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ACTIONS OF SYMPLECTIC GROUPS ON A PRODUCT OF QUATERNION PROJECTIVE SPACES

Dedicated to Professor Minoru Nakaoka on his 60th birthday

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0. Introduction

We shall study smooth actions of symplectic group $\boldsymbol{Sp}(n)$ on a closed orientable manifold X such that $X \sim \boldsymbol{P}_a(\boldsymbol{H}) \times \boldsymbol{P}_b(\boldsymbol{H})$, under the conditions: $a+b \leqq 2n-2$ and $n \geqq 7$. Our result is stated in §2 and proved in §5. Typical examples are given in §1. Similar result on smooth actions of special unitary group $\boldsymbol{SU}(n)$ on a closed orientable manifold X such that $X \sim \boldsymbol{P}_a(\boldsymbol{C}) \times \boldsymbol{P}_b(\boldsymbol{C})$ is stated in the final section.

Throughout this paper, let $H^*(\ )$ denote the singular cohomology theory with rational coefficients, and let $\boldsymbol{P}_n(\boldsymbol{H})$, $\boldsymbol{P}_n(\boldsymbol{C})$ and $\boldsymbol{P}_n(\boldsymbol{R})$ denote the quaternion, complex and real projective n-space, respectively. By $X \sim X'$, we mean that $H^*(X) \cong H^*(X')$ as graded algebras.

1. Typical examples

1.1. We regard S^{4k-1} as the unit sphere of the quaternion k-space H^k with the right scalar multiplication. Let Y be a compact $\boldsymbol{Sp}(1)$ manifold. By the diagonal action, $\boldsymbol{Sp}(1)$ acts freely on the product manifold $S^{4k-1} \times Y$. Here we consider the cohomology ring of the orbit manifold $(S^{4k-1} \times Y)/\boldsymbol{Sp}(1)$ for the case $Y \sim \boldsymbol{P}_b(\boldsymbol{H})$.

Consider the fibration: $Y \to (S^{4k-1} \times Y)/\boldsymbol{Sp}(1) \to \boldsymbol{P}_{k-1}(\boldsymbol{H})$. By the Leray-Hirsch theorem, $H^*((S^{4k-1} \times Y)/\boldsymbol{Sp}(1))$ is freely generated by $1, u, u^2, \cdots, u^b$ as an $H^*(\boldsymbol{P}_{k-1}(\boldsymbol{H}))$ module for an element $u \in H^4((S^{4k-1} \times Y)/\boldsymbol{Sp}(1))$. If u can be so chosen as $u^{b+1}=0$, then we see that $(S^{4k-1} \times Y)/\boldsymbol{Sp}(1) \sim \boldsymbol{P}_{k-1}(\boldsymbol{H}) \times \boldsymbol{P}_b(\boldsymbol{H})$.

Lemma 1.1. *Denote by F, the fixed point set of the restricted $\boldsymbol{U}(1)$ action on Y. If $F \sim \boldsymbol{P}_b(\boldsymbol{C})$, then $(S^{4k-1} \times Y)/\boldsymbol{Sp}(1) \sim \boldsymbol{P}_{k-1}(\boldsymbol{H}) \times \boldsymbol{P}_b(\boldsymbol{H})$.*

Proof. Consider the fibration: $Y \to (S^{4k-1} \times Y)/\boldsymbol{U}(1) \to \boldsymbol{P}_{2k-1}(\boldsymbol{C})$. We see that $H^*((S^{4k-1} \times Y)/\boldsymbol{U}(1))$ is freely generated by $1, v, v^2, \cdots, v^b$ as an $H^*(\boldsymbol{P}_{2k-1}(\boldsymbol{C}))$ module for an element $v \in H^4((S^{4k-1} \times Y)/\boldsymbol{U}(1))$. We shall show first that

v can be so chosen as $v^{b+1}=0$. We regard S^∞ as the inductive limit of S^{4N-1} on which $\boldsymbol{U}(1)$ acts naturally. Consider the following commutative diagram:

$$\begin{array}{ccc} H^*((S^\infty\times Y)/\boldsymbol{U}(1)) & \xrightarrow{j^*} & H^*((S^{4k-1}\times Y)/\boldsymbol{U}(1)) \\ \downarrow{\scriptstyle i_\infty^*} & & \downarrow{\scriptstyle i^*} \\ H^*(\boldsymbol{P}_\infty(\boldsymbol{C})\times F) & \xrightarrow{j_F^*} & H^*(\boldsymbol{P}_{2k-1}(\boldsymbol{C})\times F) \end{array}$$

where i, i_∞, j, j_F are natural inclusions. Since $H^{\text{odd}}(Y)=0$, we see that i_∞^* is injective [4] and j^* is surjective. Let v_∞ be an element of $H^4((S^\infty\times Y)/\boldsymbol{U}(1))$ such that $j^*(v_\infty)=v$. Let x be the canonical generator of $H^2(\boldsymbol{P}_\infty(\boldsymbol{C}))\cong H^2(\boldsymbol{P}_{2k-1}(\boldsymbol{C}))$. Then we can express

$$i_\infty^*(v_\infty) = x^2\times f_0+x\times f_1+1\times f_2$$

where $f_r\in H^{2r}(F)$ for $r=0, 1, 2$. Since $F\sim \boldsymbol{P}_b(\boldsymbol{C})$, we see that there are rational numbers a_0, a_1, a_2 and a non-zero element $y\in H^2(F)$, such that $f_r=a_r y^r$ for $r=0, 1, 2$. Then we obtain

$$i_\infty^*(v_\infty-a_0x^2)^{b+1} = (x\times f_1+1\times f_2)^{b+1} = 0\,.$$

Since i_∞^* is injective, we obtain $(v_\infty-a_0x^2)^{b+1}=0$. Put $v_1=j^*(v_\infty-a_0x^2)$. Then $v_1^{b+1}=0$, and hence

$$H^*((S^{4k-1}\times Y)/\boldsymbol{U}(1))\cong \boldsymbol{Q}[x, v_1]/(x^{2k}, v_1^{b+1});\ \deg x=2,\ \deg v_1=4\,.$$

Consider next the following commutative diagram:

$$\begin{array}{ccccc} \boldsymbol{Sp}(1)/\boldsymbol{U}(1) & \to & (S^{4k-1}\times Y)/\boldsymbol{U}(1) & \xrightarrow{p} & (S^{4k-1}\times Y)/\boldsymbol{Sp}(1) \\ \downarrow & & \downarrow & & \downarrow \\ \boldsymbol{Sp}(1)/\boldsymbol{U}(1) & \longrightarrow & \boldsymbol{P}_{2k-1}(\boldsymbol{C}) & \xrightarrow{q} & \boldsymbol{P}_{2k-1}(\boldsymbol{H})\,. \end{array}$$

Let $t\in H^4(\boldsymbol{P}_{k-1}(\boldsymbol{H}))$ be the canonical generator such that $q^*(t)=x^2$. There exist rational numbers λ, μ such that $p^*(u)=\lambda v_1+\mu x^2$. Put $u_1=u-\mu t$. Then $p^*(u_1)=\lambda v_1$, and hence $p^*(u_1)^{b+1}=0$. Since the homomorphism p^*: $H^*((S^{4k-1}\times Y)/\boldsymbol{Sp}(1))\to H^*((S^{4k-1}\times Y)/\boldsymbol{U}(1))$ is injective, we obtain $u_1^{b+1}=0$, and hence

$$H^*((S^{4k-1}\times Y)/\boldsymbol{Sp}(1))\cong \boldsymbol{Q}[t, u_1]/(t^k, u_1^{b+1});\ \deg t=\deg u_1=4\,.$$

Thus we obtain $(S^{4k-1}\times Y)/\boldsymbol{Sp}(1)\sim \boldsymbol{P}_{k-1}(\boldsymbol{H})\times \boldsymbol{P}_b(\boldsymbol{H})$. q.e.d.

1.2. We give here examples of a closed orientable $\boldsymbol{Sp}(1)$ manifold Y such that $Y\sim \boldsymbol{P}_b(\boldsymbol{H})$ and $F\sim \boldsymbol{P}_b(\boldsymbol{C})$, where F denotes the fixed point set of the restricted $\boldsymbol{U}(1)$ action on Y.

