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Osaka University
ACTIONS OF SYMPLECTIC GROUPS ON A PRODUCT OF QUATERNION PROJECTIVE SPACES

Dedicated to Professor Minoru Nakaoka on his 60th birthday

FUCHI UCHIDA

(Received November 7, 1983)

0. Introduction

We shall study smooth actions of symplectic group $Sp(n)$ on a closed orientable manifold $X$ such that $X \sim P_a(H) \times P_b(H)$, under the conditions: $a + b \leq 2n - 2$ and $n \geq 7$. Our result is stated in §2 and proved in §5. Typical examples are given in §1. Similar result on smooth actions of special unitary group $SU(n)$ on a closed orientable manifold $X$ such that $X \sim P_a(C) \times P_b(C)$ is stated in the final section.

Throughout this paper, let $H^*(\ )$ denote the singular cohomology theory with rational coefficients, and let $P_a(H)$, $P_a(C)$ and $P_a(R)$ denote the quaternion, complex and real projective $n$-space, respectively. By $X \sim X'$, we mean that $H^*(X) \approx H^*(X')$ as graded algebras.

1. Typical examples

1.1. We regard $S^{4k-1}$ as the unit sphere of the quaternion $k$-space $H^k$ with the right scalar multiplication. Let $Y$ be a compact $Sp(1)$ manifold. By the diagonal action, $Sp(1)$ acts freely on the product manifold $S^{4k-1} \times Y$. Here we consider the cohomology ring of the orbit manifold $(S^{4k-1} \times Y)/Sp(1)$ for the case $Y \sim P_4(H)$.

Consider the fibration: $Y \to (S^{4k-1} \times Y)/Sp(1) \to P_{k-1}(H)$. By the Leray-Hirsch theorem, $H^*(S^{4k-1} \times Y)/Sp(1))$ is freely generated by $1, u, u^2, \ldots, u^b$ as an $H^*(P_{k-1}(H))$ module for an element $u \in H^*(S^{4k-1} \times Y)/Sp(1))$. If $u$ can be so chosen as $u^{b+1} = 0$, then we see that $(S^{4k-1} \times Y)/Sp(1) \sim P_{k-1}(H) \times P_4(H)$.

Lemma 1.1. Denote by $F$, the fixed point set of the restricted $U(1)$ action on $Y$. If $F \sim P_4(C)$, then $(S^{4k-1} \times Y)/Sp(1) \sim P_{k-1}(H) \times P_4(H)$.

Proof. Consider the fibration: $Y \to (S^{4k-1} \times Y)/U(1) \to P_{2k-1}(C)$. We see that $H^*((S^{4k-1} \times Y)/U(1))$ is freely generated by $1, v, v^2, \ldots, v^b$ as an $H^*(P_{2k-1}(C))$ module for an element $v \in H^*((S^{4k-1} \times Y)/U(1))$. We shall show first that
\( v \) can be so chosen as \( v^{b+1} = 0 \). We regard \( S^m \) as the inductive limit of \( S^{qW-1} \) on which \( U(1) \) acts naturally. Consider the following commutative diagram:

\[
\begin{array}{ccc}
H^*((S^m \times Y)/U(1)) & \xrightarrow{j^*} & H^*((S^{qW-1} \times Y)/U(1)) \\
\downarrow i_m^* & & \downarrow i^* \\
H^*(P_m(C) \times F) & \xrightarrow{j_F^*} & H^*(P_{2a-1}(C) \times F)
\end{array}
\]

where \( i, i_m, j, j_F \) are natural inclusions. Since \( H^*(Y) = 0 \), we see that \( i_m^* \) is injective \([4]\) and \( j^* \) is surjective. Let \( v_m \) be an element of \( H^*((S^m \times Y)/U(1)) \) such that \( j^*(v_m) = v \). Let \( x \) be the canonical generator of \( H^2(P_m(C)) \cong H^2(P_{2a-1}(C)) \). Then we can express

\[
i_m^*(v_m) = x^2 + x \times f_1 + 1 \times f_2
\]

where \( f_r \in H^2(F) \) for \( r = 0, 1, 2 \). Since \( F \sim P_b(C) \), we see that there are rational numbers \( a_0, a_1, a_2 \) and a non-zero element \( y \in H^2(F) \), such that \( f_r = a_r y \) for \( r = 0, 1, 2 \). Then we obtain

\[
i_m^*(v_m - a_0 x^2)^{b+1} = (x \times f_1 + 1 \times f_2)^{b+1} = 0.
\]

Since \( i_m^* \) is injective, we obtain \( (v_m - a_0 x^2)^{b+1} = 0 \). Put \( v_1 = j^*(v_m - a_0 x^2) \). Then \( v_1^{b+1} = 0 \), and hence

\[
H^*((S^{qW-1} \times Y)/U(1)) \cong \mathbb{Q}[x, v_1]/(x^2, v_1^{b+1}); \deg x = 2, \deg v_1 = 4.
\]

Consider next the following commutative diagram:

\[
\begin{array}{ccc}
Sp(1)/U(1) & \rightarrow & (S^{qW-1} \times Y)/U(1) \\
\downarrow & & \downarrow \\
Sp(1)/U(1) & \rightarrow & P_{2a-1}(C) \xrightarrow{q} P_{2a-1}(H).
\end{array}
\]

Let \( t \in H^*(P_{2a-1}(H)) \) be the canonical generator such that \( q^*(t) = x^2 \). There exist rational numbers \( \lambda, \mu \) such that \( p^*(u) = \lambda v_1 + \mu x^2 \). Put \( u_t = u - \mu t \). Then \( p^*(u_t) = \lambda v_1 \), and hence \( p^*(u_t)^{b+1} = 0 \). Since the homomorphism \( p^*: H^*((S^{qW-1} \times Y)/Sp(1)) \rightarrow H^*((S^{qW-1} \times Y)/U(1)) \) is injective, we obtain \( u_t^{b+1} = 0 \), and hence

\[
H^*((S^{qW-1} \times Y)/Sp(1)) \cong \mathbb{Q}[t, u_t]/(t^2, u_t^{b+1}); \deg t = \deg u_t = 4.
\]

Thus we obtain \( (S^{qW-1} \times Y)/Sp(1) \sim P_{2a-1}(H) \times P_b(H) \). q.e.d.

**1.2.** We give here examples of a closed orientable \( Sp(1) \) manifold \( Y \) such that \( Y \sim P_b(H) \) and \( F \sim P_b(C) \), where \( F \) denotes the fixed point set of the restricted \( U(1) \) action on \( Y \).

Consider the \( Sp(1) \) action on \( P_b(H) = S^{qW+3}/Sp(1) \) by the left scalar multiplication. Then the fixed point set of the restricted \( U(1) \) action is naturally
diffeomorphic to $P_b(C)$, the fixed point set of the $Sp(1)$ action is naturally diffeomorphic to $P_b(R)$, and the isotropy representation at each fixed point of the $Sp(1)$ action is equivalent to $b\eta \oplus \theta^b$, where $\eta$ denotes the canonical 3-dimensional real representation of $Sp(1)$, $b\eta$ denotes the $b$-fold direct sum of $\eta$, and $\theta^b$ is the trivial representation of degree $b$.

