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LIE ALGEBRAS OF DIFFERENTIAL OPERATORS

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The purpose of the present paper is to study the algebraic structure of the Lie algebra $\mathcal{D}(M)$ that consists of all the differential operators on a smooth manifold M. $\mathcal{D}(M)$ contains the Lie algebra $\mathcal{A}(M)$ of the vector fields on M as a subalgebra, which has been studied by many authors from various standpoints. Our investigation is motivated by Gelfand-Fuks's paper [1] concerning the cohomology theory of $\mathcal{A}(M)$. Indeed, their strong algebraic tendency has led us to expect that it will be fruitful to study differential operators from the viewpoint of Lie algebra.

Our main idea lies in regarding $\mathcal{D}(M)$ as a representation space of $\mathcal{A}(M)$ through the adjoint operations. This idea applies to the following two points. The first is to establish a kind of reducibility theorem with respect to this representation, which reveals certain characteristic features of the algebraic structure of $\mathcal{D}(M)$. The second is to consider the one-dimensional coholomogy group of $\mathcal{A}(M)$ associated with the representation, which yields a sufficient knowledge of the derivation space and the automorphism group of $\mathcal{D}(M)$.

We shall describe the outline of the present paper. Section 1 deals with basic notions and certain useful lemmas. Section 2 deals with $\mathcal{A}(M)$ and refers to Pursell-Shanks [5]. We give a characterization of the subalgebra that consists of the vector fields vanishing at a point of M. Using this characterization we can show that the algebraic structure of $\mathcal{A}(M)$ uniquely determines the smooth structure of M.

A subspace of $\mathcal{D}(M)$ is called an $\mathcal{A}$-space if it is invariant under the adjoint operations of $\mathcal{A}(M)$. For example, the space $\mathcal{D}_k(M)$ consisting of all the k-th order differential operators is an $\mathcal{A}$-space. In section 3, we give a structural theorem for $\mathcal{A}$-spaces, which states that any $\mathcal{A}$-space contained in $\mathcal{D}_k(M)$ coincides with one of a finite number of canonical $\mathcal{A}$-spaces in a neighborhood of all the points of M except those which lie in a nowhere-dense subset of M. From this structural theorem we can immediately deduce the theorems concerning ideals and those subalgebras which contain $\mathcal{A}(M)$, as we show in section 4.

In section 5, we determine the derivations of $\mathcal{D}(M)$ and certain subalgebras.

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Although these results can be also attained directly by a rather elementary way, we adopt here a cohomological method, relying on Losik's paper [2] and [6]. Section 6 concerns the isomorphisms of $\mathcal{D}(M)$ and the automorphism group $\mathrm{Aut}(\mathcal{D}(M))$ of $\mathcal{D}(M)$. We prove that the algebraic structure of $\mathcal{D}(M)$ determines the smooth structure of M. We also prove that $\mathrm{Aut}(\mathcal{D}(M))$ is the product of the subgroup consisting of those automorphisms which are induced by diffeomorphisms of M and a normal subgroup which is isomorphic to a semi-direct product of the group $\Lambda^1_{cl}(M)$ of the closed 1-forms on M and Z_2, or in short,

$$\mathrm{Aut}(\mathcal{D}(M)) \simeq \mathrm{Diff}(M) \times \Lambda^1_{cl}(M) \times Z_2 .$$

Up to section 6, we confine ourselves to the differential operators of finite order to avoid complications. In section 7, it is shown that all the results in the preceding sections remain valid for the differential operators whose orders may be unbounded around the point of infinity.

1. Preliminaries

We denote by M a smooth n-dimensional manifold with a countable basis. Throughout the present paper, M is assumed to be connected. For any non-negative integer k, we write $\mathcal{D}_k(M)$ for the space of real differential operators with the k-th order which are defined on M. We have a sequence of inclusions

$$\mathcal{D}_0(M) \subset \mathcal{D}_1(M) \subset \cdots \subset \mathcal{D}_k(M) \subset \cdots .$$

Let

$$\mathcal{D}(M) = \cup_{k=0}^{\infty} \mathcal{D}_k(M) .$$

The elements of $\mathcal{D}(M)$ are called differential operators with finite orders. We shall provide $\mathcal{D}(M)$ with the structure of a Lie algebra over $\boldsymbol{R}$, by setting

$$[\varphi, \psi] = \varphi \circ \psi - \psi \circ \varphi ,$$

where $\varphi \circ \psi$ means the composition of φ and ψ as differential operators. Thus $\mathcal{D}(M)$ becomes an infinite-dimensional Lie algebra over $\boldsymbol{R}$.

$\mathcal{D}_0(M)$ is identified with $C^\infty(M)$ the space of smooth functions on M, as a vector space. For $f, g \in \mathcal{D}_0(M)$, we have $[f, g]=0$, so that $\mathcal{D}_0(M)$ is an abelian subalgebra of $\mathcal{D}(M)$. The subspace of $\mathcal{D}_0(M)$ consisting of the constant functions forms the center of $\mathcal{D}(M)$, which we identify with $\boldsymbol{R}$. We have

$$[\mathcal{D}_k(M), \mathcal{D}_{k'}(M)] \subset \mathcal{D}_{k+k'-1}(M) .$$

From this it follows that $\mathcal{D}_1(M)$ is a subalgebra of $\mathcal{D}(M)$. But, if $k>1$, it is easy to see that $\mathcal{D}_k(M)$ does not form a subalgebra.

Let

$$\pi_0 : \mathcal{D}(M) \to \mathcal{D}_0(M)$$

be the projection which assings $\varphi(1)$ to φ, where $\varphi \in \mathcal{D}(M)$ and 1 denotes the constant function identically equal to one. We write $\tilde{\mathcal{D}}(M)$ for the kernel of π_0. Then $\tilde{\mathcal{D}}(M)$ is a subalgebra of $\mathcal{D}(M)$ and we have a direct sum decomposition

$$\mathcal{D}(M) = \tilde{\mathcal{D}}(M) \oplus \mathcal{D}_0(M).$$

Put

$$\tilde{\mathcal{D}}_k(M) = \tilde{\mathcal{D}}(M) \cap \mathcal{D}_k(M), \qquad k = 0, 1, 2, \cdots.$$

Note that $\tilde{\mathcal{D}}_0(M)=0$ and $\tilde{\mathcal{D}}_1(M)$ coincides with the Lie algebra of vector fields $\mathcal{A}(M)$ over M. The direct sum decomposition

$$\mathcal{D}_1(M) = \mathcal{A}(M) \oplus \mathcal{D}_0(M)$$

gives a semi-direct product, since $\mathcal{D}_0(M)$ is an ideal of $\mathcal{D}_1(M)$. According to this decomposition, the bracket in $\mathcal{D}_1(M)$ is expressed as follows:

$$[X+f, Y+g] = [X, Y] + (Xg - Yf), \qquad X, Y \in \mathcal{A}(M), \qquad f, g \in \mathcal{D}_0(M).$$

The support of $\varphi\ (\in \mathcal{D}(M))$ as differential operator is denoted by supp φ. If we put

$$\mathcal{D}(M)_c = \{\varphi \mid \varphi \in \mathcal{D}(M), \text{ supp } \varphi \text{ is compact}\},$$

then $\mathcal{D}(M)_c$ gives an ideal of $\mathcal{D}(M)$. We shall often use the index c for indicating a subspace of $\mathcal{D}(M)_c$; for example, $\mathcal{D}_k(M)_c = \mathcal{D}_k(M) \cap \mathcal{D}(M)_c$.

For a subset S of $\tilde{\mathcal{D}}(M)$ and an open set U of M, $S \mid U$ denotes the set of all the restrictions of elements of S to U, the restrictions being considered as elements of $\mathcal{D}(U)$.

We recall that a smooth function f is *flat* at a point p if all the derivatives of f vanish at p. To any point $p \in M$ we can assign an ideal $I(p)$ of $\mathcal{D}(M)$:

$$I(p) = \{\varphi \mid \varphi \in \mathcal{D}(M), \varphi(f) \text{ is flat at } p \text{ for every } f \in C^\infty(M)\}.$$

When, on some open set U, a local coordinate system $(x_1, \cdots, x_n)$ is given, then any element φ of $\mathcal{D}(M)$ is expressed as $\sum f_\alpha(x)\partial^\alpha$ (a finite sum) on U, where α ranges over a subset of multi-indices $\alpha=(\alpha_1, \cdots, \alpha_n)$, $\partial^\alpha = \partial^{|\alpha|}/(\partial x_1)^{\alpha_1} \cdots (\partial x_n)^{\alpha_n}$, $|\alpha| = \alpha_1 + \cdots + \alpha_n$, and $f_\alpha(x) \in C^\infty(U)$. We have then

$$[f_\alpha \partial^\alpha, g_\beta \partial^\beta] = \sum_{0<\gamma\leqq\alpha} \binom{\alpha}{\gamma} f_\alpha (\partial^\gamma g_\beta) \partial^{\alpha+\beta-\gamma} - \sum_{0<\gamma'\leqq\beta} \binom{\beta}{\gamma'} g_\beta (\partial^{\gamma'} f_\alpha) \partial^{\alpha+\beta-\gamma'}.$$

We note that, when $p \in U$, then $\varphi \in I(p)$ holds if and only if all the coefficients f_α's of φ are flat at p.

If the underlying manifold M is fixed, we shall often write $\mathcal{D}$, $\mathcal{D}_k$ etc. instead of $\mathcal{D}(M)$, $\mathcal{D}_k(M)$ etc.

Let f be an element of $C^\infty(\boldsymbol{R}^n)$, where supp f lies in the open unit disk of $\boldsymbol{R}^n$. Let U be any open set satisfying $U \supset \operatorname{supp} f$. Take $g \in C^\infty(\boldsymbol{R}^n)$ such that $g \equiv 1$ on supp f and supp $g \subset U$. Moreover, by $\tilde{x}_1$ we denote a smooth function on $\boldsymbol{R}^n$ which equals x_1 on supp f and whose support lies in U. Then we have

$$(1.1) \quad \text{i)} \qquad [f\partial^{\alpha+(1)}, \tilde{x}_1] = (\alpha_1+1)f\partial^{\alpha},$$

where $\alpha=(\alpha_1, \cdots, \alpha_n)$ *and* $\alpha+(1)=(\alpha_1+1, \alpha_2, \cdots, \alpha_n)$.

$$\text{ii)} \quad \text{a)} \quad [g\partial_1, (g^m \int_0 f dx^1)\partial_1^m] = f\partial_1^m + (\text{lower order terms});$$

$$\text{b)} \quad [g\partial_1, (g^m \int_0 f dx^1)\partial_1^m \partial^\beta]$$

$$= f\partial_1^m\partial^\beta + \varphi + (\text{lower order terms}).$$

Here $\beta=(0, \beta_2, \cdots, \beta_n)$ *and* φ *consists of the terms with the form* $h\partial_1^{m+1}\partial^\gamma$, *where* $\gamma=(0, \gamma_2, \cdots, \gamma_n)$, $|\gamma|=|\beta|-1$ *and* $h \in C^\infty(\boldsymbol{R}^n)$.

An induction argument, combined with (2.1) ii), immediately yields

(1.2) *Let* $\varphi \in \tilde{\mathcal{D}}_k(\boldsymbol{R}^n)$ *(or* $\in \mathcal{D}_k(\boldsymbol{R}^n)$*) such that* supp φ *is contained in the open unit disk. Let* U *be an open set satisfying* supp $\varphi \subset U$. *Then there are an integer* $\nu(k)$ *depending only on* k *and* n, $X_\nu \in \mathcal{A}(\boldsymbol{R}^n)$ $(\nu=1, \cdots, \nu(k))$ *with* supp $X_\nu \subset U$ *and* $\psi_\nu \in \tilde{\mathcal{D}}_k(\boldsymbol{R}^n)$ *(or* $\in \mathcal{D}_k(\boldsymbol{R}^n)$*)* $(\nu=1, \cdots, \nu(k))$ *with* supp $\psi_\nu \subset U$ *such that*

$$\varphi = \sum_\nu [X_\nu, \psi_\nu].$$

We have another formula in case of $\mathcal{A}(M)$. Let f and g be as above. Take $h \in C^\infty(\boldsymbol{R}^n)$ which satisfies supp $h \subset U$ and $h \equiv 1$ on supp g. Then

$$(1.3) \qquad [[\partial_1, (h\int_0 g dx_1)\partial_i], (g\int_0 f dx_i)\partial_i] = f\partial_i.$$

From (1.1) and (1.2), we can obtain the following proposition.

$$(1.4) \quad \text{i)} \qquad \mathcal{D}(M) = [\mathcal{D}(M), \mathcal{A}(M)],\ \tilde{\mathcal{D}}(M) = [\tilde{\mathcal{D}}(M), \mathcal{A}(M)];$$

$$\text{ii)} \qquad \mathcal{D}_k(M) = [\mathcal{D}_k(M), \mathcal{A}(M)],\ \tilde{\mathcal{D}}_k(M) = [\tilde{\mathcal{D}}_k(M), \mathcal{A}(M)]$$

$$(k = 0, 1, 2, \cdots).$$

iii) *The similar formulas hold when each space is replaced by the intersection with* $\mathcal{D}(M)_c$.

Proof. Note that i) is an immediate consequence of ii). The assertion ii) in the case where M is compact and the assertion iii) are both easily checked from (1.2) and by the use of the partition of unity. For the proof of ii) in the case where M is open, we shall make use of the fact (cf. [3]) that M admits a finite open covering:

(1.5) $M=O_0\cup O_1\cup\cdots\cup O_n$ such that, for each i, $O_i=\cup_{j=1}^{\infty}O_{ij}$ where each family $\{O_{ij}\mid j=1,2,\cdots\}$ is a locally finite family of mutually disjoint open sets; each O_{ij} is realized as the open unit disk with reference to certain local coordinate system.