Consider the $\boldsymbol{Sp}(1)$ action on $\boldsymbol{P}_b(\boldsymbol{H})=S^{4b+3}/\boldsymbol{Sp}(1)$ by the left scalar multiplication. Then the fixed point set of the restricted $\boldsymbol{U}(1)$ action is naturally diffeomorphic to $\boldsymbol{P}_b(\boldsymbol{C})$, the fixed point set of the $\boldsymbol{Sp}(1)$ action is naturally diffeomorphic to $\boldsymbol{P}_b(\boldsymbol{R})$, and the isotropy representation at each fixed point of the $\boldsymbol{Sp}(1)$ action is equivalent to $b\eta\oplus\theta^b$, where η denotes the canonical 3-dimensional real representation of $\boldsymbol{Sp}(1)$, $b\eta$ denotes the b-fold direct sum of η, and θ^b is the trivial representation of degree b.

Let D^{3b} denote the unit disk of the representation space $b\eta$. Let W be a $(b+1)$-dimensional compact orientable smooth manifold which is rationally acyclic. Then the boundary $\partial(D^{3b}\times W)$ is a $4b$-dimensional compact orientable smooth $\boldsymbol{Sp}(1)$ manifold which is a rational homology sphere, and the isotropy representation at each fixed point of the $\boldsymbol{Sp}(1)$ action is equivalent to $b\eta\oplus\theta^b$. Hence we can construct an equivariant connected sum

$$Y(W)=\boldsymbol{P}_b(\boldsymbol{H})\,\#\,\partial(D^{3b}\times W)\,.$$

Denote by $F(W)$ the fixed point set of the restricted $\boldsymbol{U}(1)$ action on $Y(W)$. Then $F(W)$ is naturally diffeomorphic to $\boldsymbol{P}_b(\boldsymbol{C})\,\#\,\partial(D^b\times W)$. It is easy to see that

$$Y(W)\sim\boldsymbol{P}_b(\boldsymbol{H}),\ F(W)\sim\boldsymbol{P}_b(\boldsymbol{C})\,.$$

1.3. Let ζ be a quaternion k-plane bundle and $\zeta_{\boldsymbol{C}}$ its complexification under the restriction of the filed. Its i-th symplectic Pontrjagin class $e_i(\zeta)$ is by definition [2, §9.6]

$$e_i(\zeta)=(-1)^i c_{2i}(\zeta_{\boldsymbol{C}})\,,$$

where $c_{2i}(\zeta_{\boldsymbol{C}})$ is the $2i$-th Chern class. Denote by $\boldsymbol{P}(\zeta)$ the total space of the associated quaternion projective space bundle. Let $\hat{\zeta}$ be the canonical quaternion line bundle over $\boldsymbol{P}(\zeta)$ and put $t=e_1(\hat{\zeta})$. It is known that there is an isomorphism:

$$H^*(\boldsymbol{P}(\zeta))\cong H^*(B)\,[t]/(\textstyle\sum_{i=0}^{k}e_{k-i}(\zeta)t^i)\,, \tag{1.3}$$

where B is the base space of the bundle ζ (cf. [3, §3]).

Let ξ be the canonical quaternion line bundle over $\boldsymbol{P}_b(\boldsymbol{H})$ and ξ^* its dual line bundle. Let W be a $4b$-dimensional closed orientable smooth manifold and let f: $W\to\boldsymbol{P}_b(\boldsymbol{H})$ be a smooth mapping such that f^*: $H^*(\boldsymbol{P}_b(\boldsymbol{H}))\cong H^*(W)$. Let c be a non-negative integer such that $b\leqq c+1$. Then, there is a quaternion $(c+1)$-plane bundle ζ over W such that

$$(n+c+1)f^*\xi^*\cong\zeta\oplus\theta_{\boldsymbol{H}}^n\,,$$

where $\theta_{\boldsymbol{H}}^n$ is a trivial quaternion n-plane bundle. Put $X=\boldsymbol{P}((n+c+1)f^*\xi^*)$. Since X is diffeomorphic to $\partial(D(\zeta)\times D^{4n})/\boldsymbol{Sp}(1)$, we can act $\boldsymbol{Sp}(n)$ on X in order that the fixed point set is diffeomorphic to $\boldsymbol{P}(\zeta)$. We see that by (1.3)

$$H^*(X) \cong \boldsymbol{Q}[u, v]/(u^{n+c+1}, v^{b+1}),$$
$$H^*(\boldsymbol{P}(\zeta)) \cong \boldsymbol{Q}[t, v]/(v^{b+1}, \textstyle\sum_{i=0}^{c+1}(-1)^i\binom{n+c+1}{i}t^{c+1-i}v^i),$$

where $v=f^*e_1(\xi)$, $t=e_1(\zeta)$ and $u+v$ is the first symplectic Pontrjagin class of the canonical line bundle over $\boldsymbol{P}((n+c+1)f^*\xi^*)$.

2. Classification theorems

We shall prove the following results in this paper.

Theorem 2.1. *Let X be a closed orientable manifold on which $\boldsymbol{Sp}(n)$ acts smoothly and non-trivially. Suppose $X \sim \boldsymbol{P}_a(\boldsymbol{H}) \times \boldsymbol{P}_b(\boldsymbol{H})$; $a \geqq b \geqq 1$, $a+b \leqq 2n-2$ and $n \geqq 7$. Then there are four cases:*

(0) *$a=n-1$ and $X \cong \boldsymbol{P}_{n-1}(\boldsymbol{H}) \times Y_0$, where Y_0 is a closed orientable manifold such that $Y_0 \sim \boldsymbol{P}_b(\boldsymbol{H})$, and $\boldsymbol{Sp}(n)$ acts naturally on $\boldsymbol{P}_{n-1}(\boldsymbol{H})$ and trivially on Y_0,*

(i) *$a=n-1$ and $X \cong (S^{4n-1} \times Y_1)/\boldsymbol{Sp}(1)$, where Y_1 is a closed orientable $\boldsymbol{Sp}(1)$ manifold such that $Y_1 \sim \boldsymbol{P}_b(\boldsymbol{H})$, $\boldsymbol{Sp}(1)$ acts as right scalar multiplication on S^{4n-1}, the unit sphere of $\boldsymbol{H}^n$, and $\boldsymbol{Sp}(n)$ acts naturally on S^{4n-1} and trivially on Y_1. In addition, the fixed point set of the restricted $\boldsymbol{U}(1)$ action on Y_1 is $\sim \boldsymbol{P}_b(\boldsymbol{C})$,*

(ii) *$a=b=n-1$ and $X \cong \boldsymbol{P}_{n-1}(\boldsymbol{H}) \times \boldsymbol{P}_{n-1}(\boldsymbol{H})$ with the diagonal $\boldsymbol{Sp}(n)$ action,*

(iii) *$a \geqq n$ and $X \cong \partial(D^{4n} \times Y_2)/\boldsymbol{Sp}(1)$, where Y_2 is a compact orientable $\boldsymbol{Sp}(1)$ manifold such that* dim *$Y_2=4(a+b+1-n)$ and $Y_2 \sim \boldsymbol{P}_b(\boldsymbol{H})$, $\boldsymbol{Sp}(1)$ acts as right scalar multiplication on D^{4n}, the unit disk of $\boldsymbol{H}^n$, and $\boldsymbol{Sp}(n)$ acts naturally on D^{4n} and trivially on Y_2. In addition, the $\boldsymbol{Sp}(1)$ action on the boundary ∂Y_2 is free and the fixed point set of the restricted $\boldsymbol{U}(1)$ action on Y_2 is $\sim \boldsymbol{P}_b(\boldsymbol{C})$ or $\sim \boldsymbol{P}_b(\boldsymbol{H})$.*

REMARK. By $X \cong X'$ we mean that X is equivariantly diffeomorphic to X' as $\boldsymbol{Sp}(n)$ manifolds. In the case (iii), the fixed point set of the $\boldsymbol{Sp}(n)$ action on X is naturally diffeomorphic to the orbit manifold $\partial Y_2/\boldsymbol{Sp}(1)$.

