, $\varphi_3^\sigma = \varphi_3$, *and*

(ii-b) $\varphi_1^\tau = \varphi_2$, $\varphi_2^\tau = \varphi_3$, $\varphi_3^\tau = \varphi_1$.

(iii) $\varphi_1\varphi_2\varphi_3 \in \left(f^*\mathbf{C}(Y)^\times\right)^2$.

Then the bi-quadratic extension $K = \mathbf{C}(Z)(\sqrt{\varphi_1}, \sqrt{\varphi_2})$ is an $\mathfrak{S}_4$ extension of $\mathbf{C}(Y)$ such that $K^{V_4} = \mathbf{C}(Z)$. In particular, the K-normalization, X, of Y is an $\mathfrak{S}_4$-cover of Y with $D(X/Y, V_4) = Z$.

Conversely, if there exists an $\mathfrak{S}_4$-cover $\pi: X \to Y$ with $D(X/Y, V_4) = Z$, there exist three rational functions φ_1, φ_2, and φ_3 in $\mathbf{C}(Z)$ satisfying the three properties (i), (ii) *and* (iii) *as above.*

Proposition 3.1 gives an answer to Problem 1.1 (i) in the case of $G = \mathfrak{S}_4$, $H = V_4$. We now go on to the second question.

Proposition 3.2. *Let $f: Z \to Y$ be a smooth $\mathfrak{S}_3$-cover of Y. Suppose that there exist three different reduced divisors, D_1, D_2 and D_3 on Z as follows*:

(i) *With the same notation on* $\mathrm{Gal}(Z/Y)$ *as those in* Proposition 3.1,

(i-a) $D_1^\sigma = D_2$ *and* $D_3^\sigma = D_3$, *and*

(i-b) $D_1^\tau = D_2$, $D_2^\tau = D_3$, $D_3^\tau = D_1$.

(i-c) *there is no common component among D_1, D_2, and D_3.*

(ii) *There exists a line bundle, $\mathbf{L}$, such that $D_1 \sim 2\mathbf{L}$.*

Then there exists an $\mathfrak{S}_4$-cover $\pi: X \to Y$ satisfying (i) $D(X/Y, V_4) = Z$ *and* (ii) $\Delta(X/Z) = \mathrm{Supp}(D_1 + D_2 + D_3)$.

Proof. Choose effective divisors D_0 and D_∞ so that $\mathbf{L} \sim D_\infty - D_0$. Then we have $D_1 + 2D_0 \sim 2D_\infty$. Hence there exists a rational function, ψ, on Z such that

$$(\psi) = (D_1 + 2D_0) - 2D_\infty.$$

Define three rational functions, φ_1, φ_2 and φ_3 as follows:

$$\varphi_1 = \psi\psi^{\sigma}\psi^{\tau^2}\psi^{\sigma\tau^2}, \quad \varphi_2 = \psi\psi^{\sigma}\psi^{\tau}\psi^{\sigma\tau}, \quad \varphi_3 = \psi^{\tau}\psi^{\tau^2}\psi^{\sigma\tau}\psi^{\sigma\tau^2}.$$

Then one can easily check the following:
(i) $\varphi_1^{\sigma} = \varphi_2$, $\varphi_3^{\sigma} = \varphi_3$, $\varphi_1^{\tau} = \varphi_2$, $\varphi_2^{\tau} = \varphi_3$, $\varphi_3^{\tau} = \varphi_1$.
(ii) $\varphi_1\varphi_2\varphi_3 = (\psi\psi^{\sigma}\psi^{\tau}\psi^{\sigma\tau}\psi^{\tau^2}\psi^{\sigma\tau^2})^2 \in \left(f^*\mathbf{C}(Y)^{\times}\right)^2$.
(iii)

$$\begin{aligned}
(\varphi_1) &= D_2 + D_3 + 2(D_1 + D_0 + D_0^{\sigma} + D_0^{\tau^2} + D_0^{\sigma\tau^2}) - 2(D_\infty + D_\infty^{\sigma} + D_\infty^{\tau^2} + D_\infty^{\sigma\tau^2}) \\
(\varphi_2) &= D_1 + D_3 + 2(D_2 + D_0 + D_0^{\sigma} + D_0^{\tau} + D_0^{\sigma\tau}) - 2(D_\infty + D_\infty^{\sigma} + D_\infty^{\tau} + D_\infty^{\sigma\tau}) \\
(\varphi_3) &= D_1 + D_2 + 2(D_3 + D_0^{\tau} + D_0^{\tau^2} + D_0^{\sigma\tau} + D_0^{\sigma\tau^2}) - 2(D_\infty^{\tau} + D_\infty^{\tau^2} + D_\infty^{\sigma\tau} + D_\infty^{\sigma\tau^2}).
\end{aligned}$$

In particular, $\varphi_i \notin (\mathbf{C}(Z))^2$ $(i = 1, 2, 3)$.

Now the existence for an $\mathfrak{S}_4$ cover with property (i) follows from Proposition 3.1. The assertion on $\Delta(X/Z)$ follows from (iii). □

Conversely we have

Proposition 3.3. *Let $\pi\colon X \to Y$ be an $\mathfrak{S}_4$-cover. Suppose that* (i) *$D(X/Y, V_4)$ is smooth and* (ii) *$\Delta_{\beta_2(\pi,V_4)} \neq \emptyset$. Then there exist three effective divisors, D_1, D_2 and D_3 on $D(X/Y, V_4)$ satisfying the conditions* (i) *and* (ii) *in* Proposition 3.2.

Proof. Choose φ_1, φ_2 and φ_3 as in the second half in Proposition 3.1. We may assume that

$$(\varphi_i) = D_i + 2D_{i,0} - 2D_{i,\infty} \quad (i = 1, 2, 3),$$

where D_i is reduced and $D_{i,0}$ and $D_{i,\infty}$ are effective for each i. Then these D_1, D_2 and D_3 are the desired ones. □

4. $\mathfrak{A}_4$-covers of algebraic varieties

In this section, we consider the construction problem of $\mathfrak{A}_4$-covers. $\mathfrak{A}_4$ is the unique index-2-subgroup of $\mathfrak{S}_4$; and $V_4 \subset \mathfrak{A}_4$ such that $\mathfrak{A}_4/V_4 \cong \mathbf{Z}/3\mathbf{Z}$. Hence, after we knew how to attack $\mathfrak{S}_4$-covers, it is rather easy for us to consider the same problem for $\mathfrak{A}_4$. To describe $\mathfrak{A}_4$-covers, we simply *forget* the condition concerning σ in Proposition 3.1. Namely it is as follows:

Proposition 4.1. *Let $f\colon Z \to Y$ be a $\mathbf{Z}/3\mathbf{Z}$-cover of Y. Suppose that there exist three rational functions φ_1, φ_2 and φ_3 as follows*:
(i) $\varphi_i \notin \left(\mathbf{C}(Z)^{\times}\right)^2$ *for every i.*
(ii) *Put* $\mathrm{Gal}(\mathbf{C}(Z)/\mathbf{C}(Y)) = \langle\tau \mid \tau^3 = id\rangle$. *Then $\varphi_1^{\tau} = \varphi_2$, $\varphi_2^{\tau} = \varphi_3$.*

(iii) $\varphi_1\varphi_2\varphi_3 \in \left(f^*\mathbf{C}(Y)^\times\right)^2$.