Hence any $\varphi\in\mathcal{D}_k(M)$ is expressed as $\varphi=\varphi_0+\varphi_1+\cdots+\varphi_n$ where $\varphi_i=\sum_{j=1}^{\infty}\varphi_{ij}$ with $\operatorname{supp}\varphi_{ij}\subset O_{ij}$. Applying (1.2) to $\mathcal{D}_k(O_{ij})$, we have $\varphi_{ij}\in[\mathcal{D}_k(O_{ij})_c, \mathcal{A}(O_{ij})_c]$ and so

$$\varphi_i\in[\mathcal{D}_k(M), \mathcal{A}(M)], \qquad i=0, 1, \cdots, n.$$

The similar arguments hold for $\tilde{\mathcal{D}}_k(M)$. This completes the proof.

In the course of the above proof, we have used the following convention. For any open set O of M, there exists the canonical injective homomorphism $\mathcal{D}(O)_c\to\mathcal{D}(M)$, through which we identify $\mathcal{D}(O)_c$ with a subalgebra of $\mathcal{D}(M)$. Henceforth, we shall use this identification without any comment.

As a consequence of (1.4), we have

(1.6) *$\mathcal{D}(M)$, $\tilde{\mathcal{D}}(M)$, $\mathcal{D}_1(M)$ and $\mathcal{A}(M)$ coincide with their respective commutator subalgebras.*

2. Vector fields

We first recall the classical result.

(2.1) (Pursell-Shanks [5]). *Let I be an ideal of $\mathcal{A}(M)$. Then we have one of the following two cases:*

i) $I\subset I(p)\cap\mathcal{A}(M)$ *for some point* $p\in M$.
ii) $I\subset\mathcal{A}(M)_c$.

For the sake of completneess, we shall give a proof to (2.1).

We assume that i) does not hold. Then for any $p\in M$ we can find $X\in I$ such that $X\notin I(p)$. Take a local coordinate system around p and let $Y_1,\cdots,Y_n$ be vector fields which satisfy $Y_i=\partial_i$ near p. Then by the repeated application of $\operatorname{ad}(Y_i)$ $(i=1,2,\cdots,n)$ to X we can get $\tilde{X}\in I$ with $\tilde{X}(p)\neq 0$. This, in turn, implies that I contains a vector field which is equal to ∂_1 near p for a suitable local coordinate system. Then (1.3), together with the partition of unity, shows that the alternative ii) holds. This completes the proof.

As a result, we find that $I(p)\cap\mathcal{A}(M)$ is a maximal ideal of $\mathcal{A}(M)$; besides, any maximal ideal of $\mathcal{A}(M)_c$ is obtained in this way. Hence for any proper ideal $I\subset\mathcal{A}(M)_c$ we have $\operatorname{codim} I=\infty$.

Now we proceed to study the subalgebras of $\mathcal{A}(M)$. Let $N(p)$ be the subalgebra which consists of vector fields vanishing at $p\in M$. Note that codim $N(p)=n$.

Theorem (2.2). *$N(p)$ is characterized as a maximal subalgebra with finite codimension.*

Proof. It suffices to show that a proper subalgebra B with finite codimension is included in some $N(p)$. We assume the contrary; that is, for any $p\in M$, $B\not\subset N(p)$. We shall show that from the assumption we are led to a conclusion $B=\mathcal{A}(M)$, which is a contradiction. The proof is divided in three steps.

(i): We shall show that for any $p\in M$ there is a neighborhood U of p such that $B|U=\mathcal{A}(M)|U$. By the assumption there is $X\in B$ satisfying $X(p)\neq 0$. We may assume X is equal to ∂_1 on some neighborhood U of p in terms of a suitable local coordinate system. Put $E=\mathcal{A}(M)|U$, $F=B|U$ and $\varphi=\mathrm{ad}(X)|U$. Then we have codim $E/F<+\infty$, and $\varphi(F)\subset F$ because B is a subalgebra. Take any non-zero polynomial P over $\boldsymbol{R}$ and consider the endomorphism $P(\varphi)$ of E. Since $P(\varphi)$ is expressed as a linear differential equation in ∂_1, $P(\varphi)$ becomes a surjective endomorphism of E.

We are now in a position to apply the following fact, which is easily verified.

(2.3) *Let E be a vector space over $\boldsymbol{R}$ and F a subspace of E with finite codimension. Suppose that there is an endomorphism φ of E such that $\varphi(F)\subset F$ and, for any non-zero polynomial P, $P(\varphi)$ is a surjective endomorphism of E. Then we have $E=F$.*

Hence we have proved $B|U=\mathcal{A}(M)|U$.

(ii): We shall prove that, if $B|U=\mathcal{A}(M)|U$ holds for an open set U, then $B\supset\mathcal{A}(U)_c$. Set $B(U)_c=B\cap\mathcal{A}(U)_c$. First observe that $B(U)_c$ is an ideal of $\mathcal{A}(U)_c$. In fact, by the assumption for any $Y\in\mathcal{A}(U)_c$ we can find $\tilde{Y}\in B$ such that $\tilde{Y}=Y$ on U; we then obtain

$$\mathrm{ad}(Y)(B(U)_c)=\mathrm{ad}(\tilde{Y})(B(U)_c)\subset B(U)_c\,.$$

Since we have $\dim\mathcal{A}(U)_c/B(U)_c\leqq\dim\mathcal{A}/B<+\infty$, and $\mathcal{A}(U)_c$ does not possess any proper ideal with finite codimension, it follows $\mathcal{A}(U)_c=B(U)_c$, as we wished to prove.

(iii): Making use of the partition of unity, from i) and ii) we can get the conclusion $B\supset\mathcal{A}(M)_c$. This completes the proof when M is compact. Now, let M be open. Take a covering $M=\cup O_{ij}(i=0,1,\cdots,n;j=1,2,\cdots)$ with the properties stated in (1.5). Fix i and choose $Y_s\in\mathcal{A}(M)$ $(s=0,1,2,\cdots)$ such that $Y_s=j^s\partial_1$ on each O_{ij} $(j=1,2,\cdots)$. If codim $B=d$, we have a non-trivial linear relation

$$a_0 Y_0 + \cdots + a_d Y_d \in B$$

for suitable $a_i \in \boldsymbol{R}$. Set $Y = a_0 Y_0 + \cdots + a_d Y_d$. Note that $Y | O_{ij} = \sum_{s=0}^{d} (a_s j^s) \partial_1$. Hence, if we take j sufficiently large, say $j > j'(i)$, we have $Y | O_{ij} = C_{ij} \partial_1$ $(0 \neq C_{ij} \in \boldsymbol{R})$. The existence of such a Y implies, by the same argument as in (i), that

$$(*) \qquad B | \cup_{j > j'(i)} O_{ij} = \mathcal{A}(M) | \cup_{j > j'(i)} O_{ij} .$$

Let $M = \cup O'_{ij}$ be another open covering of M satisfying the condition (1.5) and $\bar{O}'_{ij} \subset O_{ij}$. For each i, take $Y'_s \in \mathcal{A}(M)$ $(s = 0, 1, 2, \cdots)$ such that

$$Y'_s = \begin{cases} j^s \varphi_j \partial_1 , & \text{on } O_{ij} \\ 0 , & \text{on the complement of } \cup_j O_{ij} , \end{cases}$$

where φ_j is a smooth function which equals to 1 on O'_{ij} and vanishes outside O_{ij}. Then, by a similar argument together with the fact $B \supset \mathcal{A}(M)_c$, we can find an element $Y' \in B$ and an integer $j(i)$ with $j(i) \geqq j'(i)$ for which we have $Y' | O'_{ij} = C'_{ij} \partial_1$ $(0 \neq C'_{ij} \in \boldsymbol{R})$ for $j > j(i)$ and $Y' = 0$ outside $\cup_{j > j(i)} O_{ij}$. Let Z be any element of $\mathcal{A}(M)$ with supp $Z \subset \cup_{j > j(i)} O'_{ij}$. By (1.3) there exist X_j, $X'_j \in \mathcal{A}(M)$ $(j > j(i))$ such that supp X_j, supp $X'_j \subset O'_{ij}$ and

$$[[Y', X_j], X'_j] | O_{ij} = Z | O_{ij} .$$

But, in view of (*), we can find X, $X' \in B$ with $X | O_{ij} = X_j | O_{ij}$, $X' | O_{ij} = X'_j | O_{ij}$ $(j > i(i))$. Then we have

$$Z = [[Y', X], X'] \in B .$$

Thus we have proved that any element of $\mathcal{A}(M)$ whose support lies in $\cup_{j > j(i)} O'_{ij}$ belongs to B.

Similar results hold for $i = 0, 1, \cdots, n$. These together show that there exists a compact set $K \subset M$ for which we have

$$B | K^c = \mathcal{A}(M) | K^c.$$

Combining this with the fact $B \supset \mathcal{A}(M)_c$, we have finally $B = \mathcal{A}(M)$. This completes the proof of (2.2).

Referring to (2.1) and (2.2), we find that there is no proper ideal with finite codimension. Hence we have

(2.4) *$\mathcal{A}(M)$ has no non-trivial finite-dimensional representation.*

REMARK. There is a maximal subalgebra which is not of finite codimension. In fact, take two distinct points p, q from a local coordinates neighborhood U, and set $B = \{X | X \in \mathcal{A}(M),\ \partial^\alpha f^i(p) = \partial^\alpha f^i(q)$ for $i = 1, \cdots, n$, $|\alpha| = 0, 1, 2, \cdots$, when X is written as $\sum f^i(x) \partial_i$ on $U\}$. Then B is a subalgebra which possesses the desired properties.

Using (2.2), we shall establish the following theorem which gives a generalization of Pursell-Shanks's Theorem [5].

Theorem (2.5). *Let M and N be two smooth manifolds and let Φ be a Lie algebra isomorphism from $\mathcal{A}(M)$ to $\mathcal{A}(N)$. Then there is a unique diffeomorphism Ψ from M onto N such that $\Phi=\Psi_*$.*

Proof. Φ sends any maximal subalgebra of $\mathcal{A}(M)$ with finite codimension to a subalgebra of $\mathcal{A}(N)$ with the same property. Thus from (2.2) we can get a bijection Ψ from M onto N, such that $\Phi(N(p))=N(\Psi(p))$. Note that Ψ^{-1} is similarly related to Φ^{-1}.

(i) Ψ is a homeomorphism: Let K be a closed set of M. Assume that $\Psi(K)$ is not a closed set of N. Then there is a sequence $\{q_i\}$ of $\Psi(K)$ which tends to a point q outside $\Psi(K)$. Let $p=\Psi^{-1}(q)$ and $p_i=\Psi^{-1}(q_i)$. Take an $X \in \mathcal{A}(M)$ such that $X(p)\neq 0$, $X|K=0$. Then we have $\Phi(X)(q_i)=0$ since $\Phi(X) \in N(q_i)$, so that $\Phi(X)(q)=0$. This implies $X \notin N(p)$ and $\Phi(X)\in N(\Psi(p))$ which is a contradiction. Hence Ψ maps a closed set to a closed set. Since Ψ^{-1} has the same property, Ψ is a homeomorphism.

(ii) Ψ is a diffeomorphism: For any given $p\in M$, take a local coordinate system $\{U; x_1, \cdots, x_n\}$ around p, where p corresponds to the origin $(0, \cdots, 0)$. Set $q=\Psi(p)$. Let $X_1, \cdots, X_n$ be vector fields on M such that $X_i|U=\partial/\partial x_i$. Then $[X_i, X_j]=0$ on U and so $[\Phi(X_i), \Phi(X_j)]=0$ on $\Psi(U)$. Hence, there is a local coordinate system $\{V; y_1, \cdots, y_n\}$ around q such that $\Phi(X_i)=\partial/\partial y_i$ on V. We may assume that $\Psi(U)=V$ and q has the coordinates $(0, \cdots, 0)$. If $X=Y$ on U, then we have $\Phi(X)=\Phi(Y)$ on V, so that we have only to consider the behavior of Φ on U. We write $\partial_i=\partial/\partial x_i$ and $\bar{\partial}_i=\partial/\partial y_i$ $(i=1, 2, \cdots, n)$. Since $[\partial_k, x_i\partial_j]=\delta_{ik}\partial_j$ on U, we have $[\bar{\partial}_k, \Phi(x_i\partial_j)]=\delta_{ik}\bar{\partial}_j$ on V (δ_{ik} denotes the Kronecker symbol). Hence we obtain $\Phi(x_i\partial_j)=y_i\bar{\partial}_j+C$, where C is a constant vector. Since $x_i\partial_j$ vanishes at the origin, $y_i\bar{\partial}_j+C$ has the same property, whence $C=0$ follows. Therefore we have

$$\Phi((x_i-a_i)\partial_j) = (y_i-a_i)\bar{\partial}_j\,,$$

which, in turn, implies that $\Psi|U$ sends a point $\tilde{p}$ with coordinates $(a_1, \cdots, a_n)$ to a point $\tilde{p}$ $(\in V)$ with the same coordinates. Thus Ψ is a diffeomorphism.

(iii) $\Psi_*=\Phi$. This follows immediately from the fact stated in (ii).

Since the uniqueness of Ψ is obvious, (2.5) has been proved by (i), (ii) and (iii).

REMARK. By the similar arguments we can prove that a Lie algebra isomorphism from $\mathcal{A}(M)_c$ to $\mathcal{A}(N)_c$ is induced from a diffeomorphism from M to

N. This result is due to Pursell-Shanks [5]. Their proof, however, depends upon the characterization of maximal ideals and does not seem to be directly applicable to $\mathcal{A}(M)$.

3. A-spaces

A subspace E of $\mathcal{D}(M)$ is called an *$\mathcal{A}$-space* if it is invariant under the adjoint operation of $\mathcal{A}(M)$. In other words, E is an $\mathcal{A}$-space if for any $\varphi\in E$ and any $X\in\mathcal{A}(M)$ we have $[X, \varphi]\in E$. A linear map Φ from an $\mathcal{A}$-space E to an $\mathcal{A}$-space F is called an *$\mathcal{A}$-map* if Φ is equivariant with respect to the adjoint operation of $\mathcal{A}(M)$, or in other words, $\Phi[X, \varphi]=[X, \Phi(\varphi)]$ $(X\in\mathcal{A}(M),\ \varphi\in E)$.