Let $D^{3b}$ denote the unit disk of the representation space $b\eta$. Let $W$ be a $(b+1)$-dimensional compact orientable smooth manifold which is rationally acyclic. Then the boundary $\partial(D^{3b} \times W)$ is a $4b$-dimensional compact orientable smooth $Sp(1)$ manifold which is a rational homology sphere, and the isotropy representation at each fixed point of the $Sp(1)$ action is equivalent to $b\eta \oplus \theta^b$. Hence we can construct an equivariant connected sum

$$Y(W) = P_b(H) \# \partial(D^{3b} \times W).$$

Denote by $F(W)$ the fixed point set of the restricted $U(1)$ action on $Y(W)$. Then $F(W)$ is naturally diffeomorphic to $P_b(C) \# \partial(D^{3b} \times W)$. It is easy to see that

$$Y(W) \sim P_b(H), F(W) \sim P_b(C).$$

1.3. Let $\xi$ be a quaternion $k$-plane bundle and $\xi_c$ its complexification under the restriction of the filed. Its $i$-th symplectic Pontrjagin class $e_i(\xi)$ is by definition [2, §9.6]

$$e_i(\xi) = (-1)^i c_{2i}(\xi_c),$$

where $c_{2i}(\xi_c)$ is the $2i$-th Chern class. Denote by $P(\xi)$ the total space of the associated quaternion projective space bundle. Let $\xi$ be the canonical quaternion line bundle over $P(\xi)$ and put $t = e_1(\xi)$. It is known that there is an isomorphism:

$$(1.3) \quad H^*(P(\xi)) \approx H^*(B)[t]/(\sum_{-d-k-i}(\xi)t^i),$$

where $B$ is the base space of the bundle $\xi$ (cf. [3, §3]).

Let $\xi$ be the canonical quaternion line bundle over $P_b(H)$ and $\xi^*$ its dual line bundle. Let $W$ be a $4b$-dimensional closed orientable smooth manifold and let $f: W \to P_b(H)$ be a smooth mapping such that $f^*: H^*(P_b(H)) \approx H^*(W)$. Let $c$ be a non-negative integer such that $b \leq c + 1$. Then, there is a quaternion $(c+1)$-plane bundle $\xi$ over $W$ such that

$$(n+c+1)f^*\xi^* \approx \xi \oplus \theta^n_H,$$

where $\theta^n_H$ is a trivial quaternion $n$-plane bundle. Put $X = P((n+c+1)f^*\xi^*)$. Since $X$ is diffeomorphic to $\partial(D(\xi) \times D^m)/Sp(1)$, we can act $Sp(n)$ on $X$ in order that the fixed point set is diffeomorphic to $P(\xi)$. We see that by (1.3)
\[ H^*(X) \simeq Q[u, v]/(u^{a+c+1}, v^{b+1}), \]
\[ H^*(P(\xi)) \simeq Q[t, v]/(v^{b+1}, \sum_{i=0}^{c+1} (-1)^i (n+c+1) t^{c+1-i} v^i), \]
where \( v^* = f^* e_1(\xi), t^* = e_1(\xi) \) and \( u+v \) is the first symplectic Pontryagin class of the canonical line bundle over \( P((n+c+1)f*\xi*). \)

2. Classification theorems

We shall prove the following results in this paper.

**Theorem 2.1.** Let \( X \) be a closed orientable manifold on which \( Sp(n) \) acts smoothly and non-trivially. Suppose \( X \simeq P_a(H) \times P_b(H); a \geq b \geq 1, a+b \leq 2n-2 \) and \( n \geq 7 \). Then there are four cases:

0. \( a=n-1 \) and \( X \simeq P_{n-1}(H) \times Y_0 \), where \( Y_0 \) is a closed orientable manifold such that \( Y_0 \simeq P_b(H), \) and \( Sp(n) \) acts naturally on \( P_{n-1}(H) \) and trivially on \( Y_0 \).

1. \( a=n-1 \) and \( X \simeq (S^{4n-1} \times Y_1)/Sp(1), \) where \( Y_1 \) is a closed orientable \( Sp(1) \) manifold such that \( Y_1 \simeq P_b(H), \) \( Sp(1) \) acts as right scalar multiplication on \( S^{4n-1}, \) the unit sphere of \( H^n, \) and \( Sp(n) \) acts naturally on \( S^{4n-1} \) and trivially on \( Y_1. \)

In addition, the fixed point set of the restricted \( U(1) \) action on \( Y_1 \) is \( ~P_b(C). \)

2. \( a=b=n-1 \) and \( X \simeq P_{n-1}(H) \times P_{n-1}(H) \) with the diagonal \( Sp(n) \) action,

3. \( a=n \) and \( X \simeq \partial(D^a \times Y_2)/Sp(1), \) where \( Y_2 \) is a compact orientable \( Sp(1) \) manifold such that \( \dim Y_2 = 4(a+b+1-n) \) and \( Y_2 \simeq P_b(H), \) \( Sp(1) \) acts as right scalar multiplication on \( D^a, \) the unit disk of \( H^n, \) and \( Sp(n) \) acts naturally on \( D^a \) and trivially on \( Y_2. \)

In addition, the \( Sp(1) \) action on the boundary \( \partial Y_2 \) is free and the fixed point set of the restricted \( U(1) \) action on \( Y_2 \) is \( ~P_b(C) \) or \( ~P_b(H). \)

**Remark.** By \( X \simeq X' \) we mean that \( X \) is equivariantly diffeomorphic to \( X' \) as \( Sp(n) \) manifolds. In the case (iii), the fixed point set of the \( Sp(n) \) action on \( X \) is naturally diffeomorphic to the orbit manifold \( \partial Y_2/Sp(1). \)

**Theorem 2.2.** In the case (iii) of Theorem 2.1, the cohomology ring \( H^*(\partial Y_2/Sp(1)) \) is isomorphic to one of the following:

1. \( Q[x, y]/(x^{a+1}, y^{b+1}), \)
2. \( Q[x, y]/(y^{b+1}, \sum_{i=0}^{a+1} (-1)^i (a+1) x^{a+1-i} y^i); b \leq a+1-n, \)

where \( \deg x = \deg y = 4, \) and \( x \) is the Euler class of the principal \( Sp(1) \) bundle \( \partial Y_2 \rightarrow \partial Y_2/Sp(1). \)

**Remark.** The \( Sp(n) \) action given in §1.3 is an example of the case (iii)–(2). Lemma 1.1 assures that a converse of Theorem 2.1 (i) is true.
3. Cohomology of certain homogeneous spaces

Here we consider the cohomology of $V_{n,2}/G = \mathbb{Sp}(2\mathbb{H})/\mathbb{Sp}(n-2) \times G$ for certain closed subgroups $G$ of $\mathbb{Sp}(2)$. Let $\xi$ be the canonical quaternion line bundle over $\mathbb{P}^{n-1}(\mathbb{H})$ and $\zeta$ its orthogonal complement, that is, $\xi$ is a quaternion $(n-1)$-plane bundle over $\mathbb{P}^{n-1}(\mathbb{H})$ such that its total space is

$$E(\xi) = \{(u, [v]) \in H^* \times \mathbb{P}^{n-1}(\mathbb{H}) : u \perp v\}.$$ 