Theorem 2.2. *In the case* (iii) *of Theorem* 2.1, *the cohomology ring $H^*(\partial Y_2/\boldsymbol{Sp}(1))$ is isomorphic to one of the following:*

(1) $\boldsymbol{Q}[x, y]/(x^{a+1-n}, y^{b+1})$,

(2) $\boldsymbol{Q}[x, y]/(y^{b+1}, \sum_{i=0}^{b}(-1)^i\binom{a+1}{i}x^{a+1-n-i}y^i)$; $b \leqq a+1-n$,

where $\deg x = \deg y = 4$, *and x is the Euler class of the principal $\boldsymbol{Sp}(1)$ bundle $\partial Y_2 \to \partial Y_2/\boldsymbol{Sp}(1)$.*

REMARK. The $\boldsymbol{Sp}(n)$ action given in §1.3 is an example of the case (iii)–(2). Lemma 1.1 assures that a converse of Theorem 2.1 (i) is true.

3. Cohomology of certain homogeneous spaces

Here we consider the cohomology of $V_{n,2}/G=\boldsymbol{Sp}(n)/\boldsymbol{Sp}(n-2)\times G$ for certain closed subgroups G of $\boldsymbol{Sp}(2)$. Let ξ be the canonical quaternion line bundle over $\boldsymbol{P}_{n-1}(\boldsymbol{H})$ and ζ its orthogonal complement, that is, ζ is a quaternion $(n-1)$-plane bundle over $\boldsymbol{P}_{n-1}(\boldsymbol{H})$ such that its total space is

$$E(\zeta)=\{(u,[v])\in \boldsymbol{H}^n\times \boldsymbol{P}_{n-1}(\boldsymbol{H}): u\perp v\}\ .$$

It is easy to see that the total space $\boldsymbol{P}(\zeta)$ of the associated quaternion projective space bundle is naturally diffeomorphic to $V_{n,2}/\boldsymbol{Sp}(1)\times\boldsymbol{Sp}(1)$. Since $\xi\oplus\zeta$ is a trivial bundle, we obtain $e_k(\zeta)=(-1)^k e_1(\xi)^k$. By definition, $\boldsymbol{P}(\zeta)$ is naturally identified with a subspace of $\boldsymbol{P}_{n-1}(\boldsymbol{H})\times\boldsymbol{P}_{n-1}(\boldsymbol{H})$. Let $i\colon \boldsymbol{P}(\zeta)\to\boldsymbol{P}_{n-1}(\boldsymbol{H})\times\boldsymbol{P}_{n-1}(\boldsymbol{H})$ be the inclusion. Put $\hat{\xi}=i^*(\xi^*\times 1)$. Then by (1.3) there is an isomorphism:

$$\text{(3.1)}\qquad H^*(V_{n,2}/\boldsymbol{Sp}(1)\times\boldsymbol{Sp}(1))\cong \boldsymbol{Q}[x,y]/(x^n, \textstyle\sum_i x^i y^{n-1-i})\,,$$

$\deg x=\deg y=4$, by the identification $x=i^*(1\times e_1(\xi))$ and $y=i^*(e_1(\xi)\times 1)$.

Let $p\colon V_{n,2}/\boldsymbol{Sp}(1)\times\boldsymbol{Sp}(1)\to V_{n,2}/\boldsymbol{Sp}(2)$ be the natural projection and ξ_2 the standard quaternion 2-plane bundle over $V_{n,2}/\boldsymbol{Sp}(2)$.

Lemma 3.2. *The graded algebra $H^*(V_{n,2}/\boldsymbol{Sp}(2))$ is generated by $e_1(\xi_2)$, $e_2(\xi_2)$. The algebra is isomorphic to the subalgebra of $\boldsymbol{Q}[x,y]/(x^n,\sum_i x^i y^{n-1-i})$, consisting of symmetric polynomials.*

Proof. Since the fibration p is a 4-sphere bundle and $H^{\mathrm{odd}}(V_{n,2}/\boldsymbol{Sp}(2))=0$ (cf. [1, §26]), the homomorphism $p^*\colon H^*(V_{n,2}/\boldsymbol{Sp}(2))\to H^*(V_{n,2}/\boldsymbol{Sp}(1)\times\boldsymbol{Sp}(1))$ is injective. Since $p^*(\xi_2)=i^*(\xi\times\xi)$, we obtain

$$p^*e_1(\xi_2)=i^*e_1(\xi\times\xi)=x+y\,,$$
$$p^*e_2(\xi_2)=i^*e_2(\xi\times\xi)=xy\,.$$

Then the desired result is obtained by the Leray-Hirsch theorem. q.e.d.

Corollary 3.3. $e_1(\xi_2)^{2n-4}\neq 0$ *and* $e_1(\xi_2)^{2n-3}=0$.

Proof. Put $I=(x^n,\sum_i x^i y^{n-1-i})$. It is easy to see that $y^n\in I$. In the quotient ring $\boldsymbol{Q}[x,y]/I$, we obtain

$$\begin{aligned}(x+y)^{2n-4}&=\binom{2n-4}{n-1}x^{n-1}y^{n-3}+\binom{2n-4}{n-2}x^{n-2}y^{n-2}+\binom{2n-4}{n-1}x^{n-3}y^{n-1}\\&=\left\{\binom{2n-4}{n-2}-\binom{2n-4}{n-1}\right\}x^{n-2}y^{n-2}\,,\end{aligned}$$

and hence $e_1(\xi_2)^{2n-4}\neq 0$. We obtain $e_1(\xi_2)^{2n-3}=0$ similarly. q.e.d.

4. Preliminary results

First we state the following two lemmas which are proved by a standard method (cf. [5, §5]).

Lemma 4.1. *Suppose $n\geqq 7$. Let G be a closed connected proper subgroup of $\boldsymbol{Sp}(n)$ such that $\dim \boldsymbol{Sp}(n)/G<8n$. Then G coincides with $\boldsymbol{Sp}(n-i)\times K$ ($i=1, 2, 3$) up to an inner automorphism of $\boldsymbol{Sp}(n)$, where K is a closed connected subgroup of $\boldsymbol{Sp}(i)$.*

Lemma 4.2. *Suppose $r\geqq 5$ and $k<8r$. Then an orthogonal non-trivial representation of $\boldsymbol{Sp}(r)$ of degree k is equivalent to $(\nu_r)_{\boldsymbol{R}}\oplus\theta^{k-4r}$. Here $(\nu_r)_{\boldsymbol{R}}\colon \boldsymbol{Sp}(r)\to \boldsymbol{O}(4r)$ is the canonical inclusion, and θ^t is the trivial representation of degree t.*

In the following, let X be a closed connected orientable manifold with a non-trivial smooth $\boldsymbol{Sp}(n)$ action, and suppose $n\geqq 7$ and $\dim X<8n$. Put

$$F_{(i)} = \{x\in X\colon \boldsymbol{Sp}(n-i)\subset \boldsymbol{Sp}(n)_x\subset \boldsymbol{Sp}(n-i)\times \boldsymbol{Sp}(i)\}$$
$$X_{(i)} = \boldsymbol{Sp}(n)F_{(i)} = \{gx\colon g\in \boldsymbol{Sp}(n),\ x\in F_{(i)}\}\ .$$

Here $\boldsymbol{Sp}(n)_x$ denotes the isotropy group at x. Then, by Lemma 4.1, we obtain $X=X_{(0)}\cup X_{(1)}\cup X_{(2)}\cup X_{(3)}$.

Proposition 4.3. *If $X_{(k)}$ is non-empty, then $X_{(i)}$ is empty for each $i\geqq k+2$.*

Proof. Let us denote by $F(\boldsymbol{Sp}(n-j), X_{(i)})$ the fixed point set of the restricted $\boldsymbol{Sp}(n-j)$ action on $X_{(i)}$. It is easy to see that the set is empty for each $j<i\leqq n-i$. Suppose that $X_{(k)}$ is non-empty and fix $x\in F_{(k)}$. Let σ be the slice representation at x. Then the restriction $\sigma|\boldsymbol{Sp}(n-k)$ is trivial or equivalent to $(\nu_{n-k})_{\boldsymbol{R}}\oplus\theta^t$ by Lemma 4.2. Anyhow, a principal isotropy group of the given action contains $\boldsymbol{Sp}(n-k-1)$, and hence $F(\boldsymbol{Sp}(n-k-1), X_{(i)})$ is non-empty if so is $X_{(i)}$. q.e.d.