Then the bi-quadratic extension $K = \mathbf{C}(Z)(\sqrt{\varphi_1}, \sqrt{\varphi_2})$ *is an* $\mathfrak{A}_4$ *extension of* $\mathbf{C}(Y)$ *such that* $K^{V_4} = \mathbf{C}(Z)$. *In particular, the* K*-normalization,* X*, of* Y *is an* $\mathfrak{A}_4$*-cover of* Y *with* $D(X/Y, V_4) = Z$.

Conversely, if there exists an $\mathfrak{A}_4$*-cover* $\pi\colon X \to Y$ *with* $D(X/Y, V_4) = Z$*, there exist three rational functions* φ_1*,* φ_2*, and* φ_3 *in* $\mathbf{C}(Z)$ *satisfying the three properties* (i), (ii) *and* (iii) *as above.*

We also have the geometric version as follows:

Proposition 4.2. *Let* $f\colon Z \to Y$ *be a smooth* $\mathbf{Z}/3\mathbf{Z}$*-cover of* Y*. Suppose that there exist three different reduced divisors,* D_1*,* D_2 *and* D_3 *on* W *as follows*:
(i) *With the same notation on* Gal(Z/Y) *as those in* Proposition 4.1, $D_1^\tau = D_2$, $D_2^\tau = D_3$, $D_3^\tau = D_1$*, and there is no common component among* D_1*,* D_2 *and* D_3.
(ii) *There exists a line bundle,* $\mathbf{L}$*, such that* $D_1 \sim 2\mathbf{L}$.

Then there exists an $\mathfrak{A}_4$*-cover* $\pi\colon X \to Y$ *satisfying* (i) $D(X/Y, V_4) = Z$ *and* (ii) $\Delta(X/Z) = \mathrm{Supp}(D_1 + D_2 + D_3)$.

The converse of the above proposition is as follows:

Proposition 4.3. *Let* $\pi\colon X \to Y$ *be an* $\mathfrak{A}_4$*-cover. Suppose that* (i) $D(X/Y, V_4)$ *is smooth and* (ii) $\Delta_{\beta_2(\pi, V_4)} \neq \emptyset$*. Then there exist three effective divisors,* D_1*,* D_2 *and* D_3 *on* $D(X/Y, V_4)$ *satisfying the condition* (i) *and* (ii) *in* Proposition 4.2.

We omit our proofs for Propositions 4.1, 4.2 and 4.3, since they are almost the same as those for Propositions 3.1, 3.2 and 3.3.

5. $\mathfrak{S}_4$- and $\mathfrak{A}_4$-covers of algebraic surfaces

Throughout this section, G always means $\mathfrak{S}_4$ or $\mathfrak{A}_4$. V_4 denotes the Klein group and we put $G_1 = G/V_4$. In Propositions 3.2 and 4.2, we assume that the intermediate cover is smooth. This assumption, however, seems to be too strong when we consider their application. In this section, we show that we are able to drop such assumption when Y is a surface.

Let $f\colon Z \to Y$ be a G_1-cover of Y, and let $\mu\colon \widetilde{Z} \to Z$ be the minimal resolution. Then, by the uniqueness of the minimal resolution, μ is a G_1-equivalent resolution. Namely we can consider G_1 as a finite automorphism group of $\widetilde{Z}$ over Y. Taking this into account, we can easily modify our previous results into more useful form.

Proposition 5.1. *The case* $G = \mathfrak{S}_4$*. Let* D_1*,* D_2 *and* D_3 *be reduced divisors on* $\widetilde{Z}$ *such that*
(i) (a) $D_1^\sigma = D_2$*, and* $D_3^\sigma = D_3$; (b) $D_1^\tau = D_2$*,* $D_2^\tau = D_1$ *and* $D_3^\tau = D_1$*, and*

(ii) *there exists a line bundle,* $\mathbf{L}$, *on* $\widetilde{Z}$ *such that* $D_1 \sim 2\mathbf{L}$.

Then there exists a $(\mathbf{Z}/2\mathbf{Z})^{\oplus 2}$*-cover* $g\colon \widetilde{X} \to \widetilde{Z}$ *with the following properties*:

(i) $\Delta_g = \mathrm{Supp}(D_1 + D_2 + D_3)$.

(ii) *the Stein factorization,* X, *of* $f \circ \mu \circ g$ *gives rise to an* $\mathfrak{S}_4$*-cover,* X, *of* Y *with* $\Delta(X/Y) = \Delta_f \cup f \circ \mu(\mathrm{Supp}(D_1 + D_2 + D_3))$.

The case $G = \mathfrak{A}_4$. *Just by dropping the condition* (i) (a) *in the* $\mathfrak{S}_4$ *case, the same statement holds for* $\mathfrak{A}_4$*-covers.*

Corollary 5.2. *Under the same assumption and notations as in* Proposition 5.1, *if* $\mathrm{Supp}(D_1 + D_2 + D_3)$ *is a subset of the exceptional divisor of* μ, *then there exists a* G*-cover* ($G = \mathfrak{S}_4$ *or* $\mathfrak{A}_4$) *of* Y *with branch locus* Δ_f.

The converse of Proposition 5.1 also holds:

Proposition 5.3. *Let* $\pi\colon X \to Y$ *be a* G*-cover* ($G = \mathfrak{S}_4, \mathfrak{A}_4$) *of a smooth algebraic surface* Y, *and let* $\mu\colon \widetilde{Z} \to D(X/Y, V_4)$ *be the minimal resolution of* $D(X/Y, V_4)$. *If* $\Delta_{\beta_2(\pi, V_4)} \neq \emptyset$, *then there exist three reduced divisor,* D_1, D_2 *and* D_3 *on* $\widetilde{Z}$ *such that*

(i) D_1, D_2, *and* D_3 *satisfy the conditions* (i) *and* (ii) *in* Proposition 5.1, *and*

(ii) $\Delta(X/Y) = \Delta_{\beta_1(\pi, V_4)} \cup \beta_1(\pi, V_4) \circ \mu(\mathrm{Supp}(D_1 + D_2 + D_3))$.

Propositions 5.1 and 5.3 still involve a condition concerning linear equivalences. We next rewrite them into the ones concerning only algebraic equivalences. In what follows, we always assume

($*$) The Néron-Severi group of $\widetilde{Z}$, $\mathrm{NS}(\widetilde{Z})$, is torsion free.

Note that $\mathrm{NS}(\widetilde{Z})$ is a lattice with respect to the intersection pairing under the assumption ($*$).

Let $C_1, \dots, C_r$ be irreducible divisors on $\widetilde{Z}$ satisfying the following properties:

(i) Let T be the subgroup of $NS(\widetilde{Z})$ generated by $C_1, \dots, C_r$. Then T is a sublattice of rank r. We call T a *trivial subgroup* (or *trivial sublattice*) generated by $C_1, \dots, C_r$.

(ii) T is G_1-invariant.

Since T is G_1-invariant, G_1 acts $\left(\mathrm{NS}(\widetilde{Z})/T\right)_{tor}$. Suppose that $\left(\mathrm{NS}(\widetilde{Z})/T\right)_{tor}$ contains a G_1-invariant subgroup M isomorphic to $\mathbf{Z}/2\mathbf{Z} \oplus \mathbf{Z}/2\mathbf{Z}$. This implies that we have a homomorphism $\rho\colon G_1 \to GL(2, \mathbf{Z}/2\mathbf{Z})$. Let $\widetilde{G}$ be the semi-direct product determined by ρ. If ρ is injective, then $\widetilde{G}$ is $\mathfrak{S}_4$ (resp. $\mathfrak{A}_4$) for $G_1 = \mathfrak{S}_3$ (resp. $\mathbf{Z}/3\mathbf{Z}$).