It is easy to see that the total space $\mathbb{P}(\zeta)$ of the associated quaternion projective space bundle is naturally diffeomorphic to $V_{n,2}/\mathbb{Sp}(1) \times \mathbb{Sp}(1)$. Since $\xi \oplus \zeta$ is a trivial bundle, we obtain $e_k(\xi) = (-1)^k e_k(\xi)^4$. By definition, $\mathbb{P}(\xi)$ is naturally identified with a subspace of $\mathbb{P}^{n-1}(\mathbb{H}) \times \mathbb{P}^{n-1}(\mathbb{H})$. Let $i : \mathbb{P}(\xi) \to \mathbb{P}^{n-1}(\mathbb{H}) \times \mathbb{P}^{n-1}(\mathbb{H})$ be the inclusion. Put $\xi = i^*(\xi^* \times 1)$. Then by (1.3) there is an isomorphism:

$$H^*(V_{n,2}/\mathbb{Sp}(1) \times \mathbb{Sp}(1)) \cong Q[x, y]/(x^n, \sum_i x^i y^{n-1-i}) ,$$

deg $x = \deg y = 4$, by the identification $x = i^*(1 \times e_i(\xi))$ and $y = i^*(e_i(\xi) \times 1)$.

Let $p : V_{n,2}/\mathbb{Sp}(1) \times \mathbb{Sp}(1) \to V_{n,2}/\mathbb{Sp}(2)$ be the natural projection and $\xi_2$ the standard quaternion 2-plane bundle over $V_{n,2}/\mathbb{Sp}(2)$.

**Lemma 3.2.** The graded algebra $H^*(V_{n,2}/\mathbb{Sp}(2))$ is generated by $e_i(\xi_2)$, $e_2(\xi_2)$. The algebra is isomorphic to the subalgebra of $Q[x, y]/(x^n, \sum_i x^i y^{n-1-i})$, consisting of symmetric polynomials.

Proof. Since the fibration $p$ is a 4-sphere bundle and $H^{odd}(V_{n,2}/\mathbb{Sp}(2)) = 0$ (cf. [1, §26]), the homomorphism $p^* : H^*(V_{n,2}/\mathbb{Sp}(2)) \to H^*(V_{n,2}/\mathbb{Sp}(1) \times \mathbb{Sp}(1))$ is injective. Since $p^*(\xi_2) = i^*(\xi^* \times \xi)$, we obtain

$$p^*e_i(\xi_2) = i^*e_i(\xi^* \times \zeta) = x + y ,$$

$$p^*e_2(\xi_2) = i^*e_2(\xi^* \times \zeta) = xy .$$

Then the desired result is obtained by the Leray-Hirsch theorem. q.e.d.

**Corollary 3.3.** $e_1(\xi_2)^{2n-4} = 0$ and $e_2(\xi_2)^{2n-3} = 0$.

Proof. Put $I = (x^n, \sum_i x^i y^{n-1-i})$. It is easy to see that $y^n \in I$. In the quotient ring $Q[x, y]/I$, we obtain

$$(x+y)^{2n-4} = \binom{2n-4}{n-1} x^{n-1} y^{n-3} + (2n-4) x^{n-2} y^{n-2} + (2n-4) x^{n-3} y^{n-1}$$
$$= \left\{ \begin{array}{c}
\left( \frac{2n-4}{n-2} \right) x^{n-2} y^{n-2},
\left( \frac{2n-4}{n-1} \right) x^{n-3} y^{n-1},
\end{array} \right.$$

and hence $e_1(\xi_2)^{2n-4} = 0$. We obtain $e_2(\xi_2)^{2n-3} = 0$ similarly. q.e.d.

4. Preliminary results

First we state the following two lemmas which are proved by a standard
Lemma 4.1. Suppose $n \geq 7$. Let $G$ be a closed connected proper subgroup of $\text{Sp}(n)$ such that $\dim \text{Sp}(n)/G < 8n$. Then $G$ coincides with $\text{Sp}(n-i) \times K$ $(i=1, 2, 3)$ up to an inner automorphism of $\text{Sp}(n)$, where $K$ is a closed connected subgroup of $\text{Sp}(i)$.

Lemma 4.2. Suppose $r \geq 5$ and $k < 8r$. Then an orthogonal non-trivial representation of $\text{Sp}(r)$ of degree $k$ is equivalent to $(\nu_r)_R \oplus \theta^{k-t}$. Here $(\nu_r)_R : \text{Sp}(r) \to O(4r)$ is the canonical inclusion, and $\theta^t$ is the trivial representation of degree $t$.

In the following, let $X$ be a closed connected orientable manifold with a non-trivial smooth $\text{Sp}(n)$ action, and suppose $n \geq 7$ and $\dim X < 8n$. Put

$$F(i) = \{x \in X : \text{Sp}(n-i) \subset \text{Sp}(n)x \subset \text{Sp}(n-i) \times \text{Sp}(i)\}$$

$$X(i) = \text{Sp}(n)F(i) = \{gx : g \in \text{Sp}(n), x \in F(i)\}.$$ 

Here $\text{Sp}(n)x$ denotes the isotropy group at $x$. Then, by Lemma 4.1, we obtain $X = X(0) \cup X(1) \cup X(2) \cup X(3)$.

Proposition 4.3. If $X(0)$ is non-empty, then $X(i)$ is empty for each $i \geq k+2$.

Proof. Let us denote by $F(\text{Sp}(n-j), X(i))$ the fixed point set of the restricted $\text{Sp}(n-j)$ action on $X(i)$. It is easy to see that the set is empty for each $j < i \leq n-i$. Suppose that $X(0)$ is non-empty and fix $x \in F(0)$. Let $\sigma$ be the slice representation at $x$. Then the restriction $\sigma| \text{Sp}(n-k)$ is trivial or equivalent to $(\nu_{n-k})_R \oplus \theta^k$ by Lemma 4.2. Anyhow, a principal isotropy group of the given action contains $\text{Sp}(n-k)$, and hence $F(\text{Sp}(n-k), X(i))$ is non-empty if so is $X(i)$. q.e.d.

Proposition 4.4. Suppose $X = X(0) \cup X(1+1)$. If $X(0)$ and $X(1+1)$ are non-empty, then the codimension of each connected component of $F(0)$ in $X$ is equal to $4(k+1)(n-k)$.

Proof. Fix $x \in F(0)$. Let $\sigma$ and $\rho$ denote the slice representation at $x$ and the isotropy representation of the orbit $\text{Sp}(n)x$, respectively. The restriction $\sigma| \text{Sp}(n-k)$ is equivalent to $(\nu_{n-k})_R \oplus \theta^k$ by Lemma 4.2 and the assumption that $X(1+1)$ is non-empty. On the other hand, $\rho| \text{Sp}(n-k)$ is equivalent to $k(\nu_{n-k})_R \oplus \theta^{k+t}$ by considering adjoint representations. Hence $(\sigma \oplus \rho)| \text{Sp}(n-k)$ is equivalent to $(k+1)(\nu_{n-k})_R \oplus \theta^{k+t}$.