Proposition 4.4. *Suppose $X=X_{(k)}\cup X_{(k+1)}$. If $X_{(k)}$ and $X_{(k+1)}$ are non-empty, then the codimension of each connected component of $F_{(k)}$ in X is equal to $4(k+1)(n-k)$.*

Proof. Fix $x\in F_{(k)}$. Let σ and ρ denote the slice representation at x and the isotropy representation of the orbit $\boldsymbol{Sp}(n)x$, respectively. The restriction $\sigma|\boldsymbol{Sp}(n-k)$ is equivalent to $(\nu_{n-k})_{\boldsymbol{R}}\oplus\theta^s$ by Lemma 4.2 and the assumption that $X_{(k+1)}$ is non-empty. On the other hand, $\rho|\boldsymbol{Sp}(n-k)$ is equivalent to $k(\nu_{n-k})_{\boldsymbol{R}}\oplus\theta^t$ by considering adjoint representations. Hence $(\sigma\oplus\rho)|\boldsymbol{Sp}(n-k)$ is equivalent to $(k+1)(\nu_{n-k})_{\boldsymbol{R}}\oplus\theta^{s+t}$. This shows that the codimension of $F_{(k)}$ at x is equal to $4(k+1)(n-k)$. q.e.d.

Corollary 4.5. *Suppose $X=X_{(2)}\cup X_{(3)}$. Then either $X_{(2)}$ or $X_{(3)}$ is empty.*

REMARK. $\dim \boldsymbol{Sp}(n)/\boldsymbol{Sp}(n-k)\times \boldsymbol{Sp}(k)=4k(n-k)$ and $\chi(\boldsymbol{Sp}(n)/\boldsymbol{Sp}(n-k)$

$\times \boldsymbol{Sp}(k)) = \binom{n}{k}$, where $\chi(\ \ )$ denotes the Euler characteristic, and $\binom{n}{k}$ denotes the binomial coefficient.

5. Proof of the classification theorems

Throughout this section, suppose that X is a closed orientable manifold with a non-trivial smooth $\boldsymbol{Sp}(n)$ action such that

$$(*) \qquad H^*(X) = \boldsymbol{Q}[u, v]/(u^{a+1}, v^{b+1});\ \deg u = \deg v = 4\,.$$

Moreover, suppose that $n \geqq 7$, $1 \leqq b \leqq a$ and $a+b \leqq 2n-2$. By arguments and notations in the preceding section, we see that $X = X_{(k)} \cup X_{(k+1)}$ for $k=0, 1, 2$.

5.1. We shall show first that $X \neq X_{(2)} \cup X_{(3)}$. Suppose $X = X_{(2)} \cup X_{(3)}$. Then $X = X_{(2)}$ or $X = X_{(3)}$ by Corollary 4.5. Looking at the Euler characteristic of X, we see that $X \neq X_{(3)}$.

Suppose $X = X_{(2)}$. Then $X = (V_{n,2} \times F_{(2)})/\boldsymbol{Sp}(2)$. Here we consider the following commutative diagram of natural projections:

$$\begin{array}{ccc} (V_{n,2} \times F_{(2)})/T & \xrightarrow{p_1} & V_{n,2}/T \\ \downarrow & & \downarrow q \\ X = (V_{n,2} \times F_{(2)})/\boldsymbol{Sp}(2) & \xrightarrow{p} & V_{n,2}/\boldsymbol{Sp}(2)\,, \end{array}$$

where T is a maximal torus of $\boldsymbol{Sp}(2)$. Since $\chi(F_{(2)}) \neq 0$, we see that the restricted T action on $F_{(2)}$ has a fixed point, and hence the projection p_1 has a cross-section. Therefore $p_1^*: H^*(V_{n,2}/T) \to H^*((V_{n,2} \times F_{(2)})/T)$ is injective. On the other hand, $q^*: H^*(V_{n,2}/\boldsymbol{Sp}(2)) \to H^*(V_{n,2}/T)$ is injective, because $H^{\mathrm{odd}}(V_{n,2}/\boldsymbol{Sp}(2)) = H^{\mathrm{odd}}(\boldsymbol{Sp}(2)/T) = 0$ (cf. [1, §26]). Consequently, we see that $p^*: H^*(V_{n,2}/\boldsymbol{Sp}(2)) \to H^*(X)$ is injective. In particular, we obtain $a+b \geqq 2n-4$. If $a+b = 2n-4$, then $X = V_{n,2}/\boldsymbol{Sp}(2)$. Because rank $H^4(X) = 2$ and rank $H^4(V_{n,2}/\boldsymbol{Sp}(2)) = 1$, we get a contradiction.

Syppose $a+b \geqq 2n-3$, and put $p^*e_1(\xi_2) = \alpha u + \beta v$; $\alpha, \beta \in \boldsymbol{Q}$. Since $e_1(\xi_2)^{2n-3} = 0$ by Corollary 3.3, we obtain

$$0 = p^*e_1(\xi_2)^{a+b} = \binom{a+b}{a}(\alpha u)^a(\beta v)^b\,,$$

and hence $\alpha\beta = 0$. On the other hand, $e_1(\xi_2)^{2n-4} \neq 0$ by Corollary 3.3, and hence $p^*e_1(\xi_2)^{2n-4} \neq 0$. Thus we obtain $a = 2n-4$. Looking at the Euler characteristic of $F_{(2)}$, we get a contradiction.

5.2. We consider now the case $X = X_{(1)} \cup X_{(2)}$. Suppose that both $X_{(1)}$ and $X_{(2)}$ are non-empty. We see that codim $F_{(1)} = 8n-8$ by Proposition 4.4. Since $\dim X \leqq 8n-8$, we obtain $\dim F_{(1)} = 0$ and $a+b = 2n-2$.

Fix $x\in F_{(1)}$. Since $X_{(2)}$ is non-empty, we see that the slice representation σ at x is equivalent to $\nu_{n-1}\otimes_{\boldsymbol{H}}\nu_1^*$ or $(\nu_{n-1})_{\boldsymbol{R}}\pi$ by Lemma 4.2, where π is a natural projection of $\boldsymbol{Sp}(n-1)\times\boldsymbol{Sp}(1)$ onto $\boldsymbol{Sp}(n-1)$. Then the principal isotropy group is of the form $\boldsymbol{Sp}(n-2)\times K$, where $K=\Delta\boldsymbol{Sp}(1)$ (resp. $1\times\boldsymbol{Sp}(1)$) for $\sigma=\nu_{n-1}\otimes_{\boldsymbol{H}}\nu_1^*$ (resp. $\sigma=(\nu_{n-1})_{\boldsymbol{R}}\pi$). Here $\Delta\boldsymbol{Sp}(1)$ (resp. $1\times\boldsymbol{Sp}(1)$) is a closed subgroup of $\boldsymbol{Sp}(2)$ consisting of the matrices of the form $\begin{pmatrix} q & 0 \\ 0 & q\end{pmatrix}$ (resp. $\begin{pmatrix} 1 & 0 \\ 0 & q\end{pmatrix}$). Anyhow, we see that the $\boldsymbol{Sp}(n)$ action on X has a codimension one orbit, and hence X is a union of closed invariant tubular neighborhoods of just two non-principal orbits (cf. [6]). We already see that one of the non-principal orbits is $\boldsymbol{P}_{n-1}(\boldsymbol{H})$. Looking at the Euler characteristic of X, we see that $a=b=n-1$ and another non-principal orbit is $V_{n,2}/\boldsymbol{Sp}(1)\times\boldsymbol{Sp}(1)$.