If E and F are $\mathcal{A}$-spaces, then $E+F$ is an $\mathcal{A}$-space. If $\{E_\lambda\}_{\lambda\in\Lambda}$ is a family of $\mathcal{A}$-spaces, then $\cap_{\lambda\in\Lambda}E_\lambda$ is an $\mathcal{A}$-space. It is clear that $\mathcal{D}_k(M)$, $\tilde{\mathcal{D}}_k(M)$ $(k=0, 1, 2, \cdots)$ and $I(p)$ $(p\in M)$ give examples of $\mathcal{A}$-spaces. Hence, for any subset $S\subset M$,

$$\mathcal{D}_k(M)\cap(\cap_{p\in S}I(p)),\quad \tilde{\mathcal{D}}_k(M)\cap(\cap_{p\in S}I(p))$$

also are $\mathcal{A}$-spaces. If Φ is an $\mathcal{A}$-map from E to F, then both Ker Φ and Im Φ become $\mathcal{A}$-spaces.

In the special case $n=1$, as we shall show below, there exists an interesting $\mathcal{A}$-map $\natural$, by the use of which certain $\mathcal{A}$-spaces can be constructed. Since a one-dimensional connected manifold M is isomorphic either to the circle or to the real line, the differentiation ∂ by the usual coördinate function gives rise to a vector field which does not vanish at any point of M. Let φ^* denote the formal adjoint of $\varphi\in\mathcal{D}(M)$, which is defined by using the standard measure on M. Then we have

$$\partial^* = -\partial\,.$$

Since every element of $\tilde{\mathcal{D}}(M)$ can be uniquely expressed as the product $\varphi\partial$ for $\varphi\in\mathcal{D}(M)$, we can define the involutive mapping $\varphi\partial\to(\varphi\partial)^\natural$ of $\tilde{\mathcal{D}}(M)$ to itself by putting

$$(\varphi\partial)^\natural = \varphi^*\partial\,.$$

For any φ, $\psi\in\mathcal{D}(M)$, we have

$$\{(\varphi\partial)(\psi\partial)\}^\natural = (\varphi\partial\psi)^*\partial = -\psi^*\partial\varphi^*\partial = -(\psi\partial)^\natural(\varphi\partial)^\natural$$

and hence

$$[\varphi\partial, \psi\partial]^\natural = [(\varphi\partial)^\natural, (\psi\partial)^\natural]\,,$$

which implies that $\natural$ is an automorphism of $\mathcal{D}(M)$. Since $f^*=f$ for every $f\in\mathcal{D}_0(M)$, the restriction of $\natural$ to $\mathcal{A}(M)$ is the identity. It follows that $\natural$ is an $\mathcal{A}$-map from $\tilde{\mathcal{D}}(M)$ to itself.

Since $\natural$ is involutive, we have a direct sum decomposition

$$(*)\qquad \tilde{\mathcal{D}}(M) = \tilde{\mathcal{D}}^+ \oplus \tilde{\mathcal{D}}^- ,$$

where

$$\tilde{\mathcal{D}}^+ = \{\varphi \,|\, \varphi^\natural = \varphi\}, \qquad \tilde{\mathcal{D}}^- = \{\varphi \,|\, \varphi^\natural = -\varphi\} .$$

It is clear that both $\tilde{\mathcal{D}}^+$ and $\tilde{\mathcal{D}}^-$ are $\mathcal{A}$-spaces.

For later use, we shall give an explicit description of the elements belonging to $\tilde{\mathcal{D}}^+$ and $\tilde{\mathcal{D}}^-$. Observe that any element of $\tilde{\mathcal{D}}(M)$ is uniquely expressed in the form

$$[\partial^k, f_k\partial]+[\partial^{k-1}, f_{k-1}\partial]+\cdots+[\partial, f_1\partial] \qquad (f_i \in C^\infty(M))$$

Since $(\partial^{2k+1})^\natural = \partial^{2k+1}$ and $(\partial^{2k})^\natural = -\partial^{2k}$, we have

$$\tilde{\mathcal{D}}^+ = \{\varphi \,|\, \varphi = [\partial^{2k+1}, f_{2k+1}\partial]+[\partial^{2k-1}, f_{2k-1}\partial]+\cdots;\ k = 0, 1, 2, \cdots\} ,$$
$$\tilde{\mathcal{D}}^- = \{\varphi \,|\, \varphi = [\partial^{2k}, f_{2k}\partial]+[\partial^{2k-2}, f_{2k-2}\partial]+\cdots;\ k = 1, 2, \cdots\} .$$

Hence if we write

$$\mathcal{P}_k = \{\varphi \,|\, \varphi = [\partial^k, f_k\partial]+[\partial^{k-2}, f_{k-2}\partial]+[\partial^{k-4}, f_{k-4}\partial]+\cdots\}$$

for $k=1, 2, \cdots$, then each $\mathcal{P}_k$ is an $\mathcal{A}$-space and we have

$$\tilde{\mathcal{D}}^+ = \cup_{k=0}^\infty \mathcal{P}_{2k+1}, \qquad \tilde{\mathcal{D}}^- = \cup_{k=1}^\infty \mathcal{P}_{2k} .$$

Now we shall begin with the study on the local features of $\mathcal{A}$-spaces. For this purpose, it is useful to introduce a linear map $T_{i,m}$ of $\mathcal{D}(\boldsymbol{R}^n)$ to itself, defined for each integer m and $i=1, 2, \cdots, n$ by setting

$$T_{i,m}(\varphi) = [[\varphi, x_i\partial_i], x_i\partial_i]-[[\varphi, x_i^2\partial_i], \partial_i]$$
$$-[\varphi, x_i\partial_i]-m(m+1)\varphi, \qquad \varphi \in \mathcal{D}(\boldsymbol{R}^n) .$$

Then, for any multi-index $\alpha=(\alpha_1, \cdots, \alpha_n)$ we have

$$(**)\quad T_{i,m}(f\partial^\alpha) = \{\alpha_i(\alpha_i+1)-m(m+1)\} f\partial^\alpha+\alpha_i(\alpha_i-1)(\partial_i f)\partial^{\alpha-(i)} ,$$

where $\alpha-(i)$ denotes $(\alpha_1, \cdots, \alpha_{i-1}, \alpha_i-1, \alpha_{i+1}, \cdots, \alpha_n)$. Here the first term of the right hand side vanishes if and only if $m=\alpha_i$ and the second term vanishes when $\alpha_i=0$ or $\alpha_i=1$ or f is constant.

Let $\varphi \in \tilde{\mathcal{D}}_{k_0}(\boldsymbol{R}^n)$. Denote by $E(\varphi)$ the smallest $\mathcal{A}$-space of $\tilde{\mathcal{D}}(\boldsymbol{R}^n)$ containing φ. That is, $E(\varphi)$ is the subspace of $\tilde{\mathcal{D}}(\boldsymbol{R}^n)$ spanned by the elements with the form $\mathrm{ad}(X_1)\cdots\mathrm{ad}(X_s)\varphi$, where $X_1, \cdots, X_s \in \mathcal{A}(\boldsymbol{R}^n)$ and $s=0, 1, 2, \cdots$. We can find an integer k such that φ can be written as

$$\varphi = \sum_{k_0 \geqq |\alpha| > k} b_\alpha(x)\partial^\alpha + \sum_{|\alpha| \leqq k} a_\alpha(x)\partial^\alpha ,$$

where $b_\alpha(x)$ are all flat at 0 and some of $a_\alpha(x)$ ($|\alpha| = k$) is not flat at 0. It should be noted that k is invariant under the change of local coordinates. The integer k is called the *essential order* of φ at 0. Since $\mathrm{ad}(X)$ ($X \in \mathcal{A}(\boldsymbol{R}^n)$) does not increase the essential order k, we know that any element belonging to $E(\varphi)$ has the essential order $k'(\leqq k)$ at 0. Moreover, it is clear that $E(\varphi) \subset \tilde{\mathcal{D}}_{k_0}(\boldsymbol{R}^n)$.

The following proposition is crucial.

(3.1) *There is a neighborhood U of* 0 *such that*

$$E(\varphi)|U \supset \mathcal{P}_k|U, \qquad \textit{if } n = 1;$$
$$E(\varphi)|U \supset \tilde{\mathcal{D}}_k|U, \qquad \textit{if } n > 1 .$$

Proof. First note that $T_{i,m}(\psi) \in E(\varphi)$ for any $\psi \in E(\varphi)$. We shall take a multi-index $\beta = (\beta_1, \cdots, \beta_n)$ with $|\beta| = k$ such that $a_\beta(x)$ is not flat at 0. Let $\alpha' = (\alpha'_1, \cdots, \alpha'_n)$ be any multi-index with $|\alpha'| = k_0$. If $k_0 > k$, then $|\alpha'| > |\beta|$, so that we can find α'_i with $\alpha'_i \neq \beta_i$. Then the application T_{i,α'_i} to φ yields an element of $E(\varphi)$ whose principal part is $\sum_{|\alpha| = k_0} c_\alpha b_\alpha(x)\partial^\alpha (c_\alpha \in \boldsymbol{R})$ with $c_{\alpha'} = 0$, and whose essential order is k at 0. Hence, starting from φ and applying suitable $T_{i,m}$'s successively, we can eliminate all the terms of φ occuring from the order k_0 to $k+1$, and in this procedure the essential order at 0 is kept invariant. Thus we arrive at the conclusion that there exists an element φ_1 of $E(\varphi)$ such that both the order and the essential order of φ_1 are k.

Consider a multi-index α with $|\alpha| = k$, such that the coefficient of ∂^α in the expression of φ_1 is not flat at 0. After a finite number of suitable applications of $\mathrm{ad}(\partial_i)$'s to φ_1 ($i = 1, 2, \cdots, n$), we can get $\varphi_2 \in E(\varphi)$ for which the coefficient f of ∂^α does not vanish at 0. Now, for any i with $\alpha_i \neq 0$, we can find a smooth function g satisfying

$$\frac{\partial f}{\partial x_i} g - \alpha_i f \frac{\partial g}{\partial x_i} = 1$$

in a neighborhood U of 0. Let $\varphi_3 = \mathrm{ad}(g\partial_i)\varphi_2$. Then the principal part of $\varphi_3|U$ contains a term with the form ∂^α. Actually U is the required neighborhood of 0. In order to prove this, we may assume without losing generality $U = \boldsymbol{R}^n$.

Let

$$\varphi_3 = \partial^\alpha + \sum_{\substack{|\gamma| = k \\ \gamma \neq \alpha}} \tilde{a}_\gamma(x)\partial^\gamma + (\text{lower order terms}).$$

Take any multi-index β with $\alpha \neq \beta$ and $|\beta| = k$. Let i be any integer such that $\alpha_i \neq \beta_i$. Then $T_{i,\beta_i}(\varphi_3)$ has the form

$$T_{i,\beta_i}(\varphi_3) = c_\alpha \partial^\alpha + \sum_{\substack{|\gamma| = k \\ \gamma \neq \alpha, \beta}} c_\gamma \tilde{a}_\gamma(x)\partial^\gamma + (\text{lower order terms}),$$

where c_α, c_γ are real numbers, and moreover $c_\alpha \neq 0$. Thus T_{i,β_i} eliminates from φ_3 the term associated to ∂^β. Note that $T_{i,\beta_i}(\partial^\alpha)=c_\alpha\partial^\alpha$. Therefore, repeating the similar procedure, we can eliminate all the terms of φ_3 except ∂^α. Thus we can conclude $\partial^\alpha \in E(\varphi)$. Next, the applications of $\mathrm{ad}(x_i\partial_j)$'s $(i, j=1, \cdots, n)$ to ∂^α shows that $\partial^\beta \in E(\varphi)$ for every β with $|\beta|=k$.

Now we shall deal with the case $n=1$ and $n>1$ separately. In case $n=1$, we have

$$(k+1)k(k-1)(k-2)\partial^{k-2} = 2(2k-1)[\partial^k, x^3\partial]-3[[\partial^k, x^2\partial], x^2\partial]\,,$$

while, in case $n>1$

$$2\partial_2^{k-1} = [\partial_1^2\partial_2^{k-2}, x_1^2\partial_2]-2[\partial_1\partial_2^{k-1}, x_1^2\partial_1]\,.$$

It follows from these that, in case $n=1$,

$$\partial^{k-2} \in E(\varphi) \qquad \text{for } k \geqq 3$$

and, in case $n>1$

$$\partial_1^{k-1} \in E(\varphi) \qquad \text{for } k \geqq 2\,.$$

Applying the same argument successively, we can conclude that

$$\partial^k, \partial^{k-2}, \cdots \in E(\varphi) \qquad \text{in case } n=1$$
$$\partial^\beta \in E(\varphi) \qquad \text{for every } \beta \text{ with } |\beta| \leqq k, \text{ in case } n>1.$$

This however, completes the proof in the case $n=1$. In case $n>1$, consider further the relation

$$\mathrm{ad}(g\partial_i)(\partial^\alpha) = -\alpha_i \frac{\partial x}{\partial x_i}\partial^\alpha + (\text{lower order terms})$$

where $g \in C^\infty(\boldsymbol{R}^n)$. From this we can easily deduce the desired result $\tilde{\mathscr{D}}_k \subset E(\varphi)$, which completes the proof.

We have actually also proved the following proposition.

(3.2) *In case* $n=1$, $E(\partial^k, \partial^{k-1}) = \tilde{\mathscr{D}}_k(\boldsymbol{R}^1)$ for $k=1, 2, \cdots$
In case $n>1$, $E(\partial_i{}^k) = \tilde{\mathscr{D}}_k(\boldsymbol{R}^n)$ *for* $i=1, \cdots, n$ *and* $k=1, 2, \cdots$.

Here we write $E(\partial^k, \partial^{k-1})$ *for the* $\mathcal{A}$*-space of* $\tilde{\mathscr{D}}(\boldsymbol{R}^1)$ *generated by* ∂^k *and* ∂^{k-1}, *and we set* $\partial^0=0$.