This shows that the codimension of $F(0)$ at $x$ is equal to $4(k+1)(n-k)$. q.e.d.

Corollary 4.5. Suppose $X = X(0) \cup X(0)$. Then either $X(0)$ or $X(0)$ is empty.

Remark. $\dim \text{Sp}(n)/\text{Sp}(n-k) \times \text{Sp}(k) = 4k(n-k)$ and $\chi(\text{Sp}(n))/\text{Sp}(n-k)$
\( \times Sp(k) = \left( \begin{array}{c} n \\ k \end{array} \right) \), where \( \chi(\ ) \) denotes the Euler characteristic, and \( \left( \begin{array}{c} n \\ k \end{array} \right) \) denotes the binomial coefficient.

5. **Proof of the classification theorems**

Throughout this section, suppose that \( X \) is a closed orientable manifold with a non-trivial smooth \( Sp(n) \) action such that

\[
(*) \quad H^*(X) = \mathbb{Q}[u, v]/(u^{a+1}, v^{b+1}); \quad \deg u = \deg v = 4.
\]

Moreover, suppose that \( n \geq 7, 1 \leq b \leq a \) and \( a + b \leq 2n - 2 \). By arguments and notations in the preceding section, we see that \( X \neq X_{(k)} \cup X_{(k+1)} \) for \( k = 0, 1, 2 \).

5.1. **We shall show first that** \( X \not\supset X_{(2)} \cup X_{(3)} \). Suppose \( X = X_{(2)} \cup X_{(3)} \). Then \( X = X_{(2)} \) or \( X = X_{(3)} \) by Corollary 4.5. Looking at the Euler characteristic of \( X \), we see that \( X \neq X_{(3)} \).

Suppose \( X = X_{(2)} \). Then \( X = (V_{n,2} \times F_{(2)})/Sp(2) \). Here we consider the following commutative diagram of natural projections:

\[
X = (V_{n,2} \times F_{(2)})/Sp(2) \rightarrow V_{n,2}/Sp(2),
\]

where \( T \) is a maximal torus of \( Sp(2) \). Since \( \chi(F_{(2)}) \neq 0 \), we see that the restricted \( T \) action on \( F_{(2)} \) has a fixed point, and hence the projection \( p_1 \) has a cross-section. Therefore \( p^*_1: H^*(V_{n,2}/T) \rightarrow H^*((V_{n,2} \times F_{(2)})/T) \) is injective. On the other hand, \( q^*: H^*(V_{n,2}/Sp(2)) \rightarrow H^*(V_{n,2}/T) \) is injective, because \( H^\text{odd}(V_{n,2}/Sp(2)) = H^\text{odd}(Sp(2)/T) = 0 \) (cf. [1, §26]). Consequently, we see that \( p^*: H^*(V_{n,2}/Sp(2)) \rightarrow H^*(X) \) is injective. In particular, we obtain \( a + b \geq 2n - 4 \). If \( a + b = 2n - 4 \), then \( X = V_{n,2}/Sp(2) \). Because rank \( H^t(X) = 2 \) and rank \( H^t(V_{n,2}/Sp(2)) = 1 \), we get a contradiction.

Suppose \( a + b \geq 2n - 3 \), and put \( p^*e_1(\xi_2) = \alpha u + \beta v; \alpha, \beta \in \mathbb{Q} \). Since \( e_1(\xi_2)^{2n-3} = 0 \) by Corollary 3.3, we obtain

\[
0 = p^*e_1(\xi_2)^{a+b} = \left( \frac{a+b}{a} \right) (\alpha u)^{a} (\beta v)^{b},
\]

and hence \( \alpha \beta = 0 \). On the other hand, \( e_1(\xi_2)^{2n-4} \neq 0 \) by Corollary 3.3, and hence \( p^*e_1(\xi_2)^{2n-4} \neq 0 \). Thus we obtain \( a = 2n - 4 \). Looking at the Euler characteristic of \( F_{(2)} \), we get a contradiction.

5.2. **We consider now the case** \( X = X_{(2)} \cup X_{(3)} \). Suppose that both \( X_{(1)} \) and \( X_{(2)} \) are non-empty. We see that \( \text{codim} \ F_{(1)} = 8n - 8 \) by Proposition 4.4. Since \( \text{dim} \ X \leq 8n - 8 \), we obtain \( \text{dim} \ F_{(1)} = 0 \) and \( a + b = 2n - 2 \).
Fix \( x \in F(0) \). Since \( X(0) \) is non-empty, we see that the slice representation \( \sigma \) at \( x \) is equivalent to \( v_{n-1} \otimes \nu^* \pi \) or \( (v_{n-1})^* \pi \) by Lemma 4.2, where \( \pi \) is a natural projection of \( Sp(n-1) \times Sp(1) \) onto \( Sp(n-1) \). Then the principal isotropy group is of the form \( Sp(n-2) \times K \), where \( K = \Delta Sp(1) \) (resp. \( 1 \times Sp(1) \)) for \( \sigma = v_{n-1} \otimes \nu^* \pi \) (resp. \( \sigma = (v_{n-1})^* \pi \)). Here \( \Delta Sp(1) \) (resp. \( 1 \times Sp(1) \)) is a closed subgroup of \( Sp(2) \) consisting of the matrices of the form \( \begin{pmatrix} q & 0 \\ 0 & q \end{pmatrix} \) (resp. \( \begin{pmatrix} 1 & 0 \\ 0 & q \end{pmatrix} \)).

Anyhow, we see that the \( Sp(n) \) action on \( X \) has a codimension one orbit, and hence \( X \) is a union of closed invariant tubular neighborhoods of just two non-principal orbits (cf. [6]). We already see that one of the non-principal orbits is \( P_{n-1}(H) \). Looking at the Euler characteristic of \( X \), we see that \( a = b = n-1 \) and another non-principal orbit is \( V_{n-1}/Sp(1) \times Sp(1) \).

Suppose \( K = 1 \times Sp(1) \). Then the normalizer of the principal isotropy group is connected, and hence such an \( Sp(n) \) manifold is unique up to equivariant diffeomorphism (cf. [6, §5.3]). On the other hand, the product manifold \( P_{n-1}(H) \times P_{n-1}(H) \) with the diagonal \( Sp(n) \) action is such one. Therefore \( X \) is equivariantly diffeomorphic to \( P_{n-1}(H) \times P_{n-1}(H) \) with the diagonal \( Sp(n) \) action.

Suppose next \( K = \Delta Sp(1) \). Then the normalizer of the principal isotropy group has just two connected components, and its generator corresponds to the antipodal involution of the slice representation at a point of \( V_{n-1}/Sp(1) \times Sp(1) \). Hence such an \( Sp(n) \) manifold is unique up to equivariant diffeomorphism (cf. [6, §5.3]). Here we construct such one. Let \( \xi \) be the canonical quaternion line bundle over \( P_{n-1}(H) \) and \( \xi \) its orthogonal complement (see §3). Then \( Sp(n) \) acts naturally on the total space \( E(\xi) \) as the bundle mappings.