Under these circumstances, we have the following:

Theorem 5.4. *If* ρ *is injective, then there exists a* $(\mathbf{Z}/2\mathbf{Z})^{\oplus 2}$*-cover* $g\colon \widetilde{X} \to \widetilde{Z}$ *such that*

(a) *the Stein factorization,* X, *of* $f \circ \mu \circ g$ *gives rise to an* $\mathfrak{S}_4$*-cover,* X, *of* Y *with* $D(X/Y, V_4) = Z$, *and*

(b) $\Delta_g \subset \sum_{\sigma \in G_1} \mathrm{Supp}(C_1^\sigma + \cdots + C_r^\sigma)$

Proof. We only prove for the case of $G_1 = \mathfrak{S}_3$, as our proof for the $\mathbf{Z}/3\mathbf{Z}$ case goes almost the same way as that for the $\mathfrak{S}_3$ case.

Let α_1, α_2, α_3 be three non-trivial elements of M. We put $\mathfrak{S}_3 = \langle \sigma, \tau \mid \sigma = (12), \tau = (123) \rangle$ and may assume that $\mathfrak{S}_3$ acts on α_1, α_2, α_3 by the permutation of the subindices. Choose a divisor, L, on $\widetilde{Z}$ so that L gives α_1. Since $2L \in T$ and $L \notin T$, $2L$ is represented by a divisor in the form of

$$D = C_{i_1} + \cdots + C_{i_s} + 2D', \quad D' \in T.$$

By replacing L by $L - D'$, we may assume that $2L$ is represented by the reduced divisor $D = C_{i_1} + \cdots + C_{i_s}$. Since M is $\mathfrak{S}_3$-invariant, L^σ, L^τ, L^{τ^2}, $L^{\sigma\tau}$ and $L^{\sigma\tau^2}$ give rise to non-trivial elements in M and correspond to α_2, α_2, α_3, α_3 and α_1, respectively. By replacing L by a suitable algebraically equivalent one, if necessary, we may assume that

$$D \sim 2L.$$

Hence there exists a rational function ψ in $\mathbf{C}(\widetilde{Z})$ such that

$$(\psi) = D - 2L.$$

Put

$$\varphi_1 = \psi^\tau \psi^{\sigma\tau}, \quad \varphi_2 = \psi^{\tau^2} \psi^{\sigma\tau^2}, \quad \varphi_3 = \psi\psi^\sigma.$$

Then φ_i ($i = 1, 2, 3$) satisfy
(i) $\varphi_1^\sigma = \varphi_2$, $\varphi_3^\sigma = \varphi_3$.
(ii) $\varphi_1^\tau = \varphi_2$, $\varphi_2^\tau = \varphi_3$.

Let D_1, D_2 and D_3 be the reduced part of $D^\tau + D^{\sigma\tau}$, $D^{\tau^2} + D^{\sigma\tau^2}$ and $D + D^\sigma$, respectively. Then we have the following:

CLAIM. D_1, D_2 and D_3 are distinct and satisfy (i) $D_1^\sigma = D_2$, $D_3^\sigma = D_3$ and (ii) $D_1^\tau = D_2$, $D_2^\tau = D_3$.

Proof of Claim. By the definition of φ_i ($i = 1, 2, 3$),

$$\begin{aligned} (\varphi_1) &= D^\tau + D^{\sigma\tau} - 2(L^\tau + L^{\sigma\tau}) \\ (\varphi_2) &= D^{\tau^2} + D^{\sigma\tau^2} - 2(L^{\tau^2} + L^{\sigma\tau^2}) \\ (\varphi_3) &= D + D^\sigma - 2(L + L^\sigma) \end{aligned}$$

We first show $D_i \neq \emptyset$ for every i. It is enough to show that $D_1 \neq \emptyset$ as $D_2 = D_1^\tau$, $D_3 = D_2^\tau$. Suppose that $D_1 = \emptyset$. Then $D^\tau + D^{\sigma\tau} = 2D''$, $D'' \in T$. As we assume

$\mathrm{NS}(\widetilde{Z})$ is torsion-free, it implies that $L^\tau + L^{\sigma\tau} \approx D''$, i.e., $\alpha_2 + \alpha_3 = 0$ in M. But this contradicts to $\alpha_2 + \alpha_3 = \alpha_1 \neq 0$. We next see D_1, D_2 and D_3 are distinct. It is enough to show that $D_1 \neq D_2$. Suppose that $D_1 = D_2 (= D_1^\tau)$. Then by considering the divisor of the rational function (φ_1/φ_2), we have

$$-2(L^\tau + L^{\sigma\tau}) + 2(L^{\tau^2} + L^{\sigma\tau^2}) \in 2T;$$

and it implies

$$-(L^\tau + L^{\sigma\tau}) + (L^{\tau^2} + L^{\sigma\tau^2}) \in T.$$

This implies $-(\alpha_2 + \alpha_3) + (\alpha_3 + \alpha_1) = 0$. But this is again contradiction, as $\alpha_1 + \alpha_2 = \alpha_3 \neq 0$.

We go back to prove Theorem 5.4. By the definition of D_1, there exists a line bundle $\mathbf{L}$ such that $2\mathbf{L} \sim D_1$, and $\mathrm{Supp}(D_1 + D_2 + D_3) \subset \mathrm{Supp}\left(\sum_{\sigma \in G_1} (C_{i_1}^\sigma + \cdots + C_{i_s}^\sigma)\right)$. Hence by Proposition 5.1 and Claim, we have Theorem 5.4. □

6. Examples

In this section, we consider several examples for $\mathfrak{S}_4$- and $\mathfrak{A}_4$-covers.

EXAMPLE 6.1. Let $S^4(\mathbf{P}^1)$ be the symmetric product of $\mathbf{P}^1$ of degree 4. $S^4(\mathbf{P}^1)$ is canonically identified with $\mathbf{P}^4$ and the canonical projection $\pi\colon \mathbf{P}^1 \times \mathbf{P}^1 \times \mathbf{P}^1 \times \mathbf{P}^1 \to \mathbf{P}^4$ is an $\mathfrak{S}_4$ cover of $\mathbf{P}^4$. The branch locus of π is known as the discriminant hypersurface. In this particular case, it is a hypersurface of degree 6. Let $\Sigma (= \mathbf{P}^2)$ to be a generic 2-plane in $\mathbf{P}^4$. The restriction of π to Σ gives rise to an $\mathfrak{S}_4$ cover, S, of $\mathbf{P}^2$ branched at a sextic curve, B, with 4 nodes and 6 cusps. The ramification index along B is 2. $D(S/\Sigma, V_4)$ is a $K3$ surface with $12A_1$ singularities, while $D(S/\Sigma, \mathfrak{A}_4)$ is a $K3$ surface with $6A_2$ and $4A_1$ singularities. We will look into this example from more general view point in Part II.

EXAMPLE 6.2. Let C be a hyperelliptic curve of genus g; and let $S^3(C)$ be the symmetric product of degree 3. The canonical projection $f\colon C \times C \times C \to S^3(C)$ gives an $\mathfrak{S}_3$-cover. We show that there exist $2^{2g} - 1$ distinct $\mathfrak{S}_4$-cover, $\pi\colon X \to S^3(C)$ so that $D(X/S^3(C), V_4) = C \times C \times C$, $\beta_1(\pi, V_4) = f$.