Since the notion of the essential order introduced above is invariant under the change of local coordinates, we can define for any $\varphi \in \tilde{\mathscr{D}}(M)$ the *essential order* of φ at every point $p \in M$, by using the local expression of φ around p. We denote this by ess.ord.$_p\varphi$. In case $n=1$, according to the splitting $\tilde{\mathscr{D}}(M)=\tilde{\mathscr{D}}^+\oplus\tilde{\mathscr{D}}^-$ given in (*), we can write any element φ as $\varphi=\varphi^++\varphi^-$. Then we define

$$\text{ess.ord.}_p{}^+\varphi=\text{ess.ord.}_p\varphi^+,\qquad \text{ess.ord.}_p{}^-\varphi=\text{ess.ord.}_p\varphi^-\,.$$

Let E be an $\mathscr{A}$-space of $\tilde{\mathscr{D}}_{k_0}(M)$. Define

$$\begin{aligned} O_{2s,2t-1} = \{p\,|\,p\in M;\ &\text{There exists a neighborhood } U \text{ of } p \text{ such that}\\ &E\,|\,U(p)=(\mathscr{P}_{2s}+\mathscr{P}_{2t-1})\,|\,U\} \qquad \text{in case } n=1\,, \end{aligned}$$

($\mathscr{P}_{-1}=\{0\}$) and

$$\begin{aligned} O_l = \{p\,|\,p\in M;\ &\text{There exists a neighborhood of } p \text{ such that}\\ &E\,|\,U=\tilde{\mathscr{D}}_l(M)\,|\,U\} \qquad \text{in case } n>1\,, \end{aligned}$$

where l ranges over $0, 1, \cdots, k_0$, s over $0, 1, \cdots, [k_0/2]$ and t over $0, 1, \cdots, [(k_0+1)/2]$. Then the family of $\{O_{2s,2t-1}\}$ or $\{O_l\}$ forms a family of disjoint sets open of M. For every point $p\in M$, put

$$k(p)=\operatorname*{Max}_{\varphi\in E}\ \text{ess.ord.}_p\varphi;$$

moreover, in case $n=1$ put

$$k^+(p)=\operatorname*{Max}_{\varphi\in E}\ \text{ess.ord.}_p{}^+\varphi,\qquad k^-(p)=\operatorname*{Max}_{\varphi\in E}\ \text{ess.ord.}_p{}^-\varphi\,.$$

A global version of (3.1) now can be formulated as follows:

Theorem (3.3). *Let E be an $\mathscr{A}$-space of $\tilde{\mathscr{D}}_{k_0}(M)$. Then*

i) *Both the open sets $\bigcup_{s,t} O_{2s,2t-1}$ and $\bigcup_{l=0}^{k_0} O_l$ are dense in M.*

ii) *In case $n=1$,*

$$(\mathscr{P}_{2s}+\mathscr{P}_{2t-1})(O_{2s,2t-1})_c\subset E\,|\,O_{2s,2t-1};$$

in case $n>1$,

$$\tilde{\mathscr{D}}_l(O_l)_c\subset E\,|\,O_l\,.$$

iii) *For each point $p\in M$, there is a neighborhood U of p, on which we have in case $n=1$*

$$\mathscr{P}_{k^+(p)}+\mathscr{P}_{k^-(p)}+(I(p)\cap E)\supset E\supset\mathscr{P}_{k^+(p)}+\mathscr{P}_{k^-(p)};$$

in case $n>1$

$$\tilde{\mathscr{D}}_{k(p)}(M)+(I(p)\cap E)\supset E\supset\tilde{\mathscr{D}}_{k(p)}\,.$$

Proof. i) Suppose, for example, $\bigcup O_l$ is not dense. Let Q be the open set which consists of the interior points of $(\bigcup O_l)^c$. Let

$$k' = \operatorname*{Max}_{\substack{\varphi \in E \\ p \in Q}} \text{ess.ord.}_p \varphi .$$

Referring to the definition of the essential order, we find that $E|Q \subset \mathcal{D}_{k'}(Q)$ holds. Assume that for $\varphi_0 \in E$ and for a point $p_0 \in Q$ we have really $k' = \text{ess.ord.}_{p_0} \varphi_0$. Then, applying (3.1) to φ_0 and p_0, we can deduce that there exists a neighborhood $U(\subset Q)$ of p_0 such that $E|U = \mathcal{D}_{k'}(M)|U$. This implies $p_0 \in O_{k'}$, which is a contradiction. In case $n=1$, using ess.ord.$^{\pm}$ instead of ess.ord. itself, we can apply arguments similar to the above.

ii) This follows from Proposition (1.2), together with the fact that $\mathcal{P}_l$ and $\tilde{\mathcal{D}}_l$ are $\mathcal{A}$-spaces.

iii) This follows immediately from (3.1).

(3.4) i) *Let E be an $\mathcal{A}$-space of $\mathcal{D}_0(M)$. Then there is an open dense set O with the properties:*

For any point $p \in O$, there exists a neighborhood U of p such that $E|U$ coincides with one of the $\mathcal{A}$-spaces O, $\boldsymbol{R}|U$ and $\mathcal{D}_0(M)|U$.

ii) *Let E be an $\mathcal{A}$-space of $\mathcal{D}_{k_0}(M)$. For any point $p \in M$, there exists a neighborhood U of p such that*

$$E|U = (E \cap \tilde{\mathcal{D}}_{k_0}(M))|U \oplus (E \cap \mathcal{D}_0(M))|U .$$

Proof. i) Assume that E contains an element f which is not constant. Take an open set U where $\partial f/\partial x_i$ does not vanish for some i. Then for any $g \in \mathcal{D}_0$ we have

$$g = \left[g \left(\frac{\partial f}{\partial x_i} \right)^{-1} \partial_i, f \right] \in E \qquad \text{on } U ,$$

hence E coincides with $\mathcal{D}_0(M)$ on U. Let Q be the open set consisting of those points p such that the germs of E at p coincide with the germ of $\mathcal{D}_0(M)$. Set $O = Q \cup \bar{Q}^c$. Then from the above it is easily checked that O satisfies the conditions stated in i). This completes the proof.

ii) Let U be a local coordinate neighborhood of p, and let us consider only the behavior of E on U. Take any element $\varphi \in E|U$ and write

$$\varphi = \varphi_1 + \varphi_2 \qquad (\varphi_1 \in \tilde{\mathcal{D}}_{k_0}(M)|U,\ \varphi_2 \in \mathcal{D}_0(M)|U) ,$$

according to the decomposition $\mathcal{D}_{k_0}(M)|U = \mathcal{D}_{k_0}(M)|U \oplus \mathcal{D}_0(M)|U$. We can find a finite sequence of $\{T_{i,m}\}$ such that the successive application of $T_{i,m}$ to φ diminishes the order of φ_1 and at last makes φ_1 vanish. It turns out that the element of $E|U$ obtained at the final step of this procedure is nothing but φ_2 multiplied by a non-zero constant. Hence we have $\varphi_2 \in E|U$. This proves ii).

By virtue of (3.3) and (3.4), the structure of $\mathcal{A}$-spaces contained in $\mathcal{D}_{k_0}(M)$

has been considerably clarified. As to the $\mathcal{A}$-maps, we have the following theorem.

Theorem (3.5). *Let Φ be an $\mathcal{A}$-map from $\mathcal{A}_k(M)$ to $\tilde{\mathcal{D}}(M)$. Then, in case $n=1$, there exists real numbers c_1 and c_2 such that*

$$\Phi(\varphi) = c_1\varphi \qquad \text{for } \varphi \in \tilde{\mathcal{D}}_k(M) \cap \tilde{\mathcal{D}}^+$$
$$\Phi(\varphi) = c_2\varphi \qquad \text{for } \varphi \in \tilde{\mathcal{D}}_k(M) \cap \tilde{\mathcal{D}}^- ;$$

in case $n>1$, there exists a real number c such that

$$\Phi(\varphi) = c\varphi \qquad \text{for every } \varphi \in \tilde{\mathcal{D}}_k(M) .$$

Proof. We first note that for any $\varphi \in \tilde{\mathcal{D}}_k(M)$ we have

$$\operatorname{supp} \Phi(\varphi) \subset \operatorname{supp} \varphi .$$

In fact, this is checked from a simple fact that, for any $X \in \mathcal{A}(M)$ with $\operatorname{supp} X \cap \operatorname{supp} \varphi = \phi$, we have $[X, \Phi(\varphi)] = \Phi[X, \varphi] = 0$. Take a local coordinate system $(x_1, \cdots, x_n)$ on a neighborhood U. Then the differential operators $\partial_i^k (i=1, 2, \cdots, n)$, defined on U, are completely characterized up to a multiplicative constant by the following formulas:

$$[\partial_j, \partial_i^k] = 0, \qquad j = 1, 2, \cdots, n,$$
$$[x_j\partial_j, \partial_i^k] = -k\delta_{ij}\partial_i^k .$$

Since Φ is an $\mathcal{A}$-map, this implies that the restriction of Φ to U sends ∂_i^k to $c\partial_i^k$ where c is a suitable real number. Hence, in view of (3.2), together with the assumption on Φ, we find that, at least locally, Φ must have the form stated in the theorem. Since M is assumed to be connected, the assertion now immediately follows.

For later use, we state a consequence of Theorem (3.5).

(3.6) *Let Φ be an $\mathcal{A}$-map from $\tilde{\mathcal{D}}_k(M)$ to $\tilde{\mathcal{D}}(M)$ such that $\Phi | \mathcal{A}(M)=0$. Then we have*

$$\Phi | \tilde{\mathcal{D}}_k(M) \cap \tilde{\mathcal{D}}^+ \equiv 0, \qquad \text{in case } n = 1;$$
$$\Phi \equiv 0, \qquad \text{in case } n > 1.$$

Furthermore, in case $n=1$, if we put the supplementary condition that $k \geqq 3$ and for every $\varphi, \psi \in \tilde{\mathcal{D}}_2(M)$

$$\Phi[\varphi, \psi] = [\Phi\varphi, \psi] + [\varphi, \Phi\psi]$$

holds, then we have $\Phi \equiv 0$.

It is only necessary to check the final part. Confine our consideration to a local coordinate neighborhood. The application of Φ to the identity $[\partial^2, [\partial^2, x^2\partial]] = 8\partial^3$ yields $\Phi(\partial^2)=0$, from which $\Phi | \tilde{\mathscr{D}}_k(M) \cap \tilde{\mathscr{D}}^- \equiv 0$ follows. Hence we have $\Phi \equiv 0$.

4. Subalgebras

Theorem (4.1). i) *For every ideal I of $\tilde{\mathscr{D}}(M)$, we have $I \subset I(p) \cap \tilde{\mathscr{D}}(M)$ for some point $p \in M$ or $I \supset \tilde{\mathscr{D}}(M)_c$.*

ii) *For every ideal I of $\mathscr{D}(M)$, we have $I \subset I(p) + \boldsymbol{R}$ for some point $p \in M$ or $I \supset \mathscr{D}(M)_c$.*

Proof. i) Take $p \in M$ and assume that I contains an element φ such that $\text{ess.ord.}_p \varphi \neq 0$. Note that this assumption is equivalent to the requirement $I \not\subset I(p)$. Since I is an ideal, we may assume $\text{ess.ord.}_p \varphi$ is odd. (This assumption is only necessary in case $n=1$.) Consider the smallest $\mathscr{A}$-space $E(\varphi)$ containing φ. Then $E(\varphi) \subset I$ since I is an ideal, and, moreover, by (3.1) we can find a neighborhood U of p such that $E(\varphi) | U \supset \mathscr{A}(M) | U$. Then, by virtue of (1.2), we can conclude that $E(\varphi) \supset \tilde{\mathscr{D}}(U)_c$. From these facts, i) immediately follows.

ii) Using (3.4) and (1.1) i), we can apply similar reasonings to show the validity of ii).

As a result, we find that $\tilde{I}(p) = I(p) \cap \tilde{\mathscr{D}}(M)$ is a maximal ideal of $\tilde{\mathscr{D}}(M)$ and $I(p) + \boldsymbol{R}$ is a maximal ideal of $\mathscr{D}(M)$ for each point $p \in M$. Moreover, $\tilde{I}(p) \cap \tilde{\mathscr{D}}(M)_c$ and $(I(p) + \boldsymbol{R}) \cap \mathscr{D}(M)_c$ yield all the maximal ideals of $\tilde{\mathscr{D}}(M)_c$ and $\mathscr{D}(M)_c$, respectively.

Corresponding to (2.4), we have the following proposition.

(4.2) *$\mathscr{D}(M)$ and $\tilde{\mathscr{D}}(M)$ have no non-trivial finite-dimensional representations.*

Proof. In view of (1.4) i), we find that, if Φ is non-trivial representation of $\mathscr{D}(M)$ or $\tilde{\mathscr{D}}(M)$, then the restriction of Φ to $\mathscr{A}(M)$ gives rise to a non-trivial representation of $\mathscr{A}(M)$. Thus it turns out that (4.2) is an immediate consequence of (2.4).

The following proposition often provides us with a reduction principle for the investigation of $\mathscr{D}(M)$.

(4.3) *Assume that a subalgebra B of $\tilde{\mathscr{D}}(\boldsymbol{R}^n)$ satisfies the conditions*

i) $\mathscr{A}(\boldsymbol{R}^n) \subset B$,

ii) $\partial_i^2 \in B$ *for some i.*

Then B necessarily coincides with $\tilde{\mathscr{D}}(\boldsymbol{R}^n)$.

Proof. The simple formulas

$$[\partial_i^2, x_i\partial_j] = 2\partial_i\partial_j, \qquad [\partial_i\partial_j, x_i\partial_j] = \partial_j^2$$

yield that $\partial_j^2 (j=1, 2, \cdots, n)$ belong to B. On the other hand, we have for any multi-index α with $|\alpha| = k$ $(\geqq 1)$

$$[\partial_j^2, f\partial^\alpha] \equiv 2\frac{\partial f}{\partial x_j}\partial^{\alpha+(j)} \qquad (\text{mod } \tilde{\mathcal{D}}_k(\boldsymbol{R}^n)).$$

From these facts we can easily deduce $B = \tilde{\mathcal{D}}(\boldsymbol{R}^n)$, which completes the proof.

The following family of subalgebras of $\mathcal{D}(M)$ gives rise to a "locally generic" family for the subalgebras containing $\mathcal{A}(M)$:

(S_1) (In case $n=1$): $\mathcal{A}(M)$, $\boldsymbol{R}+\mathcal{A}(M)$, $\mathcal{D}_1(M)$, $\tilde{\mathcal{D}}^+(M)$, $\boldsymbol{R}+\tilde{\mathcal{D}}^+(M)$, $\tilde{\mathcal{D}}(M)$, $\boldsymbol{R}+\tilde{\mathcal{D}}(M)$, $\mathcal{D}(M)$.