Suppose $K=1\times\boldsymbol{Sp}(1)$. Then the normalizer of the principal isotropy group is connected, and hence such an $\boldsymbol{Sp}(n)$ manifold is unique up to equivariant diffeomorphism (cf. [6, §5.3]). On the other hand, the product manifold $\boldsymbol{P}_{n-1}(\boldsymbol{H})\times\boldsymbol{P}_{n-1}(\boldsymbol{H})$ with the diagonal $\boldsymbol{Sp}(n)$ action is such one. Therefore X is equivariantly diffeomorphic to $\boldsymbol{P}_{n-1}(\boldsymbol{H})\times\boldsymbol{P}_{n-1}(\boldsymbol{H})$ with the diagonal $\boldsymbol{Sp}(n)$ action.

Suppose next $K=\Delta\boldsymbol{Sp}(1)$. Then the normalizer of the principal isotropy group has just two connected components, and its generator corresponds to the antipodal involution of the slice representation at a point of $V_{n,2}/\boldsymbol{Sp}(1)\times\boldsymbol{Sp}(1)$. Hence such an $\boldsymbol{Sp}(n)$ manifold is unique up to equivariant diffeomorphism (cf. [6, §5.3]). Here we construct such one. Let ξ be the canonical quaternion line bundle over $\boldsymbol{P}_{n-1}(\boldsymbol{H})$ and ζ its orthogonal complement (see §3). Then $\boldsymbol{Sp}(n)$ acts naturally on the total space $E(\zeta)$ as the bundle mappings. Denote by $\theta_{\boldsymbol{H}}^1$ a trivial quaternion line bundle. We see that the $\boldsymbol{Sp}(n)$ action on the total space $\boldsymbol{P}(\zeta\oplus\theta_{\boldsymbol{H}}^1)$ of the associated quaternion projective space bundle is the desired one. On the other hand, we see that by (1.3)

$$H^*(\boldsymbol{P}(\zeta\oplus\theta_{\boldsymbol{H}}^1))\cong\boldsymbol{Q}[x, y]/(x^n, \textstyle\sum_i x^i y^{n-i});\ \deg x=\deg y=4\,.$$

Hence the cohomology ring of $\boldsymbol{P}(\zeta\oplus\theta_{\boldsymbol{H}}^1)$ is not isomorphic to that of $\boldsymbol{P}_{n-1}(\boldsymbol{H})\times\boldsymbol{P}_{n-1}(\boldsymbol{H})$.

5.3. We consider next the case $X=X_{(0)}\cup X_{(1)}$ for $c<n$. We shall show first that $X_{(0)}$ is empty.

Suppose that $X_{(0)}$ is non-empty. Let U be an invariant closed tubular neighborhood of $X_{(0)}$ in X, and put $E=X-\operatorname{int}U$. Put $W=E\cap F_{(1)}$. Then W is a compact connected orientable manifold with non-empty boundary ∂W, and $\boldsymbol{Sp}(1)$ acts naturally on W. Since there is a natural diffeomorphism $E=(S^{4n-1}\times W)/\boldsymbol{Sp}(1)$, we obtain

$$\dim W=4(a+b+1-n)=4k,\ \ k\leqq b\leqq a<n\,.$$

Let $i\colon E\to X$ be the inclusion. Then $i^*\colon H^i(X)\to H^i(E)$ is an isomorphism for each $t \leqq 4n-2$, because the codimension of each connected component of $X_{(0)}$ is $4n$ by Lemma 4.2. By the Gysin sequence of the principal $\boldsymbol{Sp}(1)$ bundle $S^{4n-1}\times W \to E$ and the cohomology ring of X, we obtain rank $H^{4k}(W)$ $-$ rank $H^{4k-1}(W)=1$. On the other hand, we see that $H^{4k}(W)\cong H_{4k}(W)=0$ and rank $H^{4k-1}(W)\geqq 0$; this is a contradiction. Thus we see that $X_{(0)}$ is empty.

Consequently, we obtain $X=X_{(1)}=(S^{4n-1}\times F_{(1)})/\boldsymbol{Sp}(1)$. Put $Y=F_{(1)}$. We see that

$$\dim Y = 4(a+b+1-n) = 4k,\ k\leqq b\leqq a<n\leqq a+b\,.$$

We shall show next that $a=n-1$ and $Y\sim \boldsymbol{P}_b(\boldsymbol{H})$.

By the Gysin sequence of the principal $\boldsymbol{Sp}(1)$ bundle p: $S^{4n-1}\times Y\to X$, we obtain $H^{4i+1}(S^{4n-1}\times Y)=H^{4i+2}(S^{4n-1}\times Y)=0$ and an exact sequence:

$$0\to H^{4i-1}(S^{4n-1}\times Y)\to H^{4i-4}(X)\overset{\mu}{\to} H^{4i}(X)\overset{p^*}{\to} H^{4i}(S^{4n-1}\times Y)\to 0$$

for any i, where μ is the multiplication by $e_1(p)$, the first symplectic Pontrjagin class of the quaternion line bundle associated with the $\boldsymbol{Sp}(1)$ bundle p. We can represent $p^*u=1\times u_1$, $p^*v=1\times v_1$ for u_1, $v_1\in H^4(Y)$. Then we see that $H^{\mathrm{odd}}(Y)=0$ and $H^*(Y)$ is generated by at most two elements u_1, v_1. We can represent $e_1(p)=\alpha u+\beta v$; α, $\beta\in \boldsymbol{Q}$. By definition, the $\boldsymbol{Sp}(1)$ bundle p is a pull-back of a bundle over $\boldsymbol{P}_{n-1}(\boldsymbol{H})$, and hence $e_1(p)^n=0$. Since $n\leqq a+b$, we see that $\alpha\beta=0$. Suppose $e_1(p)=0$. Then p^* is injective, and hence $1\times u_1^a v_1^b \neq 0$. Thus we get a contradiction. Therefore we see that $e_1(p)=\alpha u$ $(\alpha\neq 0)$ or $e_1(p)=\beta v$ $(\beta\neq 0)$, and hence $u_1=0$ or $v_1=0$, respectively. Looking at the Euler characteristic of X we see that $a=n-1$ and $Y\sim \boldsymbol{P}_b(\boldsymbol{H})$.

When $b<n-1$, we see that $e_1(p)=\alpha u$ $(\alpha\neq 0)$ and $H^*(Y)\cong \boldsymbol{Q}[v_1]/(v_1^{b+1})$. When $b=n-1$, interchanging u and v if necessary we can assume that $e_1(p)=\alpha u$ $(\alpha\neq 0)$ and $H^*(Y)\cong \boldsymbol{Q}[v_1]/(v_1^n)$. It remains to consider the $\boldsymbol{Sp}(1)$ action on $Y=F_{(1)}$. We shall show that either $F\sim \boldsymbol{P}_b(\boldsymbol{C})$ or the $\boldsymbol{Sp}(1)$ action on Y is trivial, where F denotes the fixed point set of the restricted $\boldsymbol{U}(1)$ action on Y.

Put $w=\pi^*(v)$, where π is a natural projection of $(S^{4n-1}\times Y)/\boldsymbol{U}(1)$ onto $X=(S^{4n-1}\times Y)/\boldsymbol{Sp}(1)$. Consider the fibration: $Y\to (S^{4n-1}\times Y)/\boldsymbol{U}(1)\to \boldsymbol{P}_{2n-1}(\boldsymbol{C})$. We see that $w^{b+1}=0$ and $H^*((S^{4n-1}\times Y)/\boldsymbol{U}(1))$ is freely generated by $1, w, w^2, \cdots, w^b$ as an $H^*(\boldsymbol{P}_{2n-1}(\boldsymbol{C}))$ module. Consider next the following commutative diagram:

$$\begin{array}{ccc} H^r((S^\infty\times Y)/\boldsymbol{U}(1)) & \overset{j^*}{\longrightarrow} & H^r((S^{4n-1}\times Y)/\boldsymbol{U}(1)) \\ \big\downarrow i_\infty^* & & \big\downarrow i^* \\ H^r(\boldsymbol{P}_\infty(\boldsymbol{C})\times F) & \overset{j_F^*}{\longrightarrow} & H^r(\boldsymbol{P}_{2n-1}(\boldsymbol{C})\times F) \end{array}$$

where i, i_∞, j, j_F are natural inclusions.Since $H^{\mathrm{odd}}(Y)=0$, we see that [4] i_∞^* is injective for each r and surjective for each $r>4b$ and j^* is surjective. Let

w_∞ be an element of $H^4((S^\infty\times Y)/\boldsymbol{U}(1))$ such that $j^*(w_\infty)=w$. Let x be the canonical generator of $H^2(\boldsymbol{P}_\infty(\boldsymbol{C}))\cong H^2(\boldsymbol{P}_{2n-1}(\boldsymbol{C}))$. Then we can express

$$i_\infty^*(w_\infty) = x^2\times f_0+x\times f_1+1\times f_2$$

where $f_t\in H^{2t}(F)$ for $t=0, 1, 2$. It is known that [4] $F_0\sim\boldsymbol{P}_d(\boldsymbol{C})$ or $F_0\sim\boldsymbol{P}_d(\boldsymbol{H})$ $(0\leqq d\leqq b)$ for each connected component F_0 of F. We shall show that F is connected.