Let p_i denote the projection from $C \times C \times C$ to the i-th factor. Let $e_1, \dots, e_g$ and $\mathbf{o}$ be points such that $\sum_i e_i - g\mathbf{o} \not\sim 0$ but $2(\sum_i e_i - g\mathbf{o}) \sim 0$, i.e., $\sum_i e_i - g\mathbf{o}$ is a 2-torsion on $\mathrm{Pic}^0(C)$. Let h be a rational function on C such that $(h) = \sum_i e_i - g\mathbf{o}$. Put $\varphi_i = p_j^* h p_k^* h$, $\{i, j, k\} = \{1, 2, 3\}$. Then these three rational functions, φ_1, φ_2 and φ_3 satisfy the following conditions with respect to the $\mathfrak{S}_3$ action on $C \times C \times C$:

(i) $\varphi_i \notin \left(\mathbf{C}(C \times C \times C)^\times\right)^2$ for every i.

(ii) $\varphi_1^{(12)} = \varphi_2$, $\varphi_3^{(12)} = \varphi_3$; and $\varphi_1^{(123)} = \varphi_2$, $\varphi_2^{(123)} = \varphi_3$, $\varphi_3^{(123)} = \varphi_1$.

(iii) $\varphi_1\varphi_2\varphi_3 = (p_1^* h p_2^* h p_2^* h)^2 \in \left(\mathbf{C}(C \times C \times C)^\times\right)^2$.

Let $K = \mathbf{C}(C \times C \times C)(\sqrt{\varphi_1}, \sqrt{\varphi_2})$ and let X be the K-normalization of $C \times C \times C$. Then by Proposition 3.2, X is an $\mathfrak{S}_4$-cover of $S^3(C)$. Since there are $2^{2g} - 1$ distinct non-trivial 2-torsions on $\mathrm{Pic}^0(C)$, there exist $2^{2g} - 1$ distinct $\mathfrak{S}_4$ covers of $S^3(C)$.

EXAMPLE 6.3. Let $(E, \mathbf{o})$ be an elliptic curve given by the affine equation $y^2 = x^3 + ax + b$, $\mathbf{o} = [0 : 1 : 0]$. Let l denote a line. The divisor $\mathfrak{d}$ cut out by l is linearly equivalent to $3\mathbf{o}$. $\mathfrak{d}$ consists of three distinct points if l meets E transversely, while $\mathfrak{d}$ has non reduced point if l is tangent to E. Let $E^\vee$ be the dual curve of E. The above fact implies that any point in $\mathbf{P}^2 \setminus E^\vee$ corresponds to a reduced divisor linearly equivalent to $3\mathbf{o}$, while any point in $E^\vee$ corresponds to a non-reduced divisor linearly equivalent to $3\mathbf{o}$. In other word, if we let $S^3(E)$ be the symmetric product of E of degree 3 and let $\phi\colon S^3(E) \to E$ be the Abel-Jacobi map, then $\phi^{-1}(\mathbf{o}) = \mathbf{P}^2$ and $\phi^{-1}(\mathbf{o}) \cap \Delta = E^\vee$, where Δ is the branch locus of the $\mathfrak{S}_3$-cover $f\colon E \times E \times E \to S^3(E)$. Let X be the $\mathfrak{S}_4$-cover in Example 6.2 for $f\colon E \times E \times E \to S^3(E)$. Then the restriction of X to $\phi^{-1}(\mathbf{o})$ gives an $\mathfrak{S}_4$-cover of $\mathbf{P}^2$. Example 6.2 assures that there exist three distinct $\mathfrak{S}_4$-covers of $\mathbf{P}^2$ branched at $2E^\vee$.

REMARK 6.4. A branched cover $\pi\colon S \to \mathbf{P}^2$ of degree n is called a *generic n-plane* if it satisfies

(i) S is smooth and the branch locus Δ_π is an irreducible curve with only nodes and cusp as its singularities.

(ii) $\pi^*\Delta_\pi = 2R + \Gamma$; and $\pi|_R\colon R \to \Delta_\pi$ gives the normalization of Δ_π.

Example 6.3 implies that there exist three distinct 4 generic plane with branch locus $E^\vee$. This fact is classically known (see [2] for detail).

The topology of the complements to plane sextic curves

7. Automorphisms of order 2 or 3 and the rational quotients by them

Let X be a surface, and let σ be an automorphism of order 2 or 3 of X with only isolated fixed points, $Q_1, \ldots, Q_k$. Let G be the group of generated by σ. Let $\overline{Y} = X/G$ and let $\pi\colon X \to \overline{Y}$ be the quotient map. $\overline{Y}$ has quotient singularities at the points $P_i = \pi(Q_i)$. Let $\mu\colon Y \to \overline{Y}$ be the minimal resolution of $\overline{Y}$. We call the induced rational map $X \cdots \to Y$ the rational quotient map and call Y the rational quotient of X by G. Let $\widetilde{X}$ be the $\mathbf{C}(X)$-normalization of Y. It is a cyclic covering of degree $\sharp(G)$ branched along at most the exceptional set of $Y \to \overline{Y}$. In what follows, we look into the relation among X, $\widetilde{X}$ and Y.

CASE 1. $\sharp(G) = 2$. One obtains $\widetilde{X}$ from X by blowing-up at $Q_1, \ldots, Q_k$. For details, see [13, §3].

CASE 2. $\sharp(G) = 3$. In this case, the action of G around each fixed point is divided into two types. Namely, if we choose a small neighborhood, $U : (x, y) \subset \mathbf{C}^2$, $Q_i = (0, 0)$ appropriately, then we may assume that the action of σ is given either (i) $(x, y) \mapsto (\varepsilon x, \varepsilon y)$, or (ii) $(x, y) \mapsto (\varepsilon x, \varepsilon^2 y)$, where $\varepsilon = \exp(2\pi i/3)$. Hence P_i is a cyclic quotient singularity of type $(1, 3)$ for (i), while it is one of type $(2, 3)$ for (ii), i.e., a rational double point of type A_2. We relabel the Q_i's so that $P_1, \dots, P_t$ are type $(1, 3)$ and $P_{t+1}, \dots, P_k$ are type $(2, 3)$. To obtain $\widetilde{X}$ from X, we first consider a successive blowing-ups of X in the following way:

(i) Blow up at Q_i one time for $i = 1, \dots, t$, and

(ii) Blow up at Q_i three times for $i = t + 1, \dots, k$ so that the induced automorphism from σ has no isolated fixed point. One can easily see that the exceptional set is tree of three $\mathbf{P}^1$ and that the self intersection number of the middle component is -3, while those of the remaining two is -1.

We then contract the $k - t$ (-3) curves arising from (ii). Then we obtain $\widetilde{X}$.

We next consider how we obtain $\widetilde{X}$ from Y. Let $C_1, \dots, C_t$ be the exceptional curves for $P_1, \dots, P_t$ and let $C_{i,1}$ and $C_{i,2}$ $(t + 1 \geq i \geq k)$ be the exceptional curves for $P_{t+1}, \dots, P_k$. Since $\widetilde{X} \to Y$ is a cyclic triple covering of Y, the branch locus is a line bundle L on Y such that

$$3L \sim \sum_{i=1}^{t} C_i + \sum_{i=t+1}^{k} (C_{i,1} + 2C_{i,2}).$$

REMARK 7.1. A divisor in the form of $C_{i,1} + C_{i,2}$ $(i \geq t + 1)$ does not appear in the right hand side, since $C_{i,1}(C_{i,1} + C_{i,2}) = -1$ is not divisible by 3.