(S) (In case $n>1$): $\mathcal{A}(M)$, $\boldsymbol{R}+\mathcal{A}(M)$, $\mathcal{D}_1(M)$, $\tilde{\mathcal{D}}(M)$, $\boldsymbol{R}+\tilde{\mathcal{D}}(M)$, $\mathcal{D}(M)$.

More precisely, we have

Theorem (4.4). *For any subalgebra B of $\mathcal{D}(M)$ containing $B(M)$, we can find a dense open set O of M which has the following property: For any point $p \in O$, there exists a neighborhood U of p such that $B|U$ coincides with one of subalgebras listed in $(S)_1$ or (S), according to the case $n=1$ or $n>1$.*

Proof. Since B is an $\mathcal{A}$-space, (3.4) applies to B which shows that we have only to consider $B \cap \tilde{\mathcal{D}}(M)$. Set $B_2 = B \cap \tilde{\mathcal{D}}_2(M)$. Consider first the case $n>1$. Notations being the same as in (3.3), the dense open sets associated to the $\mathcal{A}$-space B_2 of $\tilde{\mathcal{D}}_2(M)$ are either $O_2 \cup O_1$ or O_1, because $B_2 \supset \mathcal{A}(M)$. If the first case occurs, then we have

$$\tilde{\mathcal{D}}(O_2)_c \subset B \cap \tilde{\mathcal{D}}(M)|O_2$$
$$\mathcal{A}(O_1)_c \subset B \cap \tilde{\mathcal{D}}(M)|O_1 \subset \mathcal{A}(O_1),$$

by virtue of (3.3) and (4.3). On the other hand, in the second case, we have $B \cap \tilde{\mathcal{D}}(M) = \mathcal{A}(M)$. Similar reasoning applies to the case $n=1$, where we have only to note that $\tilde{\mathcal{D}}^+$ forms a subalgebra, but $\tilde{\mathcal{D}}^+ + \mathcal{P}_{2k} (k=1, 2, \cdots)$ do not. From these the theorem directly follows.

In contrast with Theorem (2.2), we obtain

Theorem (4.5). *In $\mathcal{D}(M)$ there is no proper subalgebra with finite codimension.*

We first prove a local version of (4.5).

(4.6) *Let B be a subalgebra of $\mathcal{D}(\boldsymbol{R}^n)$ with finite codimension. Then there is a neighborhood U of the origin such that*

$$B|U = \mathcal{D}|U.$$

Proof of (4.6). Set

$$B^{\sharp}=\{\varphi\in B\,|\,\mathrm{ad}(\varphi)\mathcal{D}\subset B\}\ .$$

It is easily verified that $B^{\sharp}$ is an ideal of B and the codimension of $B^{\sharp}$ in $\mathcal{D}$ is finite. Hence the codimension of $B^{\sharp\sharp}=(B^{\sharp})^{\sharp}$ in $\mathcal{D}$ is finite and so there is a linear combination

$$y=x_1{}^{m}+\lambda_1 x_1{}^{m+1}+\cdots+\lambda_k x_1{}^{m+k}\qquad(m\geqq 1)\,,$$

such that $y\in B^{\sharp\sharp}$. If we take a small neighborhood U of the origin, then we may assume that

$$x_1{}^{m}\in B^{\sharp\sharp}\,|\,U$$

after a suitable change of local coordinates on U. Now we restrict our considerations only to the behaviour of B on U, and write x for x_1 and ∂ for $\partial/\partial x_1$.

Put $u=x\partial$. Then we have the formulas:

$$\begin{aligned} xu&=(u-1)x, \qquad \partial u=(u+1)\partial\\ x^n\partial^n&=(u-n+1)(u-n+2)\cdots u\\ \partial^n x^n&=(u+n)(u+n-1)\cdots(u+1)\,. \end{aligned}$$

Since $x^m\in B^{\sharp\sharp}$, we obtain $\mathrm{ad}(x^m)D\subset B^{\sharp}$, whence

$$[u^l x^k,\ x^m]=(u^l-(u-m)^l)x^{k+m}\in B^{\sharp}$$

for all intergers $l,\ k\geqq 0$. It follows that

$$u^l x^s\in B^{\sharp}\qquad \text{for } l\geqq 0 \text{ and } s\geqq m\,.$$

Hence we have $\mathrm{ad}(u^l x^s)\mathcal{D}\subset B$. In particular,

$$(*)\qquad \begin{aligned}[]
[u^a x^s,\ u^b\partial^{s+1}]=\{&u^a(u-s)^b(u-s+1)\cdots u\\ &-(u+s+1)^a u^b(u+s+1)\cdots(u+2)\}\,\partial \end{aligned}$$

belongs to B, where $a,b\geqq 0$ and $s\geqq m$. Put

$$P=(u-m+1)\cdots u,\qquad Q=(u+m+1)\cdots(u+2)\,.$$

Then we can deduce from (*):

In case $a=b=0,\ s=m,$ $\quad (P-Q)\partial\in B;$

In case $a=b=0,\ s=m+1,$ $\quad \{(u-m)P-(u+m+2)Q\}\,\partial\in B;$

In case $a=1,\ b=0,\ s=m,$ $\quad \{uP-(u+m+1)Q\}\,\partial\in B\,.$

From these it turns out that

$$P\partial,\ Q\partial,\ u(P-Q)\partial\in B\,.$$

We wish to prove that the following assertions $(\mathrm{i})_k$, $(\mathrm{ii})_k$ hold true for all $k=1, 2, \cdots$:

$$(\mathrm{i})_k \quad u^k(P-Q)\partial \in B\,,$$
$$(\mathrm{ii})_k \quad u^l P\partial,\ u^l Q\partial \in B \qquad \text{for } l=0, 1, 2, \cdots, k-1\,.$$

We note that for $k=1$ the assertions have already been verified. Assume that for some k, $(\mathrm{i})_k$ and $(\mathrm{ii})_k$ be true. We shall again use (*). In case $s=m$, $a=k+1$, $b=0$, we have

$$(**) \qquad \{u^{k+1}P-(u+m+1)^{k+1}Q\}\partial \in B;$$

in case $s=m$, $a=0$, $b=k+1$, we have

$$\{(u-m)^{k+1}P-u^{k+1}Q\}\partial \in B\,.$$

Hence, taking the difference and using $(\mathrm{ii})_k$, we have

$$\{(k+1)u^k mP-(k+1)u^k(m+1)Q\}\partial \in B\,.$$

In view of $(\mathrm{i})_k$, we can deduce

$$u^k P\partial, \qquad u^k Q\partial \in B\,,$$

which, together with (**), yields

$$u^{k+1}(P-Q)\partial \in B\,.$$

Thus, we have obtained $(\mathrm{i})_{k+1}$, $(\mathrm{ii})_{k+1}$, so that, by the induction, the $(\mathrm{i})_k$, $(\mathrm{ii})_k$ $(k=1, 2, \cdots)$ are all valid.

In conclusion, for any $k=0, 1, 2, \cdots$, we have

$$u^k P\partial, \qquad u^k Q\partial \in B\,.$$

Since P and Q are mutually prime polynomials in u, we can find two polynomials $\tilde{P}(u)$ and $\tilde{Q}(u)$ such that

$$\tilde{P}(u)P(u)+\tilde{Q}(u)Q(u)\equiv 1\,.$$

It follows that

$$\partial = (\tilde{P}\cdot P+\tilde{Q}\cdot Q)\partial \in B\,.$$

Now we are in a position to apply Lemma (2.3) to $E=\mathcal{D}\,|\,U$, $F=B\,|\,U$ and $\varphi=\mathrm{ad}(\partial)$, which yields $B\,|\,U=\mathcal{D}\,|\,U$. This completes the proof of (4.6).

Proof of Theorem (4.5). Assume that there be given a subalgebra B of $\mathcal{D}(M)$ whose codimension is finite. We shall adopt the ideas which we have used in the proof of Theorem (2.2). In fact, for such B, the conclusion obtained from the first step of that proof just corresponds to (4.6) which we have proved above. The second and the third steps remain valid without any essential alteration. Hence, referring to the proof of Theorem (2.2), we can immediately arrive at the conclusion $B=\mathcal{D}$. This, however, establishes (4.5).

5. Derivations

Let B be a subalgebra of $\mathcal{D}(M)$. A linear map δ of B to $\mathcal{D}(M)$ is called a *derivation* if

$$\delta[\varphi, \psi] = [\delta\varphi, \psi]+[\varphi, \delta\psi] \qquad (\varphi, \psi\in B)$$

holds. If $\delta=\mathrm{ad}\,\varphi_0$ for some $\varphi_0\in\mathcal{D}(M)$, then δ is called *inner* in $\mathcal{D}(M)$. Suppose that, for a given derivation δ of B to $\mathcal{D}(M)$, there exists a subspace E of $\mathcal{D}(M)$ such that $\delta(B)\subset E$. Then we say that δ is a derivation of B to E.

(5.1) i) *Assume that B coincides with one of $\mathcal{A}(M)$, $\mathcal{D}_1(M)$, $\tilde{\mathcal{D}}(M)$ and $\mathcal{D}(M)$. Then any derivation δ of B to $\mathcal{D}(M)$ has the support-preserving property* (i.e., δ *satisfies* supp $\delta(\varphi)\subset$ supp φ *for* $\varphi\in B$).

ii) *Assume that B is a subalgebra of $\mathcal{D}(M)$ which contains $\mathcal{A}(M)$. Then any derivation δ of B to $\tilde{\mathcal{D}}(M)$ has the support-preserving property.*

Proof i): We shall prove the assertion in the case $B=\mathcal{A}(M)$. The other cases will be treated in a similar way. Take any $X\in\mathcal{A}(M)$. Let U be any open set with supp $X\subset U$. By virtue of (1.6), X can be written as $X=\sum[Y_i, Z_i]$, where Y_i, Z_i are a finite number of elements of $\mathcal{A}(M)$. Referring to the proof of (1.4), we find that Y_i, Z_i may be so chosen that supp Y_i, supp Z_i are contained in U. Then we have

$$\delta(X) = \sum[\delta(Y_i), Z_i]+\sum[Y_i, \delta(Z_i)]\,,$$

and hence supp $\delta(X)\subset U$. Therefore δ has the support-preserving property.

ii): By i) $\delta\,|\,\mathcal{A}(M)$ has the support-preserving property. Take any $\varphi\in B$. Choose $X\in\mathcal{A}(M)$ such that supp $X\cap$ supp $\varphi=\phi$. Then

$$0 = \delta[X, \varphi] = [\delta X, \varphi]+[X, \delta\varphi] = [X, \delta\varphi]\,,$$

since supp $\delta X\cap$ supp $\varphi=\phi$. From this it follows directly that supp $\delta\varphi\subset$ supp φ.

This completes the proof.

Related to the support-preserving property, we have

(5.2) *Let δ be a linear map from $\mathcal{D}_l(M)$ to $\mathcal{D}(M)$ which has the support-preserving property. Then there is an integer k such that*

$$\delta(\mathcal{D}_l(M))\subset\mathcal{D}_k(M)\,.$$

The similar result holds if we replace $\mathcal{D}_l$ by $\tilde{\mathcal{D}}_l$

Proof. Suppose that the first assertion is false. Then there are a sequence $\{\varphi_i\}$ ($\varphi_i \in \mathcal{D}_l(M)$) and a sequence $\{p_i\}$ ($p_i \in M$) such that a term of order k_i arising from the local expression of $\delta(\varphi_i)$ does not vanish at p_i and $k_1 < k_2 < \cdots$. We may assume that each p_i is distinct and is not a cluster point of a sequence $\{p_i\}$. Since δ is support-preserving, we may then take these $\varphi_i (i=1, 2, \cdots)$ so as to satisfy the condition that supp φ_i is compact and supp $\varphi_i \cap$ supp $\varphi_j = \phi$ for $i \neq j$. Choose a suitable sequence $\{\alpha_i\}$ of positive numbers such that $\sum_{i=1}^{\infty} \alpha_i \varphi_i$ uniformly converges to a certain $\varphi \in \mathcal{D}_l(M)$ on each compact set. We have $\delta(\varphi) \notin \mathcal{D}_{k_i - 1}$ near p_i and $k_i \to \infty$, which, however, contradicts the fact that $\delta(\varphi)$ has a finite order. This establishes the first assertion. Since the same reasoning applies to the case where δ is a map from $\tilde{\mathcal{D}}_l(M)$ to $\mathcal{D}(M)$, this completes the proof of (5.2).

We shall give some examples of derivations:

(a) Let ω be a closed differential 1-form. Take a locally finite open covering $\{U_i\}_{i \in I}$ of M, where each U_i is realized as the open unit disk through a system of local coordinates. Then Poincaré's lemma shows that there exists $f_i \in C^\infty(U_i)$ such that $\omega | U_i = df_i$. We say that $\{U_i, f_i\}_{i \in I}$ is a *Poincaré distribution* associated to ω. Note that f_i is uniquely determinied up to an additive constant. Making use of a Poincaré distribution $\{U_i, f_i\}_{i \in I}$, we shall define the derivation $[\omega]$ of $\mathcal{D}(M)$, by setting

$$[\omega]\varphi = [f_i, \varphi], \qquad \text{on } U_i .$$

Clearly, the derivation $[\omega]$ is well-defined, independent of the choice of Poincaré distribution. We note that $[\omega] | \mathcal{D}_1(M)$ gives a derivation of $\mathcal{D}_1(M)$ to $\mathcal{D}_0(M)$.

(b) We introduce a volume element v to M. Recall that div X ($X \in \mathcal{A}(M)$) is then defined by the formula

$$L_X v = (\operatorname{div} X) v ,$$

where L_X denotes the Lie derivative along X. Then the assignment

$$\operatorname{div}: \ X \to \operatorname{div} X$$

gives rise to a derivation of $\mathcal{A}(M)$ to $\mathcal{A}_0(M)$. In fact, this can be easily checked from the formula $L_{[X,Y]} = L_X L_Y - L_Y L_X$.