Consider first the case $b<n-1$. We see that $i_\infty^*(w_\infty)=x\times f_1+1\times f_2$, that is, $f_0=0$ by the relation $(x^2\times f_0+x\times f_1+1\times f_2)^{b+1}=0$ in $H^{4b+4}(\boldsymbol{P}_{2n-1}(\boldsymbol{C})\times F)$. Consequently, we can show that if F is not connected then $i_\infty^*(w_\infty^b)=0$ and hence $w^b=0$; this is a contradiction.

Consider next the case $b=n-1$. Since $j^*(w_\infty^n)=w^n=0$, we see that $w_\infty^n=\gamma x^{2n}$ for some $\gamma\in\boldsymbol{Q}$, and hence $i_\infty^*(w_\infty^n)=x^{2n}\times\gamma$. Suppose $\gamma=0$. Then $f_0=0$, and hence we can show that F is connected by the same argument as above. Suppose next $\gamma\neq0$. We shall show that $i_\infty^*(w_\infty)=x^2\times f_0$, that is $f_1=0$ and $f_2=0$. For any connected component F_0 of F, we have an equation

$$(x^2\times f_0\,|\,F_0+x\times f_1\,|\,F_0+1\times f_2\,|\,F_0)^n = x^{2n}\times\gamma$$

in $H^{4n}(\boldsymbol{P}_\infty(\boldsymbol{C})\times F_0)$. Then we see that $(f_0\,|\,F_0)^n=\gamma\neq0$ and $f_t\,|\,F_0=0$ for $t=1, 2$. Thus we obtain $i_\infty^*(w_\infty)=x^2\times f_0$ and $f_0^n=\gamma$. Let F_1 (resp. F_2) be the union of connected components F_σ of F on which $f_0\,|\,F_\sigma$ is positive (resp. negative). Since $f_0^n=\gamma$, we can regard $f_0\,|\,F_1$ and $f_0\,|\,F_2$ as constant rational numbers. Then each element of $H^r(\boldsymbol{P}_\infty(\boldsymbol{C})\times F_s)$ for $r\geqq 4n$ is expressed as a polynomial of $x\times 1$ with rational coefficients for $s=1, 2$ because $H^*((S^\infty\times Y)/\boldsymbol{U}(1))$ is generated by an element w_∞ as a graded $H^*(\boldsymbol{P}_\infty(\boldsymbol{C}))$ algebra and i_∞^* is surjective for $r\geqq 4n$. Then we see that F_s $(s=1, 2)$ consists of just one point, and hence F consists of at most two points. This is a contradiction to the fact: $\chi(F)=\chi(Y)=n\geqq 7$.

Anyhow we see that F is connected, and hence $F\sim\boldsymbol{P}_b(\boldsymbol{C})$ or $F\sim\boldsymbol{P}_b(\boldsymbol{H})$. The $\boldsymbol{Sp}(1)$ action on Y is trivial for the latter case.

5.4. Finally, we consider the case $X=X_{(0)}\cup X_{(1)}$ for $a\geqq n$. We shall show first that $X_{(0)}$ is non-empty.

Suppose that $X_{(0)}$ is empty. Then $X=X_{(1)}=(S^{4n-1}\times F_{(1)})/\boldsymbol{Sp}(1)$. By the Gysin sequence of the principal $\boldsymbol{Sp}(1)$ bundle $S^{4n-1}\times F_{(1)}\rightarrow X$, we see that $F_{(1)}\sim\boldsymbol{P}_b(\boldsymbol{H})$. Looking at the Euler characteristic of the fibration: $F_{(1)}\rightarrow X\rightarrow\boldsymbol{P}_{n-1}(\boldsymbol{H})$ we obtain $a=n-1$; this is a contradiction.

Consequently, we see that (cf. [8]) there is an equivariant decomposition $X=\partial(D^{4n}\times Y)/\boldsymbol{Sp}(1)$, where Y is a compact connected orientable manifold with a smooth $\boldsymbol{Sp}(1)$ action, and Y has a non-empty boundary ∂Y on which the $\boldsymbol{Sp}(1)$ action is free. We see that

$$\dim Y = 4(a+b+1-n)$$

and the fixed point set of the $\boldsymbol{Sp}(n)$ action on X is naturally diffeomorphic to the orbit manifold $\partial Y/\boldsymbol{Sp}(1)$. Moreover, we see that there is a natural decomposition $X=X_1\cup X_2$, where

$$X_1=(S^{4n-1}\times Y)/\boldsymbol{Sp}(1) \quad\text{and}\quad X_2=(D^{4n}\times\partial Y)/\boldsymbol{Sp}(1)\,.$$

Put $X_0=X_1\cap X_2=(S^{4n-1}\times\partial Y)/\boldsymbol{Sp}(1)$.

Let π: $\partial(D^{4n}\times Y)\to X$ be the projection of the principal $\boldsymbol{Sp}(1)$ bundle. Denote by π_s the projection of the restricted principal $\boldsymbol{Sp}(1)$ bundle over X_s. Let j_s: $X_s\to X$ and i_s: $X_0\to X_s$ be inclusions. Put $u_s=j_s^*(u)$ and $v_s=j_s^*(v)$. We can express

$$e(\pi)=\alpha u+\beta v;\ \alpha,\ \beta\in\boldsymbol{Q}\,,$$

where $e(\pi)$ is the Euler class of the principal $\boldsymbol{Sp}(1)$ bundle π. Then we obtain

$$e(\pi_s)=j_s^*e(\pi)=\alpha u_s+\beta v_s\,.$$

Since $H^r(X,X_1)\cong H^r(X_2,X_0)\cong H^{r-4n}(\partial Y/\boldsymbol{Sp}(1))$ for each r, we obtain an isomorphism j_1^*: $H^r(X)\cong H^r(X_1)$ for each $r\leqq 4n-2$. Because Y is a compact connected manifold with non-empty boundary and $\dim Y\leqq 4n-4$, we see that $\pi_1^*(u_1^{n-1})=0$ and hence $u_1^{n-1}=x'e(\pi_1)$ for some $x'\in H^{4n-8}(X_1)$. Then $u^{n-1}=xe(\pi)$ for some $x\in H^{4n-8}(X)$ by the isomorphism j_1^*. In particular we see that $\alpha\neq 0$ in the expression: $e(\pi)=\alpha u+\beta v$. Looking at the isomorphism j_1^* and the Gysin sequence of the principal $\boldsymbol{Sp}(1)$ bundle π_1, we see that $\pi_1^*(v_1^b)\neq 0$ and the algebra $H^{ev}(S^{4n-1}\times Y)$ is generated by $\pi_1^*v_1$. Hence we obtain $Y\sim\boldsymbol{P}_b(\boldsymbol{H})$. In addition, we see that $X_1\sim\boldsymbol{P}_{n-1}(\boldsymbol{H})\times\boldsymbol{P}_b(\boldsymbol{H})$ by the fibration: $Y\to X_1\to\boldsymbol{P}_{n-1}(\boldsymbol{H})$.

Since $b\leqq n-2$, by the same argument as in the second half of §5.3, we see that $F\sim\boldsymbol{P}_b(\boldsymbol{C})$ or $F\sim\boldsymbol{P}_b(\boldsymbol{H})$, where F denotes the fixed point set of the restricted $\boldsymbol{U}(1)$ action on Y.