Denote by \( \theta^1_H \) a trivial quaternion line bundle. We see that the \( Sp(n) \) action on the total space \( P(\xi \oplus \theta^1_H) \) of the associated quaternion projective space bundle is the desired one. On the other hand, we see that by (1.3)

\[
H^k(P(\xi \oplus \theta^1_H)) \cong Q[x, y]/(x^*, \sum_i x^i y^{a-i}); \deg x = \deg y = 4.
\]

Hence the cohomology ring of \( P(\xi \oplus \theta^1_H) \) is not isomorphic to that of \( P_{n-1}(H) \times P_{n-1}(H) \).

5.3. We consider next the case \( X = X_{(0)} \cup X_{(1)} \) for \( c < n \). We shall show first that \( X_{(0)} \) is empty.

Suppose that \( X_{(0)} \) is non-empty. Let \( U \) be an invariant closed tubular neighborhood of \( X_{(0)} \) in \( X \), and put \( E = X - \text{int} U \). Put \( W = E \cap F(0) \). Then \( W \) is a compact connected orientable manifold with non-empty boundary \( \partial W \), and \( Sp(1) \) acts naturally on \( W \). Since there is a natural diffeomorphism \( E = (S^{a-1} \times W)/Sp(1) \), we obtain

\[
\dim W = 4(a + b + 1 - n) = 4k, \quad k \leq b \leq a < n.
\]

Let \( i : E \to X \) be the inclusion. Then \( i^* : H^*(X) \to H^*(E) \) is an isomorphism.
for each \( t \leq 4n - 2 \), because the codimension of each connected component of \( X_\omega \) is \( 4n \) by Lemma 4.2. By the Gysin sequence of the principal \( Sp(1) \) bundle \( S^{4n-1} \times W \to E \) and the cohomology ring of \( X \), we obtain rank \( H^{4k}(W) - \text{rank } H^{4k-1}(W) = 1 \). On the other hand, we see that \( H^{4k}(W) \cong H_{4k}(W) = 0 \) and rank \( H^{4k-1}(W) \geq 0 \); this is a contradiction. Thus we see that \( X_\omega \) is empty.

Consequently, we obtain \( X = X_\omega = (S^{4n-1} \times F(\omega))/Sp(1) \). Put \( Y = F(\omega) \). We see that

\[
\dim Y = 4(a + b + 1 - n) = 4k, \quad k \leq b \leq a < n < a + b.
\]

We shall show next that \( a = n - 1 \) and \( Y \sim P_B(H) \).

By the Gysin sequence of the principal \( Sp(1) \) bundle \( p: S^{4n-1} \times Y \to X \), we obtain \( H^{4i+1}(S^{4n-1} \times Y) = 0 \) and an exact sequence:

\[
0 \to H^{4i-1}(S^{4n-1} \times Y) \to H^{4i}(X) \to H^{4i}(S^{4n-1} \times Y) \to 0
\]

for any \( i \), where \( \mu \) is the multiplication by \( e_1(p) \), the first symplectic Pontrjagin class of the quaternion line bundle associated with the \( Sp(1) \) bundle \( p \). We can represent \( p^* u = 1 \times u_1, \ p^* v = 1 \times v_1 \) for \( u_1, v_1 \in H^4(Y) \). Then we see that \( H^{4i-1}(Y) = 0 \) and \( H^4(Y) \) is generated by at most two elements \( u_1, v_1 \). We can represent \( e_1(p) = \alpha u + \beta v; \ \alpha, \ \beta \in \mathbb{Q} \). By definition, the \( Sp(1) \) bundle \( p \) is a pull-back of a bundle over \( P_{n-1}(H) \), and hence \( e_1(p)^* = 0 \). Since \( n \leq a + b \), we see that \( \alpha \beta = 0 \). Suppose \( e_1(p) = 0 \). Then \( p^* \) is injective, and hence \( 1 \times u_1, v_1 \) is an \( H^4(P_{2n-1}(C)) \) module. Thus we get a contradiction. Therefore we see that \( e_1(p) = \alpha u \) or \( e_1(p) = \beta v \), and hence \( u_1, v_1 = 0 \) when \( \alpha = 0 \) or \( \beta = 0 \), respectively. Looking at the Euler characteristic of \( X \) we see that \( a = n - 1 \) and \( Y \sim P_B(H) \).

When \( b < n - 1 \), we see that \( e_1(p) = \alpha u (\alpha = 0) \) and \( H^*(Y) \cong \mathbb{Q}[v_1](v_1^{b+1}) \). When \( b = n - 1 \), interchanging \( u \) and \( v \) if necessary we can assume that \( e_1(p) = \alpha u (\alpha = 0) \) and \( H^*(Y) \cong \mathbb{Q}[v_1](v_1^b) \). It remains to consider the \( Sp(1) \) action on \( Y \). We shall show that either \( F \sim P_B(C) \) or \( Sp(1) \) action on \( Y \) is trivial, where \( F \) denotes the fixed point set of the restricted \( U(1) \) action on \( Y \).

Put \( w = \pi^*(v) \), where \( \pi \) is a natural projection of \( (S^{4n-1} \times Y)/U(1) \) onto \( X = (S^{4n-1} \times Y)/Sp(1) \). Consider the fibration: \( Y \to (S^{4n-1} \times Y)/U(1) \to P_{2n-1}(C) \). We see that \( w^{b+1} = 0 \) and \( H^*(S^{4n-1} \times Y)/U(1)) \) is freely generated by \( 1, w, w^2, \ldots, w^b \) as an \( H^*(P_{2n-1}(C)) \) module. Consider next the following commutative diagram:

\[
\begin{array}{ccc}
H'(S^\infty \times Y)/U(1)) & j^*_Y & \to H'(S^{4n-1} \times Y)/U(1)) \\
\downarrow i^*_Y & & \downarrow i^*_Y \\
H'(P_{2n}(C) \times F) & j^*_F & \to H'(P_{2n-1}(C) \times F) 
\end{array}
\]

where \( i, i_\infty, j, j_F \) are natural inclusions. Since \( H^{	ext{odd}}(Y) = 0 \), we see that \( [4] i^*_Y \) is injective for each \( r \) and surjective for each \( r > 4b \) and \( j^*_Y \) is surjective. Let
\( w_0 \) be an element of \( H^4((S^n \times Y)/U(1)) \) such that \( j^*(w_0) = w \). Let \( x \) be the canonical generator of \( H^4(P_\infty(C)) \approx H^4(P_{2n-1}(C)) \). Then we can express

\[
i_x^*(w_0) = x^2 \times f_0 + x \times f_1 + 1 \times f_2
\]

where \( f_i \in H^{2i}(F) \) for \( t=0, 1, 2 \). It is known that \([4] F_0 \approx P_d(C) \) or \( F_0 \approx P_d(H) \) \((0 \leq d \leq b)\) for each connected component \( F_0 \) of \( F \). We shall show that \( F \) is connected.