(c) Let π_0 and π_1 be the canonical projections from $\mathcal{D}_1(M)$ onto $\mathcal{D}_0(M)$ and $\mathcal{A}(M)$, respectively. Then π_0 gives rise to a derivation of $\mathcal{D}_1(M)$ to $\mathcal{D}_0(M)$ and div $\circ \pi_1$ a derivation of $\mathcal{D}_1(M)$ to itself.

To proceed with the study on the derivations, we shall make use of the results on differential complexes [2], [6]. Let τ be the tangent bundle of M and $\mathcal{E}^1$ the trivial one-dimensional vector bundle over M. $J^k(\mathcal{E}^1)$ denotes the k-th jet bundle of $\mathcal{E}^1$. We have the canonical splitting of vector bundle

$$J^k(\mathcal{E}^1) = \mathcal{E}^1 \oplus \tilde{J}^k(\mathcal{E}^1)\,.$$

Put

$$D(k) = \mathrm{Hom}(J^k(\mathcal{E}^1), \mathcal{E}^1), \qquad \tilde{D}(k) = \mathrm{Hom}(\tilde{J}^k(\mathcal{E}^1), \mathcal{E}^1)\,.$$

Then $D(k)$ and $\tilde{D}(k)$ are smooth vector bundles over M. Note that $\tilde{D}(1)$ is canonically isomorphic to τ. In what follows, we shall identify

$$\mathcal{D}_k(M) = \Gamma(D(k)), \qquad \tilde{\mathcal{D}}_k(M) = \Gamma(\tilde{D}(k))\,.$$

Now the well-known Peetre's theorem (cf. [4])) together with (5.1) shows that any derivation δ of $\mathcal{A}(M)$ (or $\mathcal{D}_1(M)$) to $\mathcal{D}_k(M)$ ($k=0, 1, 2, \cdots$) really gives rise to a differential operator from $\Gamma(\tau)$ (or $\Gamma(D(1))$ to $\Gamma(D(k))$. According to the notations used in [6], it is possible to write this fact as

$$\delta \in C^1[\tau, D(k)]$$

(or $\delta \in C^1[D(1), D(k)]$).

Consider the differential complex $\{C^p[\tau, D(k)], d\}$ associated to the adjoint representation of $\mathcal{A}(M)$ to $\mathcal{D}_k(M)$ (cf. [6]). In this complex, the criterion that a 1-cochain becomes a derivation or an inner derivation can be simply stated as follows:

(5.3) i) *$L \in C^1[\tau, D(k)]$ is a derivation $\Leftrightarrow$ L is a cocycle;*
ii) *$L \in C^1[\tau, D(k)]$ is inner in $D(k)$ $\Leftrightarrow$ L is a coboundary.*
The similar statement holds for the derivations of $\mathcal{D}_1(M)$ when we consider the adjoint complex $\{C^p[D(1), D(k)], d\}$.

Let $H^*(\tau, D(k))$ and $H^*(D(1), D(k))$ denote the cohomology groups of the adjoint complexes $\{C^p[\tau, D(k)], d\}$ and $\{C^p[D(1), D(k)], d\}$, respectively. Concerning these cohomology groups, we know ([2], [6]):

(5.4) i) *There is a canonical isomorphism*

$$H^*(\tau, D(0)) \simeq H^*(B(\tau^C); \boldsymbol{R})\,,$$

where $B(\tau^C)$ denotes the principal $U(n)$-bundle over M, associated to $\tau \otimes \boldsymbol{C}$.
ii) *For any k, the injection $D(0) \to D(k)$ induces the isomorphism*

$$H^*(\tau, D(0)) \simeq H^*(\tau, D(k))\,.$$

iii) *There is a canonical isomorphism*

$$H^*(D(1), D(0)) \simeq H^*(B(\tau^C) \times S^1; \boldsymbol{R})\,,$$

where S^1 denotes the circle.
iv) *For any k, the injection $D(0) \to D(k)$ induces the isomorphism*

$$H^*(D(1), D(0)) \simeq H^*(D(1), D(k)).$$

In view of (5.3), more exact informations on the one-dimensional cohomology groups are necessary for our aim. Actually, we know the following facts ([2], [6]). Let $\{\omega_i\}$ $(i=1, \cdots, b_1; b_1$ may be infinite) be a family of closed 1-forms on M, whose de Rham cohomology classes give rise to a basis of $H^1(M; \boldsymbol{R})$. Note that b_1 is the first Betti number of M. Then, $\{\omega_i\}$ $(i=1, \cdots, b_1)$ and one more closed differential 1-cochain Ω give representative cocycles of a basis of $H^1(\tau, D(0))$. (Ω can be taken so as to be dependent only upon the first jet of τ, and if Ω is so chosen, then the stalk of Ω at each point of M corresponds to a generator of $H^1(U(n); \boldsymbol{R})$ in the local considerations.) On the other hand, $\{\omega_i\}$ $(i=1, \cdots, b_1)$, $\Omega \circ \pi_1$ and π_0 give representative cocycles of a basis of $H^1(D(1), D(0))$.

We shall make a remark that we may take div as Ω, if we introduce a volume element to M. In fact, this follows from the observations that by (b) and (5.3) div is regarded as a closed differential 1-cochain, and that there is no non-trivial linear relation between $\{\omega_i\}$ and div, because, for any $X \in \mathcal{A}(M)$ with $X(p)=0$, $dX(p) \neq 0$, we have $\omega_i(X)(p)=0$ and div $(X)(p) \neq 0$. Furthermore, we observe that, for any ω_i, we have $\omega_i(X)=[\omega_i](X)$ according to the notation used in (a). Now we are in a position to formulate the theorem:

Theorem (5.5). i) *Any derivation δ of $\mathcal{A}(M)$ to $\mathcal{D}(M)$ is uniquely expressed as*

$$\delta = \operatorname{ad} \varphi + \sum_{i=1}^{b_1} \mu_i[\omega_i] + \lambda \operatorname{div};$$

ii) *Any derivation δ of $\mathcal{D}_1(M)$ to $\mathcal{D}(M)$ is uniquely expressed as*

$$\delta = \operatorname{ad} \varphi + \sum_{i=1}^{b_1} \mu_i[\omega_i] + \lambda \operatorname{div} \circ \pi_1 + \kappa \pi_0;$$

iii) *Any derivation δ of $\tilde{\mathcal{D}}(M)$, or $\mathcal{D}(M)$, to $\mathcal{D}(M)$ is uniquely expressed as*

$$\delta = \operatorname{ad} \varphi + \sum_{i=1}^{b_1} \mu_i[\omega_i].$$

Here φ is an element of $\mathcal{D}(M)$, uniquely determined by δ, and μ_i, λ, κ denote real numbers; moreover, μ_i are almost all zero if $b_1=\infty$.

Proof. i), ii): In each case, if a derivation δ admits an expression as described above, then it is easy to see that φ is uniquely determined by δ. With this understood, the assertions immediately follow from (5.2) and (5.4), together with the preceding discussions.

iii): First we shall prove the assertion in the case of $\tilde{\mathcal{D}}(M)$. The restriction of δ to $\mathcal{A}(M)$ gives rise to a derivation from $\mathcal{A}(M)$ to $\mathcal{D}(M)$. Hence, by i), $\delta \mid \mathcal{A}(M)$ is uniquely written as

$$\delta \mid \mathcal{A}(M) = \text{ad}\,\varphi + \sum_{i=1}^{b_1} \mu_i[\omega_i] + \lambda\ \text{div}.$$

Set

$$\Delta = \delta - \text{ad}\,\varphi - \sum_{i=1}^{b_1} \mu_i[\omega_i]\,.$$

Then Δ is a derivation of $\mathcal{D}(M)$ to $\mathcal{D}(M)$, satisfying $\Delta \mid \mathcal{A}(M) = \lambda \cdot \text{div}$. If we can show $\lambda = 0$, then Δ becomes an $\mathcal{A}$-map vanishing on $\mathcal{A}(M)$, whence by (3.6) Δ is identically zero. (Note that (3.6) remains true if we replace $\tilde{\mathcal{D}}(M)$ by $\mathcal{D}(M)$ when Φ is support-preserving.) This will establish ii) in case of $\tilde{\mathcal{D}}(M)$.

Now we shall give a proof to $\lambda = 0$. Since this is of local character, we have only to consider on a neighborhood U, diffeomorphic to the open unit disk. Moreover, we may assume $\text{div}(\sum f^i \partial_i) = \sum \partial f^i / \partial x^i$ holds on U. Since $\Delta(\partial_j) = 0$ and $\Delta(x_j \partial_j) = 1$ $(j = 1, \cdots, n)$, we have $\Delta(\partial_i^2) = a_i \partial_i^2$ for suitable $a_i \in \boldsymbol{R}$ (cf. the proof of (3.4)). From

$$[\partial_i^2, f\partial_i] = 2(\partial_i f)\partial_i^2 + (\partial_i^2 f)\partial_i\,,$$

it follows that

$$\Delta((\partial_i f)\partial_i^2) = a_i(\partial_i f)\partial_i^2 + \left(\frac{a_i}{2} + \lambda\right)(\partial_i^2 f)\partial_i\,.$$

Using this relation, we apply Δ to both sides of

$$[x_i\partial_i^2,\ x_i^2\partial_i] = 2x_i\partial_i + 3x_i^2\partial_i^2\,.$$

Then we can easily obtain $\lambda = 0$, as we wished to prove.

Next we shall consider the case of $\mathcal{D}(M)$. From ii) and the above we know that δ admits the expressions:

$$\delta \mid \mathcal{D}_1(M) = \text{ad}\,\varphi + \sum_{i=1}^{b_1} \mu_i[\omega_i] + \lambda\ \text{div} \circ \pi_1 + \kappa\pi_0\,,$$

$$\delta \mid \tilde{\mathcal{D}}(M) = \text{ad}\,\varphi' + \sum_{i=1}^{b_1} \mu_i'[\omega_i]\,.$$

But φ, φ' μ_i, μ_i' and λ are completely determined by the behavior of δ on $\mathcal{A}(M)$ $= \mathcal{D}_1(M) \cap \tilde{\mathcal{D}}(M)$. It results that $\varphi = \varphi'$, $\mu_i = \mu_i'$ and $\lambda = 0$. Hence, if we denote by π_0 the canonical projection from $\mathcal{D}(M)$ to $\mathcal{D}_0(M)$, δ must take the form

$$\delta = \text{ad}\,\varphi + \sum_{i=1}^{b_1} \mu_i[\omega_i] + \kappa\pi_0\,.$$

Since π_0 does not give a derivation on $\mathcal{D}(M)$, we can conclude $\kappa = 0$, which completes the proof.

As a corollary, we obtain

(5.6) *Every derivation of $\mathcal{A}(M)$ to itself is inner.*

REMARK. This is also regarded as a consequence of a more general theorem that the cohomology group of the adjoint complex of $\mathcal{A}(M)$ vanishes (cf. [6; I, Th 4.3, Cor. 2]).

Finally we shall treat with the derivations defined on a subalgebra.

Theorem (5.7). *Let B be a subalgebra of $\tilde{\mathcal{D}}(M)$ containing $\mathcal{A}(M)$. Then every derivation δ of B to $\tilde{\mathcal{D}}(M)$ is uniquely written as $\delta=\mathrm{ad}\ \varphi$, where $\varphi\in\tilde{\mathcal{D}}(M)$.*

Proof. Using (3.3) and (4.4), we know that there exist mutually disjoint open sets O_1 and O_2 of M such that $O_1\cup O_2$ is dense in M and that

$$\mathcal{A}(O_1)_c\subset B\,|\,O_1\subset\mathcal{A}(O_1),\qquad \tilde{\mathcal{D}}(O_2)_c\subset B\,|\,O_2\subset\tilde{\mathcal{D}}(O_2)$$

hold. Since $\delta\,|\,\mathcal{A}(M)$ is a derivation of $\mathcal{A}(M)$ to $\tilde{\mathcal{D}}(M)$, by (5.5) i) we can find a unique $\varphi\in\tilde{\mathcal{D}}(M)$ such that $\mathrm{ad}\ \varphi=\delta\,|\,\mathcal{A}(M)$. Set $\Delta=\delta-\mathrm{ad}\ \varphi$. Then Δ gives rise to an $\mathcal{A}$-map from B to $\tilde{\mathcal{D}}(M)$, which vanishes on $\mathcal{A}(M)$. Apply (3.6) to $(B\,|\,O_2)\cap\tilde{\mathcal{D}}_k(O_2)$ for $k=1, 2, \cdots$ and to the $\mathcal{A}$-map $\Delta\,|\,O_2$. Then we obtain $\Delta\,|\,O_2\equiv 0$, whence $\Delta\equiv 0$ throughout M. Thus $\delta=\mathrm{ad}\ \varphi$, which completes the proof.

As a result, we have

(5.8) *Every derivation of $\tilde{\mathcal{D}}(M)$ to itself is inner.*

Also we can easily prove the following proposition when $\dim M=1$.

(5.9) *Every derivation of $\tilde{\mathcal{D}}^{+}(M)$ to itself is inner.*

6. Isomorphisms

Let M and N be two smooth manifolds. In this section we shall deal with Lie algebra isomorphisms from $\mathcal{D}(M)$ to $\mathcal{D}(N)$.

(6.1) *Let Φ be an isomorphism from $\mathcal{D}(M)$ to $\mathcal{D}(N)$. Then we have*

i) $\Phi(\mathcal{D}_0(M)) = \mathcal{D}_0(N)$;

ii) $\Phi(\mathcal{D}_1(M)) = \mathcal{D}_1(N)$.

This assertion will, however, become an immediate consequence of the following proposition, which gives Lie algebraic characterizations of $\mathcal{D}_0(M)$ and $\mathcal{D}_1(M)$.

(6.2) i) $\mathcal{D}_0(M)=\{\varphi\,|\,\varphi\in\mathcal{D}(M)$; For any $\psi\in\mathcal{D}(M)$ there is an integer m such that $(\mathrm{ad}\ \varphi)^m\psi=0\}$.

ii) $\mathcal{D}_1(M) = \{\varphi\,|\,\varphi\in\mathcal{D}(M);\ \mathrm{ad}\ \varphi(\mathcal{D}_0(M))\subset\mathcal{D}_0(M)\}$.