Here we complete the proof of Theorem 2.1.

REMARK. The case $\alpha\beta\neq 0$ in the expression $e(\pi)=\alpha u+\beta v$ occurs only when $b\leqq a+1-n$, because

$$(e(\pi_1)-\beta v_1)^{a+1}=(\alpha u_1)^{a+1}=0$$

in $H^*(X_1)=\boldsymbol{Q}[e(\pi_1),v_1]/(e(\pi_1)^n,v_1^{b+1})$.

5.5. In the following, we consider the cohomology of $\partial Y/\boldsymbol{Sp}(1)$. Regarding αu and βv as new u and v if necessary, we can assume that $e(\pi)=u$ if $\beta=0$ and $e(\pi)=u+v$ if $\beta\neq 0$.

Since the algebra $H^*(X_1)$ is generated by $e(\pi_1)$ and v_1, we obtain an short exact sequence:

$$0 \to H^*(X, X_1) \xrightarrow{k_1^*} H^*(X) \xrightarrow{j_1^*} H^*(X_1) \to 0 .$$

Moreover, we see that the kernel of j_1^* is an ideal generated by $e(\pi)^n$, that is, $\ker j_1^* = H^*(X)e(\pi)^n$. Let $\tau \in H^{4n}(X, X_1)$ be an element such that $k_1^*(\tau) = e(\pi)^n$. Then $H^*(X, X_1)$ is generated by τ as an $H^*(X)$ module, that is, $H^*(X, X_1) = H^*(X)\tau$.

Let $j^*: H^*(X, X_1) \cong H^*(X_2, X_0)$ be an excision isomorphism. Denote by $t \in H^{4n}(X_2, X_0)$ the Thom class of the quaternion n-plane bundle over $\partial Y/\boldsymbol{Sp}(1)$. Then $j^*(\tau) = \lambda t$ for non-zero $\lambda \in \boldsymbol{Q}$. Since $j^*(w\tau) = j_2^*(w) j^*(\tau) = \lambda j_2^*(w) t$ for each $w \in H^*(X)$, we see that $j_2^*: H^*(X) \to H^*(X_2)$ is surjective. In addition, $j_2^*(w) = 0$ if and only if $e(\pi)^n w = 0$ for $w \in H^*(X)$. Then we can show that $\{j_2^*(u^p v^q);\ 0 \leqq p \leqq a-n,\ 0 \leqq q \leqq b\}$ are linearly independent in the graded module $H^*(X_2) \cong H^*(X)/\ker j_2^*$. On the other hand, we obtain

$$\operatorname{rank} H^*(X_2) = \operatorname{rank} H^*(X) - \operatorname{rank} H^*(X_1) = (a+1-n)(b+1) .$$

Therefore the set $\{u_2^p v_2^q;\ 0 \leqq p \leqq a-n,\ 0 \leqq q \leqq b\}$ is an additive base of the graded module $H^*(X_2)$.

Suppose first $e(\pi) = u$, i.e. $\beta = 0$. Then $j_2^*(u^{a-n+1}) = 0$, and hence $H^*(X_2) \cong \boldsymbol{Q}[u_2, v_2]/(u_2^{a-n+1}, v_2^{b+1})$. Therefore $\partial Y/\boldsymbol{Sp}(1) \sim \boldsymbol{P}_{a-n}(\boldsymbol{H}) \times \boldsymbol{P}_b(\boldsymbol{H})$.

Suppose next that $b \leqq a+1-n$ and $e(\pi) = u+v$, i.e. $\beta \neq 0$. We see that

$$e(\pi)^n \sum_{i=0}^{b} (-1)^i \binom{a+1}{i} (u+v)^{a+1-n-i} v^i = ((u+v)-v)^{a+1} = 0 ,$$

hence we obtain

$$H^*(\partial Y/\boldsymbol{Sp}(1)) \cong H^*(X_2) \cong \boldsymbol{Q}[x, y] / (y^{b+1}, \sum_{i=0}^{b} (-1)^i \binom{a+1}{i} x^{a+1-n-i} y^i) ,$$

where $x = u_2 + v_2$ and $y = v_2$.

Here we complete the proof of Theorem 2.2.

6. Construction

We regard D^{4n} as the unit disk of the quaternion n-space $\boldsymbol{H}^n$ with the right scalar multiplication and the left $\boldsymbol{Sp}(n)$ action. Let Y be a compact orientable smooth $\boldsymbol{Sp}(1)$ manifold such that the $\boldsymbol{Sp}(1)$ action is free on the non-empty boundary ∂Y. By the diagonal action, $\boldsymbol{Sp}(1)$ acts freely on the boundary $\partial(D^{4n} \times Y)$. Here we consider the cohomology ring of the orbit manifold $X = \partial(D^{4n} \times Y)/\boldsymbol{Sp}(1)$ on which $\boldsymbol{Sp}(n)$ acts naturally.

Suppose that $\dim Y = 4d+4$, $Y \sim \boldsymbol{P}_b(\boldsymbol{H})$, $1 \leqq b \leqq d \leqq n-2$, and $F \sim \boldsymbol{P}_b(\boldsymbol{C})$ or $F \sim \boldsymbol{P}_b(\boldsymbol{H})$, where F denotes the fixed point set of the restricted $\boldsymbol{U}(1)$ action on Y. Moreover suppose that $\iota^*: H^4(Y) \cong H^4(\partial Y)$, where ι is an inclusion. Put $c = d - b$. In addition, we suppose that the graded algebra $H^*(\partial Y/\boldsymbol{Sp}(1))$

is isomorphic to one of the following:

$$(1)\qquad \boldsymbol{Q}[x, y]/(x^{c+1}, y^{b+1}),$$

$$(2)\qquad \boldsymbol{Q}[x, y]/\left(y^{b+1}, \sum_{i=0}^{b}(-1)^i \binom{n+c+1}{i} x^{c+1-i}y^i\right);\ b\leqq c+1,$$

where $\deg x=\deg y=4$, and x is the Euler class of the principal $\boldsymbol{Sp}(1)$ bundle $\partial Y\to\partial Y/\boldsymbol{Sp}(1)$.

Put $X_1=(S^{4n-1}\times Y)/\boldsymbol{Sp}(1)$, $X_2=(D^{4n}\times\partial Y)/\boldsymbol{Sp}(1)$ and $X_0=X_1\cap X_2=(S^{4n-1}\times\partial Y)/\boldsymbol{Sp}(1)$. Then $X=X_1\cup X_2$. Let π: $\partial(D^{4n}\times Y)\to X$ be the projection of the principal $\boldsymbol{Sp}(1)$ bundle. Let us denote by π_s the projection of the restricted principal $\boldsymbol{Sp}(1)$ bundle over X_s. Let j_s: $X_s\to X$ and i_s: $X_0\to X_s$ be the inclusions. Let p: $X_2\to\partial Y/\boldsymbol{Sp}(1)$ be the natural projection of $4n$-disk bundle, and put $p_0=p\,|\,X_0$: $X_0\to\partial Y/\boldsymbol{Sp}(1)$.

Since $d\leqq n-2$, we see that $H^*(X_0)$ is freely generated by 1, σ as an $H^*(\partial Y/\boldsymbol{Sp}(1))$ module for an element $\sigma\in H^{4n-1}(X_0)$ and i_2^*: $H^*(X_2)\to H^*(X_0)$ is injective. Put $x_0=p_0^*(x)$, $y_0=p_0^*(y)$, $x_2=p^*(x)$ and $y_2=p^*(y)$. Then $x_0=e(\pi_0)$ and $x_2=e(\pi_2)$, the Euler classes of the principal $\boldsymbol{Sp}(1)$ bundles.

By the fibration: $Y\to X_1\to \boldsymbol{P}_{n-1}(\boldsymbol{H})$ and the assumption that $F\sim\boldsymbol{P}_b(\boldsymbol{C})$ or $F\sim\boldsymbol{P}_b(\boldsymbol{H})$ and $Y\sim\boldsymbol{P}_b(\boldsymbol{H})$, we see that by Lemma 1.1,

$$H^*(X_1)=\boldsymbol{Q}[x_1, y_1]/(x_1^n, y_1^{b+1});\ \deg x_1=\deg y_1=4,$$

where $x_1=e(\pi_1)$, the Euler class of the principal $\boldsymbol{Sp}(1)$ bundle.