From the linear equivalence as above, one can obtain a cyclic triple covering, Z, of Y branched along $\mathrm{Supp}\left(\sum_{i=1}^{t} C_i + \sum_{i=t+1}^{k} (C_{i,1} + 2C_{i,2})\right)$. If we choose L in an appropriate way, $Z = \widetilde{X}$. In particular, if $\mathrm{Pic}(Y)$ has no 3-torsion, then $Z = \widetilde{X}$.

8. 2- and 3-divisible divisors on $K3$ surfaces

A $K3$ surface is a simply connected compact complex manifold of dimension 2 with trivial canonical bundle. Throughout this article, we only consider *algebraic* $K3$ surfaces.

Before we consider the rational cyclic quotient of $K3$ surfaces, we summarize some facts from lattice theory, which we need later.

DEFINITION 8.1. A lattice is a free $\mathbf{Z}$ module of finite rank equipped with $\mathbf{Z}$ valued symmetric bilinear form.

Let L_1 and L_2 be lattices. We denote the orthogonal direct sum of them by $L_1 \oplus L_2$; and L^n denotes $L \oplus \cdots \oplus L$ (n copies). The discriminant, $\mathrm{disc}\, L$, of a lattice L

is the determinant of the intersection matrix of L. A lattice is called unimodular if $\operatorname{disc} L = \pm 1$. We denote the dual lattice of L by $L^\vee$. L is embedded to $L^\vee$ by using the bilinear form as a sub lattice with same rank. The quotient group $L^\vee/L$ is a finite abelian group, which we denote by G_L.

A sublattice, M, of L is called primitive if L/M is torsion-free.

EXAMPLE 8.2. Let X be an algebraic surface and let $H^2(X,\mathbf{Z})$ be the second cohomology group. If $H^2(X,\mathbf{Z})$ is torsion-free, then $H^2(X,\mathbf{Z})$ is unimodular lattice with respect to the intersection product by Poincaré duality. The Néron-Severi group of X is a primitive sublattice of $H^2(X,\mathbf{Z})$.

Lemma 8.3. *Let L be a unimodular lattice. Let J_1 and J_2 be sublattices of L such that $J_1^\perp = J_2$ and $J_2^\perp = J_1$. Then $G_{J_1} \cong G_{J_2}$.*

For a proof, see [7, p. 4].

By Example 8.2, for a $K3$ surface X, $H^2(X,\mathbf{Z})$ is a unimodular lattice; and by the Noether formula, $\operatorname{rank} H^2(X,\mathbf{Z}) = 22$. Let $\mathrm{NS}(X)$ be the Néron-Severi group of X. As X is simply connected, $\mathrm{NS}(X) = \mathrm{Pic}(X)$.

DEFINITION 8.4. We call $\sum_{i=1}^k C_i$ p-divisible if $1/p\,(\sum_{i=1}^k C_i) \in \mathrm{NS}(Y)$, i.e., there exists L in $\mathrm{NS}(Y)$ such that $pL \approx \sum_{i=1}^k C_i$.

Lemma 8.5 ([14, Lemma 3], [13, Lemma 3.3]). *Let $C_1, \dots, C_k$ be disjoint (-2) curves on a $K3$ surface Y, and suppose $1/2\,\sum_{i=1}^k C_i \in \mathrm{NS}(Y)$. Then $k = 0$, 8 or 16.*

For a proof, see [13].

Corollary 8.6. *Let $C_1, \dots, C_l$ be disjoint (-2) curves on a $K3$ surface Y, and let L be the sublattice generated by $C_1, \dots, C_l$. Then*:
(i) *If $\left(\mathrm{NS}(Y)/L\right)_{tor} \supset \mathbf{Z}/2\mathbf{Z}$, then $l \geq 8$, and*
(ii) *If $\left(\mathrm{NS}(Y)/L\right)_{tor} \supset (\mathbf{Z}/2\mathbf{Z})^{\oplus 2}$, then $l \geq 12$.*

Proof. (i) Let $D = \sum_{i=1}^l a_i C_i$ be an element of L such that $(1/2)D \notin L$ but $(1/2)D \in \mathrm{NS}(Y)$. By replacing D by $D_1 = \sum_{i=1}^l (a_i - 2[a_i/2])C_i$, $[x]$ being the maximal integer not exceeding x, we may assume that D is a non-zero reduced effective divisor. Hence by Lemma 8.5, the number of irreducible component of D is either 8 or 16.
(ii) Suppose that $l \leq 11$ and $\left(\mathrm{NS}(Y)/L\right)_{tor} \supset (\mathbf{Z}/2\mathbf{Z})^{\oplus 2}$. Let D_1 and D_2 be elements of L such that $(1/2)D_1$ and $(1/2)D_2$ give rise to distinct elements in $\left(\mathrm{NS}(Y)/L\right)_{tor}$. Then, by Lemma 8.5 and the assumption, both D_1 and D_2 have 8 irreducible compo-

nents. Hence by relabeling C_i if necessary, one may assume

$$\begin{aligned} D_1 &= C_1 + \cdots + C_t + C_{t+1} + \cdots + C_8 \\ D_2 &= C_1 + \cdots + C_t + C_9 + \cdots + C_l, \end{aligned}$$

where $1 \leq t \leq 7$, $9 \leq l \leq 11$. Since $l \leq 11$, $t \geq 5$. Let $X_1 \cdots \to Y$ be the rational quotient map with respect to D_1 as in §7. Then the divisor $\widetilde{D}_2$ on X_1 coming from D_2 is in the form of $(C_9' + C_9'') + \cdots + (C_l' + C_l'')$; and $1/2\widetilde{D}_2 \in \mathrm{NS}(X_1)$. Hence the number of irreducible components $\widetilde{D}_2$ is either 8 or 16 by Lemma 8.5. But this is impossible as $9 \leq l \leq 11$. □

For the existence of 2-torsions, we have the following lemma.

Lemma 8.7. *With the same notations as in* Corollary 8.6, *if* $l \geq 12$, *then* $\left(\mathrm{NS}(Y)/L\right)_{tor}$ *has a 2-torsion.*

Proof. Let $L^\sharp$ be the primitive hull of L. Note that $L^\sharp/L = \left(\mathrm{NS}(Y)/L\right)_{tor}$ and both $L^\sharp$ and L are embedded in $L^\vee$ as sublattices. Let $\left(L^\sharp\right)^\perp$ be the orthogonal complement of $L^\sharp$ in $H^2(Y, \mathbf{Z})$. Then by Lemma 8.3 $G_{L^\sharp} \cong G_{(L^\sharp)^\perp}$. Suppose that $L^\sharp/L$ has no 2-torsion. As $L^\sharp/L \subset G_L \cong (\mathbf{Z}/2\mathbf{Z})^{\oplus l}$, we have $L^\sharp = L$. Hence, $G_L = G_{L^\sharp}$. Thus the 2-length of $G_{L^\sharp} = l \geq 12$. On the other hand, $\mathrm{rank}(L^\sharp)^\perp = 22 - l$; and the 2-length of $G_{(L^\sharp)^\perp} \leq 22 - l \leq 10$. This is impossible. □

Lemma 8.8. *Let* $(C_{i,1}, C_{i,2})$ $(i = 1, \ldots, k)$ *be pairs of* (-2) *curves on a* $K3$ *surface* Y *such that*
(i) $C_{i,1}C_{i,2} = 1$ *and the divisors* $C_{1,1} + C_{1,2}, \ldots, C_{k,1} + C_{k,2}$ *are disjoint.*
(ii) $1/3 \sum_{i=1}^k (C_{i,1} + 2C_{i,2}) \in \mathrm{NS}(Y)$,
then $k = 0$, 6 *or* 9.