Proof. For convenience' sake, we denote the sets stated in the right-hand sides above by $\mathcal{D}_0^*(M)$ and $\mathcal{D}_1^*(M)$, respectively.

i): It is clear that $\mathcal{D}_0(M)\subset\mathcal{D}_0^*(M)$. In order to prove the converse implication $\mathcal{D}_0(M)\supset\mathcal{D}_0^*(M)$, we take and fix an element $\varphi\in\mathcal{D}_0^*(M)$. We shall introduce the C^∞-topology of uniform convergence on each compact set to $\mathcal{D}_0(M)$ $(\simeq C^\infty(M))$ so that $\mathcal{D}_0(M)$ becomes a Fréchet space. Set

$$E_m=\{\psi\mid\psi\in\mathcal{D}_0(M),\ \mathrm{ad}(\varphi)^m\psi=0\}\qquad m=1,2\cdots.$$

Then we have $\cup E_m=\mathcal{D}_0(M)$ and each E_m is a closed subspace of $\mathcal{D}_0(M)$, whence we find an integer m such that

$$E_m=\mathcal{D}_0(M).$$

Now observe that we have

$$(*)\qquad \mathrm{ad}(\varphi)^m\psi=\sum_{k=0}^{m}\binom{m}{k}(-1)^k\varphi^{m-k}\psi\varphi^k=0,\qquad \psi\in\mathcal{D}_0(M).$$

From this relation we wish to deduce $\varphi\in\mathcal{D}_0(M)$. Since this, however, is of local character, we may assume $M=\boldsymbol{R}^n$. Set $\psi=e^{\gamma x i}$. Since

$$\partial_i e^{\gamma x i}=e^{\gamma x i}(\partial_i+\gamma),$$

we have

$$\sum_{k=0}^{m}\binom{m}{k}(-1)^k\tilde{\varphi}^{m-k}\varphi^k=0,$$

where $\tilde{\varphi}$ is a differential operator defined by

$$\tilde{\varphi}(x;\partial_1,\cdots,\partial_n)=\varphi(x;\partial_1,\cdots,\partial_{i-1},\partial_i+\gamma,\partial_{i+1},\cdots,\partial_n).$$

Arrange φ according to the order of ∂_i and write it as follows:

$$\varphi=\varphi_k\partial_i{}^k+\varphi_{k-1}\partial_i{}^{k-1}+\cdots,$$

where φ_l ($l=0,1,\cdots,k$) denote differential operators containing only the partial derivatives with respect to $\partial_1,\cdots,\partial_{i-1},\partial_{i+1},\cdots,\partial_n$. Note that

$$\tilde{\varphi}=\varphi_k(\partial_i+\gamma)^k+\varphi_{k-1}(\partial_i+\gamma)^{k-1}+\cdots.$$

Using the fact that the highest order term in γ of (*) vanishes, we can get

$$\varphi_k{}^m=0\qquad\text{if }\ k\geqq 1.$$

But it is known (or directly proved by the use of the similar argument) that the algebra of differential operators has no nilpotent element, whence $\varphi_k=0$ follows. Successive application of the similar argument shows that we have really $\varphi_k=\varphi_{k-1}=\cdots=\varphi_1=0$. By virtue of the same reason, it turns out that φ_0 contains no partial derivatives. Hence we have the desired conclusion $\varphi\in\mathcal{D}_0$, which completes the proof of i).

ii): It is clear that $\mathscr{D}_1^*(M)$ is an $\mathscr{A}$-space containing $\mathscr{D}_1(M)$. Also it is easy to verify that, locally, $\partial_i^\alpha(|\alpha|\geqq 2; i=1, 2, \cdots, n)$ do not belong to $\mathscr{D}_1^*(M)$. Hence, from (3.1) it results that $\mathscr{D}_1^*(M)$ coincides with $\mathscr{D}_1(M)$. This completes the proof of ii).

We shall describe some examples of automorphisms of $\mathscr{D}(M)$.

(A) Let Ψ be a diffeomorphism of M onto itself. Then Ψ naturally induces an automorphism Ψ_* of $\mathscr{D}(M)$.

(B) Let ω be a closed differential 1-form. Using a Poincaré distribution $\{U_i, f_i\}$ of ω, we define an automorphism (ω) of $\mathscr{D}(M)$ as follows:

On each U_i, put

$$(\omega)\varphi = \exp(-f_i)\varphi\exp(f_i)\,.$$

It is easy to verify that (ω) really gives a well-defined automorphism of $\mathscr{D}(M)$, independent of the choice of Poincaré distribution; moreover, if $\omega\neq\omega_1$, then $(\omega)\neq(\omega_1)$.

(C) Fix a volume element of M. For $\varphi\in\mathscr{D}(M)$, we denote by φ^* the formal adjoint of φ with respect to this volume element. Since $(\varphi\psi)^*=\psi^*\varphi^*$, we have an automorphism σ of $\mathscr{D}(M)$ by setting

$$\sigma(\varphi) = -\varphi^*\,.$$

It is clear that (ω) and σ are support-preserving automorphisms. We have relations

$$(\omega)\circ(\omega_1) = (\omega+\omega_1)\,,$$
$$\sigma\circ(\omega) = (-\omega)\circ\sigma\,,$$
$$\sigma^2 = 1\,,$$

which are easily verified.

Let M and N be two smooth manifolds. We shall now prove the following theorem.

Theorem (6.3). *Let Φ be an isomorphism of $\mathscr{D}(M)$ onto $\mathscr{D}(N)$. Then we have*

i) *There exists a diffeomorphism Ψ of M onto N;*

ii) *By the use of this Ψ, Φ is written as either*

$$\Phi = \Psi_*\circ(\omega), \quad \textit{or} \quad \Phi = \Psi_*\circ\sigma\circ(\omega)$$

for a suitable closed 1-form ω.

Proof. i): By (6.1) we have $\Phi(\mathscr{D}_1(M))=\mathscr{D}_1(N)$. Hence we can consider the composition of maps

$$\Phi_1\colon \mathscr{A}(M)\overset{\iota}{\to}\mathscr{D}_1(M)\overset{\Phi}{\to}\mathscr{D}_1(N)\overset{\pi_1}{\to}\mathscr{A}(N)\,,$$

where ι denotes the canonical injection. Since $\Phi(\mathcal{D}_0(M))=\mathcal{D}_0(N)$, it follows immediately that Φ_1 gives an isomorphism from $\mathcal{A}(M)$ onto $\mathcal{A}(N)$. Hence Theorem (2.5) applies to get a diffeomorphism Ψ of M onto N which satisfies $\Psi_*=\Phi_1$. This establishes i).

ii): Using the result of i), we have

$$\pi_1\circ\Psi_*^{-1}\circ\Phi(X) = X \qquad \text{for } X\in\mathcal{A}(M)\,.$$

Since $\Psi_*^{-1}\circ\Phi$ maps $\mathcal{D}_0(M)$ (or $\mathcal{D}_1(M)$) to itself, it follows that $\pi_0\circ\Psi_*^{-1}\circ\Phi$ gives rise to a derivation from $\mathcal{D}_1(M)$ to $\mathcal{D}_0(M)$, as is easily checked. As a result, $\pi_0\circ\Psi_*^{-1}\circ\Phi\,|\,\mathcal{D}_1(M)$ has the support-preserving property, whence we find that $\Psi_*^{-1}\circ\Phi\,|\,\mathcal{A}(M)$ has also the same property. By (1.4) any $\varphi\in\mathcal{D}(M)$ is expressed as $\varphi=\sum[\psi_i, X_i]$ $(\psi_i\in\mathcal{D}(M),\ X_i\in\mathcal{A}(M))$, where ψ_i, X_i can be taken such that supp ψ_i, supp X_i are contained in an arbitrary small neighborhood of supp φ. Since $\Psi_*^{-1}\circ\Phi(\varphi)=\sum[\Psi_*^{-1}\circ\Phi(\psi_i),\ \Psi_*^{-1}\circ\Phi(X_i)]$, it follows that $\Psi_*^{-1}\circ\Phi$ is a support-preserving automorphism of $\mathcal{D}(M)$. Our goal is to find an explicit form of $\Psi_*^{-1}\circ\Phi$, which is, however, reduced to finding a local expression of $\Psi_*^{-1}\circ\Phi$ because of the support-preserving property. Hence we restrict our considerations only to the local behaviour of $\Psi_*^{-1}\circ\Phi$.

Since $\pi_0\circ\Psi_*^{-1}\circ\Phi\,|\,\mathcal{D}_1(M)$ gives a derivation of $\mathcal{D}_1(M)$ to $\mathcal{D}_0(M)$, we have from Theorem (5.5) ii) that

$$(*) \qquad \Psi_*^{-1}\circ\Phi(X+f) = X+\{[X,g]+\lambda\,\mathrm{div}\,X+\kappa f\} \qquad (X+f\in\mathcal{D}_1)$$

on a certain local coordinates disk U, where λ and κ are independent of the choice of local coordinates; a smooth function g is determined up to an additive constant, so that dg is uniquely determined. Since a volument element on U is arbitrarily taken and fixed, we may assume without losing generality that

$$\mathrm{div}\,X = \sum_{i=1}^{n}\partial_i X^i$$

for $X=\sum X^i\partial_i$. We can then write

$$(**) \qquad (-dg)\circ\Psi_*^{-1}\circ\Phi(X+f) = X+\{\lambda\sum\partial_i X^i+\kappa f\}\,.$$

We shall insert the following lemma:

(6.4) *Suppose that we have an automorphism $\tilde{\Phi}$ of $\mathcal{D}(\boldsymbol{R}^n)$ which satisfies*

$$\tilde{\Phi}(\sum X^i\partial_i+f) = \sum X^i\partial_i+\{\lambda\sum\partial_i X^i+\kappa f\}\,.$$

Then

i) $\kappa = 1,\ \lambda = 0$ *or* $\kappa = -1,\ \lambda=1$.

ii) *If $\kappa=1$, then $\tilde{\Phi}$ becomes an identity.*

Proof of (6.4). Set $\tilde{\Phi}(\partial_i^2)=\sum f_\alpha \partial^\alpha$. Apply $\tilde{\Phi}$ to both sides of the identity $[\partial_i^2, x_j]=2\delta_{ij}\partial_i$ (δ_{ij} denotes Kronecker index). Then we obtain

$$[\sum f_\alpha \partial^\alpha, \kappa x_j] = 2\delta_{ij}\partial_i = \kappa \sum f_\alpha \alpha_j \partial^{\alpha-(j)}$$

where $\alpha-(j)$ means the multi-index $(\alpha_1, \cdots, \alpha_{j-1}, \alpha_j-1, \alpha_{j+1}, \cdots, \alpha_n)$. Therefore we have $\kappa \neq 0$ and

$$\tilde{\Phi}(\partial_i^2) = \frac{1}{\kappa}(\partial_i^2+h), \qquad h \in \mathcal{D}_0 .$$

Note that h really becomes a constant function; in fact, this follows from $[\partial_i^2, \partial_j] =0$ $(i, j=1, 2, \cdots, n)$.

Starting from the identities

$$\begin{aligned}
[\partial_i^2, x_i^2] &= 2x_i\partial_i+2 \\
[\partial_i^2, f\partial_i] &= 2(\partial_i f)\partial_i^2+(\partial_i^2 f)\partial_i \\
[x_i\partial_i^2, x_i^2\partial_i] &= x_i^2\partial_i^2+2x_i\partial_i ,
\end{aligned}$$

we can proceed the similar calculations, which yields the relations

$$\begin{aligned}
1 &= \kappa+2\lambda , \\
\Phi(\partial_i f\cdot\partial_i^2) &= \frac{1}{2\kappa}[2\partial_i f\cdot\partial_i^2+(1-\kappa+2\lambda)\partial_i^2 f\cdot\partial_i \\
&\qquad +\lambda(1-\kappa)\partial_i^3 f - f\cdot\partial_i h\} , \\
\lambda(\kappa+\lambda) &= 0 ,
\end{aligned}$$

Here in order to obtain the third relation, we have used the relation

$$\tilde{\Phi}(f\partial_i^2) = \frac{1}{\kappa}(f\partial_i^2+2\lambda\partial_i f\cdot\partial_i+\lambda^2\partial_i^2 f) ,$$

which can be directly obtained from the first and the second relations. Therefore, we have either $\kappa=1$ and $\lambda=0$, or $\kappa=-1$ and $\lambda=1$, which proves i).

In order to prove ii), consider the case where $\kappa=1$ and $\lambda=0$. Then we have $\tilde{\Phi}\,|\,\mathcal{D}_1(\boldsymbol{R}^n)\equiv$identity and $\tilde{\Phi}(\partial_i^2)=\partial_i^2$. In view of (4.3), we can deduce from these facts that $\tilde{\Phi}$ coincides with the identity map on $\mathcal{D}(M)$. This completes the proof of ii).

Now we come back to the proof of Theorem. Apply (6.4) to the formula (**). Then it turns out that there arise only two cases: $\kappa=1$, $\lambda=0$ and $\kappa=-1$, $\lambda=1$.

The first case: $\kappa=1$, $\lambda=0$. From ii) of (6.4) it results that $(-dg)\circ\Psi_*^{-1}\circ\Phi \equiv$identity, on U. Define a closed differential 1-form ω on M, by setting

$$\omega = dg \qquad \text{on } U .$$

Then we obtain

$$(-\varphi)\circ\Psi_*^{-1}\circ\Phi \equiv \text{identity} \qquad \text{on } M$$

whence, in this case, we have

$$\Phi = \Psi_*\circ(\omega)\,.$$

The second case: $\kappa=-1$, $\lambda=1$. We have

$$\kappa = -1 = \Psi_*^{-1}\circ\Phi(1)\,.$$

On the other hand, we have $\sigma(1)=-1$. Hence, referring to the formula (*), we have the corresponding formula

$$\sigma\circ\Psi_*^{-1}\circ\Phi(X+f) = X+\{[X, g']+\lambda' \operatorname{div} X+f\}\ ,$$

on U. This means that $\sigma\circ\Psi_*^{-1}\circ\Phi$ must satisfy the condition of the alternative case stated in (6.3) i). Therefore, as we have seen above, there is a closed differential 1-form ω' such that

$$\Phi = \Psi_*\circ\sigma\circ(\omega')\,,$$

which establishes the alternatives of ii). This completes the proof.