Consider first the case \( b < n - 1 \). We see that \( i_x^*(w_0^*) = x \times f_1 + 1 \times f_2 \), that is, \( f_0 = 0 \) by the relation \((x^2 \times f_0 + x \times f_1 + 1 \times f_2)^{k+1} = 0 \) in \( H^{4k+4}(P_{2n-1}(C) \times F) \). Consequently, we can show that if \( F \) is not connected then \( i_x^*(w_0) = 0 \) and hence \( w_0 = 0 \); this is a contradiction.

Consider next the case \( b = n - 1 \). Since \( j^*(w_0^*) = w^* = 0 \), we see that \( w_0^* = \gamma x^{2n} \) for some \( \gamma \in Q \), and hence \( i_x^*(w_0^*) = x^{2n} \times \gamma \). Suppose \( \gamma = 0 \). Then \( f_0 = 0 \), and hence we can show that \( F \) is connected by the same argument as above. Suppose next \( \gamma \neq 0 \). We shall show that \( i_x^*(w_0) = x^2 \times f_0 \), that is \( f_1 = 0 \) and \( f_2 = 0 \). For any connected component \( F_0 \) of \( F \), we have an equation

\[
(x^2 \times f_0 \mid F_0 + x \times f_1 \mid F_0 + 1 \times f_2 \mid F_0)^n = x^{2n} \times \gamma
\]

in \( H^{4n}(P_\infty(C) \times F_0) \). Then we see that \( (f_0 \mid F_0)^n = \gamma \neq 0 \) and \( f_1 \mid F_0 = 0 \) for \( t=1, 2 \). Thus we obtain \( i_x^*(w_0) = x^2 \times f_0 \) and \( f_0 = \gamma \). Let \( F_1 \) (resp. \( F_2 \)) be the union of connected components \( F_\sigma \) of \( F \) on which \( f_0 \mid F_\sigma \) is positive (resp. negative). Since \( f_0 = \gamma \), we can regard \( f_0 \mid F_1 \) and \( f_0 \mid F_2 \) as constant rational numbers. Then each element of \( H^r(P_\infty(C) \times F_1) \) for \( r \geq 4n \) is expressed as a polynomial of \( x \times 1 \) with rational coefficients for \( s=1, 2 \) because \( H^*(((S^n \times Y)/U(1)) \) is generated by an element \( w_0 \) as a graded \( H^*(P_\infty(C)) \) algebra and \( i_x^* \) is surjective for \( r \geq 4n \). Then we see that \( F_1 \) (or \( F_2 \), \( s=1, 2 \)) consists of just one point, and hence \( F \) consists of at most two points. This is a contradiction to the fact: \( \chi(F) = \chi(Y) = n \leq 7 \).

Anyhow we see that \( F \) is connected, and hence \( F \approx P_d(C) \) or \( F \approx P_d(H) \). The \( Sp(1) \) action on \( Y \) is trivial for the latter case.

5.4. Finally, we consider the case \( X = X_{(a)} \cup X_{(b)} \) for \( a \geq n \). We shall show first that \( X_{(a)} \) is non-empty.

Suppose that \( X_{(a)} \) is empty. Then \( X = X_{(b)} = (S^{a-1} \times X_{(b)}) / Sp(1) \). By the Gysin sequence of the principal \( Sp(1) \) bundle \( S^{a-1} \times F_{(b)} \to X \), we see that \( F_{(b)} \approx P_b(H) \). Looking at the Euler characteristic of the fibration: \( F_{(b)} \to X \to P_{n-1}(H) \) we obtain \( a = n - 1 \); this is a contradiction.

Consequently, we see that (cf. [8]) there is an equivariant decomposition \( X = \partial(D^{a} \times Y)/Sp(1) \), where \( Y \) is a compact connected orientable manifold with a smooth \( Sp(1) \) action, and \( Y \) has a non-empty boundary \( \partial Y \) on which the \( Sp(1) \) action is free. We see that

\[
dim Y = 4(a + b + 1 - n)
\]
and the fixed point set of the $Sp(n)$ action on $X$ is naturally diffeomorphic to the orbit manifold $\partial Y/Sp(1)$. Moreover, we see that there is a natural decomposition $X=X_1 \cup X_2$, where

$$X_1 = (S^{4n-1} \times Y)/Sp(1) \text{ and } X_2 = (D^{4n} \times \partial Y)/Sp(1).$$

Put $X_0 = X_1 \cap X_2 = (S^{4n-1} \times \partial Y)/Sp(1)$.

Let $\pi: \partial(D^{4n} \times Y) \to X$ be the projection of the principal $Sp(1)$ bundle. Denote by $\pi_i$, the projection of the restricted principal $Sp(1)$ bundle over $X_i$. Let $j_i: X_i \to X$ and $i_0: X_0 \to X$ be inclusions. Put $u_i = j_i^\ast(u)$ and $v_i = j_i^\ast(v)$.

We can express

$$e(\pi) = \alpha u + \beta v; \quad \alpha, \beta \in \mathbb{Q},$$

where $e(\pi)$ is the Euler class of the principal $Sp(1)$ bundle $\pi$. Then we obtain

$$e(\pi_0) = j_0^\ast e(\pi) = \alpha u + \beta v_i.$$

Since $H^r(X, X_0) \simeq H^r(X_0, X_0) \simeq H^{r-4n}(\partial Y/Sp(1))$ for each $r$, we obtain an isomorphism $j_i^\ast: H^r(X, X_0) \simeq H^r(X_i)$ for each $r \leq 4n-2$. Because $Y$ is a compact connected manifold with non-empty boundary and $\dim Y \leq 4n-4$, we see that $\pi_i^\ast(u_i^{4n-1}) = 0$ and hence $u_i^{4n-1} = x' e(\pi_i)$ for some $x' \in H^{4n-8}(X_i)$. Then $u_i^{4n-1} = x e(\pi)$ for some $x \in H^{4n-8}(X)$ by the isomorphism $j_i^\ast$. In particular we see that $\alpha \neq 0$ in the expression: $e(\pi) = \alpha u + \beta v$. Looking at the isomorphism $j_i^\ast$ and the Gysin sequence of the principal $Sp(1)$ bundle $\pi_i$, we see that $\pi_i^\ast(v_i^{4n-1}) = 0$ and the algebra $H^r(S^{4n-1} \times Y)$ is generated by $\pi_i^\ast v_i$. Hence we obtain $Y \sim P_n(H)$. In addition, we see that $X_i \sim P_{n-1}(H) \times P_n(H)$ by the fibration: $Y \to X_0 \to P_{n-1}(H)$.

Since $b \leq n-2$, by the same argument as in the second half of §5.3, we see that $F \sim P_n(C)$ or $F \sim P_n(H)$, where $F$ denotes the fixed point set of the restricted $U(1)$ action on $Y$.

Here we complete the proof of Theorem 2.1.

Remark. The case $\alpha \beta \neq 0$ in the expression $e(\pi) = \alpha u + \beta v$ occurs only when $b \leq a+1-n$, because

$$(e(\pi_1) - \beta v_i)^{s+1} = (\alpha u_i^{s+1}) = 0$$

in $H^s(X_1) = \mathbb{Q}[e(\pi_1), v_i]/(e(\pi_1)^s, v_i^{s+1})$.