Proof. Suppose that $k > 0$ and let $X \cdots \to Y$ be the rational quotient map of degree 3 as in §7 and let Q_i $(i = 1, \ldots, k)$ be the points lying over $C_{i,1} + 2C_{i,2}$, $(i = 1, \ldots, k)$, respectively. Then

$$\begin{aligned} \chi_{top}(X) &= \chi_{top}\left(X \setminus \{Q_1, \ldots, Q_k\}\right) + k \\ &= 3\chi_{top}\left(Y \setminus \cup_{i=1}^k (C_{i,1} \cup C_{i,2})\right) + k \\ &= 72 - 8k. \end{aligned}$$

As $K_X \sim 0$, X is either a $K3$ surface or an abelian surface. Hence $k = 6$ for the first case and $k = 9$ for the second. □

Table 9.1.

Type of (X, x)	Local equation	$\pi_1^{loc}(X, x)$	$\sharp\left(\pi_1^{loc}(X, x)\right)$
A_n	$z^2 + y^2 + x^{n+1} = 0$	cyclic group	$n + 1$
D_n $(n \geq 4)$	$z^2 + x(y^2 + x^{n-2}) = 0$	binary dihedral group	$4(n - 2)$
E_6	$z^2 + y^3 + x^4 = 0$	binary tetrahedral group	24
E_7	$z^2 + y(y^2 + x^3) = 0$	binary octahedral group	48
E_8	$z^2 + y^3 + x^5 = 0$	binary icosahedral group	120

(Note that A_0 is nothing but a smooth point.)

Table 9.2.

p	(X, x)	(Y, y)
2	A_n $(n \equiv 1 \bmod 2)$	$A_{(n-1)/2}$
2	D_n $(n :$ even$)$	A_{2n-5} or $D_{n/2+1}$
2	D_n $(n :$ odd$)$	A_{2n-5}
2	E_7	E_6
3	A_n $(n \equiv 2 \bmod 3)$	$A_{(n-2)/3}$
3	E_6	D_4
$p \geq 5$	A_n $(n + 1 \equiv 0 \bmod p)$	$A_{(n+1)/p-1}$

9. Cyclic covers of rational double points

Let (X, x) be a 2-dimensional normal singularity, i.e., X is a normal irreducible complex space having a unique singularity at x. Let (Y, y) be another 2-dimensional normal singularity and let $f : (Y, y) \to (X, x)$ be a finite morphism such that (i) $Y \setminus y \to X \setminus x$ is unramified, and (ii) $f^{-1}(x) = y$.

Such f is determined by a subgroup of finite index of the local fundamental group, $\pi_1^{loc}(X, x)$, of (X, x). For rational double points, the results in Table 9.1 is well-known.

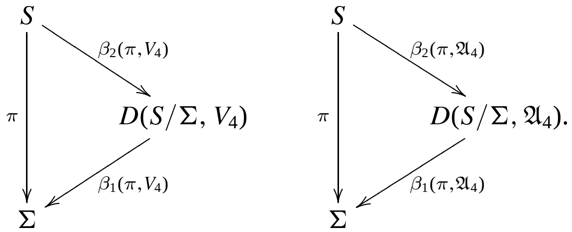
We now consider the case when f is a p-cyclic (p: odd prime) cover.

Lemma 9.1. *Let (X, x) be a rational double point. If f is a p-cyclic cover, then the pair (X, x) and (Y, y) is one of those in* Table 9.2.

Proof. If $f : (Y, y) \to (X, x)$ is a p-cyclic covering, then it corresponds to a normal subgroup of $\pi_1^{loc}(X, x)$ of index p. Our statement easily follows from the case-by-case checking. □

10. Local structure of an $\mathfrak{S}_4$-cover of a surface

We go on to study the local structure of an $\mathfrak{S}_4$-cover. Let $\pi\colon S \to \Sigma$ be an $\mathfrak{S}_4$ cover of a smooth algebraic surface Σ. As we introduced in §1, we have the commutative diagrams:



$D(S/\Sigma, V_4)$ is a $\mathbf{Z}/3\mathbf{Z}$-cover of $D(S/\Sigma, \mathfrak{A}_4)$. We denote its covering morphism by $\gamma\colon D(S/\Sigma, V_4) \to D(S/\Sigma, \mathfrak{A}_4)$.

In the following, we always assume:

(i) the branch locus $B := \Delta(S/\Sigma)$ has at most simple singularities,
(ii) π is branched at $2B$, and
(iii) $\beta_1(\pi, \mathfrak{A}_4)$ is branched along B.

Under these three conditions, one can conclude that $\beta_2(\pi, V_4)$, γ and $\beta_2(\pi, \mathfrak{A}_4)$ are branched at most singular points of the base surfaces; and all of these singularities are rational double points by Lemma 9.1. We next consider what kinds of singularities we have on S and $D(S/\Sigma, V_4)$.

Lemma 10.1. *Choose $x \in \mathrm{Sing}(D(S/\Sigma, \mathfrak{A}_4))$. Then the* 13 *cases in* Table 10.1 *occur for singularities appearing in $\beta_2(\pi, \mathfrak{A}_4)^{-1}(x)$ and $\gamma^{-1}(x)$.*

Here the coefficients of the types of singularities mean the number of singularities, e.g., $3A_n$ means three A_n singularities. Also:

(i) *if No.* 1 *occurs, $n \equiv 2 \bmod 3$,*
(ii) *if No.* 2 *occurs, $n \equiv 1 \bmod 2$, and*
(iii) *if No.* 4 *occurs, $n \equiv 0 \bmod 2$.*

Lemma 10.1 easily follows from Lemma 9.1.

11. Proof of Theorem 0.6

We keep the notations as before. Theorem 0.6 is straightforward from the following proposition:

Proposition 11.1. *Let B be as in* Theorem 0.6, *and let Z' be the double cover of $\mathbf{P}^2$ with $\Delta(Z'/\mathbf{P}^2) = B$. Suppose that there exists an $\mathfrak{S}_4$-cover $\pi\colon S \to \mathbf{P}^2$ such that* (i) *π is branched at $2B$, and* (ii) *$D(S/\Sigma, \mathfrak{A}_4) = Z'$.*

Table 10.1.

No.	Type of x	$\gamma^{-1}(x)$	$\beta_2(S/\Sigma, \mathfrak{A}_4)^{-1}(x)$
1	A_n	$A_{(n-2)/3}$	$4A_{(n-2)/3}$
2	A_n	$3A_n$	$6A_{(n-1)/2}$
3	A_n	$3A_n$	$12A_n$
4	D_n	$3D_n$	$6D_{n/2+1}$
5	D_n	$3D_n$	$6A_{2n-5}$
6	D_n	$3D_n$	$12D_n$
7	E_6	D_4	A_1
8	E_6	D_4	$2A_3$
9	E_6	D_4	$4D_4$
10	E_6	$3E_6$	$12E_6$
11	E_7	$3E_7$	$6E_6$
12	E_7	$3E_7$	$12E_7$
13	E_8	$3E_8$	$12E_8$

Then the minimal resolution, $\widetilde{S}$, of S is either an abelian surface or a $K3$ surface. Moreover, if $\widetilde{S}$ is an abelian surface (resp. $K3$ surface), then $G(R)$ contains $\mathbf{A}_2^{\oplus 9}$ (resp. $\mathbf{A}_2^{\oplus 6} \oplus \mathbf{A}_1^{\oplus 4}$).