From Theorem (6.3) we can describe the structure of the automorphism group $\mathrm{Aut}(\mathcal{D}(M))$ of $\mathcal{D}(M)$ in detail. Let Φ be an automorphism of $\mathcal{D}(M)$. We note that the expression of Φ in the form stated in Theorem (6.3) ii) is unique. This follows from the following observations: First, Ψ_* is unique, because (ω) and $\sigma\circ(\omega)$ have the support-preserving property, while Ψ_* is the identity if it is support-preserving; secondly, $(\omega)1=1$ and $\sigma 1=-1$, so that $\sigma\circ(\omega)$ does not coincide with any (ω').

Let $\mathrm{Diff}(M)$ be the diffeomorphism group of M and $\Lambda^1_{cl}(M)$ be the abelian group consisting of all closed differential 1-forms on M. We assign to any $\Phi\in\mathrm{Aut}(\mathcal{D}(M))$

$$\begin{aligned} &(\Psi,\ \omega,\ 1), && \text{if } \Phi = \Psi_*\circ(\omega)\,,\\ &(\Psi,\ \omega,\ -1), && \text{if } \Phi = \Psi_*\circ\sigma\circ(\omega)\,. \end{aligned}$$

This assignment induces a bijective map ι of $\mathrm{Aut}(\mathcal{D}(M))$ onto $\mathrm{Diff}(M)\times\Lambda^1_{cl}(M)\times Z_2$. More precisely, we have

Theorem (6.5). *There is a bijective map*

$$\iota\colon \mathrm{Aut}(\mathcal{D}(M)) \to \mathrm{Diff}(M)\times\Lambda^1_{cl}(M)\times Z_2\,.$$

Under the identification via ι, $\Lambda^1_{cl}(M)\times Z_2$ becomes a normal subgroup of $\mathrm{Aut}(\mathcal{D}(M))$, *the elements of which are characterized as the support-preserving automorphisms. The* *multiplication rule in* $\Lambda^1_{cl}(M)\times Z_2$ *is given by* $(\omega, \varepsilon)(\omega', \varepsilon')=(\omega+\varepsilon\omega', \varepsilon\varepsilon')$, *where* $\omega, \omega'\in \Lambda^1_{cl}(M)$ *and* $\varepsilon, \varepsilon'$ *are* ± 1.

From (**) we can easily prove

Theorem (6.5). *Let* Φ *be an isomorphsim of* $\mathcal{D}_1(M)$ *onto* $\mathcal{D}_1(N)$. *Then we have*

i) *There exists a diffeomorphism* Ψ *of* M *onto* N;

ii) *By the use of this* Ψ, Φ *is written as*

$$\Phi = \Psi_*\circ(\omega)\circ(\pi_1+\lambda\ \mathrm{div}\circ\pi_1+\kappa\pi_0)\,,$$

where $\kappa\neq 0$.

As to the automorphism group $\mathrm{Aut}(\tilde{\mathcal{D}}(M))$ of $\tilde{\mathcal{D}}(M)$, we have

(6.6) *Let* Φ *be an element of* $\mathrm{Aut}(\tilde{\mathcal{D}}(M))$ *for which we have* $\Phi|\mathcal{A}(M)=$ *identity. Then*

i) *In case* $n=1$, Φ *is either the identity or* $\natural$, *where* $\natural$ *is the automorphism introduced in Section* 3.

ii) *In case* $n>1$, Φ *is the identity.*

Proof. From the assumption it follows immediately that Φ becomes an $\mathcal{A}$-map from $\tilde{\mathcal{D}}(M)$ to itself. Hence, Proposition (3.5) applies to Φ, which yields that $\Phi\equiv$identity if $n>1$, and $\Phi|\tilde{\mathcal{D}}^+=$identity, $\Phi|\tilde{\mathcal{D}}^-=c_2I$ ($c_2\in \boldsymbol{R}$) if $n=1$. Let us further consider the case $n=1$. If $X, Y\in\tilde{\mathcal{D}}^-$, then $[X, Y]\in\tilde{\mathcal{D}}^+$, hence $c_2{}^2=1$. Thus if Φ is not the identity, then we have $c_2=-1$, which implies $\Phi=\natural$. This completes the proof.

This proposition involves the following: If there is a Lie algebraic characterization of $\mathcal{A}(M)$ in $\tilde{\mathcal{D}}(M)$, then

$$\begin{aligned}&\mathrm{Aut}(\tilde{\mathcal{D}}(M))\simeq\mathrm{Diff}(M)\times Z_2 &&\text{if } n=1\,,\\ &\mathrm{Aut}(\tilde{\mathcal{D}}(M))\simeq\mathrm{Diff}(M) &&\text{if } n>1\,.\end{aligned}$$

At the present, we have only succeeded in attaining this end in the one-dimensional case. So the following problems remain open.

Problem 1. Is every automorphism of $\tilde{\mathcal{D}}(M)$ induced by a diffeomorphism of M in the case $n>1$?

Problem 2. Does the structure of Lie algebra $\tilde{\mathcal{D}}(M)$ determine the underlying smooth structure of M?

7. Differential operators as support-preserving maps

Until now, we have only treated with differential operators with finite orders. But the differential operator is often defined in an alternative way to be the linear map φ from $C^\infty(M)$ to itself with supp $\varphi(f)\subset\operatorname{supp} f\,(f\in C^\infty(M))$ (cf. [4]). If we start from this definition, then the differential operator has not necessarily a finite order. Nevertheless, the space of the differential operators in this sense also has a structure of Lie algebra, the bracket being defined by $[\varphi, \psi]=\varphi\circ\psi-\psi\circ\varphi$. This Lie algebra is denoted by $\underline{\mathscr{D}}(M)$. $\underline{\mathscr{D}}(M)$ contains $\mathscr{D}(M)$ as a subalgebra. By virtue of Peetre's Theorem (cf. [4]), for any open set U with the compact closure, we have $\underline{\mathscr{D}}(M)\,|\,U=\mathscr{D}(M)\,|\,U$. Hence if M is compact, we have $\underline{\mathscr{D}}(M)=\mathscr{D}(M)$ so that there will not arise any new situation here. On the contrary, if M is open, we have $\underline{\mathscr{D}}(M)\neq\mathscr{D}(M)$; specifically, we may say that $\underline{\mathscr{D}}(M)$ consists of the differential operators which possibly take infinite order at the point of infinity.

In this section, we shall show that the results obtained hitherto for $\mathscr{D}(M)$ can be extended to the case of $\underline{\mathscr{D}}(M)$ without any essential alteration. For this purpose, we may and do assume that M is open. Corresponding to $\tilde{\mathscr{D}}(M)$, we denote by $\underline{\tilde{\mathscr{D}}}(M)$ the subalgebra of $\underline{\mathscr{D}}(M)$, consisting of those elements of $\underline{\mathscr{D}}(M)$ which have no "constant terms".

It is clear that all the arguments in the preceding sections which are concerned with local situation of $\mathscr{D}(M)$ remain valid in $\underline{\mathscr{D}}(M)$. Therefore we have to pick up propositions and theorems which are of global nature, and to verify that these results also hold in case of $\underline{\mathscr{D}}(M)$ without any essential modification. According to this plan, after a careful examination, we find that what we must do is reduced to giving the proofs to the following propositions.

(7.1) $[\underline{\mathscr{D}}(M), \mathscr{A}(M)] = \underline{\mathscr{D}}(M), \qquad [\underline{\tilde{\mathscr{D}}}(M), \mathscr{A}(M)] = \underline{\tilde{\mathscr{D}}}(M)$
(cf. Proposition (1.4) i)).

(7.2) *Every derivation of* $\underline{\mathscr{D}}(M)$, *or* $\underline{\tilde{\mathscr{D}}}(M)$, *to* $\underline{\mathscr{D}}(M)$ *has the support-preserving property* (cf. Proposition (5.1) i)).

(7.3) *Let* δ *be a linear map from* $\mathscr{D}_l(M)$ *to* $\underline{\mathscr{D}}(M)$ *which has the support-preserving property. Then, for any open set U with the compact closure, we can find an integer k such that*

$$\delta(\mathscr{D}_l(M))\,|\,U\subset\mathscr{D}_k(M)\,|\,U\,.$$

The similar result holds if we replace $\mathscr{D}_l$ *by* $\tilde{\mathscr{D}}_l$ (cf. Proposition (5.2)).

(7.4) *There are Lie algebraic characterizations of* $\mathscr{D}_0(M)$ *and* $\mathscr{D}_1(M)$ *in* $\underline{\mathscr{D}}(M)$ (cf. Proposition (6.2)).

(7.5) *Every automorphism of* $\underline{\mathscr{D}}(M)$ *which leaves any element of* $\mathscr{A}(M)$ *fixed has the support-preserving property* (cf. Proof of Theorem (6.3)).

Proof of (7.1). Since M is open, we can take an open covering $\{O_{ij}\}$ $(i=0, 1, \cdots, n;\ j=1, 2, \cdots)$ of M which has the properties stated in (1.5). For each i, we can find an imbedding γ_i of $\boldsymbol{R}^{n-1}\times\boldsymbol{R}$ to M such that $\bigcup_{j=1}^{\infty}O_{ij}$ is contained in $\gamma_i([0,1]\times\cdots\times[0,1]\times[0,\infty))$. We adopt γ_i^{-1} as a local coordinates map so that any point $p\in\gamma_i(\boldsymbol{R}^{n-1}\times\boldsymbol{R})$ has a coordinate $p=(x_1,\cdots,x_{n-1},t)$. We assume that γ_i is so chosen that, if $t\to\infty$, then $p=(x_1,\cdots,x_{n-1},t)$ tends to the infinity.

Let $\varphi\in\underline{\mathcal{D}}(M)$ be given. Using the partition of unity, we may write

$$\varphi=\varphi_0+\varphi_1+\cdots+\varphi_n$$

where supp $\varphi_i\subset\bigcup_{j=1}^{\infty}O_{ij}$. Since supp φ_i is contained in $\gamma_i(\boldsymbol{R}^{n-1}\times\boldsymbol{R})$, φ_{i1} is expressed in terms of the coordinates $(x_1,\cdots,x_{n-1},t)$. Setting

$$\begin{aligned}\psi_i(p)&=\int_0^t\varphi_i(x_1,\cdots,x_{n-1},s;\partial^{\alpha})ds, &&\text{if } p=(x_1,\cdots,x_{n-1},t)\,,\\ &=0\,, \qquad \text{if } p\notin\gamma_i(\boldsymbol{R}^{n-1}\times\boldsymbol{R})\,,\end{aligned}$$

we have $\psi_i\in\underline{\mathcal{D}}(M)$. It is clear that

$$\text{supp }\psi_i\subset\gamma_i([0,1]\times\cdots\times[0,1]\times\boldsymbol{R})\,.$$

Take $X_i\in\mathcal{A}(M)$ such that $X_i=d/dt$ on $\gamma_i([0,1]\times\cdots\times[0,1]\times R)$ and 0 outside $\gamma_i(\boldsymbol{R}^{n-1}\times\boldsymbol{R})$. Then

$$\varphi=\sum_{i=0}^{n}[X_i,\psi_i]\,,$$

whence we have proved the first assertion of (7.1). The second assertion can be proved in a similar way.

Proof of (7.2). Let δ be a derivation of $\underline{\mathcal{D}}(M)$ to itself. Letting $\varphi\in\underline{\mathcal{D}}(M)$ be given, we write $\varphi=\sum[X_i,\psi_i]$ in view of (7.1). Take any point p from the outside of supp φ. Then, referring to the proof of (7.1), we find that, in case $n>1$, we can choose X_i, ψ_i such that $p\notin\bigcup(\text{supp }X_i\cup\text{supp }\psi_i)$. From this follows $(\delta\varphi)(p)=0$, whence δ has the support-preserving property. In case $n=1$, we take a small positive number ε such that $(p-\varepsilon,p+\varepsilon)$ lies in the outside of supp φ. Let

$$\psi(t)=\begin{cases}\displaystyle\int_{p-\varepsilon}^{t}\varphi(s)ds, & \text{for } t\leqq p-\varepsilon\\ 0, & \text{for } p-\varepsilon\leqq t\leqq p+\varepsilon\\ \displaystyle\int_{p+\varepsilon}^{t}\varphi(s)ds, & \text{for } t\geqq p+\varepsilon\end{cases}$$

Also, let $\rho(t)$ be a smooth function, identically equal to 1 on $(-\infty,p-\varepsilon)\cup(p+\varepsilon,\infty)$, and 0 on a neighborhood of p. Then we have $\varphi=[\rho d/dt,\psi]$ and $\rho(p)=\psi(p)=0$, whence $(\delta\varphi)(p)=0$ by the same reasoning as above. This completes the proof.

Proof of (7.3). This can be proved in the same way as in (5.2).

Proof of (7.4). Applying the similar arguments to those used in the proof of (6.2), we find

$$\mathcal{D}_0(M)_c = \{\varphi \mid \varphi \in \underline{\mathcal{D}}(M);\ \text{For any } \psi \in \underline{\mathcal{D}}(M) \text{ there is an integer } m \text{ such that } (\mathrm{ad}\varphi)^m \psi = 0\}\ ,$$

$$\mathcal{D}_1(M) = \{\varphi \mid \varphi \in \underline{\mathcal{D}}(M);\ \mathrm{ad}\varphi(\mathcal{D}(M)_c) \subset \mathcal{D}_0(M)_c\},$$

and

$$\mathcal{D}_0(M) = \{\varphi \mid \varphi \in \underline{\mathcal{D}}(M);\ \text{For any } \psi \in \mathcal{D}_1(M) \text{ there is an integer } m \text{ such that } (\mathrm{ad}\varphi)^m \psi = 0\}.$$

Hence we have obtained characterizations of $\mathcal{D}_0(M)$ and $\mathcal{D}_1(M)$.

Proof of (7.5). Referring to the proofs of (6.3) and (7.2), we can easily verify the assertion.

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