5.5. In the following, we consider the cohomology of $\partial Y/Sp(1)$. Regarding $\alpha u$ and $\beta v$ as new $u$ and $v$ if necessary, we can assume that $e(\pi) = u$ if $\beta = 0$ and $e(\pi) = u + v$ if $\beta \neq 0$.

Since the algebra $H^s(X_1)$ is generated by $e(\pi_1)$ and $v_i$, we obtain an short exact sequence:
Moreover, we see that the kernel of $j^\#_\ast$ is an ideal generated by $e(\pi)^\ast$, that is, $\ker j^\#_\ast = H^\ast(X)e(\pi)^\ast$. Let $\tau \in H^\ast(X, X_1)$ be an element such that $k^\#_\ast(\tau) = e(\pi)^\ast$. Then $H^\ast(X, X_1)$ is generated by $\tau$ as an $H^\ast(X)$ module, that is, $H^\ast(X, X_1) = H^\ast(X)\tau$. Let $j^\#_\ast : H^\ast(X, X_1) \to H^\ast(X_2, X_0)$ be an excision isomorphism. Denote by $t \in H^\ast(X_2, X_0)$ the Thom class of the quaternion $n$-plane bundle over $\partial Y/Sp(1)$. Then $j^\#_\ast(t) = \lambda t$ for non-zero $\lambda \in Q$. Since $j^\#_\ast(w\tau) = j^\#_\ast(w)j^\#_\ast(\tau) = \lambda j^\#_\ast(w)t$ for each $w \in H^\ast(X)$, we see that $j^\#_\ast : H^\ast(X) \to H^\ast(X_2)$ is surjective. In addition, $j^\#_\ast(w) = 0$ if and only if $e(\pi)^\ast w = 0$ for $w \in H^\ast(X)$. Then we can show that \{ $j^\#_\ast(u^p\nu^q) ; 0 \leq p \leq a-n, 0 \leq q \leq b$ \} are linearly independent in the graded module $H^\ast(X_2) \approx H^\ast(X)/\ker j^\#_\ast$. On the other hand, we obtain

$$\text{rank } H^\ast(X_2) = \text{rank } H^\ast(X) - \text{rank } H^\ast(X_1) = (a+1-n)(b+1).$$

Therefore the set \( \{u^p\nu^q ; 0 \leq p \leq a-n, 0 \leq q \leq b\} \) is an additive base of the graded module $H^\ast(X_2)$.

Suppose first $e(\pi) = u$, i.e. $\beta = 0$. Then $j^\#_\ast(u^{a-n+1}) = 0$, and hence $H^\ast(X_2) \approx Q[u_2, \nu_2]/(u_2^{a-n+1}, \nu_2^{b+1})$. Therefore $\partial Y/Sp(1) \sim P_{a-n}(H) \times P_b(H)$.

Suppose next that $b \leq a+1-n$ and $e(\pi) = u+v$, i.e. $\beta \neq 0$. We see that

$$e(\pi)^\ast \sum_{i=0}^{b} (-1)^i \binom{a+1}{i} (u+v)^{a+1-n-i} \nu^i = ((u+v)-v)^{a+1} = 0,$$

hence we obtain

$$H^\ast(\partial Y/Sp(1)) \approx H^\ast(X_2) \approx Q[x, y]/(x^{a+1}, \sum_{i=0}^{b} (-1)^i \binom{a+1}{i} x^{a+1-n-i} y^i),$$

where $x = u_2 + \nu_2$ and $y = \nu_2$.

Here we complete the proof of Theorem 2.2.

6. Construction

We regard $D^a$ as the unit disk of the quaternion $n$-space $H^n$ with the right scalar multiplication and the left $Sp(n)$ action. Let $Y$ be a compact orientable smooth $Sp(1)$ manifold such that the $Sp(1)$ action is free on the non-empty boundary $\partial Y$. By the diagonal action, $Sp(1)$ acts freely on the boundary $\partial(D^a \times Y)$. Here we consider the cohomology ring of the orbit manifold $X = \partial(D^a \times Y)/Sp(1)$ on which $Sp(n)$ acts naturally.

Suppose that $\dim Y = 4d + 4$, $Y \sim P_d(H)$, $1 \leq b \leq d \leq n-2$, and $F \sim P_d(C)$ or $F \sim P_d(B)$, where $F$ denotes the fixed point set of the restricted $U(1)$ action on $Y$. Moreover suppose that $i^\ast : H^\ast(Y) \approx H^\ast(\partial Y)$, where $\iota$ is an inclusion. Put $c = d-b$. In addition, we suppose that the graded algebra $H^\ast(\partial Y/Sp(1))$
is isomorphic to one of the following:

1. \( Q[x, y]/(x^{c+1}, y^{b+1}) \),
2. \( Q[x, y]/(y^{b+1}, \Sigma_{i=0}^{b} (-1)^i (x^{c+1-i}) x^{c+1-i} y^i); b \leq c+1, \)

where \( \deg x = \deg y = 4 \), and \( x \) is the Euler class of the principal \( Sp(1) \) bundle \( \partial Y \to \partial Y/Sp(1) \).

Put \( X_1 = (S^{4n-1} \times Y)/Sp(1) \), \( X_2 = (D^{4n} \times \partial Y)/Sp(1) \) and \( X_0 = X_1 \cap X_2 = (S^{4n-1} \times \partial Y)/Sp(1) \). Then \( X = X_1 \cup X_2 \). Let \( \pi : \partial(D^{4n} \times Y) \to X \) be the projection of the principal \( Sp(1) \) bundle. Let us denote by \( \pi_1 \) the projection of the restricted principal \( Sp(1) \) bundle over \( X \). Let \( j_1 : X_1 \to X \) and \( i_2 : X_0 \to X \) be the inclusions. Let \( p : X_2 \to \partial Y/Sp(1) \) be the natural projection of \( 4n \)-disk bundle, and put \( p_0 = p|_{X_0} : X_0 \to \partial Y/Sp(1) \).

Since \( d \leq n - 2 \), we see that \( H^*(X_0) \) is freely generated by 1, \( \sigma \) as an \( H^*(\partial Y/Sp(1)) \) module for an element \( \sigma \in H^{4n-1}(X_0) \) and \( \iota^* : H^*(X_0) \to H^*(X_0) \) is injective. Put \( x_0 = p_0^*(x) \), \( y_0 = p_0^*(y) \), \( x_2 = p^*(x) \) and \( y_2 = p^*(y) \). Then \( x_0 = e(\pi_0) \) and \( x_2 = e(\pi_2) \), the Euler classes of the principal \( Sp(1) \) bundles.

By the fibration: \( Y \to X \to P_{d-1}(H) \) and the assumption that \( F \sim P_{d}(C) \) or \( F \sim P_{d}(H) \) and \( Y \sim P_{d}(H) \), we see that by Lemma 1.1,

\[
H^*(X_1) = Q[x_1, y_1]/(x_1^{c+1}, y_1^{b+1}); \quad \deg x_1 = \deg y_1 = 4,
\]

where \( x_1 = e(\pi_1) \), the Euler class of the principal \( Sp(1) \) bundle.