We need several lemmas to prove Proposition 11.1. Let us start with the following lemma:

Lemma 11.2. *Let $\widetilde{S}$ be as above. Then $\widetilde{S}$ is either an abelian surface or a $K3$ surface. Moreover, if $\mathrm{Sing}(S) \neq \emptyset$, then $\widetilde{S}$ is a $K3$ surface.*

Proof. Let $K_{Z'}$ be the canonical bundle of Z' (Note that one can define $K_{Z'}$ as we assume that Z' has only rational double singularities). By the assumption, $\beta_2(\pi, \mathfrak{A}_4)\colon S \to D(S/\Sigma, \mathfrak{A}_4)$ is branched at at most $\mathrm{Sing}(D(S/\Sigma, \mathfrak{A}_4))$. Also, by Lemma 9.1, S has again at most rational double points as its singularities. Hence we have $K_{\tilde{S}} = \mu_1^* K_S = \mu_1^* \beta_2(\pi, \mathfrak{A}_4)^* K_{Z'} = 0$, where $\mu_1\colon \widetilde{S} \to S$ denotes the minimal resolution. Hence, by the classification for algebraic surfaces, $\widetilde{S}$ is either an abelian surface or a $K3$ surface. If $\mathrm{Sing}(S) \neq \emptyset$, $\widetilde{S}$ contains at least one smooth rational curve. This implies the last assertion. □

Lemma 11.3. *If S is an abelian surface, then*:
(i) *$D(S/\Sigma, V_4)$ is an abelian surface,*
(ii) *$G(R) = \mathbf{A}_2^{\oplus 9}$, and*
(iii) *B is a nine cuspidal sextic curve.*

Proof. Suppose that $D(S/\Sigma, V_4)$ is not abelian surface. Then, by [28], $D(S/\Sigma, V_4)$ is a $K3$ surface with $16A_1$ singularities. Hence, by Lemma 9.1, singularities of $D(S/\Sigma, \mathfrak{A}_4)$ are of types either A_1, A_2 or A_5. Let n_1, n_2 and n_5 be the number of singularities of types A_1, A_2, and A_5, respectively. Then we have

$$\begin{aligned} 3n_1 + n_5 &= 16 \\ n_1 + 2n_2 + 5n_5 &\leq 19. \end{aligned}$$

Since $\gamma\colon D(S/\Sigma, V_4) \to D(S/\Sigma, \mathfrak{A}_4)$ is branched at some singularities of $D(S/\Sigma, \mathfrak{A}_4)$, by Lemmas 8.5 and 10.1, the corresponding graph $\mathbf{A}_1^{\oplus n_1} \oplus \mathbf{A}_2^{\oplus n_2} \oplus \mathbf{A}_5^{\oplus n_5}$ contains a subgraph $\mathbf{A}_2^{\oplus 6}$. Hence the only possible triplet (n_1, n_2, n_5) is $(5, 4, 1)$.

Hence one can conclude the singularities of $D(S/\Sigma, \mathfrak{A}_4)$ are $5A_1{+}4A_2{+}A_5$. Therefore the singularities of the branch locus are $5a_1 + 4a_2 + a_5$.

CLAIM. There exists no reduced sextic curve, B, with singularities $5a_1 + 4a_2 + a_5$.

Proof of Claim. Taking contribution of the genus drop from each singularity into account, we infer that B is reducible. As B has $4a_2$ singularities, it must have an irreducible component of degree 5. Put $B = B_1 + L$, where $\deg B_1 = 5$ and L is a line. Then:

Either B_1 has $4a_2 + a_5$ and B_1 meets L_1 transversely at five distinct points,

or

B_1 has $4a_2 + 3a_1$ and B_1 meets L_1 at 3 distinct points; L_1 is the tangent line at an inflection point of B_1.

In both cases, however, we see that there is no such quintic curve by considering the contribution of the genus drop from singularities.

By Claim, we have the first assertion for Lemma 11.3. We now go on to the second. As $D(S/\Sigma, V_4)$ is an abelian surface, $\chi_{top}(D(S/\Sigma, V_4)) = 0$. Hence from the argument in the proof of Lemma 8.8, we infer that $D(S/\Sigma, \mathfrak{A}_4)$ has just $9A_2$ singularities. This implies that B has nine cusps. □

Lemma 11.4. *If S is a $K3$ surface with rational double points, then $G(R)$ contains a subgraph $\mathbf{A}_2^{\oplus 6} \oplus \mathbf{A}_1^{\oplus 4}$.*

Proof. Let $\mu_1\colon \widetilde{S} \to S$ and $\mu_2\colon Z \to D(S/\Sigma, \mathfrak{A}_4) = Z'$ be the minimal resolution of S and $D(S/\Sigma, \mathfrak{A}_4)$, respectively. By the uniqueness of the minimal resolution, $\mathrm{Gal}(S/\mathbf{P}^2) \cong \mathfrak{S}_4$ is also considered as a finite automorphism group of $\widetilde{S}$, i.e., μ_1 is $\mathfrak{S}_4$-equivalent. Let $\widetilde{S}/\mathfrak{A}_4$ be the quotient surface by $\mathfrak{A}_4$. $\widetilde{S}/\mathfrak{A}_4$ is again a $K3$ surface with rational double points, and there exists a morphism $\overline{\mu}_1\colon \widetilde{S}/\mathfrak{A}_4 \to D(S/\Sigma, \mathfrak{A}_4)$

such that the following diagram commutes.

$$\begin{array}{ccc} S & \xleftarrow{\mu_1} & \widetilde{S} \\ \downarrow & & \downarrow \\ D(S/\Sigma, \mathfrak{A}_4) & \xleftarrow{\overline{\mu}_1} & \widetilde{S}/\mathfrak{A}_4 \end{array}$$

Since the canonical bundle of $\widetilde{S}/\mathfrak{A}_4$ is trivial and singularities of $\widetilde{S}/\mathfrak{A}_4$ are only A_2 and A_1, the minimal resolution of $\widetilde{S}/\mathfrak{A}_4$ is a minimal surface. Hence it is nothing but Z, and one may assume that μ_2 factors $\widetilde{S}/\mathfrak{A}_4$; and the exceptional set for $Z \to D(S/\Sigma, \mathfrak{A}_4)$ contains that of $W \to \widetilde{S}/\mathfrak{A}_4$. By [28] $\widetilde{S}/\mathfrak{A}_4$ has singularities $6A_2 + 4A_1$; and we have the assertion. □

By Lemmas 11.2, 11.3 and 11.4, we have Proposition 11.1.

An easy but interesting corollary to Proposition 11.1 is as follows:

Corollary 11.5. *Under the same notation as before, let B be a plane sextic curve with singularities $\sum_l \alpha_l a_l + \sum_m \beta_m d_m + \sum_n \gamma_n e_n$, $(\alpha_l, \beta_m, \gamma_n \in \mathbf{Z}_{\geq 0})$. Then we have*

$$G(R) = \bigoplus_l \mathbf{A}_l^{\oplus \alpha_l} \oplus \bigoplus_m \mathbf{D}_m^{\oplus \beta_m} \oplus \bigoplus_n \mathbf{E}_n^{\oplus \gamma_n}.$$

If $G(R)$ contains a subgraph neither $\mathbf{A}_2^{\oplus 9}$ nor $\mathbf{A}_2^{\oplus 6} \oplus \mathbf{A}_1^{\oplus 4}$, there is no $\mathfrak{S}_4$-cover of $\mathbf{P}^2$ branched at $2B$.