Consider the Mayer–Vietoris sequence of a triad $(X;\, X_1, X_2)$:

$$\xrightarrow{i^*} H^{r-1}(X_0)\xrightarrow{\Delta^*} H^r(X)\xrightarrow{j^*} H^r(X_1)\oplus H^r(X_2)\xrightarrow{i^*} H^r(X_0)\xrightarrow{\Delta^*}$$

where $j^*(a)=(j_1^*(a), j_2^*(a))$ and $i^*(b_1, b_2)=i_1^*(b_1)-i_2^*(b_2)$. We see that $H^r(X)=0$ for each $r\not\equiv 0 \pmod 4$ and there is the following short exact sequence for each k:

$$(*)\qquad 0\to H^{4k-1}(X_0)\xrightarrow{\Delta^*} H^{4k}(X)\xrightarrow{j_1^*} H^{4k}(X_1)\to 0\,.$$

Notice that $\dim X=4(n+d)$ and

$$(**)\qquad j_1^*\colon H^{4k}(X)\cong H^{4k}(X_1)\quad \text{for } k<n\,.$$

Let u, v be elements of $H^4(X)$ such that $j_1^*(u)=x_1$, $j_1^*(v)=y_1$. We see that $u=e(\pi)$, the Euler class of the principal $\boldsymbol{Sp}(1)$ bundle. Moreover, we see that $v^{b+1}=0$ by (**) and the assumption $b\leqq n-2$. Since $j_1^*(u^{n-1}v^b)\neq 0$, there is an element $z\in H^{4c+4}(X)$ such that $u^{n-1}v^bz\neq 0$, by the Poincaré duality. Then we see that $u^{n+c}v^b\neq 0$, by (**) and the fact $v^{b+1}=0$. In particular, we obtain $u^n\neq 0$. Looking at the exact sequence (*), we can assume that $u^n=\Delta^*(\sigma)$.

We can express $i_1^*(y_1)=\lambda x_0+\mu y_0$; λ, $\mu\in\boldsymbol{Q}$. Since $\pi_1^*(y_1)\neq 0$, we see that

$\mu \neq 0$ by the assumption $\iota^*: H^4(Y) \cong H^4(\partial Y)$. Then

$$\Delta^*(\sigma x_0^p y_0^q) = \mu^{-q} u^{n+p}(v-\lambda u)^q$$

because $\Delta^*(\sigma j_0^*(w)) = \Delta^*(\sigma) w$ for each $w \in H^*(X)$. Looking at the exact sequence (*), we see that the graded algebra $H^*(X)$ is generated by two elements u, v and rank $H^*(X) = (n+c+1)(b+1)$.

In the expression $i_1^*(y_1) = \lambda x_0 + \mu y_0$, if $\lambda = 0$ then we see that $u^{n+c+1} = 0$ in the case (1) and $(u-\mu^{-1}v)^{n+c+1} = 0$ in the case (2), and hence $X \sim \boldsymbol{P}_{n+c}(\boldsymbol{H}) \times \boldsymbol{P}_b(\boldsymbol{H})$.

Since $i_2^*: H^*(X_2) \to H^*(X_0)$ is injective, we see that $j_2^*(v) = \lambda x_2 + \mu y_2$, and hence $(\lambda x_2 + \mu y_2)^{b+1} = 0$. Then we obtain $\lambda = 0$ in the case (1), because $H^*(X_2) \cong \boldsymbol{Q}[x_2, y_2]/(x_2^{c+1}, y_2^{b+1})$.

Next we consider the case (2). We obtain a relation

$$(\gamma x_2 + y_2)^{b+1} \in I = \left(y_2^{b+1}, \textstyle\sum_{i=0}^b (-1)^i \binom{n+c+1}{i} x_2^{c+1-i} y_2^i\right),$$

where $\gamma = \lambda \mu^{-1}$. We see that $\gamma = 0$ for the case $b < c$ or $b = c \geqq 2$. Suppose $b = c+1$. Looking at the relation $(\gamma x_2 + y_2)^{c+2} \in I$, we obtain $\gamma = 0$ or

$$(A_k) \qquad \binom{c+2}{k} - (-\gamma)^k \binom{n+c+1}{k} + (n+c+1)(-\gamma)^k \binom{n+c+1}{k-1}$$
$$-(c+2)(-\gamma)^{k-1} \binom{n+c+1}{k-1} = 0$$

for each $k = 2, 3, \cdots, c+1$. Suppose $\gamma \neq 0$ and $c \geqq 2$. Then we get a contradiction from (A_2) and (A_3). Hence we obtain $\gamma = 0$ for $c \geqq 2$. Suppose $\gamma \neq 0$ and $c = 1$. We see that the quadratic equation (A_2) has a rational solution γ if and only if $3n(n+2)$ is a square number.

Summing up the above arguments, we obtain a partial converse of Theorem 2.1 (iii).

REMARK. For a positive integer n, $3n(n+2)$ is a square number if and only if $n+1$ is one of the following:

$$\textstyle\sum_{i \geqq 0} \binom{k}{2i} 2^{k-2i} 3^i; \; k = 1, 2, 3, \cdots$$

7. Concluding remark

By parallel arguments, we obtain the following result which is a generalization of a theorem [7].

Theorem 7.1. *Let X be a closed orientable manifold on which $\boldsymbol{SU}(n)$ acts smoothly and non-trivially. Suppose $X \sim \boldsymbol{P}_a(\boldsymbol{C}) \times \boldsymbol{P}_b(\boldsymbol{C})$; $a \geqq b \geqq 1$, $a+b \leqq 2n-2$ and $n \geqq 7$. Then there are three cases:*

(0) $a=n-1$ *and* $X\cong \boldsymbol{P}_{n-1}(\boldsymbol{C})\times Y_0$, *where* Y_0 *is a closed orientable manifold such that* $Y_0\sim \boldsymbol{P}_b(\boldsymbol{C})$, *and* $\boldsymbol{SU}(n)$ *acts naturally on* $\boldsymbol{P}_{n-1}(\boldsymbol{C})$ *and trivially on* Y_0,

(i) $a=b=n-1$ *and* $X\cong \boldsymbol{P}_{n-1}(\boldsymbol{C})\times \boldsymbol{P}_{n-1}(\boldsymbol{C})$ *with the diagonal* $\boldsymbol{SU}(n)$ *action,*

(ii) $a\geqq n$ *and* $X\cong \partial(D^{2n}\times Y_1)/\boldsymbol{U}(1)$, *where* Y_1 *is a compact orientable* $\boldsymbol{U}(1)$ *manifold such that* $\dim Y_1=2(a+b+1-n)$ *and* $Y_1\sim \boldsymbol{P}_b(\boldsymbol{C})$, $\boldsymbol{U}(1)$ *acts as right scalar multiplication on* D^{2n}, *the unit disk of* $\boldsymbol{C}^n$, *and* $\boldsymbol{SU}(n)$ *acts naturally on* D^{2n} *and trivially on* Y_1. *In addition, the* $\boldsymbol{U}(1)$ *action on the boundary* ∂Y_1 *is free and the fixed point set of the* $\boldsymbol{U}(1)$ *action on* Y_1 *is* $\sim \boldsymbol{P}_b(\boldsymbol{C})$.

Theorem 7.2. *In the case* (ii) *of Theorem* 7.1, *the cohomology ring* $H^*(\partial Y_1/\boldsymbol{U}(1))$ *is isomorphic to one of the following*:

$$(1) \qquad \boldsymbol{Q}[x, y]/(x^{a+1-n}, y^{b+1}),$$

$$(2) \qquad \boldsymbol{Q}[x, y]/(y^{b+1}, \textstyle\sum_{i=0}^{b}(-1)^i\binom{a+1}{i}x^{a+1-n-i}y^i);\ b\leqq a+1-n,$$

where $\deg x=\deg y=2$, *and* x *is the Euler class of the principal* $\boldsymbol{U}(1)$ *bundle* $\partial Y_1 \to \partial Y_1/\boldsymbol{U}(1)$.

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