Consider the Mayer–Vietoris sequence of a triad \( (X; X_1, X_2) \):

\[
i^* \to H^{4r-1}(X_0) \xrightarrow{\Delta^*} H^r(X) \to H^r(X_1) \oplus H^r(X_2) \xrightarrow{j^*} H^r(X) \to i^* \Delta^*
\]

where \( j^*(a) = (j^*(a), j^*(a)) \) and \( i^*(b_1, b_2) = i^*(b_1) - i^*(b_2) \). We see that \( H^r(X) = 0 \) for each \( r \neq 0 \) (mod 4) and there is the following short exact sequence for each \( k \):

\[
(*) \quad 0 \to H^{4k-1}(X_0) \xrightarrow{\Delta^*} H^{4k}(X) \xrightarrow{j^*_k} H^{4k}(X_1) \to 0.
\]

Notice that \( \dim X = 4(n + d) \) and

\[
(**) \quad j^*_k : H^{4k}(X) \cong H^{4k}(X_1) \quad \text{for } k < n.
\]

Let \( u, v \) be elements of \( H^r(X) \) such that \( j^*_k(u) = x_1, j^*_k(v) = y_1 \). We see that \( u = e(\pi) \), the Euler class of the principal \( Sp(1) \) bundle. Moreover, we see that \( v^{b+1} = 0 \) by \( (***) \) and the assumption \( b \leq n - 2 \). Since \( j^*_k(u^{c+1}v^k) \neq 0 \), there is an element \( z \in H^{4k+1}(X) \) such that \( u^{c+1}v^kz = 0 \), by the Poincaré duality. Then we see that \( u^{c+1}v^k \neq 0 \), by \( (***) \) and the fact \( v^{b+1} = 0 \). In particular, we obtain \( u^* \neq 0 \). Looking at the exact sequence \( (*) \), we can assume that \( u^* = \Delta^*(\sigma) \).

We can express \( i^*_k(y_1) = \lambda x_0 + \mu y_0 \); \( \lambda, \mu \in Q \). Since \( \pi^*_k(y_1) \neq 0 \), we see that
by the assumption $i^*: H^*(Y) \cong H^*(\partial Y)$. Then

$$\Delta^*(\sigma x^i y^j) = \mu^{-1} u^{n+c}(v-\lambda u)^t$$

because $\Delta^*(\sigma j^*(w)) = \Delta^*(\sigma)w$ for each $w \in H^*(X)$. Looking at the exact sequence (\ref{eq:exact}), we see that the graded algebra $H^*(X)$ is generated by two elements $u, v$ and rank $H^*(X) = (n+c+1)(b+1)$.

In the expression $i^*(y) = \lambda x_0 + \mu y_0$, if $\lambda = 0$ then we see that $u = 0$ in the case (1) and $(u-\mu^{-1}v)^{n+c+1} = 0$ in the case (2), and hence $X \cong P_{n+c}(H) \times P_b(H)$.

Since $i^* : H^*(X_2) \rightarrow H^*(X_3)$ is injective, we see that $j^*(v) = \lambda x_2 + \mu y_2$, and hence $(\lambda x_2 + \mu y_2)^{n+c+1} = 0$. Then we obtain $\lambda = 0$ in the case (1), because $H^*(X_2) = Q[x_2, y_2]/(x_2^{n+c+1}, y_2^{n+c+1})$.

Next we consider the case (2). We obtain a relation

$$(\gamma x_2 + y_2)^{n+c+1} \in \mathbb{I} = (y_2^{n+c+1}, \sum_{i=0}^k (-1)^i \binom{n+c+1}{i} x_2^{n+c+1-i} y_2^i),$$

where $\gamma = \lambda \mu^{-1}$. We see that $\gamma = 0$ for the case $b < c$ or $b = c \geq 2$. Suppose $b = c + 1$. Looking at the relation $(\gamma x_2 + y_2)^{n+c+1} \in \mathbb{I}$, we obtain $\gamma = 0$ or

$$(A_k) \quad \left( \frac{c+2}{k} \right) - (-\gamma) \binom{n+c+1}{k} + (n+c+1) (-\gamma) \binom{n+c+1}{k-1}$$

$$- (c+2) (-\gamma) \binom{n+c+1}{k-1} = 0$$

for each $k = 2, 3, \ldots, c+1$. Suppose $\gamma \neq 0$ and $c \geq 2$. Then we get a contradiction from $(A_2)$ and $(A_3)$. Hence we obtain $\gamma = 0$ for $c \geq 2$. Suppose $\gamma \neq 0$ and $c = 1$. We see that the quadratic equation $(A_2)$ has a rational solution $\gamma$ if and only if $3n(n+2)$ is a square number.

Summing up the above arguments, we obtain a partial converse of Theorem 2.1 (iii).

Remark. For a positive integer $n$, $3n(n+2)$ is a square number if and only if $n+1$ is one of the following:

$$\sum_{i \geq 0} \binom{k}{2i} 2^{k-2i}3^i; \quad k = 1, 2, 3, \ldots$$

7. Concluding remark

By parallel arguments, we obtain the following result which is a generalization of a theorem [7].

**Theorem 7.1.** Let $X$ be a closed orientable manifold on which $SU(n)$ acts smoothly and non-trivially. Suppose $X \cong P_\alpha(C) \times P_\beta(C); \quad a \geq b \geq 1, a+b \leq 2n-2$ and $n \geq 7$. Then there are three cases:
(0) \(a = n - 1\) and \(X \simeq P_{n-1}(C) \times Y_0\), where \(Y_0\) is a closed orientable manifold such that \(Y_0 \sim P_a(C)\), and \(SU(n)\) acts naturally on \(P_{n-1}(C)\) and trivially on \(Y_0\).

(i) \(a = b = n - 1\) and \(X \simeq P_{n-1}(C) \times P_{a-1}(C)\) with the diagonal \(SU(n)\) action,

(ii) \(a \geq n\) and \(X \simeq \partial(D^{2a} \times Y_1)/U(1)\), where \(Y_1\) is a compact orientable \(U(1)\) manifold such that \(\dim Y_1 = 2(a+b+1-n)\) and \(Y_1 \sim P_a(C)\), \(U(1)\) acts as right scalar multiplication on \(D^{2a}\), the unit disk of \(C^n\), and \(SU(n)\) acts naturally on \(D^{2a}\) and trivially on \(Y_1\). In addition, the \(U(1)\) action on the boundary \(\partial Y_1\) is free and the fixed point set of the \(U(1)\) action on \(Y_1\) is \(\sim P_b(C)\).

**Theorem 7.2.** In the case (ii) of Theorem 7.1, the cohomology ring \(H^*(\partial Y_1/U(1))\) is isomorphic to one of the following:

1. \(Q[x, y]/(x^{a+1-n}, y^{b+1})\),
2. \(Q[x, y]/(y^{b+1}, \sum_{i=0}^{b}(-1)^i(a+1)\binom{a+1}{i}x^{a+1-n-i}y^i); b \leq a+1-n\),

where \(\deg x = \deg y = 2\), and \(x\) is the Euler class of the principal \(U(1)\) bundle \(\partial Y_1 \to \partial Y_1/U(1)\).

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**References**


Department of Mathematics
Faculty of Science
Yamagata University
Yamagata 990,
Japan