12. Proof of Theorem 0.7

The goal of this section is to prove Theorem 0.7. Let us start with some setting-ups.

Let B be a reduced plane sextic curve with at most simple singularities. Let $f'\colon Z' \to \mathbf{P}^2$ be a double cover with $\Delta_{f'} = B$, and let $\mu\colon Z \to Z'$ be the canonical resolution of Z'. We denote the the subgroup of $\mathrm{NS}(Z)$ generated by the pull-back of a line of $\mathbf{P}^2$ and the irreducible components of the exceptional divisor of μ by T. As one can easily see, it has an orthogonal decomposition with respect to the intersection pairing:

$$T = \mathbf{Z}L \oplus \bigoplus_{x \in \mathrm{Sing}(Z')} R_x,$$

where L denotes the pull-back of a line, and R_x denotes the subgroup generated by all the irreducible components of the exceptional divisor for $x \in \mathrm{Sing}(Z')$.

Put $\overline{R} = \mathcal{L}(\mathbf{A}_2^{\oplus 6} \oplus \mathbf{A}_1^{\oplus 4})$ and $\overline{T} = \mathbf{Z}L \oplus \overline{R}$. Let $T^\sharp$ and $\overline{T}^\sharp$ be the primitive hull of T and $\overline{T}$ in $\mathrm{NS}(Z)$, respectively. Now let us start with the following lemma.

Lemma 12.1. *$\overline{T}^{\sharp}/\overline{T}$ has a 3-torsion. In particular, $T^{\sharp}/T$ has a 3-torsion.*

Proof. By Nikulin's theory used in [28] §1 or [23], $\overline{T}^{\sharp}/\overline{T}$ has a 3-torsion. As $\overline{T}^{\sharp} \subset T^{\sharp} \subset \mathrm{NS}(Z)$, we can find D in $T^{\sharp}$ which gives a 3-torsion in $\overline{T}^{\sharp}/\overline{T}$. We now show that this D gives a 3-torsion in $T^{\sharp}/T$, too. To see this, it is enough to show $D \notin T$. Suppose that $D \in T$ and write

$$D \sim aL + \sum_{x \in \mathrm{Sing}(Z')} \sum_i b_{i,x}\Theta_{i,x},$$

where $\Theta_{i,x}$'s denote the exceptional (-2) curves which form a basis of R_x. On the other hand, as $3D \in \overline{T}$ and $D \notin \overline{T}$, we have

$$3D \sim a'L + \sum_{x \in \mathrm{Sing}(Z')} \sum_i b'_{i,x}\Theta_{i,x},$$

where all $\Theta_{i,x} \in \overline{T}$, and at least one of a' and $b'_{i,x}$'s is not divisible by 3. Combining these two relations, we obtain a non-trivial linear relation among L and the $\Theta_{i,x}$'s, but this is impossible as they form a basis in T. □

By [26, Theorem 0.3], we have an $\mathfrak{S}_3$ covering, W', of $\mathbf{P}^2$ such that $D(W'/\mathbf{P}^2) = Z'$. Let W be the minimal resolution. $\mathrm{Gal}(W'/\mathbf{P}^2) \cong \mathfrak{S}_3$ also acts on W and let τ be an element of order 3. Then we have a commutative diagram

$$\begin{array}{ccc} W' & \longleftarrow & W \\ \downarrow & & \downarrow \\ Z' & \longleftarrow & W/\langle\tau\rangle. \end{array}$$

Since W' is a $K3$ surface with rational double points, τ has only isolated fixed points. Hence, by Lemma 8.8, $W/\langle\tau\rangle$ has singularities $6A_2$, and its minimal resolution is Z. Let $\Theta_{i,1}, \Theta_{i,2}$ $(i = 1, \dots, 6)$ be the exceptional curves. By our construction of W', these 12 curves give $\mathbf{A}_2^{\oplus 6}$ in the assumption in Theorem 0.7. Hence as $\mathbf{A}_1^{\oplus 4}$ in the assumption is disjoint from $\mathbf{A}_2^{\oplus 6}$, $\mathrm{NS}(W)$ contains 12 disjoint (-2) curves C_j $(j = 1, \dots, 12)$. Note that $\mathfrak{S}_3$ acts C_j's in such a way that, for any element, τ, of order 3, τ fixes no C_j. By Lemmas 8.5 and 8.7, if we choose 8 of the 12 C_j's, say $C_{j_1}, \dots, C_{j_8}$, appropriately, then $\sum_{k=1}^{8} C_{j_k}$ is 2-divisible in $\mathrm{NS}(W)$. Conversely, any 2-divisible member in $\bigoplus_j \mathbf{Z}C_j$ is represented in this form.

Lemma 12.2. *Let $D_1 = \sum_{l=1}^{8} C_{j_l}$ and $D_2 = \sum_{l=1}^{8} C_{k_l}$ be reduced divisor representing 2-divisible member of $\bigoplus_j \mathbf{Z}C_j$. Then either $D_1 = D_2$ or D_1 and D_2 have 4 exact common components.*

Proof. Suppose that $D_1 \neq D_2$ and let h be the number of common components. As $D_1 \neq D_2$, $D_1 + D_2 - 2$(*common components*) is also 2-divisible. Hence, by Lemma 8.5, the number of irreducible components, $16 - 2h$, is equal to 8, i.e., $h = 4$. □

Corollary 12.3. *Let τ be as above and let D be a 2-divisible reduced divisor in $\bigoplus_j \mathbf{Z}C_j$. Then D, τ^*D and $(\tau^2)^*D$ are distinct divisors. Moreover, any two of these three divisors have 4 common components.*

Proof. Suppose that $D = \tau^*D$. Then $(\tau^2)^*D = D$, and D is a τ-invariant divisor. On the other hand, as τ fixes no C_j, the number of irreducible components of any τ-invariant divisor is 3-divisible. This is impossible as the number of irreducible components of D is 8. Hence D, τ^*D and $(\tau^2)^*D$ are different to each other. The last assertion easily follows from Lemma 12.2. □

We now construct three effective reduced divisors D_1, D_2 and D_3 on W such that
(i) $\mathrm{Supp}(D_1 + D_2 + D_3) \subset \mathrm{Supp}(C_1 + \cdots + C_{12})$, and
(ii) D_1, D_2 and D_3 satisfy the conditions in Proposition 5.1.

Let τ be as before and let σ be an element of order 2 in $\mathfrak{S}_3$. Let D be any 2-divisible reduced divisor in $\bigoplus_j \mathbf{Z}C_j$. There are two possibilities: 1. $D = \sigma^*D$ and 2. $D \neq \sigma^*D$.

CASE 1. $D = \sigma^*D$.

Put $D_1 = \tau^*D$, $D_2 = (\tau^2)^*D$, and $D_3 = D$. Then these three divisors are distinct by Corollary 12.3, and satisfy
(i) $D_1^\sigma = D_2$, $D_1^\tau = D_2$, and $D_2^\tau = D_3$, and
(ii) D_1 is 2-divisible.

CASE 2. $D \neq \sigma^*D$.

Consider the divisor $D + \sigma^*D$. It is another 2-divisible divisor and is written in the form of $D' + 2D''$, where both D' and D'' are reduced and σ-invariant. Hence D' is 2-divisible as well as σ-invariant. Thus we can reduce our problem to Case 1. This finishes our proof of Theorem 0.7. □

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